International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 3 (Oct.), 2022

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 - 5708

International Journal of Mathematical Analysis and Modelling (Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 3 (October), 2022

ISSN (Print): 2682 – 5694

ISSN (Online): 2682 - 5708

Disclaimer: The articles published herein do not necessarily reflect the opinion of the editorial board of this journal or the Nigerian Society for Mathematical Biology (NSMB). The editorial board or the NSMB do not accept any legal responsibility for any errors or omissions that may be made in this volume of the International Journal of Mathematical Analysis and Modelling (Formerly Journal of the Nigerian Society for Mathematical Biology).

A publication of the Nigerian Society for Mathematical Biology

©NSMB

Editorial Board

Daniel Okuonghae, Ph.D. — Editor-in-Chief
Department of Mathematics
University of Benin
Benin City, Nigeria.
daniel.okuonghae@uniben.edu
danny.okuonghae@ccc.oxon.org
mcsb@uniben.edu

Editors

Samuel O. Okoya, Ph.D.	Obafemi Awolowo University, Ile-Ife, Nigeria
Godwin C.E. Mbah, Ph.D.	University of Nigeria, Nsukka, Nigeria
Ninuola Akinwande, Ph.D.	Federal University of Technology, Minna, Nigeria
A.R. Kimbir, Ph.D.	Federal University of Agriculture, Makurdi, Nigeria
Olufemi A. Adesina, Ph.D.	Obafemi Awolowo University, Ile-Ife, Nigeria
Philip O. Olanrewaju, Ph.D.	Federal University, Wukari, Nigeria
Augustine Iduseri, Ph.D.	University of Benin, Benin City, Nigeria
Sirajo Abdulrahman, Ph.D.	Federal University, Birnin Kebbi, Nigeria
Andrew Omame, Ph.D.	Federal University of Technology, Owerri, Nigeria
Bolarinwa Bolaji, Ph.D.	Kogi State University, Ayingba, Nigeria
Dennis Agbebaku, Ph.D.	University of Nigeria, Nsukka, Nigeria
George Akuchu, Ph.D.	University of Nigeria, Nsukka, Nigeria
Robert Okuonghae, Ph.D.	University of Benin, Benin City, Nigeria
Monday N. O. Ikhile, Ph.D.	University of Benin, Benin City, Nigeria

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 3, October 2022, pages 1 - 12

Iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mapping with application to monotone variational inclusion problems in Hilbert spaces

E. Ekuma-Okereke*, **F.M. Okoro**†‡, and **C.E. Abhulimen**†§

Abstract

Motivated by the recent result in [1], it is our aim in this paper to introduce the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex subset of real Hilbert spaces. We prove strong convergence theorem for it. Furthermore, we apply our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. In addition, we provide a numerical example to illustrate the effectiveness of our procedure. It is intended that our procedure complement our motivated result cited in the literature.

Keywords: fixed points; strictly pseudo-contractive mappings; modified Mann-type iteration; inertial term; strong convergence

1 Introduction

Throughout, we take C to represent a nonempty, closed and convex subset of real Hilbert space H with the inner product and induced norm denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. The metric projection mapping of H onto C denoted by $P_C(x)$ is defined by

$$\|x - P_C(x)\| = \inf\{\|x - z\| : z \in C\}, \tag{1}$$

and the metric projection holds true if and only if it satisfies the variational inequality

$$\langle x - P_C(x), z - P_C(x)\rangle \leq 0. \tag{2}$$

Remark 1.1 *Using this characterization, we get that a metric projection $P_C(x)$ is nonexpansive, that is $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2, \forall x, y \in X$.*

Definition 1.1 *A mapping T defined on a closed and convex subset $C \subset H$ is called*

(a) Lipschitzian if there exists $L \geq 0$ such that

$$\|Tx - Ty\|^2 \leq L\|x - y\|^2, \forall x, y \in C, \tag{3}$$

*Corresponding Author. Department of Mathematics, Federal University of Petroleum Resources, Effurun, P.M.B 1221, Nigeria; E-mail: ekuma.okereke@fupre.edu.ng

†Department of Mathematics, Ambrose Alli University, Ekpoma, Nigeria.

‡E-mail: moibi1@yahoo.com

§Email: cletusabhulimen@yahoo.co.uk

(b) *quasi nonexpansive if*

$$\|Tx - p\|^2 \leq \|x - p\|^2, \forall x \in C, p \in Fix(T). \tag{4}$$

(c) *strictly pseudocontractive [3] if there exists a constant $k \in [0, 1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in C. \tag{5}$$

or equivalently, if there exists a constant $\lambda > 0$ such that

$$\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in C. \tag{6}$$

(d) *demicontraction (or k-demicontractive) (see [1]) if there exists a constant $k \in [0, 1)$ and $Fix(T) \neq \emptyset$ such that*

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in C, p \in Fix(T). \tag{7}$$

Remark 1.2 *The mapping T is called a contraction if $0 \leq L < 1$. It is called nonexpansive if $L = 1$. It is demiclosed at a point $p \in C$ if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to $\bar{x} \in C(x_n \rightharpoonup \bar{x})$ and Tx_n converges strongly to p $(Tx_n \to p)$ and $T\bar{x} = p$.*

Numerous iterative procedures for the approximation of fixed points of nonexpansive mappings and its generalization have been constructed and studied by various authors (see [4], [10], [15] and the references therein) in the framework of Hilbert and Banach spaces. Notable among them are the Mann iteration procedure (see [13]), Ishikawa iteration procedure (see [9]), Halpern iteration procedure (see [8])and Krasnoselskij iteration (see [12]). One major setback in Mann iteration defined as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Tx_n, \tag{8}$$

is the weak convergence it only guarantees. However, in a bid to achieve strong convergence from Mann iteration, some authors on one hand, have imposed certain perceived stringent conditions on the $Fix(T)$ (see [19]) and as pointed out by [2], the unnecessary assumption like $\lim_{n\to\infty} \beta_n = 0$ used by some author to achieve convergence in their work, on the other hand. It is noteworthy to mention that it is only Halpern iteration procedure that satisfies this condition. Besides, in contributing to the volume of work on iterative procedures with inertial terms for the approximation of fixed points of various mappings and its applications, many authors have recently constructed and studied new procedures in this directions (see [16], [6], [7] [5] [14] and the references therein).

Very recently and in [1], Agbebaku, Nwokoro, Osilike, Chima and Onah, studied "The Iterative Algorithm with Inertial and Error Terms for Fixed Points of Strictly Pseudo-contractive Mappings and Zeros of Inverse Strongly Monotone Operators". They proved the following theorem:

Theorem 1.1 *Let H represent a real Hilbert space. Let $T : H \to H$, represent a $k-$ strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Choose $x_0, x_1 \in H$ arbitrarily and define the sequence $\{x_n\}$ generated from these as follows*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \beta_n y_n + \eta_n Ty_n + e_n, n \geq 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\eta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$, $\{\theta_n\}_{n=1}^{\infty}$ is a positive sequence and e_n is an error sequence satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\alpha_n + \beta_n + \eta_n = 1$, $\beta_n \geq k, \forall n \geq 1$, $\liminf_{n\to\infty} \eta_n[\beta_n - (1-\alpha_n)k] > 0$,

(iii) either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty}, \dfrac{\|e_n\|}{\alpha_n} = 0$,

(iv) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise} . \end{cases}$$

Then algorithm (9) converges strongly to metric projection of H onto $Fix(T)$.

It is important to note that their result extends the recent results of Shehu, Iyiola and Ogbuisi [16] from the class of nonexpansive mappings to the much more general class of strictly pseudocontractive mappings in Hilbert spaces.

Remark 1.3 *Observe that their map T as used in their theorem above is a self map on H. If on the contrary, the domain of their map T becomes a nonempty, closed and convex subset C of H (as it is noted to be the case in many applications) and T is a self map on C, then we can say that the iterative procedure of (9) may fail to be well-defined.*

Thus motivated by the result of Agbebaku et al. [1], it is our aim in this paper to introduce the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex set in real Hilbert spaces. We study strong convergence theorem for it. Furthermore, we apply our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. In addition, we provide a numerical example to illustrate the effectiveness of our procedure. It is intended that our procedure complement our motivated result cited in the literature.

2 Preliminaries

Lemma 2.1 *Let H be a real Hilbert space. Given $x, y \in H$, then the inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{10}$$

Lemma 2.2 *cf.([10]) Let H be a real Hilbert space. For all $x_i \in H, \alpha_i \in [0,1], i = 0,1,2,...,n$ satisfying $\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_n = 1$, then we have the equality condition as follows:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x\|^2 - \sum_{0\leq i,j\leq n} \|x_i - x_j\|^2. \tag{11}$$

Lemma 2.3 *([18]) Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$, represent a strictly pseudo-contractive mapping. Then*

(i) $(1-T)$ *is demiclosed at* 0,

(ii) $Fix(T)$ *is closed and convex subset of* C.

Lemma 2.4 *([14]) Let $\{a_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{e_n\}_{n=1}^{\infty}$ be sequences contain in $[0,\infty)$, $\{b_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{d_n\}_{n=1}^{\infty}$ be a sequence contain in $(-\infty_{n=1}, +\infty)$ such that*

$$a_{n+1} \leq [1 - b_n + c_n]a_n + d_n + e_n, n \geq 1. \tag{12}$$

If $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} e_n \infty$, *then* *(i) if* $d_n \leq Mb_n$ *for some* $M \geq 0$*, then* $\{a_n\}_{n=1}^{\infty}$ *is bounded.*

(ii) if $\lim_{n\to\infty} b_n = 0, \sum_{n=1}^{\infty} c_n = \infty$ *and* $\limsup_{n\to\infty} \dfrac{d_n}{b_n} \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 *([17]) Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0 \tag{13}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in* $(0,1)$*,* $\{\delta_n\}$ *is a sequence in* $(-\infty,+\infty)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,*

(ii) $\lim_{n\to\infty} sup\delta_n \leq 0$*. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 *([11]) Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of real numbers such that there exists a non-decreasing subsequence* $\{n_i\}$ *of* $\{n\}$ *that is* $a_{n_i} < a_{n_{i+1}}$ $\forall i \in N$*. Then there exists a non-decreasing subsequence* $\{m_k\} \subset N$ *such that* $m_k \to \infty$ *and the following properties are satisfied for all (sufficiently large number)* $k \in N$*:* $a_{m_k} \leq a_{m_{k+1}}$ *and* $a_k \leq a_{m_{k+1}}$*. In fact,* $m_k = max\{j \leq k : a_j \leq a_{j+1}\}$*.*

3 Main results

Theorem 3.1 *Let* C *represent a non-empty, closed and convex subset of a real Hilbert space* H*. Let* $T: C \to C$*, represent a strictly pseudo-contractive mapping with* $Fix(T) \neq \emptyset$*. Let the sequence* $\{x_n\} \in C$ *be generated by choosing* $x_1 < x_0 \in C$ *and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1-\alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1-\beta_n)Ty_n), n \geq 1, \end{cases} \tag{14}$$

where $\{\beta_n\}_{n=1}^{\infty} \in (k,1)$ *for any* $k \in [0,1)$*,* $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$*,* $\{\theta_n\}_{n=1}^{\infty}$ *is a positive sequence satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1-\beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \text{whenever } x_n \neq x_{n-1} \\ \tau, & \text{otherwise .} \end{cases}$$

Then $\{x_n\}$ *converges strongly to metric projection of* C *onto* $Fix(T)$*.*

Proof. We first show that the sequence $\{x_n\}$ generated by the procedure (14) is bounded. To do this, we let $p \in Fix(T)$. In view of Lemma 2.2 and the property of T, we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|P_C(\beta_n y_n + (1-\beta_n)Ty_n) - p\|^2 \\ &\leq \|\beta_n y_n + (1-\beta_n)Ty_n - p\|^2 \\ &= \|\beta_n(y_n - p) + (1-\beta_n)(Ty_n - p)\|^2 \\ &\leq \beta_n\|y_n - p\|^2 + (1-\beta_n)\|Ty_n - p\|^2 - \beta_n(1-\beta_n)\|Ty_n - y_n\|^2 \\ &\leq \beta_n\|y_n - p\|^2 + (1-\beta_n)[\|y_n - p\|^2 + k\|y_n - Ty_n\|^2] - \beta_n(1-\beta_n)\|Ty_n - y_n\|^2 \\ &\leq \|y_n - p\|^2 - (1-\beta_n)[\beta_n - k]\|y_n - Ty_n\|^2. \end{aligned} \tag{15}$$

Furthermore,

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_0 + (1-\alpha_n) z_n - p\|^2 \\
&\leq \|\alpha_n x_0 + (1-\alpha_n) z_n - p\|^2 \\
&= \|\alpha_n (x_0 - p) + (1-\alpha_n)(z_n - p)\|^2 \\
&\leq \alpha_n \|x_0 - p\|^2 + (1-\alpha_n)\|z_n - p\| - \alpha_n(1-\alpha_n)\|x_0 - z_n\|^2. \qquad (16)
\end{aligned}$$

Substituting (16) into (15), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq \alpha_n \|x_0 - p\|^2 &+ (1-\alpha_n)\|z_n - p\| - \alpha_n(1-\alpha_n)\|x_0 - z_n\|^2 \\
&- (1-\beta_n)[\beta_n - k]\|y_n - Ty_n\|^2. \qquad (17)
\end{aligned}$$

Besides

$$\begin{aligned}
\|z_n - p\|^2 &= \|x_n + \tau_n(x_n - x_{n-1}) - p\|^2 \\
&= \|(x_n - p) + \tau_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\langle \tau_n(x_n - x_{n-1}, x_n - p)\rangle \\
&\leq \|x_n - p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|.\|x_n - p\|^2 + 2\tau_n\|x_n - x_{n-1}\| \\
&= [1 + 2\tau_n\|x_n - x_{n-1}\|].\|x_n - p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|. \qquad (18)
\end{aligned}$$

Substituting (18) into (17), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_0 - p\|^2 \\
&+ (1-\alpha_n)\{\alpha_n[1 + 2\tau_n\|x_n - x_{n-1}\|]\|x_n - p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|\} \\
&- \alpha_n(1-\alpha_n)\|x_0 - z_n\|^2 - (1-\beta_n)[\beta_n - k]\|y_n - Ty_n\|^2 \\
&\leq \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\{[1 + 2\tau_n\|x_n - x_{n-1}\|].\|x_n - p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 \\
&+ 2\tau_n\|x_n - x_{n-1}\|\} \\
&\leq \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)2\tau_n\|x_n - x_{n-1}\|.\|x_n - p\|^2 \\
&+ (1-\alpha_n)\tau_n^2\|x_n - x_{n-1}\|^2 + (1-\alpha_n)2\tau_n\|x_n - x_{n-1}\| \\
&\leq \alpha_n\|x_0 - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + 2(1-\alpha_n)\alpha_n\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\|.\|x_n - p\|^2 \\
&+ (1-\alpha_n)\alpha_n^2\frac{\tau_n^2}{\alpha_n^2}\|x_n - x_{n-1}\|^2 + 2(1-\alpha_n)\alpha_n\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&= [(1-\alpha_n) + \alpha_n\{2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\|\}]\|x_n - p\|^2 \\
&+ \alpha_n\{\|x_0 - p\|^2 + 2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n - x_{n-1}\| + (1-\alpha_n)\alpha_n\frac{\tau_n^2}{\alpha_n^2}\|x_n - x_{n-1}\|^2\} \\
&\leq [(1-\alpha_n) + \alpha_n\{2(1-\alpha_n)\frac{\theta_n}{\alpha_n}\}]\|x_n - p\|^2 \\
&+ \alpha_n\{\|x_0 - p\|^2 + 2(1-\alpha_n)\frac{\theta_n}{\alpha_n} + (1-\alpha_n)\alpha_n\frac{\theta_n^2}{\alpha_n^2}\}. \qquad (19)
\end{aligned}$$

By our hypothesis, we can find a number $M > 0$ such that

$$2\{\|x_0 - p\|^2 + 2(1-\alpha_n)\frac{\theta_n}{\alpha_n} + (1-\alpha_n)\alpha_n\frac{\theta_n^2}{\alpha_n^2}\} \leq M, \text{ for all } n \geq 1.$$

Also there exists $n(\epsilon) > 0$ such that

$$2(1-\alpha_n)\frac{\theta_n}{\alpha_n} \leq \frac{1}{2}, \forall n \geq n(\epsilon).$$

Hence, (19) becomes

$$\|x_{n+1}-p\|^2 \leq (1-\eta_n)\|x_n-p\|^2+\eta_n M, \forall n \geq 1, \text{ where } \eta_n=\frac{\alpha_n}{2}. \tag{20}$$

This shows that $\{\|x_{n+1}-p\|\}_{n=1}^{\infty}$ is bounded. In view of Lemma 2.4, we get that the sequence $\{x_n\}$ is bounded.

We next show that the sequence $\{x_n\}$ generated by the procedure (14) converges to the metric projection mapping of C onto $Fix(T)$. To do this, we let $p=P_{Fix(T)}(x_0)$. In view of the construction of procedure (14), Lemma 2.1 and the approach used to get (19) we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|y_n-p\|^2=\|\alpha_n x_0+(1-\alpha_n)z_n-p\|^2\\ &\leq (1-\alpha_n)\|z_n-p\|^2+2\alpha_n\langle x_0-p, \alpha_n(x_0-p)+(1-\alpha_n)(z_n-p)\rangle.\end{aligned} \tag{21}$$

Observe from the definition of z_n, that

$$\|z_n-p\|^2 \leq \|x_n-p\|^2+2\tau_n\|x_n-x_{n-1}\|^2+\tau_n[\|x_n-p\|^2-\|x_{n-1}-p\|^2]. \tag{22}$$

So that (21) becomes

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq \|y_n-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2+2(1-\alpha_n)\tau_n\|x_n-x_{n-1}\|^2+(1-\alpha_n)\tau_n[\|x_n-p\|^2\\ &-\|x_{n-1}-p\|^2]+2\alpha_n^2\|x_0-p\|^2+\alpha_n(1-\alpha_n)\langle x_0-p, z_n-p\rangle\\ &\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n(2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|^2\\ &+(1-\alpha_n)\frac{\tau_n}{\alpha_n}[\|x_n-p\|^2-\|x_{n-1}-p\|^2]+2\alpha_n\|x_0-p\|^2\\ &+(1-\alpha_n)\langle x_0-p, z_n-p\rangle).\end{aligned} \tag{23}$$

Next, we consider two cases to complete the proof of our Theorem.

Case 1. If we suppose to find $n_0 \in N$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is decreasing, then we can get $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is convergent and hence

$$\|x_n-p\|-\|x_{n+1}-p\| \to 0, n \to \infty. \tag{24}$$

In view of (15), (24) and conditions (i) and (ii) of our hypothesis, we get

$$\begin{aligned}\|y_n-p\|^2-\|Ty_n-p\|^2 &= \|y_n-p\|^2-\|x_{n+1}-p\|^2+\|x_{n+1}-p\|^2-\|Ty_n-p\|^2\\ &\leq \|y_n-p\|^2-\|x_{n+1}-p\|^2+\|y_n-p\|^2-(1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2\\ &-\|Ty_n-p\|^2\end{aligned}$$

$$(1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2 \leq \alpha_n(\|x_0-p\|^2-\|x_n-p\|^2)+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \to 0, n \to \infty. \tag{25}$$

Thus,

$$\|y_n-Ty_n\| \to 0, n \to \infty. \tag{26}$$

Now from the definition of z_n and condition (i), we get

$$\begin{aligned}\|z_n-x_n\|^2 &= \|x_n+\tau_n(x_n-x_{n-1})-x_n\|^2\\ &= \|\tau_n(x_n-x_{n-1})\|^2\\ &\leq \tau_n\|x_n-x_{n-1}\|^2\\ &\leq \alpha_n\frac{\theta_n}{\alpha_n} \to 0, n \to \infty.\end{aligned}$$

Thus,

$$\|z_n-x_n\| \to 0, n \to \infty. \tag{27}$$

Moreover, in view of condition (i) and (27), we get

$$\|y_n - x_n\|^2 \leq \alpha_n \|x_0 - x_n\|^2 + (1-\alpha_n)\|z_n - x_n\| - \alpha_n(1-\alpha_n)\|x_0 - z_n\|^2 \to 0, n \to \infty.$$

Thus,

$$\|y_n - x_n\| \to 0, n \to \infty. \tag{28}$$

Since $\{z_n\}$ is bounded, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in C$ and

$$\begin{aligned} \limsup_{n\to\infty} \langle x_0 - p, z_n - p\rangle &= \limsup_{i\to\infty} \langle x_0 - p, z_{n_i} - p\rangle \\ &= \langle x_0 - p, z - p\rangle. \end{aligned} \tag{29}$$

Now, in view of (27) and (28), it follows that $y_{n_i} \rightharpoonup z \in C$. From (26) and Lemma 2.3 , we get $z \in Fix(T)$. By invoking (23), (29) and Lemma 2.5 we find that $\lim_{n\to\infty} \|x_n - p\| = 0$. Consequently, $x_n \to p = P_{Fix(T)}(x_0)$.

Case 2. Suppose we find a subsequence n_i of $n \neq n_0$ such that $\{\|x_n - p\|\}_{n=n_i}^{\infty}$ is decreasing, that is

$$\|x_{n_i} - p\| \leq \|x_{n_i+1} - p\|, \forall i \in N.$$

Now invoking Lemma 2.6, and following the same line of argument as in Case 1, we find that $x_n \to p = P_{Fix(T)}(x_0)$.

Corollary 3.1 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$, represent a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1-\beta_n) T y_n), n \geq 1, \end{cases} \tag{30}$$

where $\{\beta_n\}_{n=1}^{\infty} \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\theta_n\}_{n=1}^{\infty}$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1-\beta_n)\beta_n] > 0,$

(iii) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise} . \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(T)$.

4 Application

4.1 Monotone variational inclusion problem (MVIP)

By using our main result, we can approximate a solution of monotone variational inclusion problem (shortly, MVIP) for the sum of two mappings in real Hilbert spaces. Let $B : C \to 2^H$ be a set-valued mapping and let mapping $A : C \to H$ be a single-valued mapping. The MVIP is to find $x \in C$ such that

$$0 \in B(x) + A(x) \tag{31}$$

The solution set of (31) is denoted $MVIP(B, A)$.

Definition 4.1 *A mapping $A : dom(A) \subset C \to 2^H$ is*

(a) Monotone if for all $x, y \in dom(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{32}$$

(b) Maximal monotone if it is monotone and is not subset of another maximal monotone operator.

Now, given any set-valued mapping $A : dom(A) \subset C \to 2^H$, the resolvent and anti-resolvent of A is the mapping $A : C \to 2^H$ defined by $J^A_\lambda = (I + \lambda A)^{-1}$ and $H^A_\lambda = (I - \lambda A)$ respectively and I is identity.

Definition 4.2 *A mapping $A : C \to H$ is inverse strongly monotone (ism) or (α−ism) if there exists $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C. \tag{33}$$

From this inequality (33) and that of (6), we can easily obtain that a mapping A is α-ism if and only if the mapping $T = I - A$ is $(1 - \alpha)$-strictly pseudo-contractive. Hence, the fixed point for this class of mapping is connected with the zero of α-ism mapping. That is $Fix(T) = \{x \in C : Tx = x\}$ is same as $A^{-1}0 = \{x \in C : Ax = 0\}$. Now let $B : dom(A) \subset C \to 2^H$ be a maximal monotone and $A : dom(A) \subset C \to 2^H$ be an α-inverse strongly monotone, then $J^B_\lambda = (I+\lambda B)^{-1}$ is known to be single-valued and nonexpansive and consequently $J^B_\lambda(I - \lambda A)$ is nonexpansive and hence $(1-\alpha)$-strictly pseudo-contractive such that $x \in (B+A)^{-1}0 \iff Fix(J^B_\lambda(I-\lambda A))$. Following from these facts, we have

Theorem 4.1 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$, represent an α-inverse strongly monotone mapping and let $B : dom(A) \in C \to 2^H$, represent a maximal monotone set-valued mapping with $Fix(J^B_\lambda(I - \lambda A)) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1 - \alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1 - \beta_n)J^B_\lambda(I - \lambda A)y_n), n \geq 1, \end{cases} \tag{34}$$

where $\{\beta_n\}^\infty_{n=1} \in (k, 1)$ for any $k \in [0, 1)$, $\{\alpha_n\}^\infty_{n=1} \in (0, 1)$, $\{\theta_n\}^\infty_{n=1}$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum^\infty_{n=1} \alpha_n = \infty, \sum^\infty_{n=1} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1 - \beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0, 1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise} . \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(J^B_\lambda(I - \lambda A))$.

4.2 Variational inequality problem (VIP)

Similarly, using our main result, we can approximate a solution of variational inequality problem (shortly, VIP) for a single-valued mapping in real Hilbert spaces. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let C be a non-empty, closed and convex subset of $dom(A)$. Now the variational inequality problem corresponding to A is to find $x \in C$ such that

$$\langle Ax, z - x\rangle \geq 0, \forall z \in C. \tag{35}$$

The solution set of (35) is denoted by $VI(C, A)$. Let $H^A_\lambda = (I - \lambda A)$. If C is a non-empty, closed and convex subset of $dom(A)$, then following (2), $VI(C, A) = Fix(Proj_C \circ H^A_\lambda)$. Since, $Proj_C$ is nonexpansive, it is also a strictly pseudo-contractive. Thus, $Proj_C \circ H^A_\lambda$ is strictly pseudo-contractive. Following from these facts, we have

Theorem 4.2 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$, represent an α-inverse strongly monotone mapping with $Fix(Proj_C \circ (I-\lambda A)) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1 - \alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1 - \beta_n)Proj_C \circ (I - \lambda A)y_n), n \geq 1, \end{cases} \tag{36}$$

where $\{\beta_n\}^\infty_{n=1} \in (k, 1)$ for any $k \in [0, 1)$, $\{\alpha_n\}^\infty_{n=1} \in (0, 1)$, $\{\theta_n\}^\infty_{n=1}$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum^\infty_{n=1} \alpha_n = \infty, \sum^\infty_{n=1} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1 - \beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0, 1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise} . \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(Proj_C \circ (I - \lambda A))$.

5 Numerical illustration

In this section we would demonstrate the effectiveness, implementation and strong convergence results of our procedure (14) of Theorem 3.1 as discussed in section 3. All codes are written in Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz.

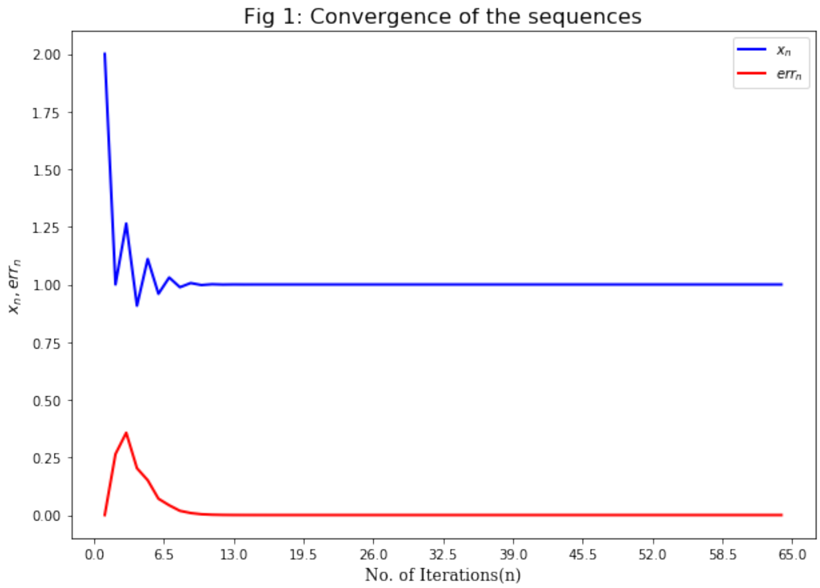
Example 5.1 *c.f [2]. Let $H = R$ denote the reals with the usual norm $|.|$ and the inner product $\langle x, y\rangle = xy$. Let $C = [\frac{1}{2}, 2]$ and define $T : C \to C$ by $T(x) = \frac{1}{x}$. Then T is strictly pseudo-contractive with $0 \leq k < 1$. Let $x_0 = 2, x_1 = 1, \alpha_n = \frac{1}{400(n+1)}, \beta_n = \frac{n+1}{2n+7}, \tau = \frac{2}{3}$, and $\theta_n = \alpha^2_n$ such that our hypothesis are satisfied. Also, the procedure of (14) is as follows:*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \frac{1}{400(n+1)}x_0 + (1 - \frac{1}{400(n+1)})z_n, \\ x_{n+1} = \begin{cases} \frac{1}{2} \; if \; x < \frac{1}{2} \\ 2 \; if \; x > 2 \\ (\frac{n+1}{2n+7})y_n + (1 - \frac{n+1}{2n+7})\frac{1}{y_n}, & \textit{otherwise} . \end{cases} \end{cases} \tag{37}$$

n	10	15	20	30	40	50	55	61
x_n	1.0009862	0.9999745	0.9999876	0.9999941	0.9999965	0.9999977	0.9999999	1.00000000
$\|x_{n+1} - x_n\|$	0.00014017	0.0000086	0.0000010	0.0000003	0.0000002	0.0000001	0.0000000	0.0000000

Table 1: The numerical values of the sequences $\{x_n\}$ and $\|x_{n+1} - x_n\| = err_n$.

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(T) = 1$. Moreover, the results are displayed below.



Remark 5.1 *We observe that the choice of the control sequences of $\{\alpha_n\}, \{\beta_n\}$ and $\{\theta_n\}$,satisfying the convergence hypothesis, influenced to a great deal, the strong convergence of the procedure (14) to the fixed point of the mapping which is $\{1\}$. If the stopping criteria is $\|x_{n+1}-x_n\| \leq 10^{-7}$, then the computation can be truncated after 50 iterations. If the stooping criteria is higher at $\|x_{n+1} - x_n\| \leq 10^{-5}$, then truncating the computation early may have a negligible computational error. Consequently, $\|x_n - p\| = \|x_n - 1\| \to 0$ as $n \to \infty$.*

6 Conclusion

We introduced the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex subset of real Hilbert spaces. We proved strong convergence theorem for it. Furthermore, we applied our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. We supported our main result with a numerical example to illustrate the effectiveness of our procedure. We observed that our procedure performed favourably with a "good" stopping criteria. Our procedure and the given example is found to complement the very recent result of Agbebaku et al [1] in the sense that our introduced procedure is different from theirs and which approximates the fixed point of strict pseudo-contractive self mapping on a closed and convex subset of a real Hilbert spaces. Therefore, our procedure or algorithm is recommended for approximation fixed point problem of this class of mapping in this direction.

References

[1] Agbebaku, D.F., Nwokoro, P.U., Osilike, M.O., Chima, E.E., and Onah, A.C. (2021).The iterative algorithm with inertial and error terms for fixed points of strictly pseudo-contractive mappings and zeros of inverse strongly monotone operators. Appl. Set-Valued Anal. Optim. 3: 95-107.

[2] Berinde, V. and Berinde, M. (2005). The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings. Carpathian J. Math. 21(1-2): 13-20.

[3] Browder, F.E. and Petryshyn, W.V. (1967). Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. 20: 197-228.

[4] Chidume, C.E. and Shahzad, N. (2010). Weak convergence theorems for a finite family of strict pseudocontractions. Nonlinear Anal. 72: 1257-1265.

[5] Chidume, C.E., Ikechukwu, S.I., and Adamu, A. (2018). Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps. Fixed Point Theory and Applications, 9.

[6] Ekuma-Okereke, E. and Okoro, F.M. (2020). Common Solution for Nonlinear Operators in Banach Spaces, GUJ Sci 33(3):737-749.

[7] Ekuma-Okereke, E., Okoro, F., and Abhulimen, C (2021). Two modified hybrid inertial algorithms for Bregman weak relatively nonexpansive mapping in Banach spaces. Nigerian Journal of Mathematics and Applications, 31:149-174.

[8] Halpern, B. (1968). Fixed points of non-expanding maps. Bull. Amer. Math. Soc., 73:957-961.

[9] Ishikawa, S. (1974). Fixed point by a new iteration method. Proc. Amer. Math. Soc. 44:P147-150.

[10] Jeong, J.U. (2015). Convergence theorems of common elements for pseudo-contractive mappings and monotone mappings. Abstract and Applied Analysis, http://dx.doi.org/10.1155/2015/383579.

[11] Mainge, P.E. (2008). Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. 16:899-912.

[12] Mainge, P.E. and Maruster, S. (2011). Convergence in norm of modified Krasnoselkii-Mann iterations for fixed points of demicontractive mappings. Appl. Maths. Comput. 217: 9864-9874.

[13] Mann, W.R. (1953). Mean value methods in iteration. Proc. Amer. Math. Soc. 4:506-510.

[14] Nwokoro, P.U., Osilike, M.O., Agbebaku, D.F., Chima, E.E., and Onah, A.C. (2020). A new Halpern-type averaging algorithm with inertial and error terms for fixed points of asymptotically nonexpansive maps, J. Math. Comput. Sci. 10:1538-1558.

[15] Shehu, Y. (2016). Iterative procedure for left Bregman strongly relatively nonexpansive mappings with application to equilibrium problems. Fixed Point Theory, 17(1):173-188.

[16] Shehu, Y., Iyiola, O.S., and Ogbuisi, F.U. (2020). Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. Numer. Algor. 83:1321-1347.

[17] Xu, H.K. (2002). Another control condition in an iterative method for nonexpansive mappings. Bulletin of the Australian Mathematical Society, 65(1):109–113.

[18] Zhang, Q.B. and Cheng, C.Z. (2008). Strong convergence theorem for a family of Lipschitz pseudocontractive mappings in a Hilbert space. Mathematical and Computer Modelling, 48(3-4):480–485.

[19] Zegeye, H., Shahzad, N., and Alghamdi, M.A. (2011). Convergence of Ishikawa's iteration method for pseudocontractive mappings, Nonlinear Analysis, 74:7304-7311.

An approximation of linear evolution equation with stochastic partial derivative in Sobolev space

B.O. Osu[*], I.U. Amadi[†], and I. Davies[‡]

Abstract

In this paper, the existence and uniqueness of classical solution of stochastic elliptic Partial Differential Equation was considered. The analytical approach of weak solutions is presented in details which has enough financial implications in time-varying investments. Finally, the computations were basically on invocation of ellipticity; which gave analytical estimates from the structural algebraic ellipticity assumption in Sobolev space.

Keywords: Sobolev spaces; stochastic PDE; smooth function; weak solution; elliptic PDE

1 Introduction

Partial Differential Equations (PDEs) arises in every filed of science and technology. These equations have been used in mathematics to solve some societal problems. Consequently, the desire to understand the solutions of these equations and its roles in real-late situation is of great practical concern to mathematicians. PDEs has inspired so many fields of study such as Sciences, Engineering and Medical fields etc. in mathematical modeling PDEs of elliptic type has played crucial roles in making real life changes.

However, most mathematicians are more interested in solution which has topological and analytical characteristics namely: continuity, infinite differentiability, continuous derivatives etc.; these types of solutions exist and are not trivial to find, but require that which is loosen the conditions for one function to be the derivative of another. Therefore, the theory of Sobolev spaces is based on this type of redefinition of differentiability. In particular, a Sobolev space gives weak solutions to differential equations and built on the basis of weak derivative.

More so, so many scholars have used the formation of PDEs in different ways such as [5] studied the existences of unique solution of an elliptic PDE. Result showed that the key proofs lie in obtaining a

[*] Corresponding Author. Department of Mathematics, Abia State University , Uturu Abia State, Nigeria;
E-mail: Osu.bright@abiastateuniversity.edu.ng
[†] Department of Mathematics/ Statistics Captain Elechi Amadi Polytechnic,
Rumuola, Port Harcourt, Nigeria.
[‡] Department of Mathematics, Rivers State University, Nkpolu Oroworokwo, Port Harcourt, Nigeria

priori estimates. [1] Considered the existence of solution for a PDE involving a singularity with a general non negative radon measure as its nonhomogeneous terms. In [3] have considered quasilinear elliptic equation involving the p-Laplacian and singular nonlinearities; they deduced a few comparison principles and have proved some unique results. In another dimension,[6] obtained some new existence, uniqueness and stability results for ODEs with coefficients Sobolev spaces. In their results linear transport equations were analyzed by the method of renormalization solutions.

The work in [4] looked at solution to nonlinear Black Scholes equations. They proved the existence of weak solutions in a bounded domain and extend the results to the whole domain using a diagonal process. [11] Considered a weak solution of nonlinear Black Scholes Equation with transaction cost and portfolio risk in Sobolev space. In their result they obtained a weak solution that is characterized by Fourier transform.

Previous studies have therefore considered a weak solution of non-linear Black-Scholes equation with transaction cost and portfolio risk in Sobolev space. In particular [11] but did not consider stochastic elliptic partial differential equation in Sobolev space which grantees the existence of weak solutions. In this study, we considered the analytical solution of stochastic elliptic PDE in Sobolev space. Apart from correctly obtaining weak formulation of the problem in Sobolev space, we also derive, in detail, closed form analytical solutions that are characterized by weak results. To the best of our knowledge this is the study that has solved elliptic PDE in Sobolev spaces.

This paper is arranged as follows: Section 2.1 Mathematical preliminaries, Section 2.1.1 presents Problem formulation, 2.1.2 Definitions, Assumptions and Main Results. The paper is concluded in Section 3.

2 Mathematical preliminaries

Let $u\left(s_{t,}V_{t,}t\right)$ represents the value of the underlying asset whose stock dynamics is given by:

$$\frac{\partial u}{\partial t}+VS^2\frac{\partial^2 u}{\partial S^2}+\rho\theta VS\frac{\partial^2 u}{\partial s\,\partial v}+\frac{1}{2}V\sigma^2\frac{\partial^2 u}{\partial v^2}+\gamma s\frac{\partial u}{\partial s}+\{k\,[\theta-V_t]-\lambda(St_1Vt_1t)\}$$

$$\frac{\partial u}{\partial V}-\gamma u=0, t>0, \qquad (1.1)$$

where $\lambda\,(St_1Vt_1t)$ demotes the price of volatility risk which is independent of a particular asset.

$$u\,(St_1Vt_1t)=f\,(S,V). \qquad (1.2)$$

Setting $[(S,V)\epsilon\,\mathbb{R}^2\colon\lambda(S,V,t)=0]$ (with different probability $\overline{\mathbb{p}}$ equivalent to the original$\overline{\mathbb{p}}$ [8].
In finance, for a contingent claim on one asset, the generic PDE can be written as follows:

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial^2 u}{\partial x^2}+b\left(x_,t\right)\frac{\partial u}{\partial x}+c\left(x_,t\right)u=0, \qquad (1.3)$$

where t represents time to maturity. x represents either the value of the underlying asset or some monotonic function of it, cd u is the value of the claim (as a function of $\mathcal{X}$ and t). The terms a (.), b (.) and c (.) are the diffusion, convention reaction Coefficients respectively and this type of PDE is called a convention diffusion PDE. Equation (1.3) can also be rewritten as

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}\right) + b(x,t)\ \frac{\partial}{\partial x}\ (\beta(x,t)u) + c(x,t)u = 0. \tag{1.4}$$

To eliminate the price process in (1.1) slightly modifies on stochastic elliptic PDE as:

$$\frac{1}{2}\ vs^2\ \frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\ v\sigma^2\ \frac{\partial^2 u}{\partial V^2} + K\ [\theta - V_t]\frac{\partial u}{\partial V} - \gamma u = 0. \tag{1.5}$$

In this paper we shall explor method of solving Stochastic Partial Differential Equation (SPDE) in Sobolev spaces. Writing (1.5) which is the boundary- value problem under study in the space format gives;

$$\begin{cases} L_u = f\ in\ \varphi \\ u = 0\ on\ \partial\varphi' \end{cases} \tag{1.6}$$

where u= 0 on $\partial\varphi$ is known as Dirichlet's boundary condition φ is open bounded subset of $\mathbb{R}^N$ and u: $\varphi \to \mathbb{R}$ is the unknown, u= $u(s,v)$. we have f: $u \to$. $\mathbb{R}$The linear second order PDE is denoted by L which is 2nd partial differential operator of the form:

$$L \equiv -\sum_{ij=1}^{N}\frac{\partial}{\partial s_i}\left(\frac{1}{2}vs^2\right)^{ij}(s,v)\frac{\partial}{\partial\, s_i}\Big) + \sum_{ij=1}^{N}\frac{\partial}{\partial v_i}\left(\frac{1}{2}\ v\sigma^2\ \right)^{ij}\left(s,v\right)\frac{\partial}{\partial v_i}\Big)$$
$$+\sum_{i=1}^{N}(\theta - V_t)^i\left(s,v\right)\frac{\partial}{\partial v_i}\ -\ r\left(s,v\right)u \tag{1.7}$$

or in non-divergence form of it

$$L = -\sum_{i=1}^{N}\left(\frac{1}{2}\ vs^2\right)^{ij}\left(s,v\right)\frac{\partial^2}{\partial s_i\partial s_j} + \sum_{ij=1}^{N}\left(\frac{1}{2}\ v\sigma^2\right)^{ij}\left(s,v\right)\frac{\partial^2}{\partial v_i\partial v_j} +$$
$$\sum_{i=1}^{N}(\theta - V_t)^i\ \left(s,v\right)^i\ \frac{\partial}{\partial v_i}\ -\ \mathrm{r}\left(s,v\right)u, \tag{1.8}$$

with the following deficient function $\left(\frac{1}{2}vs^2\right)^{ij}, \left(\frac{1}{2}\ v\sigma^2\right)^{ij}\ (\theta - V_t)^i,\ \ r(i,j = 1,2,\ldots,N)$.

We can comfortably say that the PDE L = f is the form of divergence assuming L is given by (3) also in non-divergence form on the premise that L is given by (1.8).

Observe that, if the coefficient $\left(\frac{1}{2}\ vs^2\right)^{ij}$ and $\left(\frac{1}{2}\ v\sigma^2\right)^{ij}$ respectively are of class C^1 functions, we can write an operator in divergence form (1.7) in the non-divergence form (1.8). We can indeed assume that the coefficients $\left(\frac{1}{2}\ vs^2\right)^{ij}$ and $\left(\frac{1}{2}\ v\sigma^2\right)^{ij}$ one symmetric by nature such that $\left(\frac{1}{2}\ vs^2\right)^{ij} = \left(\frac{1}{2}\ vs^2\right)^{ij}$ and $\left(\frac{1}{2}\ Vs^2\right)^{ij} = \left(\frac{1}{2}\ vs^2\right)^{ji}$ which ranges for 1$\leq i,j \leq N$.

So, because of the above assertion we shall assume that the partial differential operator L satisfies a condition of been uniform elliptic; if there exists u constant $\theta >$ o such that

$$\sum_{ij=1}^{N}\left(\frac{1}{2}\,vs^2\right)^{ij}(s,v)\mathcal{E}_i\mathcal{E}_j \geq \theta\,/\,\xi\,/^2 + \sum_{ij=1}^{N}\left(\frac{1}{2}\,v\sigma^2\right)^{ij}(s,v)\mathcal{E}_i\mathcal{E}_j; \geq \theta\,/\,\xi\,/^2 \qquad (1.9)$$

$V_{\mathcal{E}} \in \mathbb{R}^N$, almost everywhere (a.e), s, v, $\in$ Ellipicity implies that in each point s, v, $\in \varphi$, then the symmetric matrix N×N is $A_1\,(s,v) = \left(\frac{1}{2}\,vs^2\right)^{ij}(s,v)$ and $A_2\,(s,v) = \left(\frac{1}{2}\,v\sigma^2\right)^{ij}(s,v)$ are positive definite with eigenvalues $\geq \theta$.

It is quite obvious = $\left(\frac{1}{2}\,vs^2\right)^{ij} \equiv \left(\frac{1}{2}\,v\sigma^2\right)^{ij} = \delta_{ij}$, $(\theta - V_t)^i \equiv o, r \equiv o,$ in my case the partial differential operator L is $-\Delta$.

2.1 Problem formulation

Weak solution of the financial Dirichlet problem.

Firstly, we consider (1.6) when L has divergence form (1.7), then we explicitly define and construct an appropriate weak solution u of (1.6) and also examined the smoothness as well as other properties of u.

Let $\left(\frac{1}{2}vs^2\right)^{ij}, \left(\frac{1}{2}\,v\sigma^2\right)^{ij}, (\theta - V_t), r \in L^{\infty}(\varphi), ij = 1, \cdots,$N
and

$$f \in L^2(\varphi) \qquad (1.11)$$

We shall motivate the definition of weak solution on the assumption that the moment u is a smooth enough, then we multiply the PDE Lu = f by a smooth test function $z\epsilon\, C_c^{\infty}(\varphi)$ and integrate over u to obtain (1.12)

$$\int_{\varphi}\left(\sum_{ij=1}^{N}\left(\frac{1}{2}\,V\sigma^2\right)^{ij}u_{(s,v)i}\,v_{(s,v)i} + \sum_{ij}^{N}\left(\frac{1}{2}\,v\sigma^2\right)^{ij}u_{(s,v)i}\;v_{(s,v)i} + \sum_{i=1}^{N}(\theta - V_t)^i\;u_{(s,v)i}\,v + ruvd(s,v)\right)\int_{\varphi} fvd = \int_{\varphi} fvdsv + \int_{\varphi} fvdsv \qquad (1.12)$$

The first term on the left hand side when we have integrated by parts; there are no boundary term hence v=o on $\partial\varphi$. In term of approximation we observe the same identity satisfies following the smooth function ν replaced by any $\nu\,\epsilon\, H_0^1\,(\varphi)$ and resulting identity makes enough sense when u ϵ $H_0^1\,(\varphi)$. We choose the space $H_0^1\,(\varphi)$ to incorporate the boundary condition from (1.6) that u=0 on $\partial\varphi$. The following definition is introduced.

2.1.1 Definitions, assumptions and main results

Definition (1.1): The bilinear from B[,] corresponding with divergence form elliptic operator L defined in (1.7) is as follow:

$$B\,[u,v] := \int_{\varphi}\left(\sum_{ij=1}^{N}\left(\frac{1}{2}\,vs^2\right)^{ij} u_{(s,v)i}\, v_{(s,v)j} + \sum_{ij}^{N}\left(\frac{1}{2}\,v\sigma^2\right)^{ij} u_{(s,v)i}\, v_{(s,v)j} + \sum_{i=1}^{N}(\theta - V_t)^i\, u_{(s,v)i}\, vruvd(s,v), u, v\, \epsilon H_0^1(\varphi) \quad (1.13)\right)$$

Definition 1.2: A function $u, v\, \epsilon H_0^1(\varphi)$ is said to be weak solution of (1.6) if the condition below holds

$$B\,[u,v] = (f,v) \tag{1.14}$$

For all $u, v\, \epsilon H_0^1(\varphi)$, where (,) represents the linear product in L^2 (φ).
Note that identify (1.14) is otherwise called the variational formulation of (1,7).

In general, we have the boundary-value problem

$$\begin{cases} L_{u\ = f^{\,o}} \\ \quad u = \ 0\ on\ \partial \end{cases} - \sum_{i=1}^{N} f^i\ (s_i v) + \sum_{i=1}^{N} f^i\ (s_i v)i \quad in \qquad \varphi, \tag{1.15}$$

where L is linear defined in (3) and $f^i\, \epsilon\, L^2$ (φ)
i=v ⋯, N). In view of this we see the RHS term as f=f^0 - $\sum_i^N f'\,(siv)_i + \sum_{i=1}^N f'(s_i v)^i$
belongs to H$^-(\varphi)$, the dual space of $H_0^1(\varphi)$

Existence and uniqueness of weak solution
Here we assume H is a real Hilbert space with norm $\|,\|$ and inner product (,). We also let $<,>$ represent the pairing of it with its dual space [7].

So, we introduce Lax-Milgram. Assume that B: H x H → ℝ is bilinear mapping, for which here exist constants $\ltimes, \beta > o$ such that

$$B[u,v] \leq \propto \|\ u\ \|\ \|\ v\ \|,\ (u, v\, \epsilon\, H\,)$$
$$\beta\ \|\ u\ \|^2\ \leq B[u,u]. \qquad\qquad (u, \in\ H)$$

Lastly, let f : H⟶ ℝ be a bound linear functional on H. Then exists a unequal element, $u\ \in H$ such that

$$B[u,v] = < f,v > \text{ for all } v \in\ H \tag{1.16}$$

Proof 1: for ach fixed element $u \in H$,the mapping V→ B$[u, v]$ is a bounded linear functional on H; when the R_iesz representation theonem asserts the existence of a unique element W ϵ H satisfying B$(v_, v) = (w_, v), (v \epsilon H)$. (1.17)

Let us write $A_u = w$ whenever (2) holds, so that

B$[u, v] = (A_{u,} v), (u_, v \epsilon H)$ (1.18)

Proof 2: we claim A : H→ H is a bounded linear operation indeed if $\lambda_1 \lambda_2 \epsilon \mathbb{R}$ and u_1, u_2 we see for each $r. \epsilon H$ that

$(A(\lambda_1 u_1 + \lambda_2 u_2), v) =$ B$[\lambda_1 u_1 + \lambda_2 u_2 v]$ By (1.19)

$(A (\lambda_1 u_1 + \lambda_2 u_2), v) =$ B$[\lambda_1 u_1 + \lambda_2 u_2 v]$

$= \lambda_1 B[u_1, v] + \lambda_2 B [u_2 v] = \lambda_1 (Au_1 v) + \lambda_2 (Au_2, v)$

$= (\lambda_1 Au_1 + \lambda_2 Au_2, v)$.

The equality obtains for each $v, \epsilon H$ and so A is linear.

More so, $\| Au \|^2 = (Au_1 Au) = B[u_1 Au] \leq \propto \| u \| \| Au \|$.
Consequently $\| Au \| \leq \alpha \| u \|$ for all $u, \epsilon H,$ and so A is bounded

Proof 3: Next we assert

$$\begin{cases} A \text{ is one} - \text{to} - \text{one and} \\ R(A), \text{the range of } A \text{ is closed in } H \end{cases}$$

To prove the above, we complete the following

$2\beta \| u \|^2 \leq \beta[u_1 u] = (Au_1 u) \leq \| Au \| \| u \|$
Hence $\beta \| u \| \leq \| Au \|$. (1.19)
This inequality easy implies
R (A) = H (1.20)

For if not, them, since R(A) is closed, there would exist a nonzero element, $\omega \in H$ with $\omega \in R(A)^2$. But this fact in term implies the contraction.

$$\beta \| \omega \|^2 \leq B[\omega_, \omega] = (A, \omega) = 0.$$

Proof 4: Next, we observe once none from the Rieze representation theorem that
$< f_1 v > = (w_, v) + (w_, v) for\ all\ v \in H.$

Finally, we show there in at most one element $u \in H$varying (1). For if both B$[u_,$v] = <$f_,$v> and B $(\bar{u}$,v)= <$f_,$v> + <$f_,$v> then B (u, $\bar{u}, u, v$) = 0, $v \in H$. We set v = u- $\bar{u}$
to find $\beta \| u - \bar{u} \|^2 \leq B(u - \bar{u}, u - \bar{u}) = 0.$

Energy Estimates

We take a look of a particular bilinear form B[,] defined

$$B[u,v] = \int_u \sum_{ij=1}^{N} \left(\frac{1}{2}VS^2\right)^{ij} u_{(s,v)i}\, V_{(s,v)j} + \sum_{ij=1}^{N} \left(\frac{1}{2}VS^2\right)^{ij} u_{(s,v)i}\, V_{(s,v)j}$$
$$+ \sum_{i=1}^{N} (\theta - V_t)^i\, u_{(s,v)i}\, v + ruvd\,(s,v).$$

For $u_, v \in H_0^1(\varphi)$ and try to verify the hypothesis of Lax-Milgram theorem [7].

Theorem 1 (Energy Estimates). There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$/\, B[u,v] \leq \propto \|\, U \,\|\, H_0^1(\varphi)\ \|\, V\ \|\, H_0^1(\varphi) + \alpha\ \|\, u \,\|\, H_0^1(\varphi)\ \|\, V\, \|_0^1\ (\varphi)$$
$$\beta //u//^1{}_0(\varphi) \leq\ B[u,u] +\ r//u//^2 L^2(u) + \beta[u,u] + B[u,u] + r//u//^2 L^2(u) + \beta[u,u]$$
$$for\ u_,\ v_\in H^1{}_0\ (\varphi).$$

Proof: We prove the equalities by taking note of holder inequality, which follows that for each u, v∈ $H^1{}_0\,(\varphi)$

$$/B[u,v]\,/ \leq \left(\sum_{ij=1}^{N} //\left(\frac{1}{2}vs^2\right)^{ij} //_{L^\bowtie} + \int_\varphi \sum_{ij=1}^{N} //\left(\frac{1}{2}v\sigma^2\right)^{ij} //_{L^\infty} \right) \int_\varphi /Du/Dv/d\ (s,v)$$
$$+ \sum_{i=1}^{N} //\,(\theta - V_t)//L^\infty \int_\varphi /Du/\ /v/d(s,v) + //r\,//L^\infty \int_\varphi /u//v/d(s,v)$$

$\leq \propto //\, u\ \ //\, H^1{}_0(\varphi) //\, v\, //\, H^1{}_0(\varphi)$ for some appropriate constants α.

In proving the 2nd inequality of ellipticity condition (4) therefore

$$\theta \int_\varphi /Du/^2 d(s,v) \leq \int_\varphi \sum_{i=1}^{N} \left(\frac{1}{2}vs^2\right)^{ij} u_{(s,v)i}\, u_{\ (s,v)j}\ d(s,v)$$
$$+ \int_\varphi \sum_{i=1}^{N} \left(\frac{1}{2}VO^{-2}\right)^{ij} u(s,v)i\, u(s,v)j\ d(s,v)$$
$$= B\ [u,u] - \sum_{i=1}^{N} (\theta - V_t)^i\ u(s,v)iu + ru^2\ d(s,v) \quad (6)$$

$\leq B\ [u,v] + \sum_{i=1}^{N} (\theta - V_t)^i\ //L^\infty \int_\varphi /Du//u/\ d(s,v) + //\gamma//L^\infty \int_\varphi{}^{u^2}\ \ d(s,v).$

Recall that Cauchy's inequality is given as $\in>0$ for each a,b, >0 which states ab$\leq \in a^2 + \frac{b^2}{4\in}$
Using the inequality when a= /Du/ and b=/u/ then integrating over φ yields
$\int_\varphi /Du//u/\leq \int_\varphi \left(\in/Du/^2+\frac{1}{4\in}U^2\right)$.

Using the last inequality in (6) with $\in$ such that
$0 < \in < \frac{\theta}{2\sum_{i=1}^{N}//\theta-V_t//_\infty}$.

We can conclude that appropriate for an constant C

$\frac{\theta}{2}\int_\varphi /Du/^2\, d(s,v) \leq B\,[u,u] + C\int_\varphi {}^{u^2} d(s,v)$.

According to point care's inequality, we know that

$$//u//L^2(u) \leq C^1 //Du//L^2(\varphi)\ For\ constant C^1 > 0$$

Thus, we finally obtain that for some suitable constant

$$\beta > o, r \geq o.$$

Theorem 2. (Existence theorem for weak solution). There is a number $\gamma \geq 0$ such that for each

$$\mu \geq \gamma$$

And each function

$$f \in L^2(\varphi)$$

There exists a unique weak solution $u \in H_0^1(\varphi)\ of\ boundary\ value\ problem$

$$\oint \begin{matrix} L_u + \mu_u = f & in\ \varphi \\ u = o & on\ \partial\varphi \end{matrix}$$

Proof of 1: Take r from Theorem 1.1, let $\mu \geq \gamma$ and define then the bilinear form

$$B_\mu\,[u,v] := B[u,v] + \mu(u,v) \qquad \left(u, v \in H_0^1(\varphi)\right),$$

which corresponds to the operator L, μu: = μu, (1.16) means the inner product in $L^2\ (\varphi)$. Then B_μ [1.16] Satisfies the hypothesis the Lax-Milgram theorem.

Proof of 2. Now fix f E $L^2(\varphi)$ and set <f,v> : = (f, v) L^2 (φ).
This is a bounded linear functional on L^2 (φ) and thus on H_0^1 (φ). we apply the Lax-Milgram theorem to find unique function u ∈ $H_0^{'}$ (φ) satisfying.

$$B_\mu[u,v] = < f, v >$$

For all v∈ $H_0^{'}$ (φ); u is consequently the unique weak solution of (8)
Note: we can also show that for all f' ∈ L^2 (φ) (I=0, …, N) there exists a unique weak solution is of the PDE

$$\{Lu + \mu u = f^0 - \sum_{i=1}^{N}$$

$$Indeed, it\ is\ enough\ to\ note\ < f, v > = \int_\varphi f, v + \sum_{i=1}^{N} f^i\ (\theta - vt)\ v_{\ (s,v)i}\ d_{\ (s,v)}$$

is bounded linear functional on $H_0^{'}$ (φ).

In particular, we deduce that the mapping

$L_\mu; = L + \mu \mathrm{I}: H_0^{'}(\varphi)\ n \to H^{-1}(\varphi)$ $(\mu \geq r)$ is an isomorphism.

Examples: in the case Lu = -Δu so that B[u, v]= $\int_\varphi Du.Dv\ d\ (s,v)$
We easily check using point care's inequality that theorem 3.2 holds with γ= 0, A similar assertation holds for the general operator

$$L_u = -\sum_{i=1}^{N}\left(\frac{1}{2}\ vs^2\right)^{ij} u(s,v)i + \sum_{i=1}^{N}\left(\frac{1}{2} vo^{-2}\right)^{i} u(s,v)i$$
$$+ ru, provided\ r \geq o\ in\ \varphi.$$

Regularity

Here, we regularized to know whether a weak solution u is of the PDE.

$$\Longrightarrow L_u = f\ in\ \varphi$$

Is in fact smooth: it is the regularity issues for weak solutions. We shall ensure that a weak solution might be better 1 a typical function in $H_0^{'}$ (φ). Considering the problem.

$$-\Delta u = f\ \ in\ \mathbb{R}^N$$

In the sequel, we assume for heuristic purposes that is smooth and variables completely as /x/ → ∞ for justify the calculation as follows: Therefore, we calculate.

$$\int_{\mathbb{R}^N} f^2\ d\ (s,v) = \int_{\mathbb{R}^N} (\Delta u)^2\ d = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2} vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$+ \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ v\sigma^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v) +$$
$$\sum_{ij=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ VS^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)\ d(s,v) +$$

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\, V0^2\right) u_{(s,v)i(s,v)i}\, u_{(s,v)j\,(s,v)j}\; d(s,v)$$
$$= \int_{\mathbb{R}^N} /D^2u\,/^2\; d(s,v) + = \int_{\mathbb{R}^N}\; /D^2u/^2\; d(s,v)$$

It can be seen that L^2 – norm of the 2nd derivative of u can be estimated to give the L^2-norm of f. Also, the PDE (1.6) can be differentiated to obtain.

$$-\,\Delta\bar{u} = \bar{f},$$

Hence we have $\bar{u}$ = u$(s,v)_k$ and $\bar{f}$: f $(s,v)_k$, (k=1,…,N) using the same procedure, we find out L^2-norm of the third derivative of u is estimated via the first derivatives of f. Moving in the same manner, we notice the L^2-norm of the (m+2)nd derivatives of u is been controlled by the L^2-norm of mth derivative of f,m = 0, 1,⋯

The above computations is suggestive of Poisson's equation (1.6) therefore, we expect a weak solution u ∈ $H_0^{\prime}$ to belong to H^{m+2} as soon as the inhomogeneous term f belongs to H^m (m, = 1, ⋯). Informally we can conclude to say that u has "two more derivatives in L^2 than f". In particular, this will interestingly form = ∞ which is when u belong to H^m for all m = 1, . . . and it will be in the space C^{∞}. It can be seen that the computations did not justify a proof. u was assumed to be smooth or be in a special C^3, in order to carry out the computation (1.18); however, if we start these computations with a weak solution in the space $H_0^{\prime}$, the computation, cannot be justifies immediately. Otherwise, we rely upon an analysis of some certain difference quotients.

According to the computations they are technically difficult and obviously yielding powerful results and useful assertions in respect to the smoothness of weak solutions. The computations in geared toward the innovation of ellipticity; which of cause is to derive an analytical estimates from the structure algebraic ellipticity assumption.

3 Conclusion

The definitions, assumptions and analytical approach of stochastic of elliptic partial differential equation in Sobolev space has been illustrated; which gave powerful results and useful assertions in respect to the smoothness of weak solutions.

References

[1] Akasmika, P., Sekhar, G., and Debajyoti, C. (2019). Elliptic PDE involving a singularity radon measure. Arxiv: 1709.00905v4 [math.AP]

[2] Brezis, H. (2010). Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer.

[3] Canino, I., Grandinetti, M., and Scianzi, B. (2013). Symmetry of solutions and some semilinear Elliptic equation with singular nonlinearities. Journal of Differential Equations, 255(12):4437-4447.

[4] Christina, M., Emmanuel, K., and Ncheuguim, I.S. (2011). Solution to a non-linear Black Scholes equation. Electronic Journal of Des, 158:11-10.

[5] Diane, D. (2020). Existence of Unique Solution to an Elliptic PDE when the average value is known. Aims Mathematics, 6(1):518-531.

[6] Diperna, R. J. and Lions, P. L. (1989). Ordinary differential equations, transport theory and

Sobolev Spaces. Invention Maths. 98:511-547.
[7] Evans, L.C. (2010). Partial Differential Equations,2nd edition, American Mathematical Society.
[8] Heston, S.I. (1993). A closed form solution for options with stochastic volatility with application to bond and currency option, Rev. Financial Studies, 6:327-343.
[9] Imanol, P.A. (2016). Sobolev Spaces and Partial Differential Equations, Final Degree Dissertation in Mathematics.
[10] Okoroator, A.C. and Osu, B.O. (2008). The solution by stochastic iteration of an evolution equation in Hilbert space. Asian Journal of Mathematics/Statistics, 1(2):126-131.
[11] Osu, B.O. and Olunkwa, C. (2014). Weak solution of Black Scholes Equation option pricing with transaction costs. International Journal of Applied Mathematics, 1:43-55.
[12] Osu, B.O., Eze, E.O., and Obi, C.N. (2020). The impact of stochastic volatility process on the values of assets. Scientific African, 9, www.elsevier.com/locate/scient.
[13] Richard, W. (2013). Numerical solutions to PDEs with financial applications. Open quantitative research.

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 3, October 2022, Pages 24 – 43

Solving parabolic equations by Olayiwola's generalized polynomial approximation method

R.O. Olayiwola*

Abstract

Polynomial approximation methods are in wide use today for approximately solving partial differential equations of mathematical physics. An evolution of the polynomial approximation methods has led to the development of Olayiwola's generalized polynomial approximation method (OGPAM) presented in this paper, which can be applied to solve parabolic equations with constant or variable coefficients in the cases of considering slab, cylindrical or spherical geometries. Different examples have been solved using the OGPAM. The result of the present analysis has been compared with earlier analytical results. A good agreement has been obtained between the present prediction and the available results.

Keywords: cylindrical geometry; OGPAM; parabolic equations; polynomial solution; slab; spherical geometry.

1 Introduction

Partial differential equations are the chief means of providing mathematical models in science, engineering, and other fields. Generally, these models must be solved numerically and in some special cases, analytically [2]. In many practical problems that involved Cartesian, Cylindrical, and Spherical coordinates the polynomial approximation method is used.

The polynomial approximation method is one of the simplest, and in some cases, accurate methods used to solve transient conduction problems. The method involves two steps: first, selection of the proper guess temperature profile, and second, converting a partial differential equation into an equation. This can then be converted into an ordinary differential equation, where the dependent variable is the average temperature and the independent variable is time. The steps are applied to the dimensionless governing equation [1].

The present work deals with the analytical solution of parabolic equations in different geometries using Olayiwola's generalized polynomial approximation method (OGPAM).

*Department of Mathematics, Federal University of Technology, Minna, Nigeria.
E-mail.: olayiwola.rasaq@futminna.edu.ng

2 Olayiwola's generalized polynomial approximation method

In this section, Olayiwola's generalized polynomial approximation method (OGPAM) for the solution of parabolic partial differential equations with constant or variable coefficients in different geometries is presented.

The parabolic equations are defined by

$$\frac{\partial \phi}{\partial t}=\frac{k}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial \phi}{\partial r}\right)+F(r,t,\phi), \qquad t>0, \quad r\in\Omega \quad \left(\Omega\subset R^1, R^2 or\, R^3\right), \tag{1}$$

with the initial condition

$$\phi(r,0)=f(r), \tag{2}$$

and the boundary conditions

$$\alpha_1\left.\frac{\partial \phi}{\partial r}\right|_{r=a}+\beta_1\left.\phi\right|_{r=a}=g_1(t), \qquad \alpha_2\left.\frac{\partial \phi}{\partial r}\right|_{r=b}+\beta_2\left.\phi\right|_{r=b}=g_2(t), \tag{3}$$

where $F(r,t,\phi)$ are the other terms in equation (1), $n=0$ for a slab, $n=1$ for cylinder, $n=2$ for sphere, $f(r)$, $g_1(t)$, $g_2(t)$ are three known functions, $n,k,\alpha_1,\alpha_2,\beta_1,\beta_2$ are constants, and $a\le r\le b$ is the boundary of Ω.

To obtain an approximate solution of the partial differential equation (1) satisfying (2) and (3), the following steps are involved:

Case 1: When $\alpha_1\neq 0$ and $\alpha_2\neq 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\left(1+\frac{b^2\beta_1}{2(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\frac{b(2a-b)\beta_2}{2(b-a)\alpha_2}\right)+\\ \frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a\beta_2}{(b-a)\alpha_2}-\frac{b\beta_1}{(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)+\\ \frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{2(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\frac{\beta_2}{2(b-a)\alpha_2}\right)\end{array}\right)$$

$$B=\left(\begin{array}{l}\left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{b^2}{2(b-a)\alpha_1}\right)+\\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{b}{(b-a)\alpha_1}-\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)\right)+\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{1}{2(b-a)\alpha_1}\right)\end{array}\right)$$

$$C=\left(\begin{array}{l}\left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{b(2a-b)}{2(b-a)\alpha_2}\right)-\\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a}{(b-a)\alpha_2}+\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)+\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{1}{2(b-a)\alpha_2}\right)\end{array}\right)$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$\phi(r,t)=\left(\begin{array}{l}\left(\begin{array}{l}\left(1+\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\dfrac{b(2a-b)\beta_2}{2(b-a)\alpha_2}\right)\phi|_{r=b}+\\ \left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{b^2}{2(b-a)\alpha_1}\right)g_1+\\ \left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{b(2a-b)}{2(b-a)\alpha_2}\right)g_2\end{array}\right)+\\ \left(\begin{array}{l}\left(\dfrac{a\beta_2}{(b-a)\alpha_2}-\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)\phi|_{r=b}+\\ \left(\dfrac{b}{(b-a)\alpha_1}-\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)\right)g_1-\\ \left(\dfrac{a}{(b-a)\alpha_2}+\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)g_2\end{array}\right)r+\\ \left(\begin{array}{l}\left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\dfrac{\beta_2}{2(b-a)\alpha_2}\right)\phi|_{r=b}+\\ \left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{1}{2(b-a)\alpha_1}\right)g_1+\\ \left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{1}{2(b-a)\alpha_2}\right)g_2\end{array}\right)r^2\end{array}\right) \tag{4}$$

where

$$\phi|_{r=a}=\left(\begin{array}{l}\left(\dfrac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\phi|_{r=b}+\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)g_1\\ +\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)g_2\end{array}\right)$$

Step 3: By using (4), evaluate

$$\frac{(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\int_a^b r^n F(r,t,\phi)dr \tag{5}$$

and express the result in the form:

$$p_1\phi\big|_{r=b}+q_1(t) \tag{6}$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi\big|_{r=b}=e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx+f(r)e^{-\int p(t)dt} \tag{7}$$

where

$$p(t)=\frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{b^n\beta_2}{\alpha_2}-\frac{a^n\beta_1}{\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)-p_1\right)$$

$$q(t)=\frac{1}{A}\left(\begin{array}{l}\dfrac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a^n\beta_1}{\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{a^n}{\alpha_1}\right)g_1-\\ \dfrac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a^n\beta_1}{\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{b^n}{\alpha_2}\right)g_2+\\ q_1(t)-B\dfrac{\partial g_1}{\partial t}-C\dfrac{\partial g_2}{\partial t}\end{array}\right)$$

Case 2: When $\alpha_1=0$ and $\alpha_2\neq 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\left(\dfrac{a(a-2b)}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)+\dfrac{a^2\beta_2}{2\alpha_2 b}\right)+\\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\dfrac{2b}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)-\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{1}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)+\dfrac{\beta_2}{2\alpha_2 b}\right)\end{array}\right)$$

$$B=\left(\begin{array}{l}\dfrac{1}{\beta_1}-\left(\dfrac{a(a-2b)}{\left(a(a-2b)+b^2\right)\beta_1}\right)-\dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{2b}{\left(a(a-2b)+b^2\right)\beta_1}\right)+\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{1}{\left(a(a-2b)+b^2\right)\beta_1}\right)\end{array}\right)$$

$$C=\left(\begin{array}{l}\left(\left(\dfrac{a(2b-a)}{\left(a(a-2b)+b^2\right)}\right)\left(\dfrac{\left(b^2-a^2\right)}{2\alpha_2 b}\right)-\dfrac{a^2}{2\alpha_2 b}\right)-\dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\dfrac{2b}{\left(a(a-2b)+b^2\right)}\left(\dfrac{\left(b^2-a^2\right)}{2\alpha_2 b}\right)\\ -\dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{1}{2\alpha_2 b}\left(1+\dfrac{\left(b^2-a^2\right)}{\left(a(a-2b)+b^2\right)}\right)\right)\end{array}\right)$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$\phi(r,t)=\left(\begin{array}{l}\left(\begin{array}{l}\left(\dfrac{a^2\beta_2}{2\alpha_2 b}+\left(\dfrac{a(a-2b)}{a(a-2b)+b^2}\right)\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)\right)\phi\big|_{r=b}+\\ \dfrac{1}{\beta_1}\left(1-\left(\dfrac{a(a-2b)}{a(a-2b)+b^2}\right)\right)g_1+\left(\left(\dfrac{a(a-2b)}{a(a-2b)+b^2}\right)\left(\dfrac{a^2-b^2}{2\alpha_2 b}\right)-\dfrac{a^2}{2\alpha_2 b}\right)g_2\end{array}\right)+\\ \left(\begin{array}{l}\left(\left(\left(\dfrac{2b}{a(a-2b)+b^2}\right)\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)\right)\phi\big|_{r=b}-\dfrac{1}{\beta_1}\left(\dfrac{2b}{a(a-2b)+b^2}\right)g_1+\right.\\ \left.\left(\dfrac{2b}{a(a-2b)+b^2}\right)\left(\dfrac{a^2-b^2}{2\alpha_2 b}\right)g_2\right)\end{array}\right)r+\\ \left(\begin{array}{l}-\left(\left(\left(\dfrac{1}{a(a-2b)+b^2}\right)\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)+\dfrac{\beta_2}{2\alpha_2 b}\right)\phi\big|_{r=b}+\right.\\ \left.\dfrac{1}{\beta_1}\left(\dfrac{1}{a(a-2b)+b^2}\right)g_1+\left(\dfrac{1}{2\alpha_2 b}-\left(\dfrac{1}{a(a-2b)+b^2}\right)\left(\dfrac{a^2-b^2}{2\alpha_2 b}\right)\right)g_2\right)\end{array}\right)r^2\end{array}\right) \tag{8}$$

Step 3: By using (8), evaluate

$$\frac{(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\int_a^b r^n F(r,t,\phi)dr\ , \tag{9}$$

and express the result in the form:

$$p_1\phi\big|_{r=b}+q_1(t) \tag{10}$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi\big|_{r=b}=e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx+f(r)e^{-\int p(t)dt} \tag{11}$$

where

$$p(t)=\frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^{n+1}-a^{n+1}\right)\beta_2}{\alpha_2 b}+\frac{a^n(b-a)\left(2\alpha_2 b+\left(b^2-a^2\right)\beta_2\right)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)-p_1\right)$$

$$q(t)=\frac{1}{A}\left(\begin{array}{l}\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^{n+1}-a^{n+1}\right)}{\alpha_2 b}+\frac{a^n(b-a)\left(b^2-a^2\right)\left(2\alpha_2 b+\left(b^2-a^2\right)\beta_2\right)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)g_2\\ +\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a^n(b-a)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)g_1+q_1(t)-B\frac{\partial g_1}{\partial t}-C\frac{\partial g_2}{\partial t}\end{array}\right)$$

Case 3: When $\alpha_1 \neq 0$ and $\alpha_2 = 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\frac{b(b-2a)}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)-\\ \frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{\alpha_1}+\frac{2a}{(b-2a)}\frac{\beta_1}{\alpha_1}-\frac{2a}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)\right)+\\ \frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{(b-2a)\alpha_1}-\frac{1}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)\right)\end{array}\right)$$

$$B=\left(\begin{array}{l}-\frac{ab(b-2a)}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)+\\ \frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{1}{\alpha_1}+\frac{2a}{(b-2a)\alpha_1}-\frac{2a^2}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\right)+\\ \frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)-\frac{1}{(b-2a)\alpha_1}\right)\end{array}\right)$$

$$C=\left(\begin{array}{l}\frac{a^2}{\left(b(b-2a)+a^2\right)}-\frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\frac{2a}{b(b-2a)\beta_2}\left(1-\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)+\\ \frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\frac{1}{b(b-2a)\beta_2}\left(1-\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)\end{array}\right)$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$
\begin{aligned}
\phi(r,t) = \Bigg(& \left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right)\left.\phi\right|_{r=a} - \frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)g_1 + \left(\frac{a^2}{b(b-2a)\beta_2}\right)g_2\right) + \\
& \Bigg(\left(\frac{2a}{b(b-2a)}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right) - \frac{2a\beta_1}{(b-2a)\alpha_1} - \frac{\beta_1}{\alpha_1}\right)\left.\phi\right|_{r=a} + \\
& \left(\frac{1}{\alpha_1} + \frac{2a}{(b-2a)\alpha_1} - \frac{2a^2}{b(b-2a)\alpha_1}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{ab}{b(b-2a)}\right)\right)g_1 \\
& + \left(\frac{2a}{b(b-2a)}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\frac{a^2}{b(b-2a)\beta_2}\right) - \left(\frac{2a}{b(b-2a)\beta_2}\right)\right)g_2 \Bigg) r + \\
& \frac{1}{b(b-2a)}\Bigg(\left(\frac{b\beta_1}{\alpha_1} - \left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right)\right)\left.\phi\right|_{r=a} + \\
& \left(\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right) - \frac{b}{\alpha_1}\right)g_1 + \\
& \left(\frac{1}{\beta_2} - \left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\frac{a^2}{b(b-2a)\beta_2}\right)\right)g_2 \Bigg) r^2 \Bigg)
\end{aligned}
\tag{12}
$$

Step 3: By using (12), evaluate

$$\frac{(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\int_a^b r^n F(r,t,\phi)dr\,, \tag{13}$$

and express the result in the form:

$$p_1\left.\phi\right|_{r=b} + q_1(t) \tag{14}$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi\big|_{r=a} = e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx + f(r)e^{-\int p(t)dt} \tag{15}$$

where

$$p(t) = \frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\begin{array}{l}\frac{\left(b^n-a^n\right)\beta_1}{\alpha_1}+\frac{2b^n(a-b)}{(b-2a)\alpha_1}+\frac{2b^n(b-a)}{\left(b(b-2a)+a^2\right)}+\\ \frac{2ab^n(b-a)(b(b-2a)+a)\beta_1}{\left(b(b-2a)+a^2\right)b(b-2a)\alpha_1}\end{array}\right)-p_1\right)$$

$$q(t) = \frac{1}{A}\left(\begin{array}{l}\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^n-a^n\right)}{\alpha_1}+\frac{2b^n(a-b)}{(b-2a)\alpha_1}+\frac{2ab^n(b-a)(b(b-2a)+a)}{\left(b(b-2a)+a^2\right)b(b-2a)\alpha_1}\right)g_1+\\ \frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\frac{2b^n(b-a)}{b(b-2a)\beta_2}\left(1+\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)g_2+q_1(t)-B\frac{\partial g_1}{\partial t}-C\frac{\partial g_2}{\partial t}\end{array}\right)$$

Case 4: When $\alpha_1 = 0$ and $\alpha_2 = 0$

In this case, no polynomial approximation solution to equation (1) exists. This assertion is proved by the following theorem:

Theorem 1: Let $\alpha_1 = 0$ and $\alpha_2 = 0$. Then no polynomial approximation solution of (1) that satisfies (2) and (3) exists.
Proof: Substitute $\alpha_1 = 0$ and $\alpha_2 = 0$ into (4), (8), and (12), gives $\phi(r,t) \to \infty$.
Hence, $\phi(r,t)$ does not exists. This completes the proof.

3 Application of OGPAM

This section provides the results obtained when solving partial derivatives equations in time-dependent (parabolic), using the OGPAM as proposed in this paper. Three sets of examples (the first one corresponding to slab, the second one corresponding to cylindrical geometry, and the third one corresponding to spherical geometry) are given.

3.1 Set 1: Slab

Here, two examples are considered.

3.1.1 Problem 1 **(Kewat and Verma [3]):**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) \tag{16}$$

with the initial condition

$$\theta(x,0) = 1 \tag{17}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1} = -B^*\theta\big|_{x=1} \tag{18}$$

Compare (16) – (18) with (1) – (3), we have
$k=1,\ n=0,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=1,\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B^*,\ g_2(t)=0,$
$a=0,\ b=1,\ r=x,\ \ t=\tau$

Then
$$A=\frac{3+B^*}{3}, \qquad B=-\frac{1}{6}, \qquad C=-\frac{1}{3}.$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\theta\big|_{x=1}-\frac{B^*}{2}\theta\big|_{x=1}x^2. \tag{19}$$

Then $p_1=0$ and $q_1(t)=0$ since $F(r,t,\phi)=0$. So,
$$p(\tau)=\frac{B^*}{A}=\frac{3B^*}{3+B^*}, \qquad q(\tau)=0.$$

Then
$$\theta\big|_{x=1}=e^{-\frac{3B^*}{3+B^*}\tau}, \tag{20}$$

as obtained by the authors in [3].

Thus,
$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)e^{-\frac{3B^*}{3+B^*}\tau}-\frac{B^*}{2}e^{-\frac{3B^*}{3+B^*}\tau}x^2. \tag{21}$$

3.1.2 Problem 2:

Let us consider (in dimensionless form) the equation
$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)-\lambda_1(1+x)e^{-\lambda_3 t} \tag{22}$$
with the initial condition
$$h(x,0)=1 \tag{23}$$
and the boundary conditions
$$h(0,t)=1, \qquad \frac{\partial h}{\partial x}\bigg|_{x=1}=0 \tag{24}$$

Compare (22) – (24) with (1) – (3), we have
$k=1,\ n=0,\ F(r,t,\phi)=-\lambda_1(1+x)e^{-\lambda_3 t},\ \phi=h,\ f(r)=1,\ \alpha_1=0,\ \beta_1=1,\ g_1(t)=1,\ \alpha_2=1,$
$\beta_2=0,\ g_2(t)=0,\ a=0,\ b=1,\ r=x$

Then
$$A=\frac{2}{3},\qquad B=\frac{1}{3},\qquad C=\frac{5}{6}.$$

Assume a polynomial solution of the form (8) as

$$h(x,t)=1+2\left(h\big|_{x=1}-1\right)x-\left(h\big|_{x=1}-1\right)x^2. \tag{25}$$

Then $p_1=0$ and $q_1(t)=-\frac{3}{2}\lambda_1 e^{-\lambda_3 t}$. So,
$$p(t)=0,\qquad q(t)=-\frac{9}{4}\lambda_1 e^{-\lambda_3 t}.$$

Then
$$h\big|_{x=1}=\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right). \tag{26}$$

Thus,
$$h(x,t)=1+2\left(\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right)-1\right)x-\left(\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right)-1\right)x^2. \tag{27}$$

3.2 Set 2: Cylindrical Shape

Here, two examples are considered.

3.2.1 Problem 1:

Let us consider (in dimensionless form) the equation
$$\frac{\partial\theta}{\partial t}=\frac{Le}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\delta\psi^{\alpha}\phi^{\beta}\left(1+(e-2)\theta\right)-\delta_1\psi\left(1+(e-2)\theta\right)^m \tag{28}$$
with the initial condition
$$\theta(r,0)=0 \tag{29}$$
and the boundary conditions
$$\left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0,\qquad \left.\frac{\partial\theta}{\partial r}\right|_{r=1}=-Nu\,\theta\big|_{r=1} \tag{30}$$
Compare (28) – (30) with (1) – (3), we have
$k=Le,\ n=1,\ F(r,t,\phi)=\delta\psi^{\alpha}\phi^{\beta}\left(1+(e-2)\theta\right)-\delta_1\psi\left(1+(e-2)\theta\right)^m,\ \phi=\theta,\ f(r)=0,\ \alpha_1=1,$
$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Nu,\ g_2(t)=0,\ a=0,\ b=1$

Then

$$A=\left(1+\frac{Nu}{4}\right),\qquad B=-\frac{1}{12},\qquad C=-\frac{1}{4}.$$

Assume a polynomial solution of the form (4) as

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta\big|_{r=1}-\frac{Nu}{2}\theta\big|_{r=1}r^2. \tag{31}$$

Then $p_1=(e-2)\left(1+\frac{Nu}{4}\right)\left(\delta\psi^{\alpha}\phi^{\beta}-m\delta_1\psi\right)$ and $q_1(t)=\left(\delta\psi^{\alpha}\phi^{\beta}-\delta_1\psi\right)$. So,

$$p(t)=\left(\frac{2LeNu-(e-2)\left(1+\frac{Nu}{4}\right)\left(\delta\psi^{\alpha}\phi^{\beta}-m\delta_1\psi\right)}{1+\frac{Nu}{4}}\right),\qquad q(t)=\left(\frac{\delta\psi^{\alpha}\phi^{\beta}-\delta_1\psi}{1+\frac{Nu}{4}}\right).$$

Then

$$\theta\big|_{r=1}=\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right). \tag{32}$$

Thus,

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right)-\frac{Nu}{2}\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right)r^2. \tag{33}$$

3.2.2 Problem 2:

Let us consider (in dimensionless form) the equation

$$\frac{\partial\rho}{\partial t}+\frac{u}{r}\frac{\partial}{\partial r}(r\rho)=\frac{1}{Pem\ r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right)+\frac{1}{Pem\ r}\frac{\partial\rho}{\partial r} \tag{34}$$

with the initial condition

$$\rho(r,0)=1 \tag{35}$$

and the boundary conditions

$$\left.\frac{\partial\rho}{\partial r}\right|_{r=0}=0,\qquad \left.\frac{\partial\rho}{\partial r}\right|_{r=1}=-Sh\,\rho(1,t) \tag{36}$$

Compare (34) – (36) with (1) – (3), we have

$k=\frac{1}{Pem}$, $n=1$, $F(r,t,\phi)=\frac{1}{Pem\ r}\frac{\partial\rho}{\partial r}-\frac{u}{r}\frac{\partial}{\partial r}(r\rho)$, $\phi=\rho$, $f(r)=1$, $\alpha_1=1$, $\beta_1=0$, $g_1(t)=0$, $\alpha_2=1$, $\beta_2=Sh$, $g_2(t)=0$, $a=0$, $b=1$

Then

$$A=\left(1+\frac{Sh}{4}\right), \qquad B=-\frac{1}{12}, \qquad C=-\frac{1}{4}.$$

Assume a polynomial solution of the form (4) as

$$\rho(r,t)=\left(1+\frac{Sh}{2}\right)\rho\big|_{r=1}-\frac{Sh}{2}\rho\big|_{r=1}r^2. \tag{37}$$

Then $p_1=-\left(2u+\dfrac{Sh}{Pem}\right)$ and $q_1(t)=0$. So,

$$p(t)=\left(\frac{\left(2u+\dfrac{3Sh}{Pem}\right)}{1+\dfrac{Sh}{4}}\right), \qquad q(t)=0.$$

Then

$$\rho\big|_{r=1}=e^{-p(t)t}. \tag{38}$$

Thus,

$$\rho(r,t)=\left(1+\frac{Sh}{2}\right)e^{-p(t)t}-\frac{Sh}{2}e^{-p(t)t}r^2. \tag{39}$$

3.3 Set 3: Spherical Shape

Here, four examples are considered.

3.3.1 Problem 1 **(Mohanty and Dalai [4]):**
Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) \tag{40}$$

with the initial condition

$$\theta(x,0)=f(x) \tag{41}$$

and the boundary conditions

$$\frac{\partial\theta}{\partial x}\bigg|_{x=\varepsilon}=-Q, \qquad \frac{\partial\theta}{\partial x}\bigg|_{x=1}=-B^*\theta\big|_{x=1} \tag{42}$$

Compare (40) – (42) with (1) – (3), we have
$k=1,\ n=2,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=f(x),\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=-Q,\ \alpha_2=1,\ \beta_2=B^*,$
$g_2(t)=0,\ a=\varepsilon,\ b=1,\ r=x,\ t=\tau$

Then

$$A=\left(1+B^*\left(\frac{30\varepsilon(1-\varepsilon^4)-12(1-\varepsilon^5)+20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\right),$$

$$B=\left(\frac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right),\quad C=\left(\frac{12(1-\varepsilon^5)-30\varepsilon(1-\varepsilon^4)-20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)$$

Assume a polynomial solution of the form (4) as

$$\begin{aligned}\theta(x,\tau)=&\left(\left(1+\frac{(1-2\varepsilon)B^*}{2(1-\varepsilon)}\right)\theta\Big|_{x=1}+\frac{Q}{2(1-\varepsilon)}\right)+\left(\frac{\varepsilon B^*}{(1-\varepsilon)}\theta\Big|_{x=1}-\frac{Q}{(1-\varepsilon)}\right)x-\\&\left(\frac{B^*}{2(1-\varepsilon)}\theta\Big|_{x=1}-\frac{Q}{2(1-\varepsilon)}\right)x^2\end{aligned}\tag{43}$$

Then $p_1=0$ and $q_1(t)=0$ since $F(r,t,\phi)=0$. So,

$$p(\tau)=\frac{3B^*}{(1-\varepsilon^3)A},\qquad q(\tau)=\frac{3\varepsilon^2 Q}{(1-\varepsilon^3)A}.$$

Then

$$\theta\Big|_{x=1}=\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right).\tag{44}$$

Thus,

$$\begin{aligned}\theta(x,\tau)=&\left(\left(1+\frac{(1-2\varepsilon)B^*}{2(1-\varepsilon)}\right)\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)+\frac{Q}{2(1-\varepsilon)}\right)+\\&\left(\frac{\varepsilon B^*}{(1-\varepsilon)}\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{Q}{(1-\varepsilon)}\right)x-\\&\left(\frac{B^*}{2(1-\varepsilon)}\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{Q}{2(1-\varepsilon)}\right)x^2\end{aligned}\tag{45}$$

3.3.2 Problem 2 **(Prakash and Mahmood [7]):**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right)+Q \tag{46}$$

with the initial condition

$$\theta(x,0)=1 \tag{47}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1}=-B^*\theta|_{x=1}. \tag{48}$$

Compare (46) – (48) with (1) – (3), we have

$k=1,\ n=2,\ F(r,t,\phi)=Q,\ \phi=\theta,\ f(r)=1,\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B^*,\ g_2(t)=0,$
$a=0,\ b=1,\ r=x,\ t=\tau$

Then

$$A=\left(1+\frac{B^*}{5}\right)=\left(\frac{B^*+5}{5}\right),\qquad B=-\frac{1}{20},\qquad C=-\frac{1}{5}.$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\theta|_{x=1}-\frac{B^*}{2}\theta|_{x=1}x^2. \tag{49}$$

Then $p_1=0$ and $q_1(t)=Q$. So,

$$p(\tau)=\frac{3B^*}{A}=\left(\frac{15B^*}{B^*+5}\right),\qquad q(\tau)=\frac{Q}{A}=\left(\frac{5Q}{B^*+5}\right).$$

Then

$$\theta|_{x=1}=\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right) \tag{50}$$

Thus,

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{B^*}{2}\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)x^2 \tag{51}$$

3.3.3 Problem 3 **(Pradhan *et al.* [6]):**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) \tag{52}$$

with the initial condition

$$\theta(x,0)=f(x) \tag{53}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=\varepsilon}=B_1\theta|_{x=\varepsilon},\qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1}=-B_2\theta|_{x=1}. \tag{54}$$

Compare (52) – (54) with (1) – (3), we have
$k=1,\ n=2,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=f(x),\ \alpha_1=1,\ \beta_1=-B_1,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B_2,$
$g_2(t)=0,\ a=\varepsilon,\ b=1,\ r=x,\ t=\tau$

Then

$$A=\begin{pmatrix}1+B_2\left(\dfrac{30\varepsilon(1-\varepsilon^4)-12(1-\varepsilon^5)+20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+\\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{\left(2(1-\varepsilon)-\left(2\varepsilon-(1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\end{pmatrix},$$

$$B=\begin{pmatrix}\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+\\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{\left(2\varepsilon-(1+\varepsilon^2)\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\end{pmatrix},$$

$$C=\begin{pmatrix}\left(\dfrac{12(1-\varepsilon^5)-30\varepsilon(1-\varepsilon^4)-20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+\\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{\left(2\varepsilon-(1+\varepsilon^2)\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\end{pmatrix}$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\begin{pmatrix}\left(1-\dfrac{B_1}{2(1-\varepsilon)}\left(\dfrac{\left(2(1-\varepsilon)-\left(2\varepsilon-(1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)+\dfrac{(1-2\varepsilon)B_2}{2(1-\varepsilon)}\right)\theta|_{x=1}+\\ \left(\dfrac{\varepsilon B_2}{(1-\varepsilon)}+\dfrac{B_1}{(1-\varepsilon)}\left(\dfrac{\left(2(1-\varepsilon)-\left(2\varepsilon-(1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\right)\theta|_{x=1}x-\\ \left(\dfrac{B_2}{2(1-\varepsilon)}+\dfrac{B_1}{2(1-\varepsilon)}\left(\dfrac{\left(2(1-\varepsilon)-\left(2\varepsilon-(1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\right)\theta|_{x=1}x^2\end{pmatrix} \tag{55}$$

where

$$\theta|_{x=\varepsilon}=\left(\frac{\left(2(1-\varepsilon)-\left(2\varepsilon-(1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon)+\left((1+\varepsilon^2)-2\varepsilon\right)B_1\right)}\right)\theta|_{x=1}$$

Then $p_1 = 0$ and $q_1(\tau) = 0$ since $F(r,t,\phi) = 0$. So,

$$p(\tau) = \frac{3}{(1-\varepsilon^3)A}\left(B_2 + \varepsilon^2 B_1\left(\frac{\left(2(1-\varepsilon) - \left(2\varepsilon - (1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon) + \left((1+\varepsilon^2) - 2\varepsilon\right)B_1\right)}\right)\right), \qquad q(\tau) = 0$$

Then

$$\theta\big|_{x=1} = f(x)e^{-p(\tau)\tau} \tag{56}$$

Thus,

$$\theta(x,\tau) = \left(\begin{array}{l} \left(1 - \frac{B_1}{2(1-\varepsilon)}\left(\frac{\left(2(1-\varepsilon) - \left(2\varepsilon - (1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon) + \left((1+\varepsilon^2) - 2\varepsilon\right)B_1\right)}\right) + \frac{(1-2\varepsilon)B_2}{2(1-\varepsilon)}\right) f(x)e^{-p(\tau)\tau} + \\ \left(\frac{\varepsilon B_2}{(1-\varepsilon)} + \frac{B_1}{(1-\varepsilon)}\left(\frac{\left(2(1-\varepsilon) - \left(2\varepsilon - (1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon) + \left((1+\varepsilon^2) - 2\varepsilon\right)B_1\right)}\right)\right) f(x)e^{-p(\tau)\tau} x - \\ \left(\frac{B_2}{2(1-\varepsilon)} + \frac{B_1}{2(1-\varepsilon)}\left(\frac{\left(2(1-\varepsilon) - \left(2\varepsilon - (1+\varepsilon^2)\right)B_2\right)}{\left(2(1-\varepsilon) + \left((1+\varepsilon^2) - 2\varepsilon\right)B_1\right)}\right)\right) f(x)e^{-p(\tau)\tau} x^2 \end{array}\right) \tag{57}$$

3.3.4 Problem 4 **(Olayiwola *et al.* [5]):**

Let us consider (in dimensionless form) the equation

$$\left.\begin{array}{c} \frac{\partial u}{\partial t} = \frac{1}{Da\,\mathrm{Re}}u + \frac{4}{3\mathrm{Re}}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) - \frac{2}{3\mathrm{Re}}\frac{1}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \\ \frac{\partial \phi}{\partial t} = \frac{1}{Pem}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) \\ \frac{\partial \theta}{\partial t} = \frac{1}{Pe}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) + \frac{\beta}{Pem}\frac{\partial \phi}{\partial r}\frac{\partial \theta}{\partial r} + \frac{4Ec}{3\mathrm{Re}}g\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)^2 \end{array}\right\} \tag{58}$$

with the initial condition

$$u(r,0) = 1, \qquad \phi(r,0) = \frac{1}{2}, \qquad \theta(r,0) = \frac{1}{2} \tag{59}$$

and the boundary conditions

$$\left.\begin{array}{ll} \frac{\partial u}{\partial r}\Big|_{r=0} = 0, & \frac{\partial u}{\partial r}\Big|_{r=1} = \gamma \\ \frac{\partial \phi}{\partial r}\Big|_{r=0} = 0, & \frac{\partial \phi}{\partial r}\Big|_{r=1} = -Sh\,\phi\big|_{r=1} \\ \frac{\partial \theta}{\partial r}\Big|_{r=0} = 0, & \frac{\partial \theta}{\partial r}\Big|_{r=1} = -Nu\,\theta\big|_{r=1} \end{array}\right\} \tag{60}$$

Compare (58) – (60) with (1) – (3), we have
For the first equation in (58):

$$k=\frac{4}{3\mathrm{Re}},\ n=0,\ F(r,t,\phi)=\frac{1}{Da\,\mathrm{Re}}u-\frac{2}{3\mathrm{Re}}\frac{1}{r}\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right),\ \phi=u,\ f(r)=1,\ \alpha_1=1,$$
$$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=0,\ g_2(t)=\gamma,\ a=0,\ b=1$$

For the second equation in (58):

$$k=\frac{1}{Pem},\ n=2,\ F(r,t,\phi)=0,\ \phi=\phi,\ f(r)=\frac{1}{2},\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Sh,$$
$$g_2(t)=0,\ a=0,\ b=1$$

For the third equation in (58):

$$k=\frac{1}{Pe},\ n=2,\ F(r,t,\phi)=\frac{\beta}{Pem}\frac{\partial\phi}{\partial r}\frac{\partial\theta}{\partial r}+\frac{4Ec}{3\mathrm{Re}}g\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)^2,\ \phi=\theta,\ f(r)=\frac{1}{2},\ \alpha_1=1,$$
$$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Nu,\ g_2(t)=0,\ a=0,\ b=1$$

Then, for

First equation: $A=1,\quad B=\frac{1}{6},\quad C=-\frac{1}{3}$

Second equation: $A=\left(1+\frac{Sh}{5}\right),\quad B=-\frac{1}{20},\quad C=-\frac{1}{5}$

Third equation: $A=\left(1+\frac{Nu}{5}\right),\quad B=-\frac{1}{20},\quad C=-\frac{1}{5}$

Assume polynomial solutions of the form (4) as

$$u(r,t)=\left(u|_{r=1}-\frac{\gamma}{2}\right)+\frac{\gamma}{2}r^2 \tag{61}$$

$$\phi(r,t)=\left(1+\frac{Sh}{2}\right)\phi|_{r=1}-\frac{Sh}{2}\phi|_{r=1}r^2 \tag{62}$$

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta|_{r=1}-\frac{Nu}{2}\theta|_{r=1}r^2. \tag{63}$$

Then, for

First equation: $p_1=\left(\frac{3-2Da}{3Da\,\mathrm{Re}}\right)$ and $q_1(t)=-\frac{\gamma}{3Da\,\mathrm{Re}}$

Second equation: $p_1=0$ and $q_1(t)=0$

Third equation: $p_1=\frac{\beta ShNu}{Pem}\phi|_{r=1}$ and $q_1(t)=\frac{2Ecg}{3\mathrm{Re}}\gamma\left(1-\frac{\gamma}{3}\right)$.

So, for

First equation: $p(t)=\frac{1}{A}\left(\frac{2Da-3}{3Da\,\mathrm{Re}}\right),\qquad q(t)=-\frac{1}{A}\left(\frac{4Da+\gamma}{3Da\,\mathrm{Re}}\right)$

Second equation: $p(t)=\frac{1}{A}\left(\frac{3Sh}{Pem}\right),\qquad q(t)=0$

Third equation: $p(t)=\frac{1}{A}\left(\frac{3Nu}{Pe}-\frac{\beta ShNu}{Pem}\phi\big|_{r=1}\right),\qquad q(t)=\frac{2Ecg}{3\,\mathrm{Re}\,A}\gamma\left(1-\frac{\gamma}{3}\right).$

Then

$$u\big|_{r=1}=\frac{q(t)}{p(t)}+\left(1-\frac{q(t)}{p(t)}\right)e^{-p(t)t} \tag{64}$$

$$\phi\big|_{r=1}=\frac{1}{2}e^{-p(t)t} \tag{65}$$

$$\theta\big|_{r=1}=e^{-\left(ct+\frac{c_1}{p(t)}e^{-p(t)t}\right)}\left(\frac{1}{2}+q(t)\left(\frac{1}{c}\left(e^{\left(ct+\frac{c_1}{p(t)}e^{-p(t)t}\right)}-1\right)-\frac{c_1}{c}\left(\left(1-e^{\left(t+\frac{c_1}{p(t)}e^{-p(t)t}\right)}\right)+\frac{c_1\sqrt{\pi}}{\sqrt{-c_1p(t)}}erf\left(\frac{\sqrt{-c_1p(t)}}{p(t)}\left(e^{-t}-1\right)\right)\right)\right)\right) \tag{66}$$

where

$$c=\frac{3Nu}{PeA},\qquad c_1=\frac{\beta ShNu}{2PemA}$$

Thus,

$$u(r,t)=\left(\frac{q(t)}{p(t)}+\left(1-\frac{q(t)}{p(t)}\right)e^{-p(t)t}-\frac{\gamma}{2}\right)+\frac{\gamma}{2}r^2 \tag{67}$$

$$\phi(r,t)=\left(1+\frac{Sh}{2}\right)\frac{1}{2}e^{-p(t)t}-\frac{Sh}{2}\frac{1}{2}e^{-p(t)t}r^2 \tag{68}$$

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta\big|_{r=1}-\frac{Nu}{2}\theta\big|_{r=1}r^2. \tag{69}$$

4 Discussion

Readers should note that:

The p_1 as used in this paper could be a constant or a function. For cases 1 and 2, $b\neq 2$ and case 3, $a\neq 2$. When $b=2$ in cases 1 and 2 or $a=2$ in case 3, a new space variable needs to be introduced.

$\phi|_{r=a}$ as used in this paper, means $\phi(a,t)$. It meant $\phi(r,t)$ at $r=a$.

$\phi|_{r=b}$ as used in this paper, is the same as $\phi(b,t)$. It meant $\phi(r,t)$ at $r=b$.

$\phi(r,t)\neq\phi(a,t)$ and $\phi(r,t)\neq\phi(b,t)$ as being used by some authors in the literature. Most especially those that worked on the lumped model for transient heat conduction using the polynomial approximation method.

5 Conclusion

The use of Olayiwola's generalized polynomial approximation method (OGPAM) is important for solving partial differential equations. Also, the derivation of the generalized polynomial approximation method for solving hyperbolic equations by the author is ongoing and it will be made available to readers as soon as is ready.

Different examples have been solved using OGPAM in different geometries. The simplified relations obtained in this study can be used for science and engineering calculations in many conditions.

References

[1] Behera P. (2009). Analysis of transient heat conduction in different geometries. M.Tech.Thesis, National Institute of Technology, Rourkela.

[2] Benito J.J., Urena F., and Gavete L. (2007). Solving parabolic and hyperbolic equations by the generalized finite difference method. Journal of Computational and Applied Mathematics, Vol. 209: 208-233.

[3] Kewat S. K., and Verma S. (2017). Analysis of lumped system in slab. International Journal of Engineering Science and Computing, Vol. 7(6): 13004-13007.

[4] Mohanty S., and Dalai B. (2018). Modified lumped model for transient heat conduction in a hollow sphere. International Journal for Research in Engineering Application and Management, Vol. 4(7): 660-663.

[5] Olayiwola R.O., Cole A.T., Shehu M.D., Oguntolu F.A., Iyeme E.E., and Abubakar A.W. (2022). Modelling heat and mass transfer of a CO2 binary mixture: a mathematical approach. International Journal of Mathematical Analysis and Modelling, Vol. 5(2):191-215.

[6] Pradhan S.K., Behuria S.N., Mohanty S., and Suchismita S. (2019). Improved lumped model for transient heat conduction in a hollow sphere for different heat conduction problems. International Journal of Applied Engineering Research, Vol. 14(13): 31-34.

[7] Prakash A., and Mahmood S. (2013). Modified lumped model for transient heat conduction in spherical shape. American International Journal of Research in Science, Technology, Engineering and Mathematics, 2(2): 155-159.

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 3, October 2022, Pages 44 – 64

A series solution to asset valuation and pricing of linear evolution equations with stochastic partial derivatives

I.U. Amadi*, I. Davies†, and B.O. Osu‡

Abstract

The achievement of any investment depends mainly on the value of asset which impels the entire financial power of every company and differential equations are well known prevailing mathematical tool used for the prediction of stock prices. Therefore, this paper, considered four investment equations with stochastic and stock return rates parameters in the model. Detailed conditions are achieved which govern asset price return rates through multiplicative effects, additive invers effects, multiplicative inverse effects, stock volatility and asset growth-rates of assets parameter respectively. Consequently, the impacts on the value of asset prices of investors in capital market were analyzed; and graphical results of stock variables and the effect of relevant parameters were discussed for the purpose of investment plans.

Keywords: stock price; asset pricing; series solution; stochastic analysis; return rates

1 Introduction

Making short or long term business plans needs financial assets which is geared towards bringing returns to the investors over time. Financial assets are the basis with through which individual or group of persons generate wealth to become comfortable in life. In any time-varying investment, cash flows acquired by the investor as cash dividends over the period as the rate of return is expected to cover-up unforeseen risks of the investments. More so, an investor must take into account on par the safety and investment rates of returns to avoid event risks; like market crashes, flood disasters and political crisis which may directly or indirectly impact on investments. However, without capital investment human needs may be difficult to actualize but, even the most secure stocks can encounter losses for investors or other stakeholders, therefore an investor may need to rebalance his portfolio of investment or diversify his asset allocation in order to reduce overall portfolio risk and improve expected returns to cover-up loop-holes in the investment.

*Department of Mathematics/ Statistics, Captain Elechi Amadi Polytechnic, Rumuola, Port Harcourt, Nigeria. Email address: innocent.amadi@portharcourtpoly.edu.ng

†Department of Mathematics, Rivers State University, Nkpolu Oroworokwo, Port Harcourt, Nigeria

‡ Corresponding Author. Department of Mathematics, Abia State University , Uturu Abia State, Nigeria; E-mail: Osu.bright@abiastateuniversity.edu.ng

It is worthy of note that, in modeling the value of the rate of return on stocks and other relevant stochastic parameters involved in making investment decisions that accounts for certain level of unpredicted randomness, an accurate analytical method of solution is to be adopted based on the specifics of the problem in order to stimulate realistic stock quantities or variables; that is why differential equation models with stochastic parameters are most frequently adopted to capture other environmental effects.

A differential equation [11] is an important tool for harnessing different components into a single system and analyzing the inter-relationships that exists between these components which otherwise would continue to be independent of each other. Differential equations are one of the most frequently used tools for mathematical modeling in engineering and sciences [8-9]. Thus, differential equations have been used in different ways and independent results obtained. For instance,[10] considered the unstable nature of stock market forces using proposed differential equation model. [13] studied the variation in capital market changes where they developed a differential equation with stochastic parameter in the model obtaining an analytical solution which determined the equilibrium prices.

However, [16] considered a set of simultaneous differential equations of stock prices of the share in both A and H stock markets. They applied iteration method and results were obtained. [2] Studied the application of differential equation on stock market prices where they modeled two competing growth rates and there carrying capacity on stock market prices which was chosen based on minimum variance criteria. Similarly, [11] investigated a problem using differential equation in stock market analysis; the two competing growth rates and it's carrying capacity were considered. In [12], ordinary differential equations with variable coefficients problem were determined; the Elzaki transform was applied and results obtained. See for example [3],[4],[6],[7],[14],[15], and [18] for more research works on stock market price modeling.

This research work considers four systems of second order ordinary differential equations in time varying investments with stochastic parameter whose rate of returns is assumed to follow: multiplicative effects, additive inverse effects, multiplicative inverse effects; all having quadratic functions over time. These investment equations were solved analytically by adopting series solution of Ferobenius method and four solutions were obtained. To verify the validity of the analytical solutions, the value of assets and relevant parameters in the model were assed using detailed conditions of predicting stock returns and asset valuations; and the effects of rate of returns, asset valuation and stock volatility displayed graphically. This paper extends the works of [5] by solving the problem in second order ordinary differential equation domain.

This paper is organized as follows: Section 2.1 presents the Mathematical framework, problem formulation is seen in Subsection 2.1.1, method of solution is seen in Subsection 2.1.2, Results and discussion are seen in Section 3.1 and paper is concluded in Section 4.1.

2 Mathematical framework

Thus, let $S(t)$ be the price of some risky asset at time t ,and μ , an expected rate of returns on the stock and dt as a relative change during the trading days such that the stock follows a random walk which is govern by a stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t \tag{1}$$

where, μ is drift and σ the volatility of the stock, W_t is a Brownian motion or Wiener's process on a probability space $(\Omega, \xi, \wp)$, ξ is a $\sigma-\text{algebra}$ generated by $W_t, t \geq 0$.

Theorem 1: (Ito's formula) Let $(\Omega, \beta, \mu, \mathrm{F}(\beta)$ be a filtered probability space $X = \{X, t \geq 0\}$ be an adaptive stochastic process on $(\Omega, \beta, \alpha, \mathrm{F}(\beta)$ possessing a quadratic variation (X) with SDE defined as:

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t) \tag{2}$$

$t \in \Re$ and for $u = u(t, X(t) \in C^{1\times2}(\Pi \times \Box))$

$$du(t, X(t)) = \left\{\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial x} + \frac{1}{2}f^2\frac{\partial^2 u}{\partial x^2}\right\}d\tau + f\frac{\partial u}{\partial x}dW(t) \tag{3}$$

The solution to stochastic differential equation (1) can be obtained easily using Theorem 1 and is given by:

$$S(t) = S_0 \exp\left\{\sigma dW(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}, \forall t \in [0,1].$$

From (1) Black-Scholes Partial Differential Equation (PDE) is given as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{4}$$

Following the method of [14]

$$dS(t) = \hat{\alpha}S(t)dt + \sigma S(t)dW(t), \hat{\alpha} = \alpha + \lambda \tag{5}$$

where λ is the market price of risk. The equation governing stock option is backward Black-Scholes partial differential equation given (in one variable) as Osu (2008), Osu, et al. (2009).

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (\alpha - \lambda)S\frac{\partial V}{\partial S} - ru = 0. \tag{6}$$

Assume $\lambda = 0$, (6) gives

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \alpha S\frac{\partial V}{\partial S} - rV = -\frac{\partial V}{\partial t}. \tag{7}$$

From (7) $V \neq V(t)$ hence V is a function of S alone and we arrived at the following 2nd order nonhomogeneous differential equation by also defining the index price of the form: $-\varphi - GR_t$ which replaces RHS of (7) below.

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + \alpha S \frac{dV}{dS} - rV = -\varphi - GR_t \tag{8}$$

where φ represents constant and GR_t is growth of the underlying asset.

Suppose the interest rate $r = 1$ in the left hand side of (8) becomes:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + \alpha S \frac{dV}{dS} - V = -\varphi - GR_t\,. \tag{9}$$

Dividing homogeneous part of (9) by $\dfrac{\sigma^2 S}{2}$ gives

$$S\frac{d^2V}{dS^2} + \frac{2\alpha}{\sigma}\frac{dV}{dS} - \frac{2rV}{S\sigma^2} = 0\,. \tag{10}$$

Suppose the trading of shares or adjustment of portfolio is allowed to take place. We therefore adjust the stock variables following the method of [13] by setting:

$\dfrac{2\alpha}{\sigma} = 1,\ \dfrac{2r}{S\sigma^2} = S\sigma^2 V,$ Since the rate of change of investment output value depends on the investment output at present time t ,[7] (10) becomes

$$S\frac{d^2V}{dS^2} + \frac{dV}{dS} - S\sigma^2 V = 0 \tag{11}$$

The last term of LHS of (11) is the aggregate intrinsic value of stock, [7] hence it comprises the volatility of the underlying asset throughout the trading days. Secondly, risk factors S describe any kind of risk and uncertainty present in the financial market such as stock prices, interest rate, strike price etc. So, combining (11) and (LHS) of (10) gives a complete 2nd non-homogenous ordinary differential equation below:

$$S\frac{d^2V}{dS^2} + \frac{dV}{dS} - S\sigma^2 V = -\varphi - GR_t\,. \tag{12}$$

2.1 Problem Formulation

Now, we consider capital investment plans where investors will be faced with some decisions on how to value assets and its return rates. So, investors at all seasons receive its wages at different time intervals. So, considering four different investments selected stocks (companies) to be represented as $S_1.S_2, S_3$ and S_4 be defined in the system of variable coefficient differential equations, the factors underlying price changes are very uncertain and are described in probability terms. Hence the rate of returns is defined following the method of [3] and [5] respectively:

$$R_t := \left(\lambda_1 \lambda_2\right)^2, \ldots \text{where } t = 1, 2\ldots \tag{13}$$

$$R_t := \left(\lambda_1 + \lambda_2\right)^2, .., \text{where } t = 1, 2\ldots \tag{14}$$

$$R_t := \left\{\left(\lambda_1 + \lambda_2\right)^{-1}\right\}^2, .., \text{where } t = 1, 2\ldots \tag{15}$$

$$R_t := \left\{\left(\lambda_1 \lambda_2\right)^{-1}\right\}^2, .., \text{where } t = 1, 2\ldots \tag{16}$$

Using (13)-(16) in (12) independently gives the following investment equations for four different stocks (companies):

$$S_1 \frac{d^2 V_{1(t)}}{dS_1^{\,2}} + \frac{dV_{1(t)}}{dS_1} - (\lambda_1 \lambda_2)^2 S_1 \sigma^2 V_{1(t)} = 0 \tag{17}$$

$$S_2 \frac{d^2 V_{2(t)}}{dS_2^{\,2}} + \frac{dV_{2(t)}}{dS_2} - \left\{(\lambda_1 + \lambda_2)^{-1}\right\}^2 S_2 \sigma^2 V_{2(t)} = 0 \tag{18}$$

$$S_3 \frac{d^2 V_{3(t)}}{dS_3^{\,2}} + \frac{dV_{3(t)}}{dS_3} - \left\{(\lambda_1 + \lambda_2)^{-1}\right\}^2 S_3 \sigma^2 V_{3(t)} = -\varphi - GR_t V_{4(t)} \tag{19}$$

$$S_4 \frac{d^2 V_{4(t)}}{dS_4^{\,2}} + \frac{dV_{4(t)}}{dS_4} - \left\{(\lambda_1 \lambda_2)^{-1}\right\}^2 S_4 \sigma^2 V_{4(t)} = 0\,. \tag{20}$$

With the following boundary conditions

$$\left.\begin{array}{l} V_{1(t)} = V_1\alpha,\ V_{2(t)} = V_2\alpha,\ V_{3(t)} = 0, V_{4(t)} = V_4\alpha \\ \dfrac{dV_{1(t)}}{dS_1} = \dfrac{dV_{2(t)}}{dS_2} = \dfrac{dV_{3(t)}}{dS_3} = \dfrac{dV_{4(t)}}{dS_4} = 0 \end{array}\right\} \tag{21}$$

2.2 Method of Solution

The propose model (17)-(20) consist of a system of variable coefficient differential equations whose solutions are not trivial. We adopts the method of Ferobenius in solving for $V_{1(t)}(S), V_{2(t)}(S)$,$V_{3(t)}(S)$ and $V_{4(t)}(S)$, Firstly, we solve equation (17) and (18), then solve (20) and substituting the result in equation (19) to obtain following stock quantities. To tackle this problems adequately we note that $\mathrm{V}_{1(t)}(\mathrm{S}),\ \mathrm{V}_{2(t)}(\mathrm{S}), \mathrm{V}_{3(t)}(\mathrm{S}), \mathrm{V}_{4(t)}(\mathrm{S}) \ < \infty$ for all $S \in [0,1)$.

Solving Investment Equation when stock returns follow multiplicative effects:

From the homogenous part of (17)

$$\text{Let } V_{1(t)} = \sum_{m=0}^{\infty} a_m S_1^{\,m+c} = S_1^{\,c} \sum_{m=0}^{\infty} a_m S_1^{\,m} \tag{22}$$

$$V_{1(t)}' = \sum_{m=0}^{\infty} a_m \left(m+c\right) S_1^{\,m+c-1} = S_1^{\,c=1} \sum_{n=0}^{\infty} a_m \left(m+c\right) S_1^{\,m} \tag{23}$$

$$V_{1(t)}'' = S_1^{\,c-2} \sum_{m=0}^{\infty} a_m \left(m+c\right)\left(m+c-1\right) S_1^{\,m} = S_1^{\,c-1} \sum_{m=0}^{\infty} a_m \left(m+c\right)\left(m+c-1\right) S_1^{\,m} \tag{24}$$

substituting (18)-(20) into (17) after long algebraic simplifications gives

$$V_{1(t)}(S) = v_1 = \alpha_1 \left\{ \begin{array}{l} 1 + \dfrac{(\lambda_1\lambda_2)^2 \sigma^2 S^2}{2^2} + \dfrac{(\lambda_1\lambda_2)^4 \sigma^4 S^4}{2^2 \times 4^2} + \dfrac{(\lambda_1\lambda_2)^6 \sigma^6 S^6}{2^2 \times 4^2 \times 6^2} + \dfrac{(\lambda_1\lambda_2)^8 \sigma^8 S^8}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \dfrac{(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} + \ldots \end{array} \right\} \tag{25}$$

Another is given by $v_2 = \dfrac{dV_{1(t)}}{dc}$

$$\frac{dV_{1(t)}}{dc} = a_0 S^c lnr \left\{ 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S^2}{(c+2)^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S^4}{(c+2)^2 (c+4)^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S^6}{(c+2)^2 (c+4)^2 (c+6)^2} \right.$$
$$\left. + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S^8}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2} + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2 (c+10)^2} .. \right\}$$
$$+ a_0 S^c \frac{d}{dc} \left\{ 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S^2}{(c+2)^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S^4}{(c+2)^2 (c+4)^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S^6}{(c+2)^2 (c+4)^2 (c+6)^2} \right.$$
$$\left. + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S^8}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2} + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2 (c+10)^2} + \ldots \right\}$$

$$\frac{dV_{1(t)}}{dc} = a_0 S^c \ln S \left\{ 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S^2}{(c+2)^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S^4}{(c+2)^2 (c+4)^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S^6}{(c+2)^2 (c+4)^2 (c+6)^2} \right.$$
$$\left. + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2} + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{(c+2)^2 (c+4)^2 (c+6)^2 (c+8)^2 (c+10)^2} + \ldots \right\}$$
$$+ a_0 r^c \left\{ 0 - \frac{2(\lambda_1\lambda_2)^2 \sigma^2 S^2}{(c+2)^3} - \frac{4(\lambda_1\lambda_2)^4 \sigma^4 S^4 (c+3)}{(c+2)^3 (c+4)^3} - \frac{8(\lambda_1\lambda_2)^6 \sigma^6 S^6}{(c+2)^3 (c+4)^3 (c+6)^3} + \frac{16(\lambda_1\lambda_2)^8 \sigma^8 S^8}{(c+2)^3 (c+4)^3 (c+6)^3 (c+8)^3} \right.$$
$$\left. - \frac{32(\lambda_1\lambda_2)^{10} \sigma^{10} S^{10}}{(c+2)^3 (c+4)^3 (c+6)^3 (c+8)^3} + \ldots \right\}$$

$$V_{2(t)}(S_2) = v_1 = \phi_1 \left\{ \begin{array}{l} 1 + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^2 \sigma^2 S_2{}^2}{2^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^4 \sigma^4 S_2{}^4}{2^2 \times 4^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^6 \sigma^6 S_2{}^6}{2^2 \times 4^2 \times 6^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^8 \sigma^8 S_2{}^8}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_2{}^{10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} + \ldots \end{array} \right\}$$

$$V_{1(t)} = v_2 = \alpha_2 \left\{ \begin{array}{l} \ln S_1 \left(\begin{array}{l} 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S_1^{\,2}}{2^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S_1^{\,4}}{2^2 \times 4^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S_1^{\,6}}{2^2 \times 4^2 \times 6^2} + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S_1^{\,8}}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S_1^{\,10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} \end{array} \right) \\ + a_0 S_1^{\,c} \left(\begin{array}{l} - \frac{(\lambda_1\lambda_2)^2 \sigma^2 S_1^{\,2}}{2^2} - \frac{(\lambda_1\lambda_2)^4 \sigma^4 S_1^{\,4}}{2^3 \times 4^2} - \frac{(\lambda_1\lambda_2)^6 \sigma^6 S_1^{\,6}}{4^3 \times 6^3} + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S_1^{\,8}}{2^5 \times 6^3 \times 8^3} \\ - \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S_1^{\,10}}{4^2 \times 6^3 \times 8^3 \times 10^3} + \ldots \end{array} \right) \end{array} \right\} \tag{26}$$

A linear combination (25) and (26) gives the complete solution

$$\begin{array}{l} V_{1(t)}(S_1) = \alpha_1 \left\{ \begin{array}{l} 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S_1^{\,2}}{2^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S_1^{\,4}}{2^2 \times 4^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S_1^{\,6}}{2^2 \times 4^2 \times 6^2} + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S_1^{\,8}}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S_1^{\,10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} + \ldots \end{array} \right\} \\ + \alpha_2 \left\{ \begin{array}{l} \ln S_1 \left(\begin{array}{l} 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 S_1^{\,2}}{2^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 S_1^{\,4}}{2^2 \times 4^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 S_1^{\,6}}{2^2 \times 4^2 \times 6^2} + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S_1^{\,8}}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} S_1^{\,10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} \end{array} \right) \\ - \frac{(\lambda_1\lambda_2)^2 \sigma^2 S_1^{\,2}}{2^2} - \frac{(\lambda_1\lambda_2)^4 \sigma^4 S_1^{\,4}}{2^3 \times 4^2} - \frac{(\lambda_1\lambda_2)^6 \sigma^6 S_1^{\,6}}{4^3 \times 6^3} + \frac{(\lambda_1\lambda_2)^8 \sigma^8 S_1^{\,8}}{2^5 \times 6^3 \times 8^3} - \frac{(\lambda_1\lambda_2)^2 \sigma^{10} S_1^{\,10}}{4^2 \times 6^3 \times 8^3 \times 10^3} + \ldots \end{array} \right\} \end{array}$$

(27)

Applying the boundary conditions in (21) and setting $\alpha_2 = 0$ gives $\alpha_1 = \frac{V_1 \alpha}{y_1(1)}$

$$y_1(\alpha) = \left(\begin{array}{l} 1 + \frac{(\lambda_1\lambda_2)^2 \sigma^2 \alpha^2}{2^2} + \frac{(\lambda_1\lambda_2)^4 \sigma^4 \alpha^4}{2^2 \times 4^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^6 \alpha^6}{2^2 \times 4^2 \times 6^2} + \frac{(\lambda_1\lambda_2)^6 \sigma^8 \alpha^8}{2^2 \times 4^2 \times 6^2 \times 8^2} \\ + \frac{(\lambda_1\lambda_2)^{10} \sigma^{10} \alpha^{10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} + \ldots \end{array} \right)$$

$$V_{1(t)}(S_1)=\frac{V_1\alpha}{y_1(\alpha)}\left\{\begin{array}{l}1+\frac{\left(\lambda_1\lambda_2\right)^2\sigma^2 S_1^{\ 2}}{2^2}+\frac{\left(\lambda_1\lambda_2\right)^4\sigma^4 S_1^{\ 4}}{2^2\times4^2}+\frac{\left(\lambda_1\lambda_2\right)^6\sigma^6 S_1^{\ 6}}{2^2\times4^2\times6^2}+\frac{\left(\lambda_1\lambda_2\right)^6\sigma^8 S_1^{\ 8}}{2^2\times4^2\times6^2\times8^2}\\ +\frac{\left(\lambda_1\lambda_2\right)^{10}\sigma^{10} S_1^{\ 10}}{2^2\times4^2\times6^2\times8^2\times10^2}+\ldots\end{array}\right\}$$

(28)

Solving Investment Equation when stock returns follow additive effects:

Solving (18) from left hand side using Frobenius method

$$\text{Let } V_{2(t)}=\sum_{m=0}^{\infty}a_m S_2^{\ m+c}=S_2^{\ c}\sum_{m=0}^{\infty}a_m S_2^{\ m} \tag{29}$$

$$V'_{2(t)}=\sum_{m=0}^{\infty}a_m\left(m+c\right)S_2^{\ m+c-1}=S_2^{\ c-1}\sum_{m=0}^{\infty}a_m\left(m+c\right)S_2^{\ m} \tag{30}$$

$$V''_{2(t)}=S_2^{\ c-2}\sum_{m=0}^{\infty}a_m\left(m+c\right)\left(m+c-1\right)S_2^{\ m}=S_2^{\ c-1}\sum_{m=0}^{\infty}a_m\left(m+c\right)\left(m+c-1\right)S_2^{\ m} \tag{31}$$

substituting (29)-(31) into (18) and after long algebraic expressions gives

$$V_{2(t)}=v_2=\phi_2\left\{\begin{array}{l}\ln S_2\left(\begin{array}{l}1+\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2^{\ 2}}{2^2}+\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2^{\ 4}}{2^2\times4^2}+\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2^{\ 6}}{2^2\times4^2\times6^2}\\ +\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8 S_2^{\ 8}}{2^2\times4^2\times6^2\times8^2}+\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2^{\ 10}}{2^2\times4^2\times6^2\times8^2\times10^2}\end{array}\right)\\ +a_0 S_2^{\ c}\left(\begin{array}{l}-\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2^{\ 2}}{2^2}-\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2^{\ 4}}{2^3\times4^2}-\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2^{\ 6}}{4^3\times6^3}\\ +\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8 S_2^{\ 8}}{2^5\times6^3\times8^3}-\frac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2^{\ 10}}{4^2\times6^3\times8^3\times10^3}+\ldots\end{array}\right)\end{array}\right\} \tag{32}$$

A linear combination (32) and (33) then applying the boundary conditions (21) and setting $\alpha_2=0$ gives the complete solution

$$V_{2(t)}(S_2)=\phi_1\left\{\begin{array}{l}1+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2{}^2}{2^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2{}^4}{2^2\times 4^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2{}^6}{2^2\times 4^2\times 6^2}\\+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8 S_2{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots\end{array}\right\}$$

$$+\phi_2\left\{\begin{array}{l}\ln S_2\left(\begin{array}{l}1+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2{}^2}{2^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2{}^4}{2^2\times 4^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2{}^6}{2^2\times 4^2\times 6^2}\\+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8 S_2{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}\end{array}\right)\\-\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2{}^2}{2^2}-\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2{}^4}{2^3\times 4^2}-\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2{}^6}{4^3\times 6^3}\\+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8 S_2{}^8}{2^5\times 6^3\times 8^3}-\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2{}^{10}}{4^2\times 6^3\times 8^3\times 10^3}+\ldots\end{array}\right\} \tag{33}$$

$$\phi_1=\frac{V_2\alpha}{y_2(\alpha)},$$

where

$$y_2(\alpha)=\left\{\begin{array}{l}1+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2\alpha^2}{2^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4\alpha^4}{2^2\times 4^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6\alpha^6}{2^2\times 4^2\times 6^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^8\sigma^8\alpha^8}{2^2\times 4^2\times 6^2\times 8^2}\\+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10}\alpha^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots\end{array}\right\}$$

$$V_{2(t)}(S_2)=\frac{V_2\alpha}{y_2(\alpha)}\left\{\begin{array}{l}1+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\sigma^2 S_2{}^2}{2^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^4\sigma^4 S_2{}^4}{2^2\times 4^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^6 S_2{}^6}{2^2\times 4^2\times 6^2}\\+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^6\sigma^8 S_2{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\dfrac{\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^{10}\sigma^{10} S_2{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots\end{array}\right\} \tag{34}$$

Solving investment equation when stock returns follows additive inverse effects :

$$\text{Let } V_{3(t)} = \sum_{m=0}^{\infty} a_m S_3^{\ m+c} = S_3^{\ c} \sum_{m=0}^{\infty} a_m S_3^{\ m} \tag{35}$$

$$V'_{3(t)} = \sum_{m=0}^{\infty} a_m (m+c) S_3^{\ m+c-1} = S_3^{\ c=1} \sum_{n=0}^{\infty} a_m (m+c) S_3^{\ m} \tag{36}$$

$$V''_{3(t)} = S_3^{\ c-2} \sum_{m=0}^{\infty} a_m (m+c)(m+c-1) S_3^{\ m} = S_3^{\ c-1} \sum_{m=0}^{\infty} a_m (m+c)(m+c-1) S_3^{\ m}, \tag{37}$$

substituting (35)-(37) into (19) and long algebraic simplifications gives

$$V_{3(t)}(S_3) = v_1 = \alpha_1 \left\{ \begin{aligned} & 1 + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^2 \sigma^2 S_3^{\ 2}}{2^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^4 \sigma^4 S_3^{\ 4}}{2^2 \times 4^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^6 \sigma^6 S_3^{\ 6}}{2^2 \times 4^2 \times 6^2} \\ & + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^8 \sigma^8 S_3^{\ 8}}{2^2 \times 4^2 \times 6^2 \times 8^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_3^{\ 10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} + \ldots \end{aligned} \right\} \tag{38}$$

$$V_{3(t)} = v_2 = \alpha_2 \left\{ \begin{aligned} & \ln S_3 \left(\begin{aligned} & 1 + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^2 \sigma^2 S_3^{\ 2}}{2^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^4 \sigma^4 S_3^{\ 4}}{2^2 \times 4^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^6 \sigma^6 S_3^{\ 6}}{2^2 \times 4^2 \times 6^2} \\ & + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^8 \sigma^8 S_3^{\ 8}}{2^2 \times 4^2 \times 6^2 \times 8^2} + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_3^{\ 10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} \end{aligned} \right) \\ & + a_0 S_3^{\ c} \left(\begin{aligned} & - \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^2 \sigma^2 S_3^{\ 2}}{2^2} - \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^4 \sigma^4 S_3^{\ 4}}{2^3 \times 4^2} - \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^6 \sigma^6 S_3^{\ 6}}{4^3 \times 6^3} \\ & + \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^8 \sigma^8 S_3^{\ 8}}{2^5 \times 6^3 \times 8^3} - \frac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_3^{\ 10}}{4^2 \times 6^3 \times 8^3 \times 10^3} + \ldots \end{aligned} \right) \end{aligned} \right\} \tag{39}$$

A linear combination (38) and (39) then applying the boundary conditions (21) and setting $\alpha_2 = 0$ gives the complete solution

$$V_{3(t)}(S_3) = \left\{ \begin{array}{l} 1 + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^2 \sigma^2 {S_3}^2}{2^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^4 \sigma^4 {S_3}^4}{2^2\times 4^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^6 \sigma^6 {S_3}^6}{2^2\times 4^2\times 6^2} \\ + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^8 \sigma^8 {S_3}^8}{2^2\times 4^2\times 6^2\times 8^2} + \dfrac{\left\{(\lambda_1+\lambda_2)^{-1}\right\}^{10} \sigma^{10} v^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2} + \ldots \end{array} \right\} \quad (40)$$

Solving investment equation when stock returns follow multiplicative inverse effects:

From (20)

$$\text{Let } V_{4(t)} = \sum_{m=0}^{\infty} a_m {S_4}^{m+c} = {S_4}^c \sum_{m=0}^{\infty} a_m {S_4}^m \quad (41)$$

$$V'_{4(t)} = \sum_{m=0}^{\infty} a_m (m+c) {S_4}^{m+c-1} = {S_4}^{c=1} \sum_{n=0}^{\infty} a_m (m+c) {S_4}^m \quad (42)$$

$$V''_{4(t)} = {S_4}^{c-2} \sum_{m=0}^{\infty} a_m (m+c)(m+c-1) {S_4}^m = {S_4}^{c-1} \sum_{m=0}^{\infty} a_m (m+c)(m+c-1) {S_4}^m, \quad (43)$$

substituting (41)-(43) into (21) and after long algebraic simplificatins gives

$$V_{4(t)}(S_4) = v_1 = \alpha_1 \left\{ \begin{array}{l} 1 + \dfrac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 {S_4}^2}{2^2} + \dfrac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 {S_4}^4}{2^2\times 4^2} + \dfrac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 {S_4}^6}{2^2\times 4^2\times 6^2} \\ + \dfrac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^8 \sigma^8 {S_4}^8}{2^2\times 4^2\times 6^2\times 8^2} + \dfrac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} {S_4}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2} + \ldots \end{array} \right\} \quad (44)$$

$$V_{4(t)} = v_2 = \alpha_2 \left\{ \begin{array}{l} \ln S_4 \left(\begin{array}{l} 1+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 S_4{}^2}{2^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 S_4{}^4}{2^2\times 4^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 S_4{}^6}{2^2\times 4^2\times 6^2} \\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^8 \sigma^8 S_4{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_4{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2} \end{array} \right) \\ +a_0 S_4{}^c \left(\begin{array}{l} -\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 S_4{}^2}{2^2}-\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 S_4{}^4}{2^3\times 4^2}-\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 S_4{}^6}{4^3\times 6^3} \\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^8 \sigma^8 S_4{}^8}{2^5\times 6^3\times 8^3}-\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_4{}^{10}}{4^2\times 6^3\times 8^3\times 10^3}+\ldots \end{array} \right) \end{array} \right\} \quad (45)$$

A linear combination (44) and (45) then applying the boundary conditions in(21) and setting $\alpha_2 = 0$ gives the complete solution $\alpha_1 = \frac{V_4\alpha}{y_3(\alpha)}$

where

$$y_3(\alpha) = \left\{ \begin{array}{l} 1+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 \alpha^2}{2^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 \alpha^4}{2^2\times 4^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 \alpha^6}{2^2\times 4^2\times 6^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^8 \sigma^8 \alpha^8}{2^2\times 4^2\times 6^2\times 8^2} \\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} \alpha^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots \end{array} \right\}$$

$$V_{4(t)}(S_4) = \frac{V_4\alpha}{y_3(1)} \left\{ \begin{array}{l} 1+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 S_4{}^2}{2^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 S_4{}^4}{2^2\times 4^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 S_4{}^6}{2^2\times 4^2\times 6^2} \\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^8 S_4{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_4{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots \end{array} \right\} \quad (46)$$

Non-homogenous part of investment equation when stock returns follow additive inverse effects

To get the particular solution of (19) which is the non-homogenous part of the problem, we solve using the method of undetermined coefficients.

$$V_{4(t)}(S_4) = \alpha_1 \left\{ \begin{array}{l} 1+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \sigma^2 S_4{}^2}{2^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^4 \sigma^4 S_4{}^4}{2^2\times 4^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^6 S_4{}^6}{2^2\times 4^2\times 6^2} \\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^6 \sigma^8 S_4{}^8}{2^2\times 4^2\times 6^2\times 8^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\right\}^{10} \sigma^{10} S_4{}^{10}}{2^2\times 4^2\times 6^2\times 8^2\times 10^2}+\ldots \end{array} \right\}$$

From the complementary function for (19)

$$V_{3(t)c}\left(S_3\right)=C_c\left[\begin{array}{l}1+\frac{\left\{(\lambda_1+\lambda_2)^{-1}\sigma S_3)\right\}^2}{2^2}+\frac{\left\{(\lambda_1+\lambda_2)^{-1}\sigma S_3)\right\}^4}{\left(2\times 4\right)^2}+\frac{\left\{(\lambda_1+\lambda_2)^{-1}\sigma S_3)\right\}^6}{\left(2\times 4\times 6\right)^2}\\ +\frac{\left\{(\lambda_1+\lambda_2)^{-1}\sigma S_3)\right\}^8}{\left(2\times 4\times 6\times 8\right)^2}+\frac{\left\{(\lambda_1+\lambda_2)^{-1}\sigma S_3)\right\}^{10}}{\left(2\times 4\times 6\times 8\times 10\right)^2}+\ldots\end{array}\right] \tag{47}$$

Consider the particular solution

$$V_p\left(S\right)=A_0+A_1S^2+A_2S^4+A_3S^6+A_4S^8+A_5S^{10} \tag{48}$$

$$V_p'\left(S\right)=2A_1S+4A_2S^3+6A_3S^5+8A_4S^7+10A_5S^9 \tag{49}$$

$$V_p''\left(S\right)=2A_1+12A_2S^2+30A_3S^4+56A_4S^6+90A_5S^8 \tag{50}$$

Putting (48)-(50) into (19) gives the following:

$$\begin{array}{l}\Rightarrow 4A_1+16A_2S^2+36A_3S^4+64A_4S^6+100A_5S^8-\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_0-\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_1S^2\\ -\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_2S^4-\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_3S^6-\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_4S^8-\left\{(\lambda_1+\lambda_2)^{-1}\sigma\right\}^2A_5S^{10}\equiv\\ -\varphi-GR_tV_{4(2)}\left(\begin{array}{l}1+\frac{\left\{(\lambda_1\lambda_2)^{-1}\sigma S\right\}^2}{2^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\sigma S\right\}^{24}}{(2\times 4)^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\sigma S\right\}^6}{(2\times 4\times 6)^2}+\frac{\left\{(\lambda_1\lambda_2)^{-1}\sigma S\right\}^8}{(2\times 4\times 6\times 8)^2}\\ +\frac{\left\{(\lambda_1\lambda_2)^{-1}\sigma S\right\}^{10}}{(2\times 4\times 6\times 8\times 10)^2}\end{array}\right)\end{array} \tag{51}$$

Taking like terms and simplifying after some algebraic expressions gives

$$\begin{array}{l}V_3\left(S_3\right)=c+A_0+\left(\frac{c((\lambda_1+\lambda_2)^{-1}\sigma)^2}{4}+A_1\right)S_3{}^2+\left(\frac{c((\lambda_1+\lambda_2)^{-1}\sigma)^4}{\left(2\times 4\right)^2}+A_2\right)S_3{}^4\\ +\left(\frac{c((\lambda_1+\lambda_2)^{-1}\sigma)^6}{\left(2\times 4\times 6\right)^2}+A_3\right)S_3{}^6+\left(\frac{c((\lambda_1+\lambda_2)^{-1}\sigma)^8}{\left(2\times 4\times 6\times 8\right)^2}+A_4\right)S_3{}^8+\left(\frac{c((\lambda_1+\lambda_2)^{-1}\sigma)^{10}}{\left(2\times 4\times 6\times 8\times 10\right)^2}+A_5\right)S_3{}^{10}\end{array} \tag{52}$$

where $A_0, A_1, A_2, A_3 A_4$ *and* A_5 are constants which is seen in appendix 1

3 Results and Discussion

In this Section, we present the computational results for the problem formulated in (17)-(21) whose solution is in (28), (34), (46) and (52). These solutions are implemented in a mathematica programming language. However, the following parameter values were explicitly used in the simulation study: $c = 1.0, \lambda_1 = 180, \lambda_2 = 120, \varphi = 1.0, V_{2\alpha} = 0.1$ and $V_\alpha = 0.1$. The non-linear 2nd order differential equations governing the problem are solved analytically for four different investment equations.

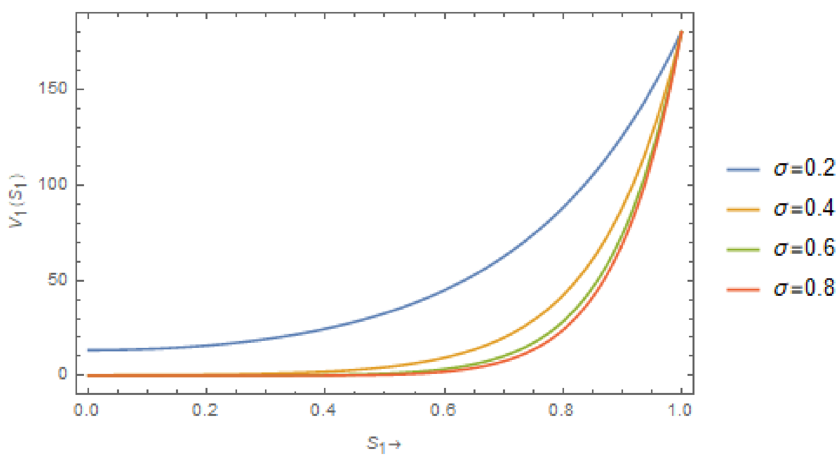


Figure 1: Stock (1) (S_1) stock return profiles against different values S of volatility(σ)

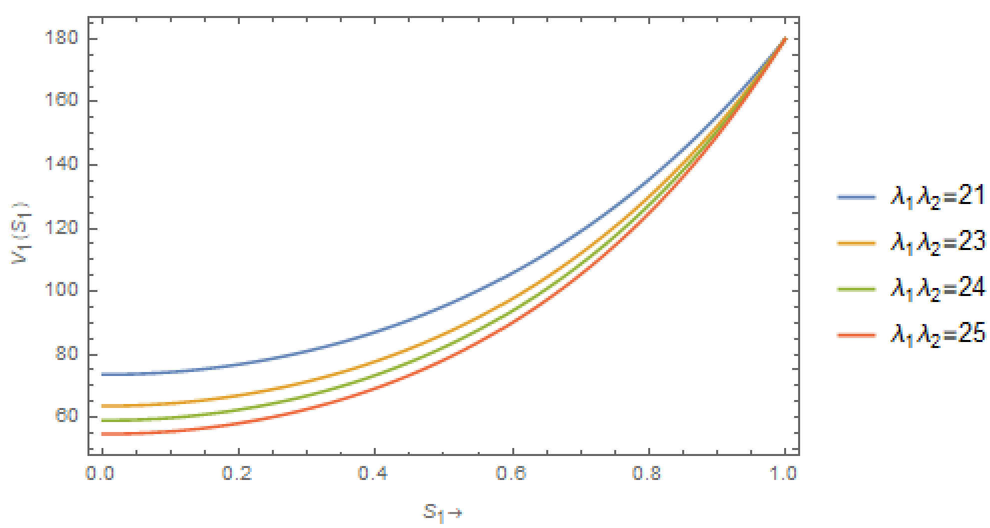


Figure 2: Plot of Multiplicative effects stock returns for assessing the value of assets for company (1)

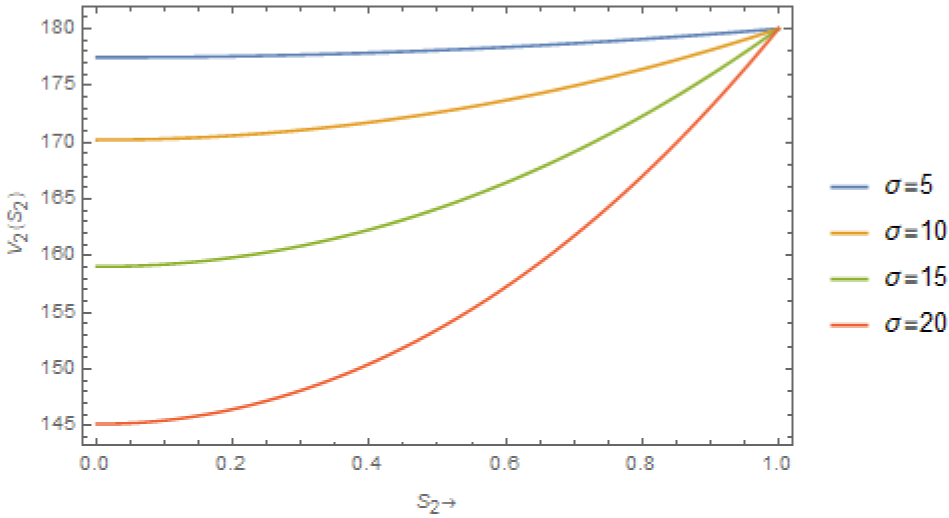


Figure 3: Stock (2) (S_2) stock return profiles against different values S of volatility(σ)

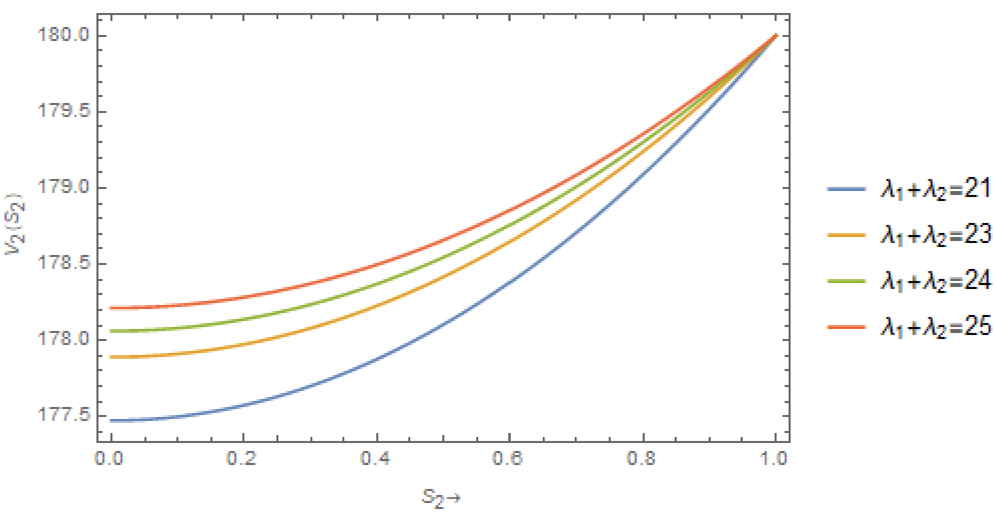


Figure 4: Plot of Additive inverse stock returns for assessing the value of assets for company (2)

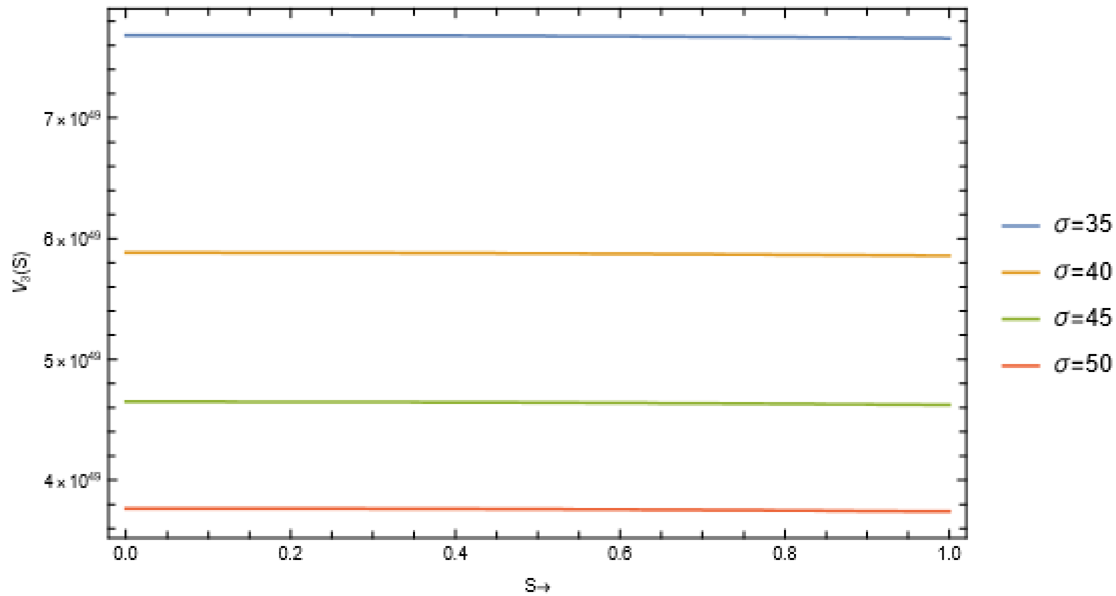


Figure 5: Stock (3) (S_3) stock return profiles against S with different values of volatility (σ)

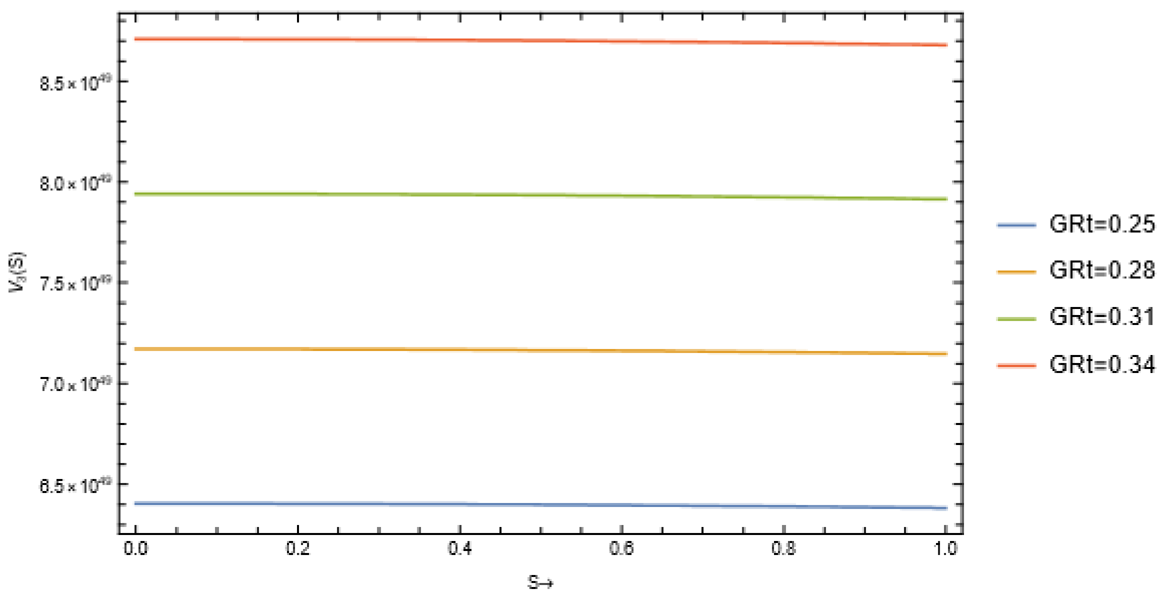


Figure 6: Stock (3) (S_3) stock return profiles against S with different values of volatility (GR_t)

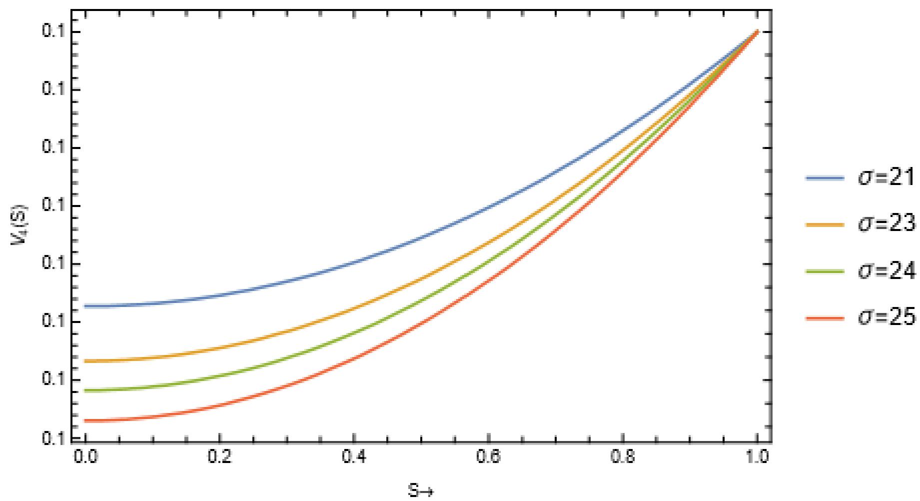


Figure 7: Stock (4) (S_4) stock return profiles against S with different values of volatility

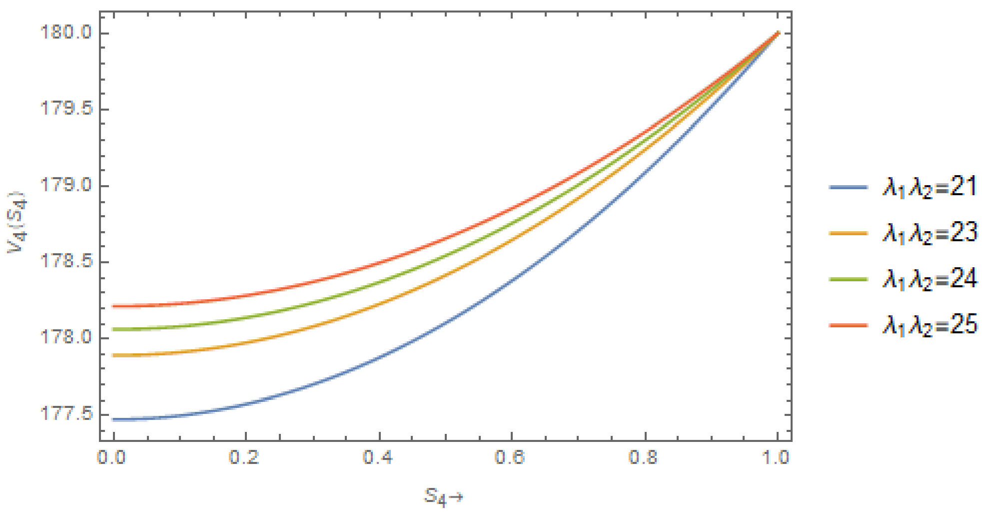


Figure 8: Plot of Multiplicative inverse stock returns for assessing the value of assets for company (4)

Figure 1 shows the variation of Stock (1) rate of return with volatility parameter. It can be seen that Stock (1) return reduces with increasing stock volatility. This is absolutely correct, because when there's high volatility in the market price, investors find it difficult to invest; thereby causing low rate of return in an investment. There will be no investor who will like to maximize loss in an investment; causing a kind of panic buying, tensions everywhere in stock trading. Visual inspection of the plot shows high levels of price changes in the capital investment.

Figure 2 shows the variation of Stock (1) rate of return which follows multiplicative effects. In this situation the results indicate increase in return rates decreases the gain of asset. This remark shows that, there are some stochastic bubbles in financial market which affected the investments. For instance, selling an auction price due to products expiring dates etc, one will be compelled to dispose the commodities on any amount in order to avoid expired products. So, in that case the levels of profit cannot be compared to the original amount of the investments. Also, the issue of over- staff in managerial cadre of the investments may causes the rate of return to decrease. This could also mean there is a serious crash worth of millions in any of the currency for capital investment. A crash simply implies a substantial droplet in the overall value of a market, creating unnecessary situation where lots of investors will not invest in capital market as a result of massive losses. The disadvantage of the scenario is that it results to panic buying, selling and deteriorating of markets which of course affects the entire masses.

Figure 3 shows the variation of Stock (2) rate of return with volatility parameter. A little increase in stock volatility parameter reduces stock return rates. That is, to say that volatility affects trading business over time in the investment which might be indexed with millions of naira. This situation informs investors on the activities of volatility in the capital investment hence it measures the price changes.

Figure 4 shows the variation of Stock (2) rate of return parameter. It is clear that increasing return rates when stock return rates follow additive inverse effect also increases the value of asset pricing throughout the stipulated trading days. It implies that additive inverse effect is profit maximizing in terms of long- and short-term investment plans.

Figure 5 shows the variation of Stock (3) rate of return with volatility parameter. It can be seen that Stock (3) rate of returns is pararrel; hence an increase in stock volatility decreases rate of returns. Therefore, profit making may not be possible due to the uniqueness of horizontal trend lines.
Figure 6 shows the variation of Stock (3) return with growth rate parameter. It is observed that a little increase in the growth-rate parameter increases the return rate of the capital investment over the trading days. This situation is quite obvious because when an investor increases its turn-over, the financial strength of the business will rise to a certain level.

Figure 7 shows the variation of Stock (4) return with volatility parameter. It is observed that increase in stock volatility reduces the rate of return of an investment. This is correct because when there's a high volatility in price formation of a commodity; investors will not be willing to invest, even up to final consumers will not also be eager to buy from the investors thereby causing low rate of return in the investment. In another scenario, the plot has an upward trend which moves separately and converges to a point during the trading days. This remark shows that rate of return in multiplicative inverse has a limit of how it can yield profit to an investor; this also describes the formative of random walk-in stochastic process.

In Figure 8, increasing return rate when stock returns follow multiplicative inverse effect also increases the value of asset pricing which is beneficial to every investor in terms of profit making. These remarks showing such pattern of trend implies high levels of progress during the period of an investments.

4 Conclusion

This paper, considered four investment equations with stochastic and stock return rates parameters in the model. Detailed conditions are achieved which govern asset price return rates through multiplicative effects, additive invers effects , multiplicative inverse effects ,stock volatility and asset growth-rates of assets parameter respectively .Therefore, the impacts on the value of asset prices of investors in capital market were analyzed; which shows: (i) Stock (1) return reduces with increasing stock volatility(ii) increasing growth rates in an investment automatically grows the financial strength of the investment over time (iii) A little increase in stock volatility parameter reduces stock return rates.(iv) increasing return rates when stock return rates follows additive inverse effect also increases the value of asset pricing throughout the stipulated trading days.(v) an increase in stock volatility decreases rate of returns(vi) a little increase in the growth-rate parameter increases the return rate of the capital investment over the trading days.

APPENDIX 1

Constants of coupled investment equations

$$\Rightarrow A_5 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[\frac{\left((\lambda_1\lambda_2)^{-1}\sigma\right)^{10} GR_t V_{4(t)}}{\left(2\times4\times6\times8\times10\right)^2}\right], \Rightarrow A_4 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[100A_5 + \frac{\left((\lambda_1\lambda_2)^{-1}\sigma\right)^{8} GR_t V_{4(t)}}{\left(2\times4\times6\times8\right)^2}\right]$$

$$\Rightarrow A_3 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[64A_4 - \frac{\left((\lambda_1\lambda_2)^{-1}\sigma\right)^{6} GR_t V_{4(t)}}{\left(2\times4\times6\right)^2}\right], \Rightarrow A_2 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[36A_3 - \frac{\left((\lambda_1\lambda_2)^{-1}\sigma\right)^{4} GR_t V_{4(t)}}{\left(2\times4\right)^2}\right]$$

$$\Rightarrow A_1 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[16A_2 - \frac{\left((\lambda_1\lambda_2)^{-1}\sigma\right)^{2} GR_t V_{4(t)}}{4}\right], \Rightarrow A_0 = -\frac{1}{\left((\lambda_1+\lambda_2)^{-1}\sigma\right)^2}\left[4A_1 + \varphi - GR_t V_{4(t)}\right]$$

Reference

[1] Aggarwal, S. Chauhan, R. and Sharma, N. (2018). Application of Elzaki transform for solving population growth and decay problems. Journal of Engineering Technologies and Innovative Research, 5(9):281-284.

[2] Amadi, I.U., Eleki, A.G. and Onwugbta, G. (2017). Application of nonlinear first order ordinary differential equations on stock market prices. Frontiers of Knowledge Journal Series, Issue 1: 2635-3393.

[3] Amadi, I.U. and Charles, A. (2022). An analytical method for the assessment of asset price changes in capital market. International Journal of Applied Science and Mathematical Theory, 8(2):1-13.

[4] Amadi, I.U, Ogbogbo, C.P and Osu, B.O. (2022). Stochastic analysis of stock price changes as Markov chain in finite states. Global Journal of Pure and Applied Sciences, 28:91-98.

[5] Amadi, I.U and Okpoye, O.T. (2022). Stochastic analysis model in estimation of stock return rates in capital market investments. International Journal of Mathematical Analysis and Modeling, 5(1):108-120.

[6] Amadi, I.U, Igbudu, R. and Azor, P.A. (2022). Stochastic analysis of the impact of growth-rates on stock market prices in Nigeria. Asian journal of Economics, Business and

Accounting, 21(24):9-21.
[7] Osu, B.O., I.U Amadi and Azor P.A. (2022). Analytical solution of time-varying investment returns with random parameters. Sebha University Journal of Pure and Applied Sciences, 21:2.
[8] Davies, I. (2005). Relative controllability of non-linear systems with delays in state and control. Journal of the Nigerian Association of Mathematical Physics, 9:239 – 246.
[9] Davies, I. (2006). Euclidean null controllability of linear systems with delays in state and control. Journal of the Nigerian Association of Mathematical Physics, 10:553 – 558.
[10] Davies, I. Amadi, I.U. and Ndu, R.I. (2019). Stability analysis of stochastic model for stock market prices. International journal of Mathematics and Computational Methods, 4:79-86.
[11] Davies, I. (2006), Euclidean null controllability of infinite neutral differential systems. Australian and New Zealand Industrial and Applied Mathematics Journal. 48:285-293.
[12] Durojaye, M.O. and Uzoma, G.I. (2020). Application of nonlinear first order ordinary differential equations on stock market analysis. Direct Research Journal of Engineering and Information Technology, 7(1):15-20.
[13] Elzaki, T.M. (2011). The new integral transform Elzaki transform. Global Journal of Pure and Applied Mathematics, 7(1):57-64.
[14] Nwobi, F.N and Azor, P.A (2022). An empirical approach for the variation in capital market price changes. International Journal of Statistics and Mathematics, 8():164-172.
[15] Osu B.A and Amadi, I.U. (2022). A stochastic analysis of stock market price fluctuations for capital market. Journal of Applied Mathematics and Computation 6(1):85-95.
[16] Osu, B.O. and Okoroafor, A.C. (2007). On the measurement of random behavior of stock price changes. Journal of Mathematical Science Dattapukur, 18(2):131-141.
[17] Tian-Quan, Y. and Tao, Y. (2010). Simple differential equations of A and H stock prices as application to analysis of equilibrium state. Journal Article Technology of Investment, http://www.scRp.org/journal/ti.
[18] Ugbebor, O.O, Onah, S.E. and Ojowo, O. (2001). An empirical stochastic model of stock price changes. Journal Nigerian Mathematical Society, 20:95-101.
[19] Wokoma, D.S.A.(Bishop), Amadi, I.U and Aboko, I.S (2022). A stochastic model for stock market price variation. Asian Research Journal of Mathematics,18(4):41-49.

International Journal of Mathematical Analysis and Modelling

Scope: The International Journal of Mathematical Analysis and Modelling (IJMAM), formerly known as the Journal of the Nigerian Society for Mathematical Biology, is the official journal of the Nigerian Society for Mathematical Biology, designed to disseminates original research findings in all areas of pure and applied mathematics. The aim of the journal is to cause to be published, original research, that shows the connection between pure and applied mathematicians, statisticians, operations researchers, physicists, engineers, computer scientists and specialists in any field of human endeavour such as in agriculture, medicine, pharmacy, life sciences, physical sciences, engineering, social sciences, humanities, arts, etc., at the crossroad of theoretical, experimental and field studies with the mathematical sciences, in order to provide solutions to problems that cut across these fields through exchange of ideas and collaborations.

Submitted papers should provide rigorous mathematical analysis of any real-world problem being discussed; at the same time, the results obtained should be useful and easily understandable by non-mathematicians. The journal will accept different kinds of articles, ranging from review articles to special invitations. All submitted manuscript will be assessed by the Editor-in-Chief before being sent out for peer-review.

Areas covered in the IJMAM include, mathematical modelling, numerical analysis, fluid mechanics, real and complex analysis, functional analysis, operator theory, statistics, algebra, computational mathematics and differential equations. Articles that show the connections between different areas of mathematics when solving real-world problems are especially welcome in the IJMAM.

Manuscript Information and guidelines for Authors

This is to invite you to submit high-quality research articles to the International Journal of Mathematical Analysis and Modelling (IJMAM), formerly known as the Journal of the Nigerian Society for Mathematical Biology.

Manuscripts should be written either using MS Word (most preferred) or in LaTeX. Initial submissions for review can be in MS Word or PDF. There will be a page restriction of a maximum of 26 pages per accepted article. The article should be concise and details of well-established mathematical methods should be omitted. Articles that do not contain significant mathematical content and results will not be accepted for publication.

Citations within the manuscript should be done using the number system i.e., “Following the idea in [7], we have that …”. Theorems, Lemmas, Propositions, Remarks, etc., should be numbered using the Section number where they appear e.g., Theorem 2.1 will appear in Section 2.

References should follow the format shown below:

[1] Anderson, R.M. and May, M. (1991). Infectious Diseases of Humans: Dynamics and Control. Oxford University Press, UK.

[2] Castillo-Chavez, C. and Song, B. (2004). Dynamical models of tuberculosis and their applications. Math. Biosci. Eng., 1:361-404.

References can be in alphabetical order in the list of references, which should be the last section in the manuscript.

Manuscripts should be submitted to the Editor-in-Chief, via the following email address: mcsb@uniben.edu. Please note that this is the **only** acceptable channel for manuscript submission at this time.

There is a publication charge for each accepted article. Payment details will be corresponded to the author(s) at the time of manuscript acceptance for publication in the IJMAM.

Subscription rates: This is subject to change from time to time. Please, contact the Editor-in-Chief for the current rates, for local and foreign subscriptions.

International Journal of Mathematical Analysis and Modelling
(Formerly Journal of the Nigerian Society for Mathematical Biology)
Volume 5, Issue 3 (October), 2022