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Mathematical modeling of two-dimensional size exclusion chromatography model comparing fully porous and core-shell particles

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Abstract

This work presents a comparison of fully porous and core-shell particles for size-exclusion chromatography. A two-dimensional general rate model is formulated for a column of cylindrical geometry. The model equations consist of a system of partial differential equations. Semi-analytical solutions are obtained for the formulated model equations by applying Hankel and Laplace transformations one after the other. This task is aimed at theoretically investigating and comparing the separation process output for both fully porous and core-shell particles for size-exclusion chromatography operations under isothermal conditions. The simulations in this work are used to demonstrate the size-exclusion effect for fully porous and core-shell particles and the results show better resolution of elution profiles are obtainable for some range of values of core radius fraction.

Keywords: general rate model; core-shell particles; semi-analytical solutions; size-exclusion chromatography

1 Introduction

Size-exclusion chromatography (SEC) is a versatile and widely used chromatographic technique (10). It operates on the principle of separating macromolecules based on their size (19). SEC utilizes a porous stationary phase (such as gel or resin) through which the sample is passed (10). Smaller molecules, which can penetrate the pores, take longer to elute, while larger molecules elute faster (10). SEC is commonly used for the analysis of macromolecules like proteins, nucleic acids, and synthetic polymers (10). It offers advantages such as simplicity, high resolution, and mild separation conditions (19). SEC has applications in various scientific disciplines, including biochemistry, biotechnology, polymer chemistry, and pharmaceutical analysis (10). It aids in determining the molar mass averages and distributions of macromolecules (24). This technique is commonly used in two-dimensional separations to determine the molar mass and chemical composition distributions of a sample (24).

A wide application of SEC can be found in several reports in the literature (1; 4; 5; 8; 17; 18; 22; 23; 24; 25; 26; 27; 28). However, practitioners must be aware of some of the pitfalls and challenges associated with SEC which include low feed volume to column volume ratio (24; 29). Despite these challenges, SEC remains an important technique as Luo and co-researchers compared the use of fully porous beads and cored beads in SEC for protein purification (16). The adsorption of poly(maleic) acid, natural organic matter and humic acid from river water on activated carbon was studied using SEC by (12). They measured the molecular weight distribution of polyelectrolytes remaining in the solution after equilibrium with activated carbon. Kaczmariski and co-workers used SEC to evaluate

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competitive adsorption of a non-ionic surfactant and non-ionic hydrophobe-modified ethoxylated urethane thickener on pre-treated rutile titanium dioxide pigments (11).

Core-shell particles, also known as fused-core, solid core, or superficially porous particles, have been widely used in liquid chromatography, particularly in high-performance liquid chromatography (HPLC) (9). These particles consist of a solid core and a porous shell, which allows for efficient and fast separation with low operating pressure (9). They are prepared using various methods and offer advantages such as low back pressure and improved performance compared to other types of silica particles (9). Core-shell particles are commonly used in reversed-phase HPLC, where they have been found to reduce the plate height by up to 1.7 times (20). Their applications in liquid chromatography are diverse and continue to be explored for further advancements in the field (9).

Several mathematical models have been suggested to describe the size exclusion chromatography (13; 14; 15; 30). These models are very useful as they eliminate the risks involved in experimental processes. A two-dimensional general rate model in which the stationary phase equation is similar to the one-dimensional model implemented by Luo et al is applied in this work (16). To the best of the knowledge of the authors, there are no reported theoretical study carried out on SEC applied a two-dimensional general rate model comparing fully porous and core-shell particles.

The use of the two-dimensional model is very important in SEC as it captures the effects of radial disturbances at the point of solutes injection into the column. This phenomenon may occur in reality for large column operations, which is a significant feature of industrial SEC separations. The model equation consists of a system of partial differential equations and closed with typical initial and the Danckwerts boundary conditions. Semi-analytical solutions are obtained by applying Hankel and Laplace transformations one after the other (2; 21). The solutions are termed semi-analytical because the process of obtaining them involves a combination of both analytical and numerical procedures.

SEC is an industrial operation that involves very large columns and modeling this process mathematically is adequately captured using a two-dimensional model equation. The main objective of this work is to theoretically compare the separation processes involving the use of fully porous particles and core-shell particles under isothermal operating conditions. To achieve this, the following are carried out:

- formulate a two-dimensional general rate model for a cylindrical column packed with core-shell particles used to simulate SEC process operating under isothermal condition.
- define appropriate initial and boundary conditions for the system of equations.
- obtain semi-analytical solutions of the model equations by applying Hankel and Laplace transforms one after the other.
- determine the performance of the chromatographic column by comparing profile outputs of fully porous particles with that obtained for core-shell particles.

2 Mathematical model

The following assumptions are made to obtain the governing equations of the system:

- the column is operated under an isothermal condition
- both radial and axial transport of solute is considered
- given t as time, solute travels along the radial coordinate r by means of dispersion and along the axial coordinate z by means of advection with a constant velocity throughout the length of the column; and by means of dispersion
- the cylindrical column has radius R_c , packed with uniform sized spherically shaped particles of radius R_p

- there is no interaction between the solutes inside the column
- the column is designed such that solute can be injected via an inner region or an outer region; and as a consequence, a new parameter $\bar{r}$ is introduced, see Figure 1
- lastly, only injection of solutes through the inner region are considered

Given the above assumptions, the governing equations of the model representing the bulk and particle phases are given as

$$\frac{\partial c_b}{\partial t} + u \frac{\partial c_b}{\partial z} - D_z \frac{\partial^2 c_b}{\partial z^2} - D_r \left(\frac{\partial^2 c_b}{\partial r^2} + \frac{1}{r} \frac{\partial c_b}{\partial r} \right) + F_b \frac{3}{R_p} k_{\text{ext}} (c_b - c_p|_{r_p=R_p}) = 0, \quad (1)$$

$$\epsilon_p^a \frac{\partial c_p}{\partial t} - \epsilon_p^a D_p \left(\frac{\partial^2 c_p}{\partial r_p^2} + \frac{2}{r_p} \frac{\partial c_p}{\partial r_p} \right) = 0. \quad (2)$$

In the above equations, u is the interstitial velocity, $F_b = \frac{(1-\epsilon_b)}{\epsilon_b}$, where ϵ_b is known as the external porosity and ϵ_p is called the internal particle porosity. D_{zi} , D_{ri} and D_{pi} are the dispersion constants for axial, radial and particle coordinates respectively. In SEC modeling, a key parameter proposed by Gu

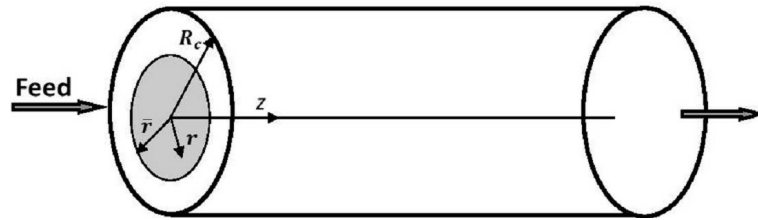


Figure 1: Schematic diagram of a chromatographic column of cylindrical geometry.

in (6) is the accessible porosity ϵ_p^a . For a very small molecule, $\epsilon_p^a = \epsilon_p$, because the particle macropores can be penetrated by the small molecule. For medium sized molecules, $0 < \epsilon_p^a < \epsilon_p$, and for large molecules, such that there is complete exclusion from the particle macropores, $\epsilon_p^a = 0$.

The above coupled system of partial differential equations (PDEs) can be nondimensionalized using the following dimensionless parameters:

$$x = \frac{z}{L}, \quad \tau = \frac{ut}{L}, \quad \rho = \frac{r}{R_c}, \quad \rho_p = \frac{r_p}{R_p}, \quad Pe_z = \frac{Lu}{D_z}, \quad Pe_\rho = \frac{R_c^2 u}{D_r L},$$

$$\zeta = \frac{k_{\text{ext}} R_p}{\epsilon_p^a D_p}, \quad \eta = \frac{\epsilon_p^a D_p L}{R_p^2 u}, \quad \xi = 3\zeta\eta F_b.$$

Hence, equations (1) and (2) become:

$$\frac{\partial c_b}{\partial \tau} + \frac{\partial c_b}{\partial x} - \frac{1}{Pe_z} \frac{\partial^2 c_b}{\partial x^2} - \frac{1}{Pe_\rho} \left(\frac{\partial^2 c_b}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial c_b}{\partial \rho} \right) + \xi [c_b - c_p|_{\rho_p=1}] = 0, \quad (3)$$

$$\epsilon_p^a \frac{\partial c_p}{\partial \tau} - \eta \left(\frac{\partial^2 c_p}{\partial \rho_p^2} + \frac{2}{\rho_p} \frac{\partial c_p}{\partial \rho_p} \right) = 0. \quad (4)$$

By following the procedure suggested by Gu in (6), for fully porous particles ρ_p ranges from 0 to 1, while for core-shell particles it ranges from $\rho_{\text{core}} = R_{\text{core}}/R_p$ to 1. Figure 1 shows a sketch of a spherical core-shell particle, with core radius fraction $\rho_{\text{core}} = R_{\text{core}}/R_p$. Thus, for arbitrary core radius fraction $\rho_{\text{core}} \neq 0$, $\rho_{\text{core}} \leq \rho_p \leq 1$ for core-shell particles. While for fully porous particles, $\rho_{\text{core}} = 0$. We proceed to replace the ρ_p -axis by $0 \leq \gamma \leq 1$, where the dimensionless radial axis is

$$\gamma = \frac{\rho_p - \rho_{\text{core}}}{1 - \rho_{\text{core}}} \quad \Rightarrow \quad \rho_p = \gamma(1 - \rho_{\text{core}}) + \rho_{\text{core}}. \quad (5)$$

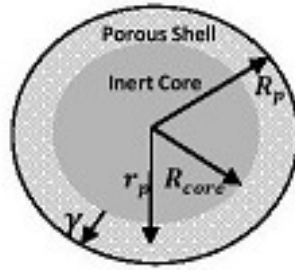


Figure 2: Schematic diagram of a core-shell particle.

For a column packed with core-shell particles, the governing equations are now obtained by substituting the above Eq. (5) into Eqs. (3) and (4) to obtain

$$\frac{\partial c_b}{\partial \tau} + \frac{\partial c_b}{\partial x} - \frac{1}{Pe_z} \frac{\partial^2 c_b}{\partial x^2} - \frac{1}{Pe_\rho} \left(\frac{\partial^2 c_b}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial c_b}{\partial \rho} \right) + \xi [c_b - c_p|_{\gamma=1}] = 0, \quad (6)$$

$$\epsilon_p^a \frac{\partial c_p}{\partial \tau} - \frac{\eta}{(1 - \rho_{core})^2} \left(\frac{\partial^2 c_p}{\partial \gamma^2} + \frac{2(1 - \rho_{core})}{(\gamma(1 - \rho_{core}) + \rho_{core})} \frac{\partial c_p}{\partial \gamma} \right) = 0. \quad (7)$$

The system of PDEs require the following initial and boundary conditions.

Initial conditions:

$$c_b(\rho, x, \tau = 0) = 0, \quad 0 \leq \rho \leq 1, \quad 0 \leq x \leq 1, \quad (8)$$

$$c_p(\gamma, \rho, x, \tau = 0) = 0, \quad 0 \leq \gamma \leq 1, \quad 0 \leq \rho \leq 1, \quad 0 \leq x \leq 1. \quad (9)$$

For the symmetry of the radial profile and impermeable wall of the column, the following boundary conditions are used for equation (6) at $\rho = 0$ and $\rho = 1$:

$$\frac{\partial c_b(\rho = 0, x, \tau)}{\partial \rho} = 0, \quad \frac{\partial c_b(\rho = 1, x, \tau)}{\partial \rho} = 0. \quad (10)$$

At both column boundaries, the Danckwerts boundary conditions are used. For pulse injection through the inner region:

$$c_b(\rho, x = 0, \tau) - \frac{1}{Pe_z} \frac{\partial c_b(\rho, x = 0, \tau)}{\partial x} = \begin{cases} c_b^{inj}, & \text{if } 0 \leq \rho \leq \bar{\rho} \text{ and } 0 \leq \tau \leq \tau_{inj}, \\ 0, & \bar{\rho} < \rho \leq 1 \text{ or } \tau > \tau_{inj}. \end{cases} \quad (11)$$

where c_b^{inj} is the injected concentration and τ_{inj} is the dimensionless time of injection. At the outlet of the column, the classic Neumann boundary conditions is applied:

$$\left. \frac{\partial c_b(\rho, x, \tau)}{\partial x} \right|_{x=1} = 0. \quad (12)$$

Lastly for the particle phase, equation (7), the following boundary conditions are used at $\gamma = 0$ and $\gamma = 1$ as

$$\left. \frac{\partial c_p}{\partial \gamma} \right|_{\gamma=0} = 0, \quad \left. \frac{\partial c_p}{\partial \gamma} \right|_{\gamma=1} = (1 - \rho_{core}) \zeta (c_b - c_p|_{\gamma=1}). \quad (13)$$

3 Obtaining semi-analytical solutions

To obtain analytical solutions, the Hankel and Laplace transforms are applied one after the other. This technique and solution procedure are similar to the one presented by by David et al in (2) and Qamar

et al in (21); hence the details are omitted here. The solution obtained after applying the Danckwerts boundary condition is given below as:

$$\bar{c}_H(x, \lambda_n, s) = \frac{[m_2 e^{m_2 + m_1 x} - m_1 e^{m_1 + m_2 x}] \left[\frac{F(\lambda_n)}{s} (1 - e^{-s\tau_{inj}}) \right]}{m_2 e^{m_2} \left(1 - \frac{m_1}{Pe_z} \right) - m_1 e^{m_1} \left(1 - \frac{m_2}{Pe_z} \right)}. \quad (14)$$

Where

$$m_{1,2} = \frac{Pe_z}{2} \left(1 \pm \sqrt{1 + \frac{4\phi(s, \lambda_n)}{Pe_z}} \right) \quad (15)$$

and m_1 is obtained by choosing the plus sign and m_2 by choosing the minus sign. Also,

$$F(\lambda_n) = \begin{cases} \frac{\bar{\rho}^2}{2}, & \text{if } \lambda_n = 0, \\ \frac{\bar{\rho}}{\lambda_n} J_1(\lambda_n \bar{\rho}), & \text{if } \lambda_n \neq 0, \end{cases} \quad (16)$$

with λ_n representing the parameter of the finite Hankel transform that can be estimated using the equation $\frac{dJ_0(\lambda_n)}{d\rho} = -J_1(\lambda_n) = 0$. Where, $J_0(\cdot)$ is the zeroth order, and $J_1(\cdot)$ is the first order Bessel functions of the first kind.

The analytical Laplace inversion of the above obtained solution is not currently achievable. The numerical inversion technique proposed by Durbin in (3) is used to get semi-analytical solutions in the actual time domain.

4 Simulation results

The parameters used in the simulation of the process are given in Table 1. These parameters fall within

Table 1: Values of parameters used in simulation.

Parameter	Symbol	Value
External porosity	ϵ_b	0.4
Internal porosity	ϵ_p	0.66
Accessible porosity	ϵ_p^a	0.5
Axial Peclet number	Pe_z	1500
Radial Peclet number	Pe_ρ	37.5
Dimensionless constant	η	2
Dimensionless constant	ζ	50
Injected concentration	c_{bi}^{inj}	1 mol/L
Time of injection	τ_{inj}	1.0

the range typically used in experimental studies of liquid chromatography (7). Where the parameters differ from what is given in Table 1, they are clearly stated. The results are plotted at the outlet of the column showing the dimensionless time of elution.

4.1 Effect of core radius fraction ρ_{core} on elution profiles

Figure 3 shows that the peak of the elution profile become sharper and with narrower tails, which implies shorter retention times as the value of ρ_{core} is increased. The reduced intraparticle diffusional mass-transfer distance caused the sharpness of the peak and a shorter time of elution. In chromatography operations, the resolution of peaks is improved by sharper peaks. However, if the retention times are too short, the overall range of retention times in an SEC chromatogram becomes too small to effectively resolve multiple peaks.

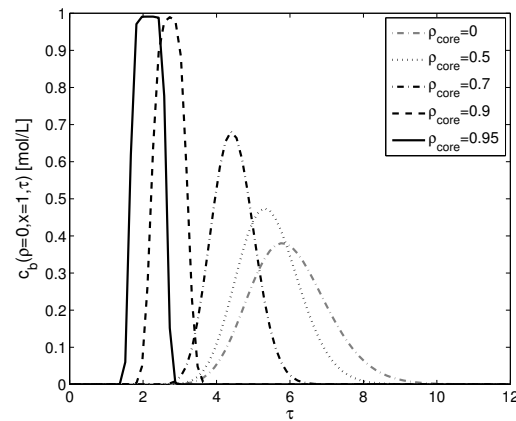


Figure 3: Effect of core radius fraction ρ_{core} on SEC

4.2 Effect of the accessible porosity parameter ϵ_p^a

The effect of the parameter ϵ_p^a is used to compare the results for both fully porous particles and core-shell particles. It can be seen in Figure 4 that for smaller values of ϵ_p^a , which implies small access for the molecules to penetrate the particles in the stationary phase, the elution is faster and thus resulting in shorter retention times. Although the retention times for both cases shown in Figure 4(a) can be said to be almost the same, significant increase in the retention times can be seen in Figures 4(b) to 4(d) as the value of ϵ_p^a is increased. A taller and narrower peak can be observed for the case where $\rho_{core} = 0.85$ compared to the fully porous particle case, that is where $\rho_{core} = 0$.

4.3 Effect of radial dispersion coefficient Pe_p

Under this subsection, $\rho_{core} = 0.6$ for the obtained results. Figure 5 shows that for smaller value of Pe_p , there is no gradient visible along the radial coordinate. This case reduces to a one-dimensional model. For increased Pe_p value, the evolution of gradients along the radial coordinate becomes evident. Thus, the radial disturbances in SEC operations is captured by the current two-dimensional model.

4.4 Effect of mass transfer parameters on band broadening and peak retention time

Again, $\rho_{core} = 0.6$ for the following results obtained here. In Figure 6(a), other parameters are fixed while Pe_z is altered to give the result shown. It can be seen that smaller values Pe_z cause a broader and tailing profile, which also leads to increased retention time. Similar phenomena are observed for the results for ζ and η as shown in Figures 6(b) and 6(c) respectively.

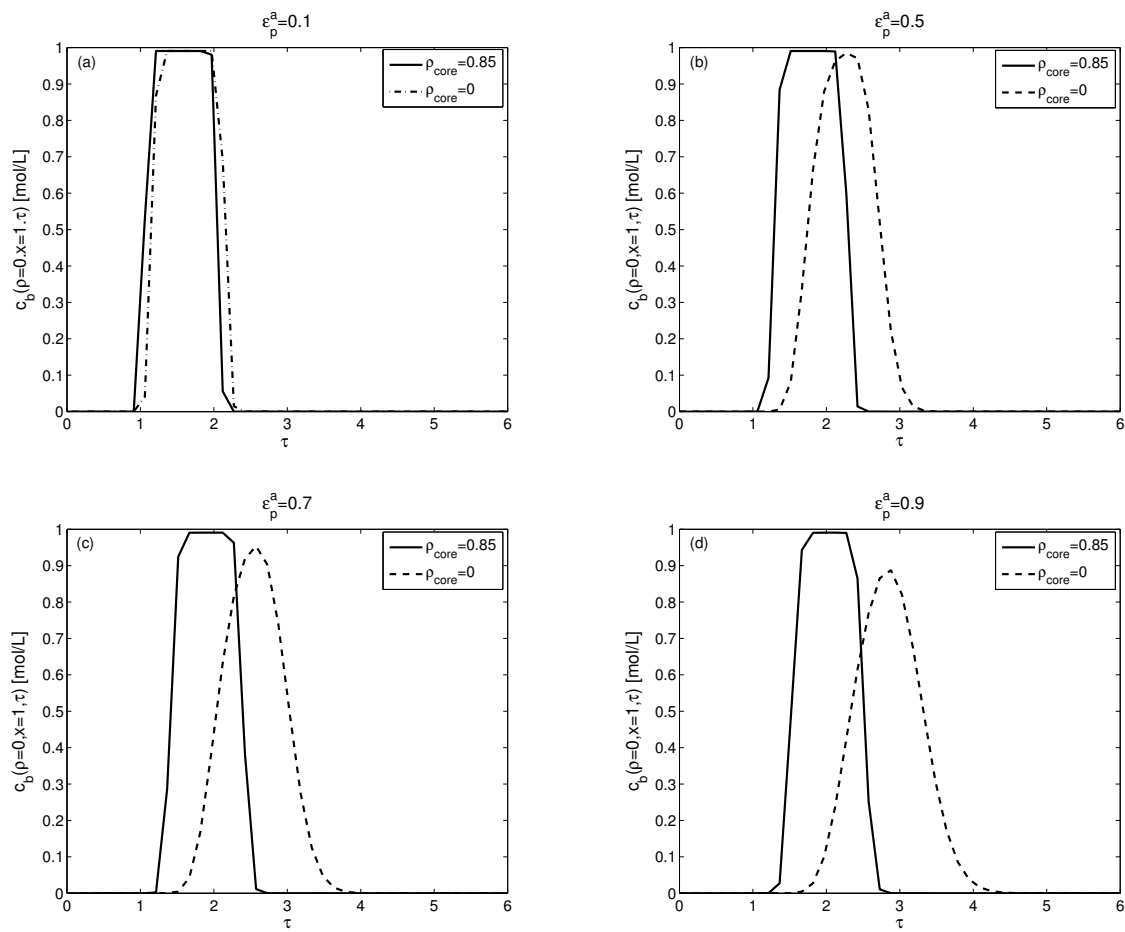


Figure 4: Size-exclusion effect for both fully porous particles and core-shell particles

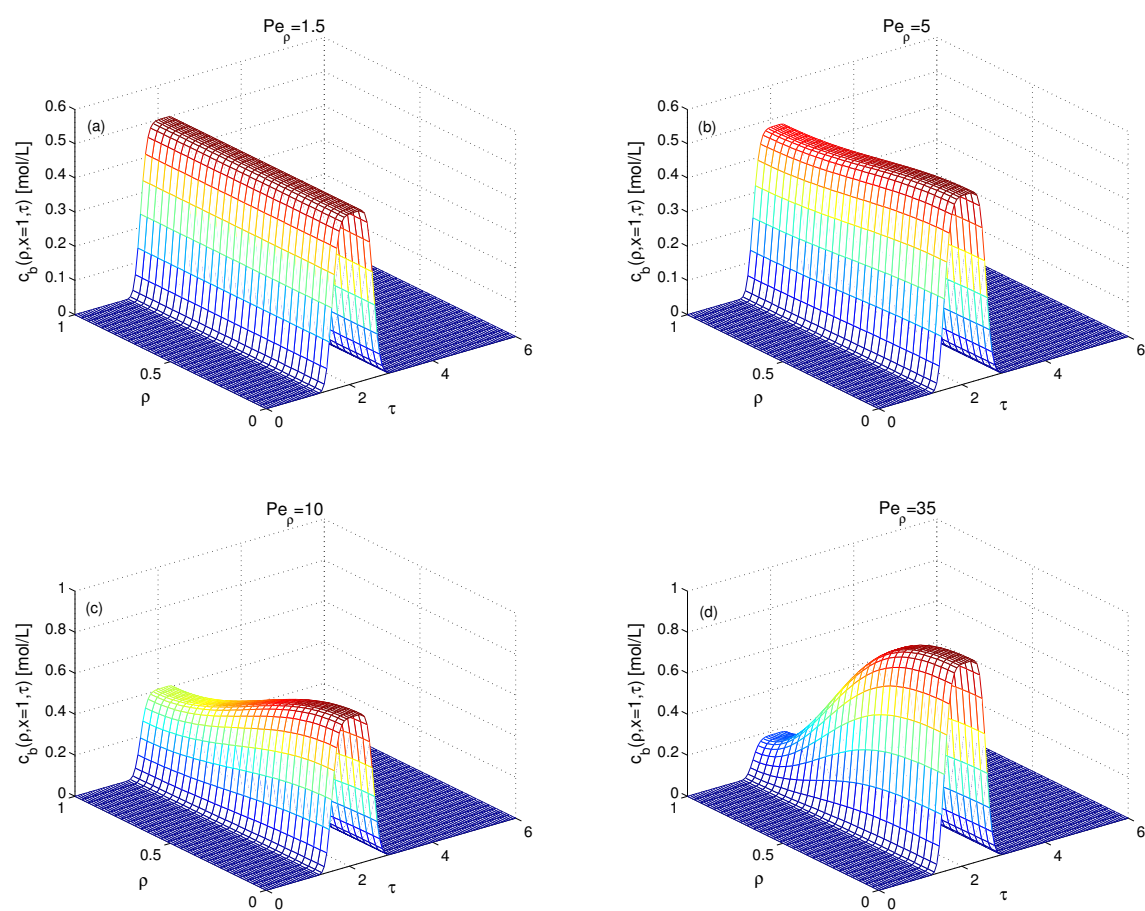


Figure 5: Effect of Pe_ρ on the elution profile

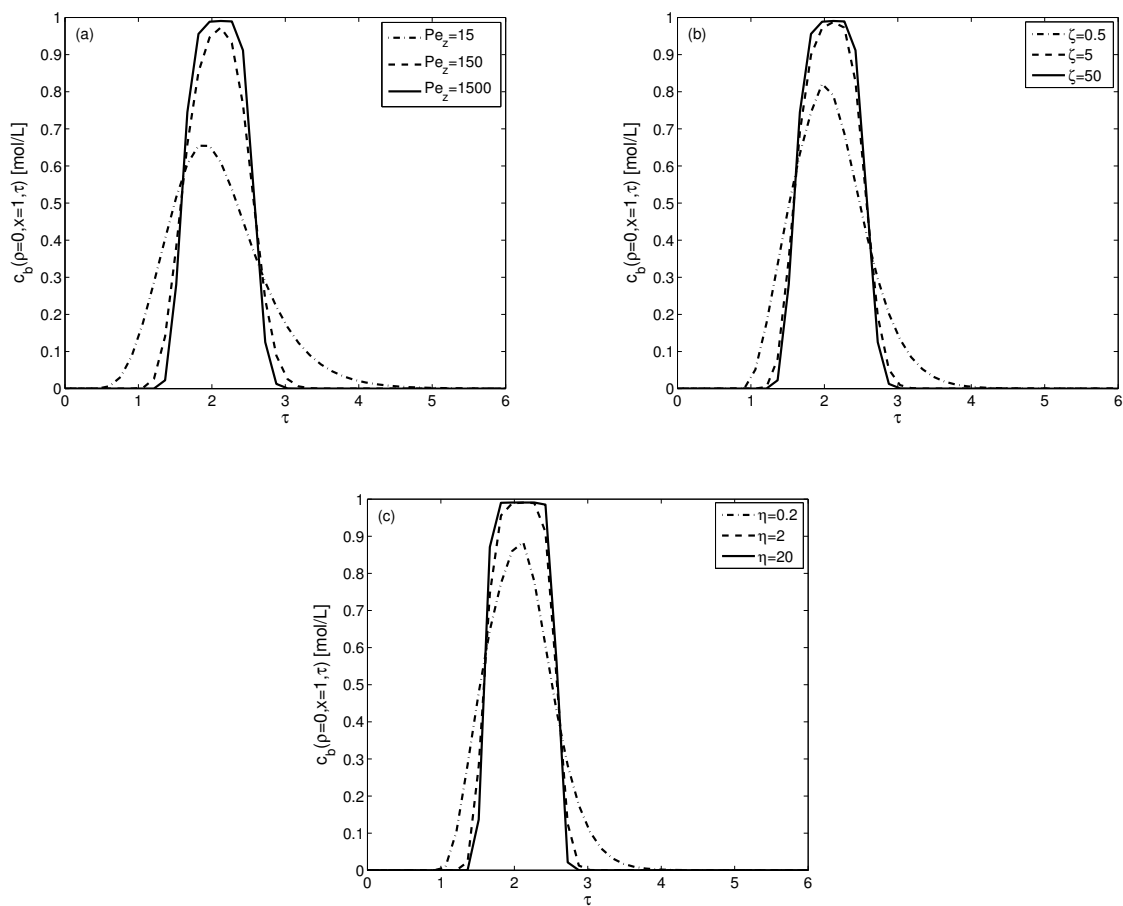


Figure 6: Effect of kinetic parameters Pe_z , ζ and η on the elution profiles

5 Conclusion

A two-dimensional general rate model was formulated and implemented to simulate size-exclusion chromatography process comparing fully porous and core-shell particles operation under isothermal conditions. Semi-analytical solutions of the formulated model equations were obtained by successively applying Hankel and Laplace transformations considering Danckwerts boundary condition. Despite the absence of any reported theoretical study on two-dimensional models, the larger column feature in SEC operations motivated its use in this study. The obtained results show that for some reasonable range of core radius fraction and accessible porosity values, better peak resolutions can be achieved in SEC processes.

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Modelling and analysis of a model for Chlamydia Trachomatis transmission dynamics

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Abstract

Chlamydia infection, one of the commonest sexually transmitted infections (STIs), remain a public health challenge in both underdeveloped and developed countries of the world. Chlamydia trachomatis has been observed to have negative health consequences hence much research work is needed to be done to curb the spread of the disease in the population. In this paper, a mathematical model for studying the impact of condom usage and treatment on the transmission dynamics and control of Chlamydia in the population is presented. Qualitative analysis of the model shows that it undergoes the phenomenon of backward bifurcation. In the absence of this phenomenon (which is shown to occur as a result of the reinfection of recovered individuals), the disease-free equilibrium of the model is globally asymptotically stable whenever the associated reproduction number is less than unity. Further, for the same scenario as above, it is shown that the unique endemic equilibrium of the model exists whenever the reproduction number is greater than unity. Numerical results show a relationship between the progression rate, treatment rate and the reproduction number. Results from the sensitivity analysis of the model, using the reproduction number, R_c reveal that the top parameters that significantly drive the dynamics of Chlamydia in the population are the efficacy of condoms, condom compliance, a fraction of treated individuals who recover due to effective treatment and treatment rate. Numerical simulations of the model suggest that infected persons after treatment should wait for at least 7 days before engaging in any form of sexual activity or, if not possible use condoms correctly (to avoid reinfection) in order to effectively control the spread of the disease in the population.

Keywords: Chlamydia; reproduction number; reinfection; stability; bifurcation

1 Introduction

Chlamydia trachomatis, one of the commonest sexually transmitted infections (STIs) is primarily transmitted through vaginal, oral and anal sex, is a preventable and curable STIs caused by the bacterium Chlamydia trachomatis [8, 9, 34]. In 2020, an estimated 128.5 million new infections associated with Chlamydia trachomatis was reported worldwide amongst adults between 15 to 49 years of age and their prevalence globally was estimated to be 4.0 percent for women and 2.5 percent for men [34]. Furthermore, reports from Statista show that in 2020 about 25.5 million people in Africa had Chlamydia. The prevalence rate was higher among females with about 14.7 million people infected while 10.8 million males were infected with Chlamydia [31]. In Chlamydia infection, about 80 percent of infected people have no symptoms [23, 34]. If symptoms occur, they may not appear for about three weeks after having sex with someone who is infected with Chlamydia. In women, common symptoms include: a change in vaginal discharge, bleeding between menstrual period or after sex, itching or burning in and around the vagina, pain or discomfort in the lower abdomen, burning sensation when urinating while in men, common symptoms include: burning sensation when urinating, discharge from the penis and

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pain or discomfort in the testicles. Further, anal infection in women and men can equally cause pain, discharge and bleeding[8, 9, 34].

When someone is diagnosed with an uncomplicated Chlamydia infection, treatment can be administered using antibiotic tablets (mainly azithromycin or doxycycline) [9]. It is important to note that treatment may fail when individuals under treatment continue to have unprotected sex with infected partners. In pregnant women, untreated Chlamydia is associated with pre-term delivery and can even spread to the newborns thus causing an eye infection or pneumonia while in men, untreated Chlamydia can cause infection of the urethra which often spread to the tube that carries sperm from the testis. This may cause pain, fever and in most cases infertility [23, 28]. Chlamydia can be prevented through consistent and correct use of condoms when having vaginal or anal sex. At the moment, there are no vaccines for the prevention of Chlamydial infection [34].

Mathematical epidemiology is the study of diseases with the sole aim of identifying factors responsible for their occurrence. Over the last two decades, mathematical models have become veritable tools in epidemiology for gaining insights into the epidemiological patterns of diseases and predicting the consequences of the introduction of public health strategies in controlling the spread of such diseases [27]. Theoretical developments on numerous epidemic models are extensively discussed in [1, 3, 10, 11, 12, 16, 22, 27, 29, 30]. Research works on the use of mathematical models in studying the transmission dynamics, prevention and control of Chlamydia are very few [28]. For example, [18, 20] developed a discrete-time population-level models while [4, 25] designed a continuous-time models. A delayed differential equation model was formulated by [5] and was used to estimate the parameters associated with the intracellular development cycle of Chlamydia trachomatis. In the works of Wilson [35, 36], several differential equation models were developed for studying different aspects of Chlamydia dynamics in vivo. Further, [19, 32] designed other forms of Chlamydia models involving the use of an individually-based stochastic model and semi-parametric threshold regression analysis [37]. Also, [28, 29, 30] developed and rigorously analyzed a two group deterministic mathematical model for the spread of Chlamydia trachomatis in a population. [21] designed a deterministic model for assessing the epidemiological consequences of chlamydia gonorrhea co-infection and qualitatively analyzed it. In another research work by [26], a delayed chlamydia epidemic model with pulse vaccination was analyzed while [27], formulated and analyzed a Chlamydia epidemic model using mass action incidence function. Their model were compartmentalized into the susceptible (S), exposed (infected but not yet infectious)(E), infectives in the asymptomatic stage (showing no symptoms of Chlamydia)(I_A), infectives in the symptomatic stage (showing symptoms of Chlamydia)(I_S) and the recovered class (infectious people who have cleared or recovered from Chlamydia infection)(R). Consequently, this model was extended to an optimal control model. The present study complements the aforementioned research works (particularly the one by [27]) by developing and analyzing an equivalent mathematical model for Chlamydia transmission dynamics by incorporating standard incidence function, condom usage and a class of treated individuals who failed treatment for the spread of Chlamydia in a population. This model on Chlamydia transmission dynamics is probably the first to incorporate the class of treated individuals who failed treatment. The model is based on the following assumptions:

- i. The present study incorporates the class of treated individuals who failed treatment. This assumption is in line with the information obtained from [9, 34].
- ii. This study incorporates the use of condoms in preventing Chlamydia infections as reported in [9, 34].
- iii. This study adopts the use of standard incidence function since it is more appropriate for large population [1].

The rest of the paper is organized as follows. The model is developed in Section 2 and analyzed in Section 3. Sensitivity analysis and numerical simulations of the model are presented in Sections 4 and 5, respectively while the concluding remarks in Section 6.

2 Model Formulation

The total sexually active population at time t , denoted by $N(t)$, is divided into the mutually exclusive compartments of the susceptible individuals ($S(t)$), exposed individuals ($E(t)$), infected individuals in the asymptomatic stage ($I_a(t)$), infected individuals in the symptomatic stage ($I_s(t)$), treated individuals who failed treatment ($Q(t)$) and recovered individuals ($R(t)$).

Infectious interactions in the population are modelled with a standard incidence function. Let Λ represent the recruitment rate of newborns and sexually active individuals (assumed susceptible) into the population while ε and ψ ($0 < \varepsilon, \psi < 1$) represents the efficacy of condoms in preventing chlamydia infection in susceptible individuals and condom compliance, respectively. Also, we further assume that β_1 is the coefficient of transmission of chlamydia infection from infectives in the asymptomatic stage to the susceptibles, β_2 is the coefficient of transmission of chlamydia infection from infectives in the symptomatic stage to the susceptibles, β_3 is the coefficient of transmission of chlamydia infection from individuals who failed treatment to the susceptibles, μ is the natural death rate associated to all the epidemiological classes. Let $\rho\eta$ be the proportion of exposed individuals who are asymptomatic while $(1-\rho)\eta$ is the proportion of exposed individuals who are symptomatic. Let η be the progression rate of individuals in the exposed class to the asymptomatic and symptomatic stages of infection, respectively.

A fraction, ϕ , of the treated individuals in the symptomatic stage of infection, will recover and move to class R, while the remaining fraction $(1-\phi)$, will fail treatment and move to the class Q. Individuals who have failed treatment will eventually be treated successfully at a rate $\sigma\tau$ and move to class R. The parameter $\sigma < 1$ accounts for the assumed decrease in the recovery rate of individuals in class Q in comparison to the successfully treated individuals. Further, recovered individuals are reinfected at a rate $\theta\lambda$ ($\theta > 0$). Recovered individuals can relapse and become infectious again, at a rate γ . Putting together these definitions and assumptions, it follows that the model for the transmission dynamics of chlamydia in a sexually active population is given by the following system of non-linear ordinary differential equations (a flow diagram of the model is shown in Figure 1, and the state variables and parameters of the model are presented in Table 1):

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - \lambda S - \mu S, \\ \frac{dE(t)}{dt} &= \lambda S - [\rho\eta + (1-\rho)\eta + \mu]E, \\ \frac{dI_a(t)}{dt} &= \rho\eta E + \theta\lambda R + \gamma R - [(1-k)m + km + \mu]I_a, \\ \frac{dI_s(t)}{dt} &= (1-\rho)\eta E + (1-k)mI_a - [(1-\phi)\tau + \phi\tau + \mu]I_s, \\ \frac{dQ(t)}{dt} &= (1-\phi)\tau I_s - (\sigma\tau + \mu)Q, \\ \frac{dR(t)}{dt} &= \phi\tau I_s + \sigma\tau Q + kmI_a - [\theta\lambda + (\mu + \gamma)]R\end{aligned}\tag{1}$$

where

$$\lambda = (1 - \varepsilon\psi) \left(\frac{\beta_1 I_a + \beta_2 I_s + \beta_3 Q}{N} \right),\tag{2}$$

and

$$N = S + E + I_a + I_s + Q + R.\tag{3}$$

Therefore, the model (1) adds to the existing models on chlamydia transmission dynamics as in the literature (see for example, [27])

- i. Allowing for the reinfection of individuals who recover from the infection.
- ii. Allowing for the relapse of individuals who recovered from the infection.

The flow diagram of the model is shown in Figure 1, and the state variables and parameters of the model are tabulated/interpreted in Table 1.

Table 1: Description of state variables/parameters in the chlamydia model (1).

Variables/parameters	Interpretation
S	Susceptible individuals
E	Exposed individuals
I_a	Infected individuals in the asymptomatic stage
I_s	Infected individuals in the symptomatic stage
Q	Treated individuals who failed treatment
R	Recovered individuals
Λ	Recruitment rate
β_1	The coefficient of transmission of chlamydia infection from infectives in the asymptomatic stage to the susceptibles
β_2	The coefficient of transmission of chlamydia infection from infectives in the symptomatic stage to the susceptibles
β_3	The coefficient of transmission of chlamydia infection from individuals who failed treatment to the susceptibles
μ	Natural death rate
ε	Efficacy of condoms in preventing chlamydia infection in susceptible individuals
ψ	Condom compliance rate
$\rho\eta$	Proportion of exposed individuals who are asymptomatic
$(1 - \rho)\eta$	Proportion of exposed individuals who are symptomatic
θ	Re-infection rate
γ	Relapse rate for individuals
σ	Accounts for the assumed decrease in the recovery rate of individuals in class Q in comparison to the successfully treated individuals
ϕ	A fraction of treated individuals who recover due to effective treatment and move to R
τ	Treatment rate
km	Accounts for the proportion of the asymptomatic individuals who recover and moved to R
λ	Force of infection
η, m	Progression rates

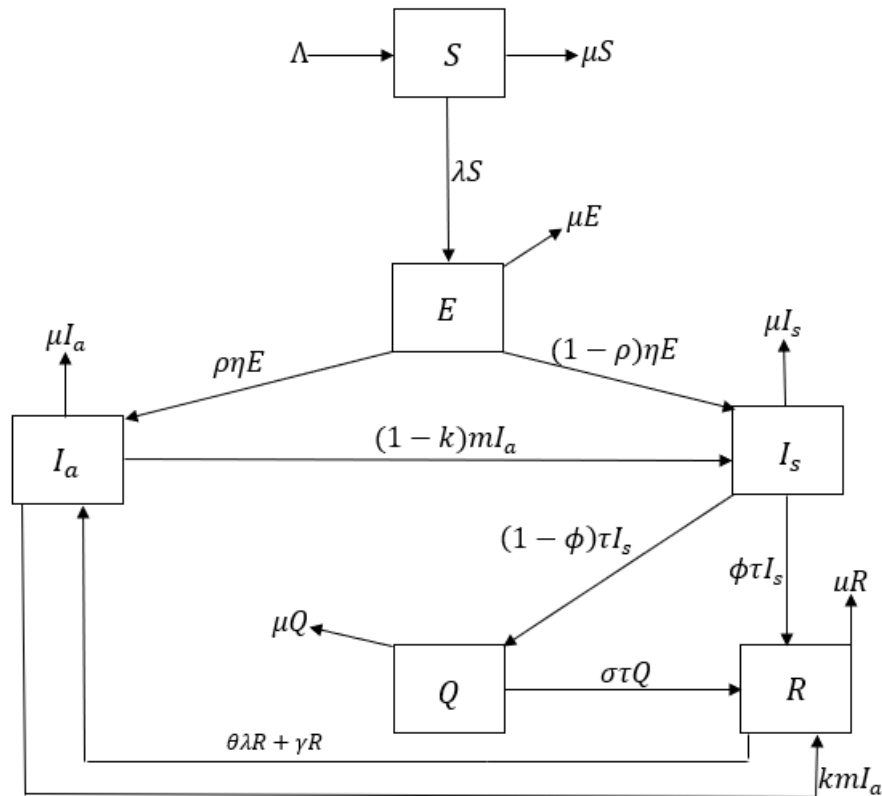


Figure 1: Schematic diagram for model (1)

2.1 Basic properties

For model (1) to be epidemiologically meaningful, it is vital to show that all its state variables are non-negative for all time, t . In order words, the solutions of model (1) with positive initial data will remain positive for all $t \geq 0$.

2.1.1 Positivity and boundedness of solution.

Since the model (1) monitors the human population, the state variables and parameters of the model are non-negative. Consider the biologically feasible region.

$$\Gamma = \{(S, E, I_a, I_s, Q, R) \in \mathbb{R}_+^6 : N \leq \frac{\Lambda}{\mu}\}. \quad (4)$$

It can be shown that the set Γ is a positively invariant set and a global attractor of this system. This implies that, any phase trajectory initiated anywhere in the non-negative region $\mathbb{R}_+^6$ of the phase space will eventually enter the region Γ and remains in Γ thereafter.

Lemma 2.1. *The region Γ is positively invariant for model (1).*

Proof. The rate of change of the total population with time is given by

$$\frac{dN}{dt} = \Lambda - \mu N. \quad (5)$$

Since the right hand side of (5) is bounded by $\Lambda - \mu N$, we can show using a standard comparison theorem [17] that

$$N(t) \leq N(0)\exp^{-\mu t} + \frac{\Lambda}{\mu}(1 - \exp^{-\mu t}). \quad (6)$$

In particular, if $N(0) \leq \frac{\Lambda}{\mu}$, then $N(t) \leq \frac{\Lambda}{\mu}$. Therefore, Γ is positively invariant. Hence, no solution path leaves through any boundary of Γ and it is sufficient to consider the dynamics of model (1) in Γ . Hence, in this region, the model can be thus considered as being mathematically and epidemiologically well posed [15].

3 Mathematical Analysis

3.1 Asymptotic Stability of Disease-free Equilibrium (DFE)

The DFE of the model (1) is given by $E_0 = (S^*, E^*, I_a^*, I_s^*, Q^*, R^*) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0)$. The local stability of E_0 will be obtained using the next generation operator method [33]. The notations as used in [33] is adopted, the non-negative matrix, F , of new infection terms and the M -matrix, V , of transition terms associated with the model (1) are

$$F = \begin{pmatrix} 0 & q\beta_1 & q\beta_2 & q\beta_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ -\rho\eta & d_2 & 0 & 0 \\ -d_5 & -d_6 & d_3 & 0 \\ 0 & 0 & -d_7 & d_4 \end{pmatrix},$$

where

$$\begin{aligned} q &= (1 - \varepsilon\psi), d_1 = (\rho\eta + d_5 + \mu), d_2 = (d_6 + km + \mu), d_3 = (d_7 + \phi\tau + \mu), \\ d_4 &= (\sigma\tau + \mu), d_5 = (1 - \rho)\eta, d_6 = (1 - k)m, d_7 = (1 - \phi)\tau, d_8 = (\mu + \gamma). \end{aligned} \quad (7)$$

It follows that the effective reproduction number of the model (1), denoted by R_c , is given by

$$R_c = \rho(FV^{-1}) = \frac{q[\beta_1\rho\eta d_3 d_4 + (\eta\rho d_6 + d_2 d_5)(\beta_2 d_4 + \beta_3 d_7)]}{d_1 d_2 d_3 d_4} \quad (8)$$

where $\rho(FV^{-1})$ is the spectral radius of the matrix (FV^{-1}) .

The result below follows from Theorem 2 in [33].

Lemma 3.1. *The DFE (E_0) of the model (1) is locally asymptotically stable if $R_c < 1$ and unstable if $R_c > 1$.*

The threshold quantity, R_c is the effective reproduction number of chlamydia [33]. By definition, it represents the average number of secondary chlamydia infections generated by a typical infected person in a completely susceptible population [33]. The implication of Lemma 3.1 epidemiologically is that when $R_c < 1$ is less than one, chlamydia can be eradicated from the population if the initial sizes of the subpopulation of the model are in the basin of attraction of the DFE (E_0). Hence, a small inflow of chlamydia-infected person into the community will not generate large chlamydia outbreaks, and the disease will die out as time progresses.

3.2 Backward bifurcation analysis

3.2.1 Existence of backward bifurcation analysis.

It is extremely important to categorize the type of bifurcation model (1) may undergo. We claim the following result.

Theorem 3.2. *Model (1) exhibits backward bifurcation at $R_c = 1$ whenever a bifurcation coefficient, denoted by a is positive.*

Proof. Suppose $E_1 = (S^{**}, E^{**}, I_a^{**}, I_s^{**}, Q^{**}, R^{**})$ represents any arbitrary endemic equilibrium of model (1) (that is an equilibrium in which at least one of the infected components is non-zero). The existence of backward bifurcation will be determined using the centre manifold theory [6, 7]. To apply

this theory, we perform the following change of variables. Let $S = x_1, E = x_2, I_a = x_3, I_s = x_4, Q = x_5$ and $R = x_6$. Also, by using the vector notation $X = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. The model (1) can be written in the form $\frac{dX}{dt} = F(X)$ where $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ as follows

$$\begin{aligned}\frac{dx_1}{dt} &\equiv f_1 = \Lambda - (\lambda + \mu)x_1, \\ \frac{dx_2}{dt} &\equiv f_2 = \lambda x_1 - d_1 x_2, \\ \frac{dx_3}{dt} &\equiv f_3 = \rho \eta x_2 + \theta \lambda x_6 + \gamma x_6 - d_2 x_3, \\ \frac{dx_4}{dt} &\equiv f_4 = d_5 x_2 + d_6 x_3 - d_3 x_4, \\ \frac{dx_5}{dt} &\equiv f_5 = d_7 x_4 - d_4 x_5, \\ \frac{dx_6}{dt} &\equiv f_6 = \phi \tau x_4 + \sigma \tau x_5 + km x_3 - \theta \lambda x_6 - d_8 x_6.\end{aligned}\tag{9}$$

where (as defined earlier),

$$\begin{aligned}d_1 &= (\rho \eta + d_5 + \mu), d_2 = (d_6 + km + \mu), d_3 = (d_7 + \phi \tau + \mu), \\ d_4 &= (\sigma \tau + \mu), d_5 = (1 - \rho) \eta, d_6 = (1 - k) m, d_7 = (1 - \phi) \tau, d_8 = (\mu + \gamma),\end{aligned}\tag{10}$$

and, the force of infection is given by

$$\lambda = \frac{q(\beta_1 x_3 + \beta_2 x_4 + \beta_3 x_5)}{N}\tag{11}$$

with $q = (1 - \varepsilon \psi)$ and $N = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$.

We consider the scenario where $\beta_1 = \beta_2 = \beta_3 = \beta^*$ is chosen as the bifurcation parameter. Solving for $\beta_1 = \beta_2 = \beta_3 = \beta^*$ from $R_c = 1$, yields

$$\beta_1 = \beta_2 = \beta_3 = \beta^* = \frac{d_1 d_2 d_3 d_4}{q[\rho \eta d_3 d_4 + (\eta \rho d_6 + d_2 d_5)(d_4 + d_7)]}\tag{12}$$

The Jacobian matrix of the transformed system (9), evaluated at the DFE (E^0) with $\beta_1 = \beta_2 = \beta_3 = \beta^*$, is given by

$$J^* = \begin{pmatrix} -\mu & 0 & -q\beta^* & -q\beta^* & -q\beta^* & 0 \\ 0 & -d_1 & q\beta^* & q\beta^* & q\beta^* & 0 \\ 0 & \rho \eta & -d_2 & 0 & 0 & \gamma \\ 0 & d_5 & d_6 & -d_3 & 0 & 0 \\ 0 & 0 & 0 & d_7 & -d_4 & 0 \\ 0 & 0 & km & \phi \tau & \sigma \tau & -d_8 \end{pmatrix}.$$

Matrix $J(E_0)$ has a right eigenvector (associated with the zero eigenvalue) given by $w = [w_1, w_2, w_3, w_4, w_5, w_6]^T$, where

$$\begin{aligned}w_1 &= -\frac{q\beta^*}{\mu}(w_3 + w_4 + w_5), \\ w_2 &= -\frac{q\beta^*}{d_1}(w_3 + w_4 + w_5), \\ w_3 &= -\frac{b_1}{d_4(b_2 - b_3)}w_4, \\ w_4 &= w_4 > 0, \\ w_5 &= \frac{d_7}{d_4}w_4 > 0, \\ w_6 &= \frac{1}{d_8}(kmw_3 + \phi \tau w_4 + \sigma \tau w_5).\end{aligned}\tag{13}$$

where $b_1 = \beta^* \eta q \rho d_8 (d_4 + d_7) + \gamma d_1 \tau (\phi d_4 + \sigma d_7)$, $b_2 = (\beta^* \eta q \rho d_8 + \gamma k m d_1)$ and $b_3 = d_1 d_2 d_3$. Furthermore, J^* has a left eigenvector, $v = (v_1, v_2, v_3, v_4, v_5, v_6)$, satisfying $v \cdot w = 1$, with

$$\begin{aligned} v_1 &= 0, v_2 = \frac{(\rho \eta v_3 + d_5 v_4)}{d_1}, v_3 = v_3 > 0, v_4 = -\frac{b_1}{d_8 (d_1 - c_1)} \\ v_5 &= \frac{(q \beta^* v_2 + \sigma \tau v_6)}{d_4}, v_6 = \frac{d_8}{\gamma} v_3, \end{aligned} \quad (14)$$

with $c_1 = \beta^* q d_5 (d_4 + d_7) > 0$ and $c_2 = d_1 d_2 d_3 > 0$. From Theorem 4.1 in [7], it can be shown, by computing the non-zero partial derivatives of F at the DFE (E_0) and simplifying, that

$$\begin{aligned} a &= \sum_{k,i,j=1}^6 v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (E_0) \\ &= -\frac{2\mu \beta^* (w_3 + w_4 + w_5) [q v_2 (w_2 + w_3 + w_4 + w_5 + w_6) + \theta w_6 (q v_6 - v_3)]}{\Lambda} \end{aligned} \quad (15)$$

Thus, the bifurcation coefficient, a , is positive whenever $(w_3 + w_4 + w_5) [q v_2 (w_2 + w_3 + w_4 + w_5 + w_6) + \theta w_6 (q v_6 - v_3)] < 0$. Furthermore, it can be shown that

$$b = \sum_{k,i=1}^6 v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta^*} (E_0) = v_2 q p (w_3 + w_4 + w_5) \quad (16)$$

The bifurcation coefficient, b , is positive if $w_4 + w_5 > w_3$ and $v_2 > 0$ if $\rho \eta v_3 > d_5 v_4$. Therefore, it follows from Theorem 4.1 in [7], that the model (1), exhibits a backward bifurcation at $R_c = 1$.

The phenomenon of backward bifurcation, which has been seen in several disease transmission dynamics (see for example, [7, 13]) is mainly characterized by the coexistence of a stable DFE and a stable endemic equilibrium when the associated reproduction number of the model is less than one. The implication of the phenomenon of backward bifurcation of model (1) is that the classical epidemiological requirement of having the reproduction number (R_c) less than one, although necessary, is no longer sufficient for the effective control of the disease.

3.2.2 Nonexistence of backward bifurcation

It is instructive to note that mathematical models that incorporate reinfection of recovered individuals are known to lose their backward bifurcation property when the reinfection parameters are set to zero (see for example, [28, 29] and [15]). Here, the role of the reinfection of recovered individuals (θ) on the phenomenon of backward bifurcation for model (1) will be investigated. For ease of reference, we denote $\widehat{R}_c = R_c | \theta = \gamma = 0$.

Theorem 3.3. *The special case of the model (1) in the absence of reinfection of recovered individuals (that is, $\theta = \gamma = 0$) does not undergo a backward bifurcation at $\widehat{R}_c = 1$.*

Proof. Setting $\theta = \gamma = 0$ in the expression for the bifurcation coefficients a and b given in equations (15) and (16) above yields (note here that $w_5 > 0$, $w_4 > 0$ but $w_2, w_6 > 0$ if $(w_4 + w_5) > \widehat{w}_3$ and $v_2 > 0$ if $\rho \eta v_3 > d_5 v_4$)

$$\begin{aligned} a &= \sum_{k,i,j=1}^6 v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (E_0) \\ &= -\frac{2\mu \beta^* (\widehat{w}_3 + w_4 + w_5) [q v_2 (w_2 + \widehat{w}_3 + w_4 + w_5 + w_6)]}{\Lambda} \end{aligned} \quad (17)$$

and

$$b = \sum_{k,i=1}^6 v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta^*}(E_0) = v_2 q p(\widehat{w}_3 + w_4 + w_5) \quad (18)$$

Therefore, from Theorem 4.1 in [17], it follows that the model (1) with $\theta = \gamma = 0$ will not undergo a backward bifurcation at $\widehat{R}_c = 1$. The epidemiological consequence of Theorem 3.3 is that the present study confirms the fact that reinfection (of recovered individuals) causes backward bifurcation in chlamydia transmission dynamics. To confirm further the absence of backward bifurcation in the model (when $\widehat{R}_c < 1$) for the special case of the model (1) with $\theta = \gamma = 0$, a global asymptotic stability result is given for the DFE E_0 .

Theorem 3.4. *The DFE for the special case of the model (1) with $\theta = \gamma = 0$ is globally asymptotically stable in Γ whenever $\widehat{R}_c \leq 1$.*

Proof. Note here that the equations for the infected classes of the model (1) with $\theta = \gamma = 0$ can be written in the matrix-vector form as follows:

$$\frac{dY(t)}{dt} = \left[(F - V) - \left(1 - \frac{S}{N}\right)M \right] Y(t), \quad (19)$$

where $Y(t) = (E(t), I_a(t), I_s(t), Q(t), R(t))^T$ and the matrices F and V are given in Section 3. Furthermore,

$$M = \begin{pmatrix} q\beta^* & q\beta^* & q\beta^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since M is a nonnegative matrix and $S \leq N$ in Γ , it follows that

$$\frac{dY(t)}{dt} \leq [(F - V)] Y(t). \quad (20)$$

Using the fact that the eigenvalues of the matrix $F - V$ all have negative real parts (that is, $\rho(FV^{-1}) < 1$ if $\widehat{R}_c < 1$), it follows that the linearized differential inequality system (20) is stable whenever $\widehat{R}_c < 1$. Thus, by comparison Theorem [17],

$$\lim_{t \rightarrow \infty} (E(t), I_a(t), I_s(t), Q(t), R(t)) = (0, 0, 0, 0, 0). \quad (21)$$

It can be shown by substituting (21) into (1) that, $S(t) \rightarrow \frac{\Lambda}{\mu}$, as $t \rightarrow \infty$. Hence,

$$\lim_{t \rightarrow \infty} (S(t), E(t), I_a(t), I_s(t), Q(t), R(t)) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0 \right) = E_0. \quad (22)$$

Thus, every solution to the equation of the model (1), with $\theta = \gamma = 0$ and initial conditions in Γ , approaches the DFE (E_0) as $t \rightarrow \infty$ whenever $\widehat{R}_c < 1$.

3.3 Existence of unique endemic equilibrium: Special case

Here, the existence of a unique endemic equilibrium of the model (1) will be examined, for the special case with no reinfection of recovered individuals (i.e., $\theta = \gamma = 0$).

To establish the existence of endemic equilibria of model (1), for the special case $\theta = \gamma = 0$, let

$E_2 = \{S^{**}, E^{**}, I_a^{**}, I_s^{**}, Q^{**}, R^{**}\}$ represent any arbitrary equilibrium of model (1). The equations in (1), with $\theta = \gamma = 0$, are solved in terms of the force of infection at the steady state to give

$$\begin{aligned} S^{**} &= \frac{\Lambda}{(\lambda^{**} + \mu)}, E^{**} = \frac{\lambda^{**}\Lambda}{d_1(\lambda^{**} + \mu)}, I_a^{**} = \frac{\lambda^{**}\rho\eta\Lambda}{d_1d_2(\lambda^{**} + \mu)}, \\ I_s^{**} &= \frac{\lambda^{**}\Lambda(\eta\rho d_6 + d_2d_5)}{d_1d_2d_3(\lambda^{**} + \mu)}, Q = \frac{\lambda^{**}d_7\Lambda(\eta\rho d_6 + d_2d_5)}{d_1d_2d_3d_4(\lambda^{**} + \mu)}, \\ R^{**} &= \frac{\lambda^{**}\Lambda[km\rho\eta d_3d_4 + \phi\tau d_4(\rho d_6 + d_2d_5) + \sigma\tau d_7(\eta\rho d_6 + d_2d_5)]}{\mu d_1d_2d_3d_4(\lambda^{**} + \mu)}. \end{aligned} \quad (23)$$

where (as defined earlier),

$$\begin{aligned} d_1 &= (\rho\eta + d_5 + \mu), d_2 = (d_6 + km + \mu), d_3 = (d_7 + \phi\tau + \mu), \\ d_4 &= (\sigma\tau + \mu), d_5 = (1 - \rho)\eta, d_6 = (1 - k)m, d_7 = (1 - \phi)\tau. \end{aligned} \quad (24)$$

The force of infection at steady state, λ^{**} is expressed as

$$\lambda^{**} = q \frac{(\beta_1 I_a^{**} + \beta_2 I_s^{**} + \beta_3 Q^{**})}{N^{**}} \quad (25)$$

with $q = (1 - \varepsilon\psi)$. Substituting the expressions in (23) into (25) gives

$$\left(1 + \frac{\lambda^{**}}{d_1} H_1\right) = \frac{q[\beta_1 \rho\eta d_3d_4 + (\eta\rho d_6 + d_2d_5)(\beta_2 d_4 + \beta_3 d_7)]}{d_1d_2d_3d_4}$$

which yields

$$\left(1 + \frac{\lambda^{**}}{d_1} H_1\right) = R_c$$

$$\lambda^{**} = \frac{d_1}{H_1} (R_c - 1) > 0,$$

with

$$\begin{aligned} H_1 &= \left[1 + \frac{\rho\eta}{d_2} + \frac{(\eta\rho d_6 + d_2d_5)}{d_2d_3} + \frac{d_7(\eta\rho d_6 + d_2d_5)}{d_2d_3d_4} \right. \\ &\quad \left. + \frac{[km\rho\eta d_3d_4 + \phi\tau d_4(\rho d_6 + d_2d_5) + \sigma\tau d_7(\eta\rho d_6 + d_2d_5)]}{\mu d_2d_3d_4} \right]. \end{aligned} \quad (26)$$

Hence, a unique endemic equilibrium (for the case $\theta = \gamma = 0$) exists when $\widehat{R}_c > 1$. The above result is summarized in the theorem below.

Theorem 3.5. *When there are no reinfection of recovered individuals (i.e., $\theta = \gamma = 0$), the model (1) has a unique endemic equilibrium whenever $\widehat{R}_c > 1$ and no endemic equilibrium if otherwise.*

Theorem 3.6. *The unique endemic equilibrium of the model (1) for the special case $\theta = \gamma = 0$, is locally asymptotically stable whenever $\widehat{R}_c > 1$ and $\widehat{R}_c$ near 1.*

Here, a contour plot of the reproduction number $\widehat{R}_c$, as a function of condom compliance (ψ) and efficacy of condom (ε) is shown in Figure 2 using the parameter values in Table 2. This graphical profile shows that no matter the level of condom compliance and its efficacy, the control or eradication of chlamydia in the population will be difficult (since $\widehat{R}_c$ is always greater than one). Furthermore, if the treatment rate is sufficiently increased, then the effective control of chlamydia in the population will be possible using condom compliance of 75 percent (based on the assumed efficacy of 80 percent)(see Figure 3). This result is similar to the one obtained by [10]. Though the result of their study was based on syphilis while the present study is based on chlamydia.

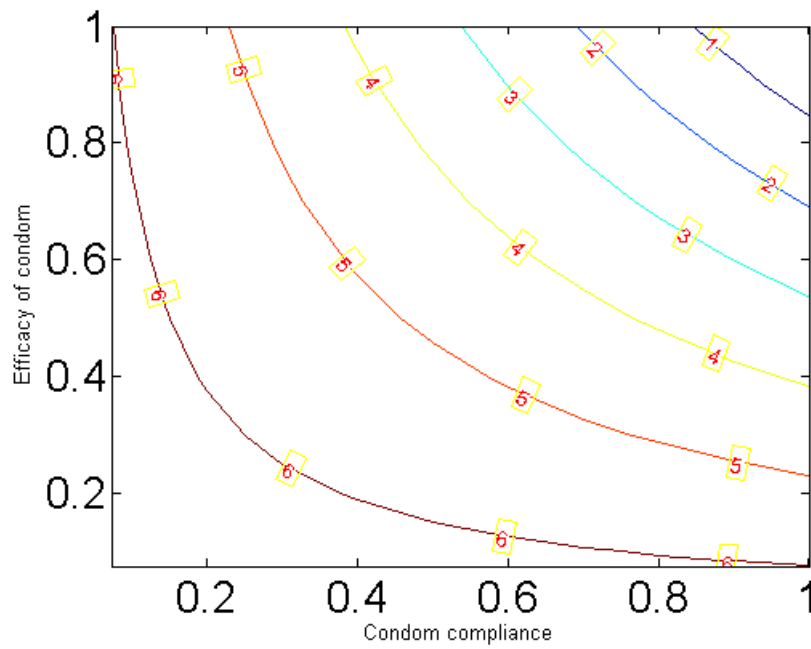


Figure 2: Contour plot of $\hat{R}_c$ as a function of condom compliance (ψ) and efficacy of condom (ϵ). Parameter values used are as in Table 2.

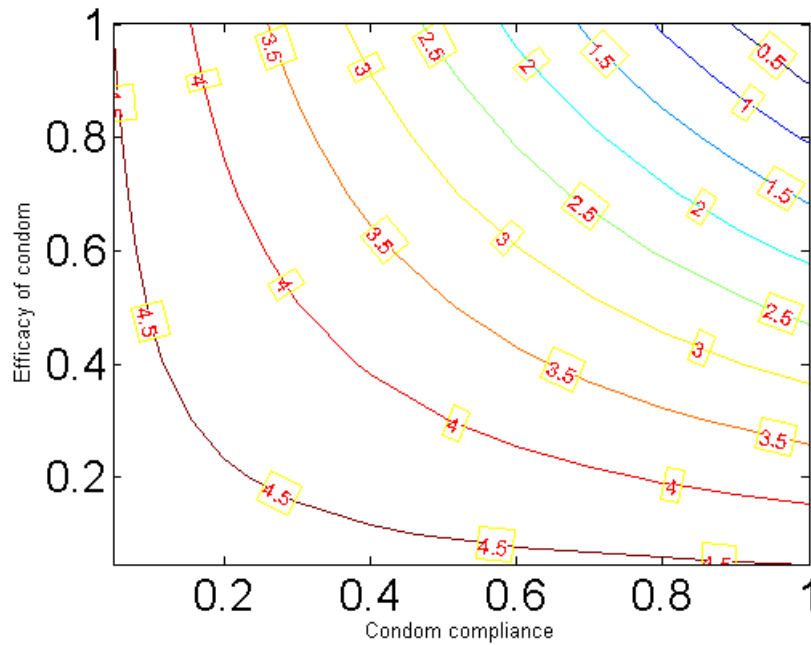


Figure 3: Contour plot of $\hat{R}_c$ as a function of condom compliance (ψ) and efficacy of condom (ϵ). Parameter values used are as in Table 2.

4 Sensitivity Analysis

In this section, we conduct sensitivity analysis on the parameters of the model associated with the reproduction number, R_c using the parameter values in Table 2. The essence of this analysis is to determine the relative importance of each parameter in the model that depicts Chlamydia transmission. A method similar to the ones outlined in the works of [2, 3, 24] were utilized to obtain the sensitivity index of all the parameters connected to the reproduction number using the formula in (26).

$$\chi_p^{R_c} = \frac{\partial R_c}{\partial \xi} \times \frac{\xi}{R_c}, \quad (27)$$

where ξ denotes model parameters contained in R_c . We obtained from (26) the sensitivity indices of the parameters associated with the reproduction number as presented in Table 3. The parameters with positive sensitivity indices signify a high impact burden of Chlamydia in the population if their values keeps increasing. In a similar manner, parameters in which their sensitivity indices are negative have a great effect in reducing Chlamydia burden in the population as their values increase while the others remain constant. Hence, as their values increase, the reproduction number decreases thus reducing the endemicity of the disease in the population.

From the results of the sensitivity analysis presented in Table 3, some of the parameters with high negative indices are the efficacy of condoms in preventing Chlamydia infection in the susceptible individuals (ε), condom compliance (ψ), a fraction of treated individuals who recover due to effective treatment (ϕ) and treatment rate (τ). These are the top parameters that significantly drive the dynamics of Chlamydia in the population. Consequently, to control the spread of the disease, these top parameters must be effectively targetted by policy makers in the health sector so that the sensitivity indices of these parameters must be kept negative for effective control of the disease. This implies that the advocacy for the use of condoms by susceptible individuals and proper treatment of infected individuals must be higher if Chlamydia must be curtailed in the population.

Table 2: The parameter values of model (1)

Parameter	Nominal value ($year^{-1}$)	Reference
Λ	1000	[29]
β_1	0.1	[27]
β_2	0.15	[27]
β_3	0.14	Implied from [27]
μ	0.15	[27]
ε	0.80	[11]
ψ	0.75	Assumed
ρ	0.70	[27]
η	0.50	[27]
θ	0.90	[27]
γ	0.80	[27]
σ	0.20	Assumed
ϕ	0.85	Assumed
τ	0.36	Assumed
k	0.70	[27]
m	0.70	[27]

Table 3: Sensitivity indices of some parameters values

Parameter	Sensitivity indices
Λ	+1.0000
β_1	+0.3255
β_2	+0.5493
β_3	+0.1248
μ	-1.3837
ε	-1.5000
ψ	-1.5000
ρ	-0.4263
η	$+1.4820 \times 10^{-10}$
σ	-0.0405
ϕ	-0.7072
τ	-0.3918
k	-0.5755
m	-0.2245

5 Numerical Simulations

In this section, numerical simulations are carried out on model (1) using the parameter values in Table 2. The model (1) is solved numerically using MATLAB ODE45 solver. Figure 4 shows the effect of the progression rate (η) of individuals in the exposed stage to the asymptomatic and symptomatic stages of infection on the number of infected individuals in some of the stages of Chlamydia infection. An increase in the progression rate (η) leads to an increase in the number of infected persons in the asymptomatic, symptomatic and individuals who failed treatment stages of Chlamydia infection. Unfortunately, increasing the treatment rate (τ) (as shown in Figure 5) does not lead to a decline rather an increase in the number of infected persons in the asymptomatic, symptomatic and treated individuals who failed treatment stages of infection. This result clearly show that treatment alone may not necessarily lead to a reduction in the number of infected persons in the population. As reported in WHO, infected persons after treatment should wait for at least 7 days before engaging in any form of sexual activity or, if not possible, use condoms correctly [34]. Violation of this report by WHO could be responsible for the increase in the number of asymptomatic, symptomatic and treated individuals who failed treatment despite the administration of drugs (azithromycin or doxycycline) (see Figure 5). Consequently, infected persons undergoing treatment or those who have successfully completed treatment are encouraged not to engage in any form of sexual activity until at least 7 days after treatment or correctly use condoms to avoid reinfection.

We demonstrate numerically in Figures 6 and 7, the relationship between the progression rate (η) and the treatment rate (τ). The reproduction number (R_c) is plotted as a function of the parameters η and τ . From Figure 6, it is observed that an increase in the progression rate from the exposed class to the asymptomatic and symptomatic stages of infection leads to an increase in the reproduction number. This implies that if concerted efforts are not put in place by relevant stakeholders in the health sector, this will trigger the number of Chlamydia cases in the population. In Figure 7 (as expected), increasing the treatment rates for individuals in the symptomatic and treated individuals who failed treatment leads to a decrease in the reproduction number. This can only be achievable if infected persons complete their treatment regime and not engage in any form of sexual activity until at least 7 days after treatment or, if not possible, use condoms correctly.

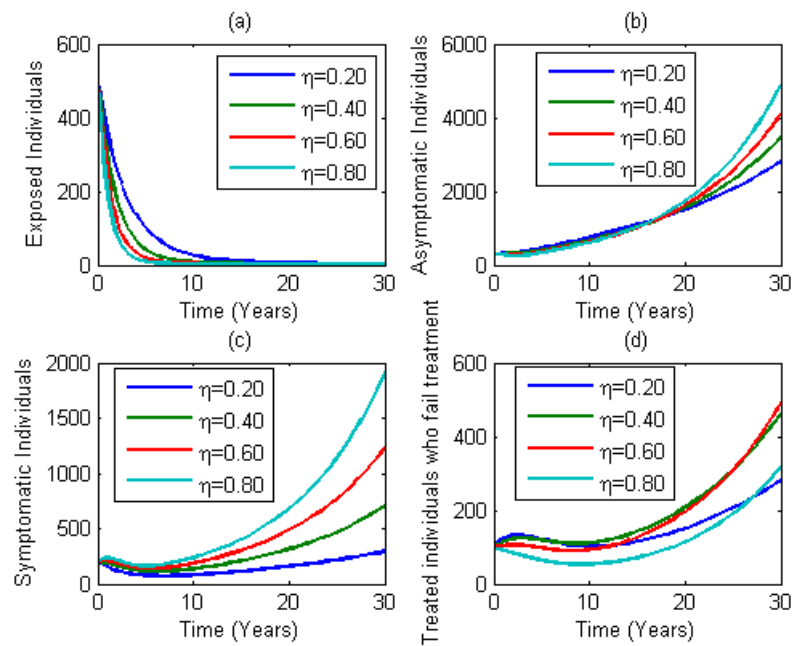


Figure 4: Simulation of model (1) showing the effect of progression rate from the exposed stage to the asymptomatic, symptomatic and treated individuals who failed treatment stages of infection. Here, η is varied from 0.20 to 0.80.

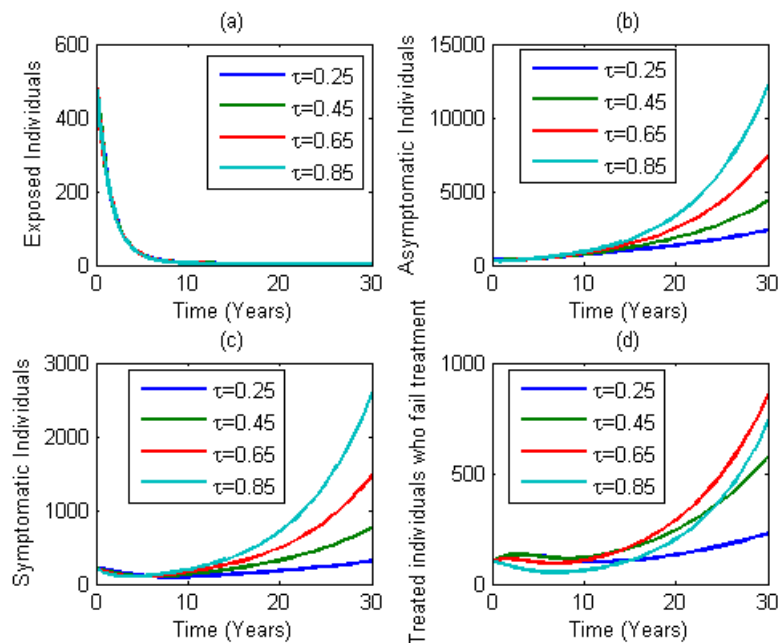


Figure 5: Simulation of model (1) showing the effect of treatment rate for individuals in the exposed, asymptomatic, symptomatic and treated individuals who failed treatment stages of infection. Here, τ is varied from 0.25 to 0.85.

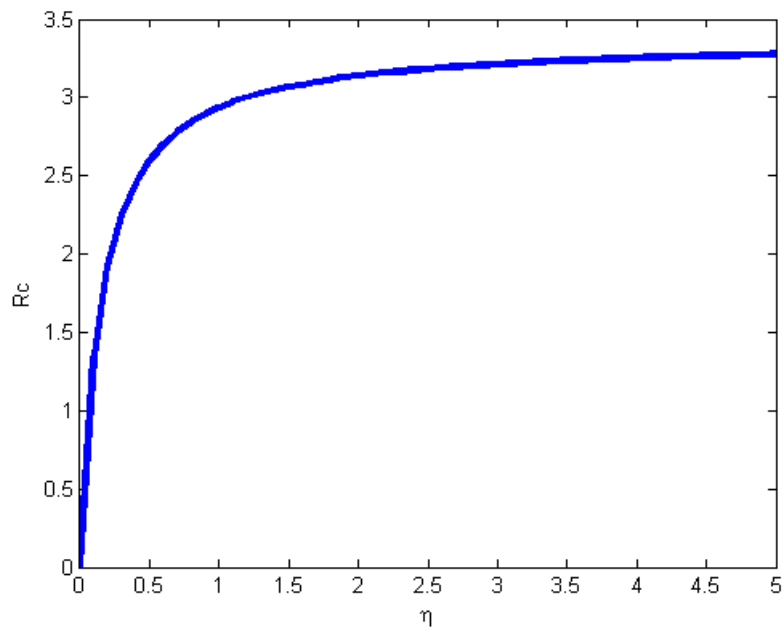


Figure 6: Plot of reproduction number, R_c as a function of progression rate, η . Parameter values used are as in Table 2.

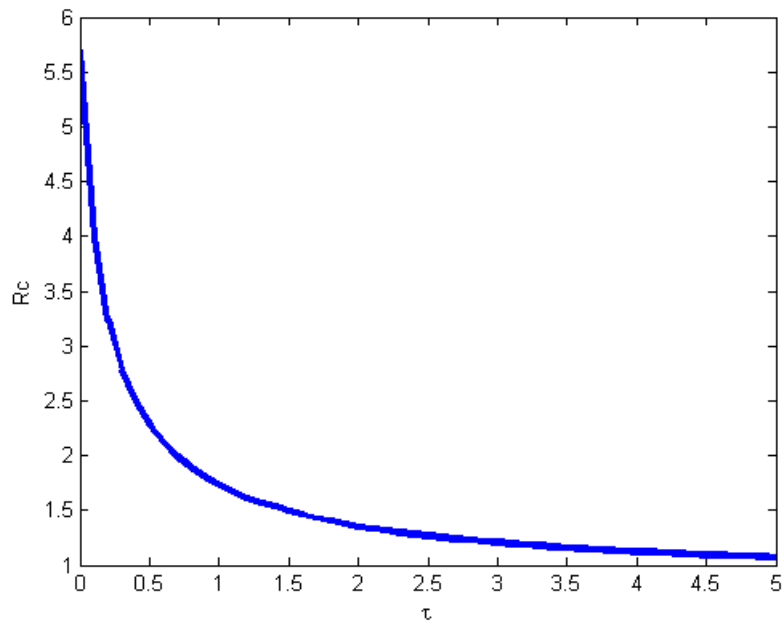


Figure 7: Plot of reproduction number, R_c as a function of treatment rate, τ . Parameter values used are as in Table 2.

6 Conclusion

This paper presents a deterministic mathematical model for gaining insights into the transmission dynamics of Chlamydia trachomatis in a population with condom usage and treatment as intervention strategies. The disease-free equilibrium (DFE) of the model (1) is locally asymptotically stable whenever the associated reproduction number is less than unity. Qualitative analysis of the model (1) show that it undergoes the phenomenon of backward bifurcation, where a stable DFE coexists with a stable endemic equilibrium. In a scenario of backward bifurcation, the effective control of Chlamydia is largely dependent on the initial sizes of the sub-population of the model. The epidemiological implication is that, the presence of backward bifurcation in the transmission dynamics of the disease makes its control extremely difficult. This phenomenon is shown to arise due to the reinfection of recovered individuals. In the absence of this phenomenon (i.e., where reinfection of recovered individuals does not occur), it is shown that the disease-free equilibrium of the model (1) is globally asymptotically stable whenever the associated reproduction number is than unity. Further, it is shown that the unique endemic equilibrium of the model exists whenever the reproduction number is greater than unity. Results from the sensitivity analysis of the model, using the reproduction number, R_c show that the top parameters that largely drive the dynamics of Chlamydia in the population are the efficacy of condoms, condom compliance, a fraction of treated individuals who recover due to effective treatment and treatment rate. Numerical simulations of the model suggest that infected persons after treatment should wait for at least 7 days before engaging in any form of sexual activity or, if not possible use condoms correctly (to avoid reinfection) in order to effectively control the spread of the disease in the population.

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Competing interests

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Towards mechanism-based interventions for auxin signaling: application of algebraic edge control to a Boolean network model

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Abstract

This study developed a Boolean network model representing auxin signaling dynamics. Control techniques identified five (5) interactions that, when applied, may modulate auxin signaling dynamics. Interpreting mechanisms by which these predicted targets regulates auxin responses generated testable hypotheses regarding novel regulators. While the simplified model omits molecular details, validating predictions experimentally can inform iterative model refinements that deepen mechanistic understanding. Integrating computational modeling, control strategies and experiment-model cycles thus shows promise for developing target-based auxin interventions. This framework may inform applications beyond auxin signaling, illustrating a blueprint for progressing from abstract biological representations to meaningful mechanisms and practical tools for purposefully manipulating complex systems.

Keywords: Boolean network; Auxin signaling dynamics; control techniques; validating predictions; experiment-model cycles; target-based interventions.

1 Introduction

Auxin signaling coordinates nearly all aspects of plant growth and development by regulating gene expression, cell division and expansion [16]. Precisely controlling auxin responses therefore holds promise for optimizing plant productivity, stress tolerance and trait modification [24]. However, the molecular mechanisms underlying auxin perception, transmission and response remain incompletely understood [8]. Systems-level modeling approaches offer a means to gain novel mechanistic insights into auxin signaling [2].

Boolean modeling provides a simplified yet effective framework for capturing network dynamics [13]. The model output suggests potential regulatory modules as candidate intervention targets [3]. However, traditional boolean models lack rigor in identifying such controllable nodes [17]. The algebraic edge control approach developed by Murrugarra et al. computationally identifies control targets within boolean networks in a systematic manner [21]. Applying this method to available auxin signaling boolean models could reveal potential mechanisms for specifically modulating auxin responses [9].

Arellano et al. constructed a boolean model of auxin signaling components and their interactions in plants [3]. Triggered by auxin, the network incorporates perception, signal transmission and response elements, resembling auxin-mediated regulation of genes [12]. Leveraging this model, we aim at applying algebraic edge control analysis to predict potential regulators of auxin responses within the boolean network, generate experimentally testable hypotheses regarding candidate mediating edges, interpret predicted controls based on current knowledge to evaluate their biological plausibility and usefulness

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and provide a foundation for iteratively refining model predictions through experiments, progressing towards mechanism-based auxin interventions.

The identified control targets are analyzed based on known functions of edges in auxin pathways [15]. Novel predictions generate experimentally verifiable hypotheses [5, 25]. Rigorously validating predictions through knockout and overexpression experiments then motivates refinement of model structures and parameters [4, 6]. This closed-loop process between modeling and experimentation may ultimately facilitate targeted manipulation of auxin responses [1, 20].

In all, applying algebraic edge control analysis to Arellano et al.'s auxin signaling Boolean model [3] aims to identify potential regulators of auxin responses within the network. Experimentally investigating predicted controls can reveal mechanisms underlying auxin signaling and motivate rational intervention design. This integrated framework generally enables mechanism-based innovation for biological systems [14].

2 Materials and Methods

A Boolean network model is a discrete dynamical system that captures the temporal evolution of a biological system [10, 19]. It consists of:

- A set of nodes representing molecular components (genes, proteins, etc.). Each node can be in one of two states: ON (1) or OFF (0) [10].
- A Boolean transition function for each node that determines its next state based on the current states of other nodes. This function is defined using Boolean operations (AND, OR, NOT) [19]. For a network with n nodes $\{y_1, y_2, \dots, y_n\}$, the Boolean transition function for node y_i is:

$$y_i(t+1) = f_i(y_1(t), y_2(t), \dots, y_n(t))$$

- A wiring diagram that specifies which nodes influence which others based on known or inferred interactions between components [10].
- A state transition table that lists the next state of each node for every possible combination of the network's states. From this table, temporal trajectories can be simulated *in silico* [19].

The transition functions are typically specified based on biological knowledge of the system, and calibrated to optimize how well model outputs match experimental observations. The Boolean abstraction, though simplistic, can reveal complex dynamical behaviors that provide insight into the system's emergent properties [10]. Arellano et al.'s Boolean auxin signaling model captures key dynamics of the pathway using a simplified, logical abstraction [3]. The model consists of 9 nodes (y_1 to y_9) representing major molecular components (see Appendix table 2), that includes:

$$\begin{aligned} y_1 &= ARF & y_2 &= AUXINS \\ y_3 &= IAA & y_4 &= JKD \\ y_5 &= MGP & y_6 &= PLT \\ y_7 &= SCR & y_8 &= SHR \\ y_9 &= WOX \end{aligned} \tag{1}$$

The Boolean transition function for node y_i is given by:

$$y_i(t+1) = f_i(y_1(t), y_2(t), \dots, y_9(t)) \tag{2}$$

Where f_i represents the boolean transition function (see Appendix table 2) determining y_i 's state from the states of other nodes at time t , in a more specific form we get;

$$\begin{aligned} f_1 &= \neg y_3 \\ f_2 &= y_2 \\ f_3 &= \neg y_2 \\ f_4 &= y_7 \wedge y_8 \\ f_5 &= y_7 \wedge y_8 \wedge \neg y_9 \\ f_6 &= y_1 \\ f_7 &= \neg y_5 \wedge y_7 \wedge y_8 \vee y_4 \wedge y_7 \wedge y_8 \\ f_8 &= y_8 \\ f_9 &= y_1 \wedge y_7 \wedge y_8 \wedge y_9 \vee y_1 \wedge \neg y_5 \wedge y_7 \wedge y_8 \end{aligned} \quad (3)$$

This model structure - comprised of nodes, interactions and update rules - allows simulating the temporal dynamics of auxin signaling under logical abstraction [11]. Though simple, the model recapitulates qualitative system-level properties while illuminating regulatory mechanisms [19]. The network diagram of the auxin signaling network [3], is displayed in Figure 1. The functions that describe the interactions between molecules in the network can be represented as polynomials over the finite field $\mathbb{F}_2$, which is defined as the set $\{0, 1\}$ with standard addition and multiplication operations. This allows the dynamical system $F = (f_1, \dots, f_n) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ to be expressed as a polynomial dynamical system, as described in reference [13]. By using polynomial representations, researchers can analyze the dynamics of Boolean networks such as the auxin signaling network in a more efficient and effective manner.

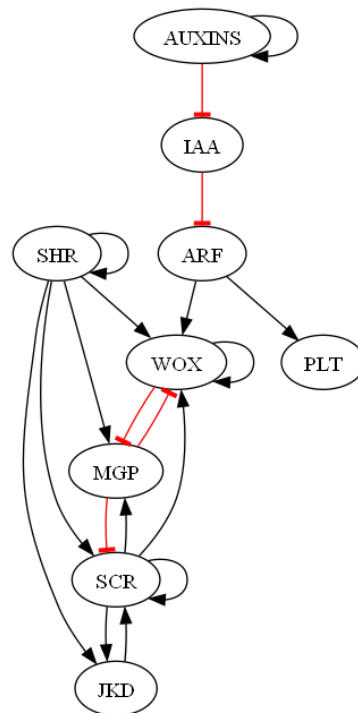


Figure 1: Auxin Signalling Network (Also see [3])

To convert Boolean functions into polynomials, we utilize a set of guidelines that dictate how Boolean operations are calculated with modulo 2 in $F_2[y_1, \dots, y_n]$. Specifically, the logical AND operation between variables x and y , which is denoted by $x \wedge y$, is equivalent to their product, xy . The logical OR operation, represented by $x \vee y$, is equivalent to their sum plus their product, $x + y + xy$.

The squaring of a variable x is equal to itself, $x^2 = x$. The multiplication of a variable x by a scalar λ is equal to 0 for any scalar $\lambda \in \mathbb{R} \setminus [-1, 1]$, $\lambda x = 0$. Finally, the logical NOT operation, denoted by $\neg x$, is equal to 1 plus the variable x , $\neg x = 1 + x$. By applying these rules to the auxin signaling network, we can obtain a set of polynomials that describe the behavior of the system.

$$\begin{aligned}
 f_1 &= 1 + y_3 \\
 f_2 &= y_2 \\
 f_3 &= 1 + y_2 \\
 f_4 &= y_7 y_8 \\
 f_5 &= y_7 y_8 (1 + y_9) = y_7 y_8 + y_7 y_8 y_9 \\
 f_6 &= y_1 \\
 f_7 &= y_4 y_5 y_7 y_8 + y_5 y_7 y_8 + y_7 y_8 \\
 f_8 &= y_8 \\
 f_9 &= y_1 y_5 y_7 y_8 y_9 + y_1 y_5 y_7 y_8 + y_1 y_7 y_8
 \end{aligned} \tag{4}$$

Determining attractors is a pivotal stride in comprehending and regulating intricate arrangements, and computational algebra presents beneficial instruments for accomplishing that [22, 23]. To unearth the unvarying conditions of the network, we must resolve a collection of equations in which $f_i = y_i$ for each unit i in the arrangement. This signifies that we are in quest of the origins of a set of equations $g_i = 0$, where $g_i = f_i - y_i$. By unraveling this assortment of equations, we can ascertain the plausible unvarying conditions of the network and acquire insights into the fundamental mechanisms that govern its conduct.

$$\begin{aligned}
 g_1 &= f_1 - y_1 \\
 g_2 &= f_2 - y_2 \\
 g_3 &= f_3 - y_3 \\
 g_4 &= f_4 - y_4 \\
 g_5 &= f_5 - y_5 \\
 g_6 &= f_6 - y_6 \\
 g_7 &= f_7 - y_7 \\
 g_8 &= f_8 - y_8 \\
 g_9 &= f_9 - y_9
 \end{aligned} \tag{5}$$

Since the system exhibits non-linear characteristics, conventional techniques like Gaussian elimination are inadequate for solving it. Nevertheless, computational algebra offers an alternative method by encoding the solutions as an algebraic entity called an ideal of polynomials. In our scenario, the ideal is denoted as $I = \{g_i\} \forall i = 1 \dots, 9$. Subsequently, we can simplify the representation of this ideal by identifying its Gröbner basis [7] utilizing mathematical software tools like Sagemath [18]. By determining the Gröbner basis of the ideal, we obtain a more streamlined system that encompasses the same solutions as the original system. This approach bestows a potent instrument for resolving non-linear systems and comprehending their behavior. Upon solving the aforementioned equation (5), we reach the following outcome:

$$\begin{aligned}
 g_1 &= 0 & g_2 &= 0 & g_3 &= 1 \\
 g_4 &= 0 & g_5 &= 0 & g_6 &= 0 \\
 g_7 &= 0 & g_8 &= 0 & g_9 &= 0
 \end{aligned} \tag{6}$$

Therefore, the value 001000000 represents a stable state of the network.

2.1 Edge Control

To control the edge $y_i \rightarrow y_j$ in the wiring diagram W , we utilize the function.

$$F_j(y, v_{i,j}) = f_j(y_1, \dots, (v_{i,j} + 1)y_i, \dots, y_n) \quad (7)$$

This function encodes the control settings for the edge $y_i \rightarrow y_j$, based on the possible values of $v_{i,j} \in \mathbb{F}_2$. The control settings for the edge $y_i \rightarrow y_j$ can be described as follows:

- If $v_{i,j} = 0$, then $F_j(y, 0) = f_j(y_1, \dots, y_i, \dots, y_n)$. In this case, the control is not active.
- If $v_{i,j} = 1$, then $F_j(y, 1) = f_j(y_1, \dots, y_i = 0, \dots, y_n)$. Here, the control is active, and it represents the removal of the edge $y_i \rightarrow y_j$.

This can be expanded to encompass the control of multiple edges, resulting in $F : \mathbb{F}_2^n \times \mathbb{F}_2^e \rightarrow \mathbb{F}_2^n$, where e denotes the number of edges in the wiring diagram. Each coordinate, $v_{i,j}$, of v in $F(y, v)$ encodes the control for an edge $y_i \rightarrow y_j$ [13]. The above is the Murrugarra et al.'s algebraic edge control method and it provides a rigorous mathematical approach for identifying controllable nodes within Boolean network models [13]. We use the Boolean functions in equation (4) defining network dynamics into multivariate polynomials over GF(2), the Galois field with two elements $\{0, 1\}$. Here:

$$\begin{aligned} f_1 &= 1 + y_3(v_{3,1} + 1) \\ f_2 &= y_2(v_{2,2} + 1) \\ f_3 &= 1 + y_2(v_{2,3} + 1) \\ f_4 &= y_7y_8(v_{7,4} + 1)(v_{8,4} + 1) \\ f_5 &= y_7y_8(v_{7,5} + 1)(v_{8,5} + 1) + y_7y_8y_9(v_{7,5} + 1)(v_{8,5} + 1)(v_{9,5} + 1) \\ f_6 &= y_1(v_{1,6} + 1) \\ f_7 &= y_4y_5y_7y_8(v_{7,7} + 1)(v_{5,7} + 1)(v_{4,7} + 1)(v_{8,7} + 1) + y_5y_7y_8(v_{5,7} + 1) \\ &\quad (v_{8,7} + 1)(v_{7,7} + 1) + y_7y_8(v_{7,7} + 1)(v_{8,7} + 1) \\ f_8 &= y_8(v_{8,8} + 1) \\ f_9 &= y_1y_5y_7y_8y_9(v_{1,9} + 1)(v_{5,9} + 1)(v_{7,9} + 1)(v_{8,9} + 1)(v_{9,9} + 1) + y_1y_5y_7y_8(v_{1,9} + 1)(v_{5,9} + 1)(v_{7,9} + 1) \\ &\quad (v_{8,9} + 1) + y_1y_7y_8(v_{1,9} + 1)(v_{7,9} + 1)(v_{8,9} + 1) \end{aligned} \quad (8)$$

Suppose that $z_0 = (z_{01}, \dots, z_{0n}) \in F_2^n$ is a desirable cell state (for instance, it could represent the state of cell senescence) but is not a fixed point, i.e., $F(z_0) \neq z_0$. The problem is then to choose a control v such that $F(z_0, v) = z_0$. We now show how this can be achieved in our framework.

After encoding our Boolean Network (BN) with control as a polynomial system $F_j(y, v) \in F_2[y, v]$ in equation (8), we consider the system of polynomial equations in the v parameters:

$$F_j(z_0, v) - z_{0j} = 0, \quad j = 1, \dots, m. \quad (9)$$

Lets consider a scenario, where our goal is to control the edges in order to achieve a steady state where $z_0 = 111001010$ (desired steady state). If we evaluate the polynomial system in equation (8) at $z_0 = 111001010$, we obtain a system of equations that involves only the control parameters v . The specific equations depend on the structure of the Boolean Network and the chosen control actions, where $(v_{i,j} + 1) = \bar{v}_{i,j}$. Then;

$$\begin{aligned} f_1 &= \bar{v}_{3,1}, & f_2 &= v_{2,2} \\ f_3 &= \bar{v}_{2,3}, & f_4 &= 0 \\ f_5 &= 0, & f_6 &= v_{1,6} \\ f_7 &= 0, & f_8 &= v_{8,8} \\ f_9 &= 0 \end{aligned} \quad (10)$$

In the above, we set $F_i = z_{0i}$ for $i = 1, \dots, 9$. By solving for the control inputs that satisfy these conditions, we find that for this to be a stable state $v_{3,1} = v_{2,3} = v_{2,2} = v_{8,8} = v_{1,6} = 1$. In other words, if we delete the edges from nodes y_i to y_j and fix them, the network will reach the steady state 111001010 and remain there. This control scheme is unique for this particular steady state.

Let's suppose we have a Boolean network with n nodes, and there is a steady state $w_0 \in \{0, 1\}^n$ that we want to avoid. This steady state could represent an undesirable cell state, for example. Our goal is to find a set of control inputs that ensure the network never reaches this steady state. Alternatively, we may want to prevent the network from transitioning between two states z_0 and w_0 . To achieve this, we can use a polynomial equation that relates the control inputs to the network's dynamics.

The equation we use is given by:

$$(F_1(z_0, v) - w_{01} + 1) \cdots (F_n(z_0, v) - w_{0n} + 1) = 0 \quad (11)$$

This equation defines a polynomial in the control inputs $v_1, \dots, v_m$. If we can find a solution to this equation, then we can ensure that the network's dynamics will never reach the undesired state w_0 , or that it will not transition from z_0 to w_0 . In other words, if we can find a set of control inputs that satisfy the equation, then we have achieved our goal. Let's consider a scenario, where we are interested in identifying edges in the Boolean network that can block the transition from state $z_0 = 111001010$ to the undesired steady state $F(z_0) = 010001010$. Then again $v_{3,1} = v_{2,3} = v_{2,2} = v_{8,8} = v_{1,6} = 1$ can achieve this too.

Table 1: Disparity in the influence of removing edges on combinatorial operations.

Solution	Controllers	Description	% of Basin Size of z_0
$v_{3,1} = 1$	IAA $\longrightarrow$ ARF	Control sets that force z_0	12.5
$v_{2,2} = 1$	AUXINS $\longrightarrow$ AUXINS	to be a fixed point and block	
$v_{2,3} = 1$	AUXINS $\longrightarrow$ IAA	its transition to 010001010	
$v_{1,6} = 1$	ARF $\longrightarrow$ PLT		
$v_{8,8} = 1$	SHR $\longrightarrow$ SHR		

3 Result

Auxin signaling is an important phytohormone pathway that regulates many aspects of plant growth and development. The dynamics and multistability of auxin-mediated gene networks allow plants to produce different phenotypes in response to environmental and genetic factors.

The results in the table [1](#) provide insights into how purposefully manipulating specific interactions within the Boolean model of auxin signaling can achieve desired control over network dynamics. Specifically, deleting the 5 edges listed as "Controllers" reduces the basin size of the initial state z_0 representing the desired state by 87.5% of its original size. This minimum control set suggests that:

- Deleting the IAA $\longrightarrow$ ARF edge may disrupt negative feedback within the network, altering phenotypes
- removing the AUXINS self-interaction likely inhibits auxin accumulation, constraining responses
- the elimination of AUXINS $\longrightarrow$ IAA edge relieves IAA inhibition of ARF, having the opposite intended effect
- the removal of ARF $\longrightarrow$ PLT edge could prevent PLT activation downstream, perturbing outputs

- erasing the SHR self-interaction may disrupt SHR's co-regulation of targets with SCR

Together, this multi-faceted perturbation - systematically deleting specified interactions - appears to synergistically reroute network dynamics in a way that reduces 87.5% of state trajectories leading to the initial state z_0 , representing the desired attractor.

In summary, a rigorous analysis of how purposefully manipulating specific interactions achieves control over this Boolean auxin signaling model's dynamics affords insights into novel regulatory mechanisms - insights which, when validated experimentally, could inform engineering of plant phenotypes for agricultural applications. These results suggest that the auxin signaling network is highly sensitive to changes in its edge structure, and that targeted removal or addition of edges can be a powerful tool for controlling the network's behavior. By understanding the network's structure and dynamics, we can identify key edges that are critical for maintaining the network's behavior and use this knowledge to develop new strategies for controlling plant growth and development.

4 Conclusion

Our framework identified potential auxin response regulators via algebraic model analysis and control techniques. Validating these interventions experimentally can deepen our mechanistic insights, informing model refinements that yield improved predictions and hypotheses. Integrating modeling, control strategies and experiment-model cycles thus shows promise for purposefully manipulating molecular networks, paving the way for beneficial auxin-based applications. This auxin case study demonstrates a blueprint for developing mechanism-based interventions across biologically and medically relevant networks.

Appendix

Table 2: Boolean Rules For the Auxin Signalling Network

Node	Boolean Transition Rules
ARF	NOT IAA
AUXINS	AUXINS
IAA	NOT AUXINS
JKD	SCR AND SHR
MGP	SCR AND SHR AND NOT WOX
PLT	ARF
SCR	NOT MGP AND SCR AND SHR OR JKD AND SCR AND SHR
SHR	SHR
WOX	ARF AND SCR AND SHR AND WOX OR ARF AND NOT MGP AND SCR AND SHR

Table 3: Nodes and their corresponding gene names

Node	Gene Name
ARF	AUXIN RESPONSE FACTOR
AUX	AUXIN SIGNALING
IAA	INDOLE-3-ACETIC ACID
JKD	JAGGED
MGP	MORE GROWTH PLANT
PLT	PLETHORA
SCR	SCARECROW
SHR	SHORT ROOT
WOX	WUSCHEL-RELATED HOMEBOX

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Numerical approximation of space fractional order wave equation using shifted Chebyshev polynomials of second and third kinds

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Abstract

In this paper, we proposed a numerical method to solve space fractional order wave equation (SFOWE). The method implements shifted Chebyshev polynomials of the second and third kinds with the fractional derivative interpreted in Caputo sense. Chebyshev collocation method is adopted to reduce the problem to a system of second order ordinary differential equation, while the finite difference method reduces that to a system of linear algebraic equations. The system of algebraic equations thus formed is solved with the aid of Mathematica 11.0. The proposed method gives results that compared favourably with existing results in the literature, while it is observed that the performance of second kind is relatively superior to that of the third kind polynomials. The results are presented in both tabular and 3D graphs, for ease of comparison.

1 Introduction

In recent years, a swift development on fractional calculus in theory and applications has become more apparent (1). Fractional calculus is a branch of mathematics that deals with generalization of differentiation and integration on non-integer orders (12). The history of fractional calculus dated back to the 17th century when mathematicians began exploring the concept of fractional order. However, the formal development of fractional calculus did not begin until 1790s, when French mathematician Joseph Liouville studied the properties of fractional order equations (14). In the mid-20th century, there was a renewed interest in fractional calculus with the advent of modern computers, as it was found to be useful in wide range of fields such as physics, engineering, and finance. Several researchers such as (12) and (11) advanced the theory of fractional calculus and its applications during this time. The fractional calculus has found its applications in the study of diffusion equations (9), wave propagation (2), signal processing (10) and other areas of mathematical physics. It has also been applied to the field of engineering, finance and biology (4). Seeking solution to fractional order differential equations remains a great task to be accomplished by the researchers, except in a very few number of these equations. Several attempts have been made by different researchers to solve the differential equation of non-integers values. Among the methods are, but not limited to Adomian Decomposition Method (ADM) (6), Homotopy Perturbation Method (HPM) (13), Collocation Method (7), Homotopy Analysis Method (HAM) (3; 15), Orthogonal Polynomial Approach (16), while (5) used Gegenbauer polynomial as an approach to the solution of fractional order partial differential equations. This paper focuses on the numerical solution of space fractional order wave equation by utilizing the accuracy of shifted Chebyshev polynomials of the second and the third kinds. Both polynomials in their shifted forms, along with the fractional derivative expressed in Caputo sense are used to transform the (SFOWE) into a system of second order ordinary differential equations while Finite Difference Method (FDM) reduces the system thus formed, to a system of linear algebraic equations which are later solved for the unknown constants.

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2 Preliminaries

2.1 Statement of the Problem

The class of problem considered in this paper is the space fractional order wave equation of the form:

$$\frac{\partial^2 \omega(x, t)}{\partial t^2} = \xi(x, t) \frac{\partial^\theta \omega(x, t)}{\partial x^\theta} + \eta(x, t), \quad (1)$$

on a finite domain $a < x < b$, $0 \leq t \leq T$. The parameter θ is the fractional order of partial differential equation (1) with $1 < \theta < 2$. The function $\eta(x, t)$ is a non-homogeneous source term and the coefficient function $\xi(x, t) \geq 0$. We also assumed the problem (1) is subject to the following initial conditions:

$$\omega(x, 0) = \omega^0(x), \quad \omega_t(x, 0) = \omega'(x) \quad (2)$$

and the Dirichlet boundary conditions

$$\omega(a, t) = \omega(b, t) = 0. \quad (3)$$

It is important to note that when $\theta = 2$, (1) becomes the classical wave equation of the form:

$$\frac{\partial^2 \omega(x, t)}{\partial t^2} = \xi(x, t) \frac{\partial^2 \omega(x, t)}{\partial x^2} + \eta(x, t) \quad (4)$$

2.2 Fractional Derivative

Let

$$f(x) = x^k \quad (5)$$

The first derivative is as usual

$$f'(x) = \frac{d}{dx} f(x) = kx^{k-1} \quad (6)$$

Repeating this process gives more general result

$$\frac{d^\alpha}{dx^\alpha} x^k = \frac{k!}{(k-\alpha)!} x^{k-\alpha}. \quad (7)$$

Replacing the factorials by the gamma function equivalent, (7) becomes

$$\frac{d^\alpha}{dx^\alpha} x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}; \quad k > 0 \quad (8)$$

For $k = 2$ and $\alpha = \frac{1}{2}$, we obtain the half derivative of the function x^2 as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x^2 = \frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{2}+1)} x^{2-\frac{1}{2}} = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}}.$$

2.3 The Caputo fractional derivative

The Caputo fractional derivative operator D^θ of order θ is defined in the following form:

$$D^\theta f(x) = \frac{1}{\Gamma(n-\theta)} \int_0^x (x-t)^{n-\theta-1} f^{(n)}(t) dt; \theta > 0 \quad (9)$$

where $n-1 < \theta < n$, $n \in \mathbb{N}$, $x > 0$.

The linearity property of Caputo fractional derivative suit with that of the integer order differential operator.

$$D^\theta(\psi f(x) + \phi g(x)) = \psi D^\theta f(x) + \phi D^\theta g(x) \quad (10)$$

where ψ and ϕ are constants. For the Caputo derivative, it is possible to obtain the following results:

$$D^\theta x^\tau = \begin{cases} 0, & \text{for } \tau \in N_0 \text{ and } \tau < \lceil \theta \rceil \\ \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-\theta)} x^{\tau-\theta}, & \text{for } \tau \in N_0 \text{ and } \tau \geq \lceil \theta \rceil, \end{cases} \quad (11)$$

where the function $\lceil \theta \rceil$ is the smallest integer greater than or equal to θ and N_0 represents the natural numbers with the inclusion of zero.

2.4 Proof:

We note that:

$$\begin{aligned} f^{(n)}(t^\tau) &= \tau f^{(n-1)} t^{\tau-1} \\ &= \tau(\tau-1) f^{(n-2)} t^{\tau-2} \\ &= \tau(\tau-1)(\tau-2) f^{(n-3)} t^{\tau-3} \end{aligned}$$

The process continues until

$$\begin{aligned} &= \frac{\tau!}{(\tau-n)!} t^{\tau-n} \\ f^{(n)}(t^\tau) &= \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)} t^{\tau-n} \end{aligned}$$

From (9) we have

$$\begin{aligned} D^\theta x^\tau &= \frac{1}{\Gamma(n-\theta)} \int_0^x (x-t)^{n-\theta-1} f^{(n)}(t^\tau) dt \\ &= \frac{1}{\Gamma(n-\theta)} \int_0^x \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)} t^{\tau-n} (x-t)^{n-\theta-1} dt \end{aligned}$$

Let $t = \mu x; 0 \leq \mu \leq 1, dt = x d\mu$

$$= \frac{1}{\Gamma(n-\theta)} \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)} x^{\tau-\theta} \int_0^1 \mu^{\tau-n} (1-\mu)^{n-\theta-1} d\mu,$$

But

$$\begin{aligned} B(m, n) &= \int_0^x x^{m-1} (1-x)^{n-1} dx \\ &= \frac{1}{\Gamma(n-\theta)} \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)} x^{\tau-\theta} B(\tau-n+1, n-\theta), \end{aligned}$$

Also note:

$$\begin{aligned} B(m, n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\ &= \frac{1}{\Gamma(n-\theta)} \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)} x^{\tau-\theta} \frac{\Gamma(\tau-n+1)\Gamma(n-\theta)}{\Gamma(\tau+1-\theta)} \\ \therefore D^\theta x^\tau &= \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-\theta)} x^{\tau-\theta} \end{aligned}$$

3 Some Properties of Second and Third Kinds Chebyshev Polynomials

The Second and Third kinds Chebyshev polynomials denoted by $U_n(x)$ and $V_n(x)$ see [8] are orthogonal polynomials of degree n in x defined on $[-1, 1]$ and are of the form:

$U_n(x) = \sin(n+1)\alpha/\sin\alpha$ and $V_n(x) = \cos(n+\frac{1}{2}\alpha)/\cos\frac{1}{2}\alpha$ respectively, with $x = \cos\alpha$ and

$\alpha \in [0, \pi]$.

The orthogonality of the second and third kinds polynomials are shown respectively as:

$$\langle U_n(x), U_m(x) \rangle = \int_{-1}^1 \sqrt{1-x^2} U_n(x)U_m(x)dx = \begin{cases} 0; & n \neq m, \\ \frac{\pi}{2}; & n = m, \end{cases} \quad (12)$$

$$\langle V_n(x), V_m(x) \rangle = \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} V_n(x)V_m(x)dx = \begin{cases} 0; & n \neq m, \\ \pi; & n = m, \end{cases} \quad (13)$$

where $(1-x^2)^{\frac{1}{2}}$ and $(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ are the weight functions respectively. The recurrence relations for these polynomials are given as:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots \quad (14)$$

with the first two terms given as $U_0(x) = 1$ and $U_1(x) = 2x$. Also,

$$V_n(x) = 2xV_n(x) - V_{n-1}(x), \quad n = 1, 2, \dots \quad (15)$$

with the first two terms given as $V_0(x) = 1$ and $V_1(x) = 2x - 1$. The second kind shifted Chebyshev polynomial on the general intervals $[a, b]$ is obtained as

$$U_n^*(x) = \left\{ \frac{2x-b-a}{b-a} \right\} U_{n-1}(x) - U_{n-2}(x), \quad x \geq 2. \quad (16)$$

For a particular case where $x \in [0, 2]$, the shifted forms of (14) and (15) are now given as

$$U_n^*(x) = 2(x-1)U_{n-1}(x) - U_{n-2}(x), \quad n \geq 2. \quad (17a)$$

$$V_{n+1}^*(x) = 2(x-1)V_n(x) - V_{n-1}(x), \quad n \geq 1. \quad (17b)$$

Thus, 12 and 13 now becomes;

$$\langle U_n^*(x), U_m^*(x) \rangle = \int_0^2 \sqrt{2x-x^2} U_n^*(x)U_m^*(x)dx = \begin{cases} 0; & n \neq m, \\ \frac{\pi}{2}; & n = m, \end{cases} \quad (18a)$$

$$\langle V_n^*(x), V_m^*(x) \rangle = \int_0^2 \sqrt{\frac{x}{2-x}} V_n^*(x)V_m^*(x)dx = \begin{cases} 0; & n \neq m, \\ \pi; & n = m, \end{cases} \quad (18b)$$

with the corresponding weight functions given as $(2x-x^2)^{\frac{1}{2}}$ and $(\frac{x}{2-x})^{\frac{1}{2}}$ respectively.

The analytical form of the second kind shifted Chebyshev polynomials of degree n in x denoted by $U_n^*(x)$ is given by

$$U_n^*(x) = \sum_{k=0}^n (-1)^k 2^{n-k} \frac{\Gamma(2n-k+2)}{\Gamma(k+1)\Gamma(2n-2k+2)} x^{n-k}, \quad n \geq 0. \quad (19a)$$

The analytical form of the third kind shifted Chebyshev polynomials of degree n in x denoted by $V_n^*(x)$ is given by

$$V_n^*(x) = \sum_{k=0}^n (-1)^k 2^{n-k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} x^{n-k}, \quad n \geq 0. \quad (19b)$$

The function $\omega(x)$ which will be required in the solution of (1) can be written as series of $U_n^*(x)$ and $V_n^*(x)$ respectively as

$$\omega(x) = \sum_{i=0}^{\infty} a_i U_i^*(x) \quad \text{and} \quad \omega(x) = \sum_{i=0}^{\infty} a_i V_i^*(x)$$

where the coefficients a_i ; $i = 0, 1, 2, \dots$ are generated respectively by

$$a_i = \frac{2}{\pi} \int_0^2 \omega(x) \sqrt{2x - x^2} U_i^*(x) dx. \quad (20a)$$

$$a_i = \frac{1}{\pi} \int_0^2 \omega(x) \sqrt{\frac{x}{2-x}} V_i^*(x) dx. \quad (20b)$$

In practice, only the $(n+1)$ -terms of shifted Chebyshev polynomials of the second and third kinds are required in the approximation. Thus, we have

$$\omega_n(x) = \sum_{i=0}^n a_i U_i^*(x) \quad (21a)$$

$$\omega_n(x) = \sum_{i=0}^n a_i V_i^*(x) \quad (21b)$$

4 Solution of the Fractional Derivative Using Shifted Chebyshev Polynomials of the Second and Third Kinds

Suppose $\omega(x)$ is approximated with the help of (19a) and (19b), and taken $\theta > 0$, combining this with the linearity property of Caputo differential operator, we get

$$\omega_n(x) = \sum_{i=0}^n a_i D^\theta (U_i^*(x)) \quad (22a)$$

$$\omega_n(x) = \sum_{i=0}^n a_i D^\theta (V_i^*(x)) \quad (22b)$$

From the Caputo derivative given in (11), it follows that;

$$D^\theta (U_i^*(x)) = 0, \quad \text{for } i = 0, 1, \dots, [\theta] - 1, \quad \theta > 0,$$

$$D^\theta (V_i^*(x)) = 0, \quad \text{for } i = 0, 1, \dots, [\theta] - 1, \quad \theta > 0,$$

And

$$D^\theta (U_i^*(x)) = \sum_{k=0}^i (-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)}{\Gamma(k+1)\Gamma(2i-2k+2)} D^\theta x^{i-k}. \quad (23a)$$

$$D^\theta (V_i^*(x)) = \sum_{k=0}^i (-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)} D^\theta x^{i-k}. \quad (23b)$$

Using (11), (23a) and (23b) becomes

$$D^\theta (U_i^*(x)) = \sum_{k=0}^{i-[\theta]} (-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta} \quad (24a)$$

$$D^\theta (V_i^*(x)) = \sum_{k=0}^{i-[\theta]} (-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta}. \quad (24b)$$

Combining (22a) and (24a), (22b) and (24b), we obtained respectively:

$$D^\theta (\omega_n(x)) = \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} a_i (-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta} \quad (25a)$$

$$D^\theta (\omega_n(x)) = \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} a_i (-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta} \quad (25b)$$

(25a) and (25b) can both be respectively written as:

$$D^\theta (\omega_n(x)) = \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} a_i M_{i,k}^{(\theta)} x^{i-k-\theta}, \quad (26a)$$

$$D^\theta (\omega_n(x)) = \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} a_i G_{i,k}^{(\theta)} x^{i-k-\theta}, \quad (26b)$$

where

$$M_{i,k}^{(\theta)} = (-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}. \quad (27a)$$

and

$$G_{i,k}^{(\theta)} = (-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}. \quad (27b)$$

The accuracy of our proposed method can be tested as follows:

Suppose $\omega(x) = x^3$, $n = 4$ and $\theta = 1.8$

Using the definition of Caputo derivative given in (11), we obtained

$$D^{1.8} x^3 = \frac{\Gamma(3+1)}{\Gamma(3+1-1.8)} x^{3-1.8} = \frac{\Gamma(4)}{\Gamma(\frac{11}{5})} x^{\frac{6}{5}}. \quad (28)$$

Now, by using our proposed method given in (26a) and (27a), we have

$$D^{1.8} x^3 = \sum_{i=2}^4 \sum_{k=0}^{i-2} a_i M_{i,k}^{(1.8)} x^{i-k-1.8} \quad (29a)$$

$$= a_2 M_{2,0}^{(\frac{9}{5})} x^{\frac{1}{5}} + a_3 M_{3,0}^{(\frac{9}{5})} x^{\frac{6}{5}} + a_3 M_{3,1}^{(\frac{9}{5})} x^{\frac{1}{5}} + a_4 M_{4,0}^{(\frac{9}{5})} x^{\frac{11}{5}} + a_4 M_{4,1}^{(\frac{9}{5})} x^{\frac{6}{5}} + a_4 M_{4,2}^{(\frac{9}{5})} x^{\frac{1}{5}}, \quad (30a)$$

where

$$M_{2,0}^{(\frac{9}{5})} = \frac{8}{\Gamma(\frac{6}{5})}, M_{3,0}^{(\frac{9}{5})} = \frac{48}{\Gamma(\frac{11}{5})}, M_{3,1}^{(\frac{9}{5})} = -\frac{48}{\Gamma(\frac{6}{5})}, M_{4,0}^{(\frac{9}{5})} = \frac{384}{\Gamma(\frac{16}{5})}, M_{4,1}^{(\frac{9}{5})} = -\frac{384}{\Gamma(\frac{11}{5})} \text{ and } M_{4,2}^{(\frac{9}{5})} = \frac{168}{\Gamma(\frac{6}{5})}.$$

We obtain the constants a_2 , a_3 and a_4 using (20a) which are then substituted back into (30a) to have

$$D^{1.8} x^3 = \frac{\Gamma(3+1)}{\Gamma(3+1-1.8)} x^{3-1.8} = \frac{\Gamma(4)}{\Gamma(\frac{11}{5})} x^{\frac{6}{5}}. \quad (31a)$$

Since $a_2 = \frac{3}{4}$, $a_3 = \frac{1}{8}$ and $a_4 = 0$

In a similar manner, using (26b) and (27b), we have:

$$D^{1.8} x^3 = \sum_{i=2}^4 \sum_{k=0}^{i-2} a_i G_{i,k}^{(1.8)} x^{i-k-1.8} \quad (29b)$$

$$= a_2 G_{2,0}^{(\frac{9}{5})} x^{\frac{1}{5}} + a_3 G_{3,0}^{(\frac{9}{5})} x^{\frac{6}{5}} + a_3 G_{3,1}^{(\frac{9}{5})} x^{\frac{1}{5}} + a_4 G_{4,0}^{(\frac{9}{5})} x^{\frac{11}{5}} + a_4 G_{4,1}^{(\frac{9}{5})} x^{\frac{6}{5}} + a_4 G_{4,2}^{(\frac{9}{5})} x^{\frac{1}{5}}, \quad (30b)$$

where

$$G_{2,0}^{(\frac{9}{5})} = \frac{8}{\Gamma(\frac{6}{5})}, G_{3,0}^{(\frac{9}{5})} = \frac{48}{\Gamma(\frac{11}{5})}, G_{3,1}^{(\frac{9}{5})} = -\frac{56}{\Gamma(\frac{6}{5})}, G_{4,0}^{(\frac{9}{5})} = \frac{384}{\Gamma(\frac{16}{5})}, G_{4,1}^{(\frac{9}{5})} = -\frac{432}{\Gamma(\frac{11}{5})} \text{ and } G_{4,2}^{(\frac{9}{5})} = \frac{216}{\Gamma(\frac{6}{5})}.$$

We obtain the constants a_2 , a_3 and a_4 using (20b) which are then substituted back into (30b) to have

$$D^{1.8}x^3 = \frac{\Gamma(3+1)}{\Gamma(3+1-1.8)}x^{3-1.8} = \frac{\Gamma(4)}{\Gamma(\frac{11}{5})}x^{\frac{6}{5}}. \quad (31b)$$

Since $a_2 = \frac{7}{8}$, $a_3 = \frac{1}{8}$ and $a_4 = 0$

5 Methodology

Consider the given fractional order wave equation (1) together with the conditions (2) and (3). To implement Chebyshev collocation method, we shall let

$$\omega_n(x, t) = \sum_{i=0}^n \omega_i(t) U_i^*(x). \quad (32a)$$

and

$$\omega_n(x, t) = \sum_{i=0}^n \omega_i(t) V_i^*(x). \quad (32b)$$

Represent the required approximations of second and third kinds Chebyshev polynomials respectively. From (1), (26a) and (27a), it follows that

$$\sum_{i=0}^n \frac{d^2 \omega_i(t)}{dt^2} U_i^*(x) = \xi(x, t) \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} \omega_i(t) M_{i,k}^{(\theta)} x^{i-k-\theta} + \eta(x, t) \quad (33a)$$

In a similar manner, from (1), (26b) and (27b), we equally get

$$\sum_{i=0}^n \frac{d^2 \omega_i(t)}{dt^2} V_i^*(x) = \xi(x, t) \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} \omega_i(t) G_{i,k}^{(\theta)} x^{i-k-\theta} + \eta(x, t). \quad (33b)$$

We shall now collocate (33a) and (33b) at the points $x_d = (n+1 - \lceil \alpha \rceil)$, $d = 0, 1, \dots, n - \lceil \theta \rceil$ as

$$\sum_{i=0}^n \ddot{\omega}_i(t) U_i^*(x_d) = \xi(x_d, t) \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} \omega_i(t) M_{i,k}^{(\theta)} x_d^{i-k-\theta} + \eta(x_d, t), \quad (34a)$$

and

$$\sum_{i=0}^n \ddot{\omega}_i(t) V_i^*(x_d) = \xi(x_d, t) \sum_{i=\lceil \theta \rceil}^n \sum_{k=0}^{i-\lceil \theta \rceil} \omega_i(t) G_{i,k}^{(\theta)} x_d^{i-k-\theta} + \eta(x_d, t), \quad (34b)$$

respectively with $\ddot{\omega}(t) = \frac{d^2 \omega}{dt^2}$.

For us to have the best choice of collocation points, we use the roots of shifted Chebyshev polynomial equations $U_{n+1-\lceil \theta \rceil}^*(x) = 0$ and $V_{n+1-\lceil \alpha \rceil}^*(x) = 0$. Applying the initial condition on (32a) and (32b), and using (20a) and (20b) respectively, we obtained the required constants ω_i , $i = 0, 1, \dots$ at the initial point $t = 0$. Furthermore, when the boundary conditions are applied in (32a) and (32b) with $x \in [0, 2]$, we get $\lceil \theta \rceil$ equations as follows:

$$\sum_{i=0}^n (-1)^i (1+i) \omega_i(t) = 0, \quad \sum_{i=0}^n (1+i) \omega_i(t) = 0 \quad (35a)$$

and

$$\sum_{i=0}^n (-1)^{i+2} (1+2i) \omega_i(t) = 0, \quad \sum_{i=0}^n (-1)^i \omega_i(t) = 0 \quad (35b)$$

(34a) and (34b) together with the $[\theta]$ equations obtained from the boundary conditions (35a) and (35b) give $(n+1)$ -system of second order ordinary differential equations which can be solved by using finite difference method to get the unknown ω_i ; $i = 0, 1, \dots, n$.

6 Numerical Experiment

Consider the fractional order wave equation of the form

$$\frac{\partial^2 \omega(x, t)}{\partial t^2} = \xi(x, t) \frac{\partial^{1.8} \omega(x, t)}{\partial x^{1.8}} + \eta(x, t), \quad (36)$$

defined on a finite domain $0 \leq x \leq 2$ and $t > 0$ with the coefficient functions

$$\xi(x, t) = \Gamma(1.2) x^{1.8}, \quad (37)$$

and the inhomogeneous source function

$$\eta(x, t) = 4e^{-t} x^2 (2-x) - 16e^{-t} x^2 + 20e^{-t} x^3, \quad (38)$$

with initial conditions

$$\omega(x, 0) = 4x^2(2-x), \quad \omega_t(x, 0) = -4x^2(2-x) \quad (39)$$

and Dirichlet boundary conditions

$$\omega(0, t) = \omega(2, t) = 0. \quad (40)$$

The exact solution for this problem is

$$\omega(x, t) = 4e^{-t} x^2 (2-x). \quad (41)$$

Solution

We shall consider an approximate solution of the form

$$\omega_n(x, t) = \sum_{i=0}^n \omega_i(t) U_i^*(x) \quad (42)$$

for $n = 3$, we have

$$\omega_3(x, t) = \sum_{i=0}^3 \omega_i(t) U_i^*(x) \quad (43)$$

Using (43) in (34a), we get

$$\sum_{i=0}^3 \frac{d^2 \omega_i(t)}{dt^2} U_i^*(x_d) = \xi(x_d, t) \sum_{i=2}^3 \sum_{k=0}^{i-2} \omega_i(t) M_{i,k}^{(\theta)} x_d^{i-k-\theta} + \eta(x_d, t) \quad d = 0, 1 \quad (44)$$

Using (44) and (35a), we arrive at the following system of second order ordinary differential equations:

$$\left. \begin{aligned} \ddot{\omega}_0(t) + c_1 \ddot{\omega}_1(t) + c_2 \ddot{\omega}_3(t) &= D_1 \omega_2(t) + D_2 \omega_3(t) + \eta(x_0, t) \\ \ddot{\omega}_0(t) + c_{11} \ddot{\omega}_1(t) + c_{22} \ddot{\omega}_3(t) &= D_{11} \omega_2(t) + D_{22} \omega_3(t) + \eta(x_1, t) \\ \omega_0(t) - \omega_1(t) + \omega_2(t) - \omega_3(t) &= 0 \\ \omega_0(t) + 3\omega_1(t) + 5\omega_2(t) + 7\omega_3(t) &= 0, \end{aligned} \right\} \quad (45)$$

where

$c_1 = U_1^*(x_0)$, $c_2 = U_3^*(x_0)$, $c_{11} = U_1^*(x_1)$, $c_{22} = U_3^*(x_1)$, $D_1 = \xi(x_0, t)M_{2,0}^{1.8}x_0^{\frac{1}{5}}$,
 $D_2 = \xi(x_0, t)\{M_{3,0}^{1.8}x_0^{\frac{6}{5}} + M_{3,1}^{1.8}x_0^{\frac{1}{5}}\}$, $D_{11} = \xi(x_1, t)M_{2,0}^{1.8}x_1^{\frac{1}{5}}$, $D_{22} = \xi(x_1, t)\{M_{3,0}^{1.8}x_1^{\frac{6}{5}} + M_{3,1}^{1.8}x_1^{\frac{1}{5}}\}$.
 To solve the system of equations 45 using finite difference method, we will use the notations $t_i = i\Delta t$
 for $i = 0, 1, \dots, N$, with $\Delta t = \frac{T}{N}$ and $T = T_{final}$. we also defined $\omega_i^n = \omega_i(t_n)$, $\eta_i^n = \eta_i(t_n)$,
 The system (45) now becomes

$$\frac{\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}}{(\Delta t)^2} + c_1 \frac{\omega_1^{n+1} - 2\omega_1^n + \omega_1^{n-1}}{(\Delta t)^2} + c_2 \frac{\omega_3^{n+1} - 2\omega_3^n + \omega_3^{n-1}}{(\Delta t)^2} = D_1\omega_2^{n+1} + D_2\omega_3^{n+1} + \eta_0^{n+1} \quad (46)$$

$$\frac{\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}}{(\Delta t)^2} + c_{11} \frac{\omega_1^{n+1} - 2\omega_1^n + \omega_1^{n-1}}{(\Delta t)^2} + c_{22} \frac{\omega_3^{n+1} - 2\omega_3^n + \omega_3^{n-1}}{(\Delta t)^2} = D_{11}\omega_2^{n+1} + D_{22}\omega_3^{n+1} + \eta_1^{n+1} \quad (47)$$

$$\omega_0^{n+1} - \omega_1^{n+1} + \omega_2^{n+1} - \omega_3^{n+1} = 0 \quad (48)$$

$$\omega_0^{n+1} + 3\omega_1^{n+1} + 5\omega_2^{n+1} + 7\omega_3^{n+1} = 0 \quad (49)$$

(46)-(49) can be written in matrix form as

$$\begin{pmatrix} 1 & c_1 & -\Delta t^2 D_1 & c_2 - \Delta t^2 D_2 \\ 1 & c_{11} & -\Delta t^2 D_{11} & c_{22} - \Delta t^2 D_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 2c_1 & 0 & 2c_2 \\ 1 & 2c_{11} & 0 & 2c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}^n - \begin{pmatrix} 1 & c_1 & 0 & c_2 \\ 1 & c_{11} & 0 & c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n-1} + (\Delta t)^2 \begin{pmatrix} \eta_0 \\ \eta_1 \\ 0 \\ 0 \end{pmatrix}^{n+1} \quad (50)$$

which is equivalent to

$$E\Omega^{n+1} = F\Omega^n - G\Omega^{n-1} + (\Delta t)^2 H^{n+1} \quad (51a)$$

or

$$\Omega^{n+1} = E^{-1}F\Omega^n - E^{-1}G\Omega^{n-1} + E^{-1}(\Delta t)^2 H^{n+1}, \quad (52a)$$

Following the same process demonstrated in (43)-(50) and using (34b) and (35b) on the same problem given in (36)-(41), we obtained

$$W\Omega^{n+1} = X\Omega^n - Y\Omega^{n-1} + (\Delta t)^2 Z^{n+1} \quad (51b)$$

or

$$\Omega^{n+1} = W^{-1}X\Omega^n - W^{-1}Y\Omega^{n-1} + W^{-1}(\Delta t)^2 Z^{n+1}, \quad (52b)$$

where

$\Omega^n = (\omega_0^n, \omega_1^n, \omega_2^n, \omega_3^n)^T$ and $H^n = ((\Delta t)^2 \eta_0^n, (\Delta t)^2 \eta_1^n, 0, 0)^T$

To obtain the initial solution Ω^0 of (52a) and 52b, we use the initial condition of the problem $\omega(x, 0)$ combine with (20a) and 20b respectively. We shall also obtain Ω^1 of (52a) and 52b, using the initial condition of the problem $\omega_t(x, 0)$ as follows

$$\omega_t(x, 0) = \frac{\omega_{i,1} - \omega_{i,0}}{\Delta t}, \quad i = 0, 1, \dots \quad (53)$$

then

$$\omega_{i,1} = \omega_{i,0} + \omega'(x_i)\Delta t, \quad i = 0, 1, \dots \quad (54)$$

hence

$$\Omega^1 = \Omega^0 + \Delta t R, \quad (55)$$

where

$$R = \omega'(x_i). \quad (56)$$

The approximate solution is then obtained by substituting the results derived above in the series (43).

7 Results, Discussion of Results and Conclusion

7.1 Results

Table 1: Results obtained using $U_n(x)$ and $V_n(x)$ for $n = 3$, $T = 1$ and $\Delta t = \frac{1}{400}$

x	Exact sol.	Approx. sol. ($U_n(x)$)	Approx. sol. ($V_n(x)$)	Approx. sol. [16]	Error($U_n(x)$)	Error($V_n(x)$)	Error[16]
0.00	0	0	0	0	0	0	0
0.20	0.2872808993	0.26861000911	0.2596317122	0.2853666054	1.667089×10^{-2}	2.764919×10^{-2}	1.914294×10^{-3}
0.40	1.0188927790	0.99784927790	0.9820922350	0.9747779627	2.104610×10^{-2}	3.680054×10^{-2}	4.411482×10^{-2}
0.60	2.0009365580	1.9925884911	1.9721847885	1.9862317000	1.348067×10^{-2}	2.875170×10^{-2}	1.470486×10^{-2}
0.80	3.0414330890	3.0577193060	3.0347125926	3.1096670040	1.628622×10^{-2}	1.720496×10^{-2}	6.823392×10^{-2}
1.00	3.9503112020	3.9981193981	3.9744788867	4.0900055000	4.780820×10^{-2}	2.416768×10^{-2}	1.396943×10^{-1}
1.20	4.5395818000	4.6186699383	4.5962868328	4.6852897270	7.908813×10^{-2}	5.670503×10^{-2}	1.457079×10^{-1}
1.40	4.6223961170	4.7242520977	4.7049397088	4.6992014000	1.018560×10^{-1}	8.254359×10^{-2}	7.752402×10^{-2}
1.60	4.0148937660	4.1197470471	4.1052407153	3.9929906310	1.048533×10^{-1}	9.034695×10^{-2}	2.190314×10^{-2}
1.80	2.5343312070	2.6100359575	2.6019930724	2.4617225700	7.570475×10^{-2}	6.766187×10^{-2}	7.260864×10^{-2}
2.00	0	0	0	0	0	0	0

Table 2: Results obtained using $U_n(x)$ and $V_n(x)$ for $n = 3$, $T = 1$ and $\Delta t = \frac{1}{4000}$

x	Exact sol.	Approx. sol. ($U_n(x)$)	Approx. sol. ($V_n(x)$)	Approx. sol. [16]	Error($U_n(x)$)	Error($V_n(x)$)	Error[16]
0.00	0	0	0	0	0	0	0
0.20	0.2872808993	0.2860144777	0.2848944100	0.2863548880	1.266423×10^{-3}	2.386489×10^{-3}	9.260113×10^{-4}
0.40	1.0188927790	1.0213496360	1.0195198087	1.0221455440	2.456857×10^{-3}	6.270297×10^{-3}	3.252965×10^{-3}
0.60	2.0009365580	2.0136756798	2.0114858745	2.0149684560	1.273912×10^{-2}	1.054932×10^{-2}	1.403190×10^{-2}
0.80	3.0414330890	3.0706628142	3.0684022857	3.0724231120	2.922972×10^{-2}	2.696920×10^{-2}	3.099002×10^{-2}
1.00	3.9503112020	3.9999812440	3.9978782089	4.0021090000	4.967004×10^{-2}	4.756701×10^{-2}	5.179780×10^{-2}
1.20	4.5395818000	4.6093011744	4.6075248584	4.6116256080	6.971937×10^{-2}	6.794305×10^{-2}	7.222980×10^{-2}
1.40	4.6223961170	4.7242520977	4.7049397088	4.6992014000	1.018560×10^{-1}	8.254359×10^{-2}	7.752402×10^{-2}
1.60	4.0148937660	4.0986263564	4.0977649540	4.1005489400	8.389669×10^{-2}	8.255426×10^{-2}	8.617630×10^{-2}
1.80	2.5343312070	2.5939720180	2.5939720180	2.5951546300	5.964081×10^{-2}	6.082342×10^{-2}	7.260864×10^{-2}
2.00	0	0	0	0	0	0	0

7.2 Discussion of Results

Tables 7.1 and 7.2 presented above compare the relationship between the exact and approximate solutions obtained using the proposed method for the case $n = 3$ and the result reported in (16) at different step sizes $\Delta t = \frac{1}{400}$ and $\Delta t = \frac{1}{4000}$ using shifted Chebyshev polynomial of the second and the third kinds. It is obvious that the result in columns 3 and 4 of table 7.2 is better than that of column 3 and 4 of table 7.1. This is based on the general understanding that whenever a differential equation is stable, the smaller the step size, the better the results. Meanwhile, the results in columns 5 of Table 7.1 and Table 7.2 are better than that of columns 3 and 4 of both tables. Also, the results in columns 3 of Table 7.1 and Table 7.2 are better than that of columns 4 of both tables. That is, fourth kind polynomial approximates better than the second kind polynomial and the second kind polynomial, approximates better than the third kind polynomial. The 3D graphs presented are for the exact solution, and the approximate solutions obtained using different step sizes. It is also worthy to note that the accuracy of our results depends on the step-size rather than the order of the given problem.

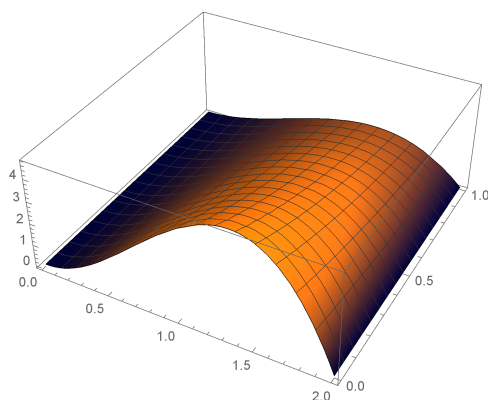


Figure 1: Exact solution graph

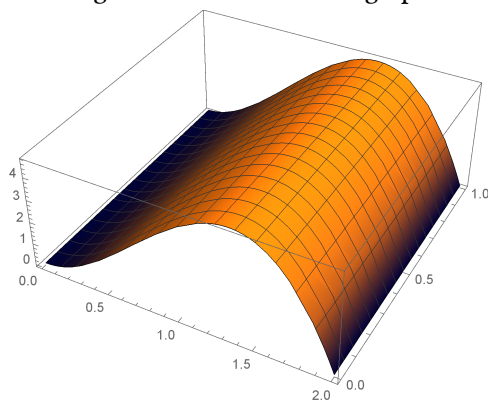


Figure 2: Approximate solution graph $U_n(x)$, for $\Delta t = \frac{1}{400}$

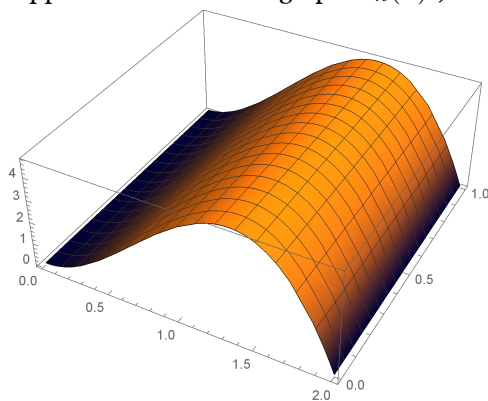


Figure 3: Approximate solution graph $U_n(x)$, for $\Delta t = \frac{1}{4000}$

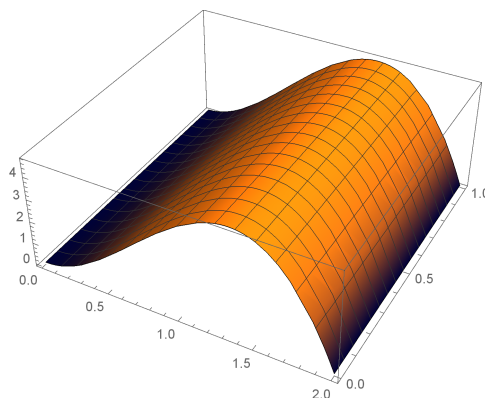


Figure 4: Approximate solution graph $V_n(x)$, for $\Delta t = \frac{1}{400}$

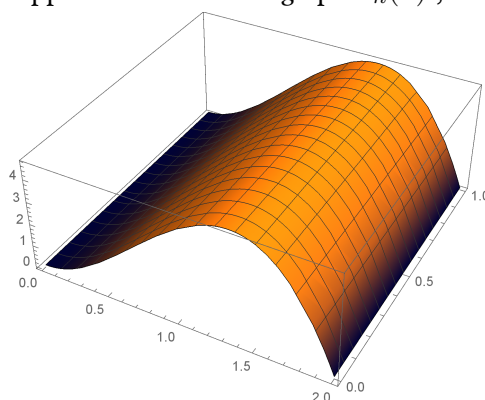


Figure 5: Approximate solution graph $V_n(x)$, for $\Delta t = \frac{1}{4000}$

7.3 Conclusion

This research paper talks about the numerical solution of SFOWE with the fractional derivate expressed in Caputo sense, and the given fractional order PDE were reduced to system of second order ODEs which are later transformed to a system of linear algebraic equations with the aid of finite difference method, and later collocated at the roots of the shifted Chebyshev polynomials of the second and third kinds respectively. The approximate solution obtained using the proposed scheme is compared with the exact solution, and it was realized that the proposed scheme is very reliable and accurate. All computations in this work are carried out using Mathematica 11.0.

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Convergence analysis of an inertial algorithm for generalized mixed equilibrium problems and Bregman quasi asymptotically nonexpansive mappings

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Abstract

The aim of this article is to introduce an inertial algorithm for approximating solution of a system of generalized mixed equilibrium problem and common fixed point of countable families of Bregman quasi asymptotically nonexpansive mappings. The strong convergence theorem has been establish in reflexive Banach spaces. Our results are significant improvement of some recent results announced in the literature.

Keywords and phrases: strong convergence; generalized mixed equilibrium problem; Bregman projections; Bregman quasi asymptotically nonexpansive mappings.

1 Introduction

An inertial-type algorithm is a techniques for accelerating the convergence of the sequence of an algorithm introduced by Polyak [27]. Many problems have been solved using inertial algorithms (for more details see, [11, 13, 16, 18, 24] and the references therein).

Let E be a reflexive Banach space with E^* as the dual space of E and C be a nonempty closed convex subset of E . Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunctions, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \longrightarrow E^*$ is a nonlinear continuous monotone mapping and $\varphi : C \longrightarrow \mathbb{R}$ be a convex and lower semi continuous function. The short form $GMEP$ is consider as generalized mixed equilibrium problem [19] mean that to find $x \in C$ such that:

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x \rangle \geq 0, \forall y \in C. \quad (1)$$

Also the short form $GMEP(\Theta, \Psi, \varphi)$ is consider as the set of solutions of generalized mixed equilibrium problem [1] is defined by

$$GMEP(\Theta, \Psi, \varphi) = \{x \in C : \Theta(x, y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x \rangle \geq 0, \forall y \in C\}.$$

Numerous problems in same areas such as nonlinear analysis, optimization, economics and variational inequality problem can be reduced to finding solutions of some equilibrium problems (see [12, 14] and the references therein). Moreover, various methods have been studied for solutions of some equilibrium problems in Hilbert spaces, (see [12, 17, 32, 36] and the references therein).

In 1967, Bregman [5] introduced an effective technique by considering the Bregman distance function D_f for designing and analyzing feasibility and optimization algorithms. The Bregman technique has applied for approximating feasibility problems, optimization problems and fixed point problem for

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nonlinear mappings (see, e.g. [3, 15, 25, 34] and the references therein).

A mapping $J : E \rightarrow 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E,$$

is considered as normalized duality mapping on E and $\langle x, x^* \rangle$ present as the pairing between element of E and that of E^* .

Let E be a reflexive Banach space, $f : E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. $\text{dom} f := \{x \in E : f(x) < +\infty\}$, is considered as the domain of f . Let $x \in \text{int}(\text{dom} f)$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}.$$

The Young - Fenchel inequality present as:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*), \forall x \in E, x^* \in E^*.$$

A function f on E is coercive [22] if the sublevel set of f is bounded :

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty.$$

A function f on E is said to be strongly coercive [35] provided that

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

The right-hand derivative of f at x in the direction y , for any $x \in \text{int dom} f$ and $y \in E$ is defined by

$$f^0(x, y) := \lim_{t \rightarrow 0^+} \frac{f(x + ty) - f(x)}{t}.$$

If $\lim_{t \rightarrow 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y , then the function f is called Gâteaux differentiable at x . Therefore, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x . The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom} f)$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in y with $\|y\| = 1$. Also f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. Recall that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int}(\text{dom} f)$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on $\text{int}(\text{dom}(f))$ (see [4, 7]).

Definition 1.1. [6] The function f is said to be:

- (i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
- (ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\text{dom} f$;
- (iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.2. Note that the following results follows, if E is a reflexive Banach space:

- (i) f is essentially smooth if and only if f^* is essentially strictly convex (see [6], Theorem 5.4).
- (ii) $(\partial f)^{-1} = \partial f^*$ (see [7]).
- (iii) f is Legendre if and only if f^* is Legendre (see [6], Corollary 5.5).
- (iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\text{ran } \nabla f = \text{dom } \nabla(f^*) = \text{int}(\text{dom } f^*)$ and $\text{ran } \nabla f^* = \text{dom } f = \text{int}(\text{dom } f)$ (see [6], Theorem 5.10), where ran stands for the range.

Consider $f : E \rightarrow (-\infty, +\infty]$ as a convex and Gâteaux differentiable function. The function $D_f : \text{dom} \times \text{intdom } f \rightarrow [-\infty, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the Bregman distance with respect to f . Further the following property [30] is from the definition of D_f :

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle$$

Definition 1.3. [37] Let $T : C \rightarrow C$ with $F(T) \neq \emptyset$ be a mapping and $F(T) = \{x \in C : Tx = x\}$ is a set of fixed points of T . Then

- (1) A point $p \in C$ is said to be an asymptotic fixed point of T if C contain a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ and $\hat{F}(T)$ denote the set of asymptotic fixed points of T ;
- (2) quasi-Bregman nonexpansive with respect to f [37], provided that

$$F(T) \neq \emptyset \text{ and } D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

- (3) Bregman relatively nonexpansive with respect to f [37], provided that

$$\hat{F}(T) = F(T) \text{ and } D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

- (4) Bregman strongly nonexpansive with respect to f and $\hat{F}(T)$ provided that

$$F(T) \neq \emptyset \text{ and } D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in \hat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n \rightarrow 0} (D_f(p, x_n) - D_f(p, Tx_n)) = 0$$

Consequently

$$\lim_{n \rightarrow 0} D_f(x_n, Tx_n) = 0.$$

- (5) Bregman quasi-nonexpansive, provided that $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

(6) Bregman quasi- asymptotically nonexpansive [37], provided that $F(T) \neq \emptyset$ and there exists a real number $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$D_f(p, T^n x) \leq k_n D_f(p, x), \quad \forall n \geq 1, x \in C, p \in F(T).$$

(7) Closed, if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x \in C$ and $Tx_n \rightarrow y$ ($y \in C$), then $y = Tx$.

Definition 1.4. [37], (1) A countable family of mappings $\{T_i\} : E \rightarrow E$ is said to be uniformly Bregman quasi- asymptotically nonexpansive, if $\Gamma = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists a real number $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$D_f(p, T_i^n x) \leq k_n D_f(p, x), \quad \forall n \geq 1, i \geq 1, x \in E, p \in \Gamma.$$

(2) A mappings $T : E \rightarrow E$ is said to be uniformly L - Lipschits continuous, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in E, \forall n \geq 1.$$

In 2013, Zhao et al. [37], developed the following algorithm for approximating a common fixed point of Bregman quasi asymptotically nonexpansive mappings. They applied their results for the solution of generalised mixed equilibrium problem in reflexive Banach spaces:

$$\begin{cases} x_1 \in E \text{ chosen arbitrary : } C_1 = C, \\ y_{n,m} = \nabla f^*[\beta_n \nabla f(x_1) + (1 - \beta_n) \nabla f(S_m^n x_n)], \quad m \geq 1, \\ C_{n+1} = \{v \in C_n : \sup_{m \geq 1} D_f(v, y_{n,m}) \leq \beta_n D_f(v, x_1) + (1 - \beta_n) D_f(v, x_n) + \xi_n, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \forall n \geq 1, \end{cases}$$

where $\xi_n = (k_n - 1) \sup_{m \geq 1} D_f(p, x_n)$, $P_{C_{n+1}}^f$ is the Bregman projection of E onto C_{n+1} . If $\Gamma = \bigcap_{i=1}^{\infty} F(T_i)$ is bounded, then $\{x_n\}$ converges strongly to $P_{\Gamma}^f(x_1)$.

In 2014, by considering shrinking projection technique, Tomizawa [33] established strong convergence theorem for Bregman quasi-asymptotically nonexpansive mappings in the intermediate sense in a reflexive Banach spaces. Darvish [20] studied strong convergence result for weak Bregman relatively nonexpansive mapping and the set of solutions of generalized mixed equilibrium problems in Banach spaces. Kazmi et al. [23] introduced hybrid iterative method to approximate a common solution for a system of a generalized equilibrium problem and a fixed point problem for a Bregman relatively non-expansive mapping in reflexive Banach spaces.

Recently, Alansari et al [1], consider the following inertial algorithm for solving a common solution for a system of variational inequality problem, generalized equilibrium problem and a common fixed point problem of relatively nonexpansive mapping in Banach space. Let the sequences $\{x_n\}$, and $\{z_n\}$ be generated by the algorithm:

$$\begin{cases} x_0 = x_1, \quad z_0 \in C, \quad C_0 := C; \\ \mu_n = x_n + \sigma_n(x_n - x_{n-1}) \\ u_n = \Pi_C J^{-1}(J\mu_n - \omega_n D\mu_n); \\ y_n = J^{-1}(\beta_n Jz_n + (1 - \beta_n) JSu_n); \\ z_{n+1} = S_{r_n} y_n, \\ M_n = \{z \in C : \phi(z, z_{n+1}) \leq \beta_n \phi(z, z_n) + (1 - \beta_n) \phi(z, \mu_n)\}; \\ N_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \leq 0\}; \\ x_{n+1} = \Pi_{M_n \cap N_n} x_0, \forall n \geq 0, \end{cases}$$

where $\{\beta_n\} \in [0, 1]$, $r_n \in [a, \infty)$ for some $a > 0$, $\{\sigma_n\} \in (0, 1)$ and $\{\omega_n\} \in (0, \infty)$. Then sequences $\{x_n\}$, converges strongly to a point $\hat{x} \in \Gamma$.

Very recently, Jantakan and Kaewcharoen [21], established strong convergence result by approximating a common solution for a mixed equilibrium problem and common fixed point of a countable family of Bregman relatively nonexpansive mappings using the algorithm below:

$$\begin{cases} x_1 \in C, S_i x_1 = z_1^i \in C; \\ y_n^i = \nabla f^*(\gamma_n \nabla f(z_n^i) + (1 - \gamma_n) \nabla f(S_i x_n)); \\ z_{n+1}^i = Res_{G, \psi}^f(y_n^i); \\ C_n^i = \{z \in C : \phi(z, z_{n+1}^i) \leq \gamma_n \phi(z, z_n^i) + (1 - \gamma_n) \phi(z, x_n)\}, \\ C_n = \cap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \forall n \geq 1, \end{cases}$$

where $\{x_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \gamma_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to $Proj_{\Omega}^f x_1$, where $Proj_{\Omega}^f x_1$ is the projection of C onto Ω

In this article, motivated by the results of Zhao et al. [37], Alansari et al [1] and Jantakan and Kaewcharoen [21] mentioned above, we consider the algorithm that combine *GMEP* with the class of Bregman nonexpansive mappings and also incorporate inertial extrapolation technique for accelerating the convergence of the sequence of an algorithm. This lead to the investigation for approximate common solution of generalized mixed equilibrium problems and fixed points problem for Bregman quasi-asymptotically nonexpansive mappings in a reflexive Banach spaces. Our result modify and improves the result of Jantakan and Kaewcharoen [21] and many recently announced results.

2 Preliminaries

Let E be a reflexive Banach space and $f : E \rightarrow (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection [5] of $x \in \text{int dom } f$ onto a nonempty, closed and convex set $C \subset \text{dom } f$ is the necessarily unique vector $proj_C^f(x)$ satisfying

$$D_f(proj_C^f(x), x) := \inf\{D_f(y, x) : y \in C\}.$$

The modulus of total convexity of f at $x \in \text{int dom } f$ is the function $v_f(x, t) : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom } f, \|y - x\| = t\}.$$

The function f is called totally convex at x , if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex, if it is totally convex at any point $x \in \text{int dom } f$ and it is said to be totally convex on bounded sets, if $v_f(B, t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $v_f : \text{int dom } f \times [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \text{dom } f\}.$$

Recall that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets [10]

Lemma 2.1. [10] Let C be a nonempty, closed and convex subset of a reflexive Banach space E . Let $f : E \rightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function and let $x \in E$. Then:

- i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$;
- ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Let $f : E \rightarrow (-\infty, +\infty]$ be a strong coercive Bregman function. Following [2] and [14], let the function $V_f : E \times E^* \rightarrow [0, +\infty)$ associated with f is defined by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^* \rangle, \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and the following assertions hold:

- (1) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $y^* \in E^*$.
- (2) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ for all $x \in E$ and $y^* \in E^*$.

Lemma 2.2. [26] Let E be a Banach space and $f : E \rightarrow (-\infty, +\infty]$ be a Gateaux differentiable function which is uniformly convex on bounded subsets of E . Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be bounded sequences in E . Then

$$\lim_{t \rightarrow \infty} D_f(x_n, y_n) = 0 \text{ if and only if } \lim_{t \rightarrow \infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. [10] If $x \in \text{dom } f$, then the following statements are equivalent:

- i) the function f is totally convex at x ;
- ii) for any sequence $\{y_n\} \subset \text{dom } f$,

$$\lim_{t \rightarrow +\infty} D_f(y_n, x) = 0 \implies \lim_{t \rightarrow +\infty} \|y_n - x\| = 0.$$

The function f is called sequentially consistent [10] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{t \rightarrow +\infty} D_f(y_n, x_n) = 0 \implies \lim_{t \rightarrow +\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. [8] Let $f : E \rightarrow (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then, f is sequentially consistent if and only if it is totally convex on bounded sets.

Lemma 2.5. [29] Let $f : E \rightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.6. [23] Let $f : E \rightarrow (-\infty, +\infty]$ be a Legendre function and C be a nonempty, closed and convex subset of $\text{int dom } f$. Let $T : C \rightarrow C$ be a Bregman quasi asymptotically nonexpansive mapping with respect to f . Then $F(T)$ is closed and convex.

Lemma 2.7. [29] Let $f : E \rightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function, $x_1 \in E$ and C be a nonempty, closed and convex subset of E . Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C . If $D_f(x_n, x_1) \leq D_f(P_C^f x_1, x_1)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ strongly converges to $P_C^f x_1$.

Lemma 2.8. [9] Let $f : E \rightarrow (-\infty, +\infty]$ be a Legendre function. Then, the function f is totally convex on bounded sets if and only if f is uniformly convex on bounded subsets of E .

Lemma 2.9. [35] Let $f : E \rightarrow \mathbb{R}$ be a strong coercive and uniformly convex on bounded subsets of E , then f^* is bounded and uniformly Frechet differentiable on bounded subsets of E^* .

Lemma 2.10. [31] Let $f : E \rightarrow (-\infty, +\infty]$ be a proper convex and lower semi-continuous Legendre function, then $f^* : E^* \rightarrow (-\infty, +\infty]$ is a proper weak* lower semi continuous and convex function.

Thus for any $z \in E$, for any $\{x_n\} \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$, the following holds

$$D_f(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)) \leq \sum_{i=1}^N t_i D_f(z, x_i).$$

Lemma 2.11. [28] Let $f : E \rightarrow (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on bounded subsets of E . Then, f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .

Consider $\Theta : C \times C \rightarrow \mathbb{R}$ as a bifunction satisfies the following assumptions [12]:

- (A₁) $\Theta(x, x) = 0, \forall x \in C$;
- (A₂) Θ is monotone, i.e, $\Theta(x, y) + \Theta(y, x) \leq 0, \forall x, y \in C$;
- (A₃) For each $x, y, z \in C, \limsup_{t \rightarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y)$;
- (A₄) For each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.12. [21] Let C be a nonempty, closed and convex subset of a real reflexive Banach space E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A₁) – (A₄), $\psi : C \rightarrow E^*$ be a continuous monotone mapping and $\varphi : C \rightarrow \mathbb{R} \cup \{\infty\}$ be proper convex and lower semi-continuous. Let $f : E \rightarrow (-\infty, +\infty]$ be a coercive Legendre function. For $x \in E$ and define a mapping $Res_{\Theta, \psi, \varphi}^f : E \rightarrow 2^C$ as follows:

$$Res_{\Theta, \psi, \varphi}^f(x) = \{z \in C : \Theta(z, y) + \langle \psi z, y - z \rangle + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C\}$$

Then, the following properties holds:

- (a) $Res_{\Theta, \psi, \varphi}^f$ is single-valued and $\text{dom}(Res_{\Theta, \psi, \varphi}^f) = E$;
- (b) $Res_{\Theta, \psi, \varphi}^f$ is Bregman firmly nonexpansive type mapping;
- (c) $F(Res_{\Theta, \psi, \varphi}^f) = GMEP(\Theta, \psi, \varphi)$;
- (d) $GMEP(\Theta, \psi, \varphi)$ is closed and convex;
- (e) $D_f(p, Res_{\Theta, \psi, \varphi}^f(x)) + D_f(Res_{\Theta, \psi, \varphi}^f(x), x) \leq D_f(p, x), \forall p \in F(Res_{\Theta, \psi, \varphi}^f), x \in E$.

It is well known that $Res_{\Theta, \psi, \varphi}^f$ is a closed Bregman quasi-nonexpansive mapping, and so is a closed Bregman quasi-asymptotically nonexpansive mapping.

3 Strong convergence theorem

Theorem 3.1. Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \rightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be bi functions which satisfying assumptions (A₁) – (A₄), $\psi : C \rightarrow E^*$ be continuous monotone mappings and $\varphi : C \rightarrow \mathbb{R}$ be convex and lower semi-continuous functions. Let $\{T_j\}_{j=1}^N : E \rightarrow E$ be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each $j \geq 1, T_j$ is uniformly L_j – Lipschitz continuous. Assume that $\Omega := (\cap_{j=1}^N F(T_j)) \cap (GMEP(\Theta, \psi, \varphi)) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j)); \\ z_{n+1}^j = Res_{\Theta, \psi, \varphi}^f(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1 - \alpha_n) D_f(z, w_n) + \beta_n \xi_n\}; \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \forall n \geq 1, \end{cases} \quad (2)$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\liminf_{n \rightarrow \infty} (1 - \alpha_n) \beta_n > 0$. $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0, 1)$ and $\xi_n = (k_n - 1) D_f(q, w_n)$. Then, $\{x_n\}$ converges strongly to

$Proj_{\Omega}^f x_1$, where $Proj_{\Omega}^f x_1$ is the Bregman projection of C onto Ω .

Proof.: Let two functions $\Gamma : C \times C \rightarrow \mathbb{R}$ and $Res_{\Theta, \psi, \varphi}^f : E \rightarrow 2^C$ be defined by

$$\Gamma(z, y) = \Theta(z, y) + \langle \psi z, y - z \rangle + \varphi(y) - \varphi(z), \forall y \in C$$

and

$$Res_{\Theta, \psi, \varphi}^f(x) = \{z \in C : \Gamma(z, y) + \langle \nabla f(x), y - z \rangle \geq 0, \forall y \in C\}, \forall x \in E$$

respectively. Consequently by Lemma 2.12 that, the function Γ satisfies assumptions $(A_1) - (A_4)$ and $Res_{\Theta, \psi, \varphi}^f$ has the properties $(a) - (e)$.

We present the proof as follows:

Step 1 : We show that Ω is closed and convex. In view of Lemma 2.6, we get that $F(T_j)$ is closed and convex for all $j = 1, 2, \dots, N$, this implies that $\cap_{j=1}^N F(T_j)$ is closed and convex and also by Lemma 2.12(c), we obtain that $GMEP(\Theta, \psi, \varphi)$ is closed and convex and therefore $\Omega := \cap_{j=1}^N F(T_j) \cap GMEP(\Theta, \psi, \varphi)$ is closed and convex.

Step 2 : We show that $C_n \cap Q_n$ is closed and convex for all n . Now, we first show that Q_n is convex for all $n \geq 1$. Assume that $a_1, a_2 \in Q_n$, $\lambda \in [0, 1]$ and $\omega = \lambda a_1 + (1 - \lambda)a_2$. Now

$$\langle \nabla f(x_1) - \nabla f(x_n), a_1 - x_n \rangle \leq 0, \quad (3)$$

and

$$\langle \nabla f(x_1) - \nabla f(x_n), a_2 - x_n \rangle \leq 0. \quad (4)$$

Multiplying λ and $(1 - \lambda)$ on both sides of (3) and (4) respectively, we conclude that

$$\langle \nabla f(x_1) - \nabla f(x_n), \lambda a_1 + (1 - \lambda)a_2 - x_n \rangle \leq 0,$$

by definition of ω , we have

$$\langle \nabla f(x_1) - \nabla f(x_n), \omega - x_n \rangle \leq 0.$$

Hence, $\omega \in Q_n$ and Q_n is convex. Let $\{\mu_m\}$ be a sequence in Q_n and $\mu_m \rightarrow \mu$ as $m \rightarrow \infty$. By the definition of Q_n , we obtain

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu_m - x_n \rangle \leq 0,$$

which gives that

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu_m - \mu \rangle + \langle \nabla f(x_1) - \nabla f(x_n), \mu - x_n \rangle \leq 0.$$

By taking limit as $m \rightarrow \infty$, we get

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu - x_n \rangle \leq 0.$$

Therefore $\mu \in Q_n$, which implies that Q_n is closed for all $n \geq 1$. Also, we show that C_n is closed for all $n \geq 1$. Assume that $\{v_m\}$ is a sequence in C_n and $v_m \rightarrow v$ as $m \rightarrow \infty$. Then $\{v_m\}$ is in C_n^j for all $j = 1, 2, \dots, N$, it follows from the definition of C_n^j in (2), that

$$\begin{aligned} D_f(v_m, z_{n+1}^j) &\leq \alpha_n D_f(v_m, Z_n^j) + (1 - \alpha_n) D_f(v_m, w_n) + \beta_n \xi_n, \\ &\forall n \geq 1, j = 1, 2, \dots, N. \end{aligned} \quad (5)$$

Now, by definition of the Bregman distance with respect to f , we present

$$\begin{aligned} f(v_m) - f(z_{n+1}^j) &= \langle \nabla f(z_{n+1}^j), v_m - z_{n+1}^j \rangle \leq \alpha_n (f(v_m) - f(z_{n+1}^j) - \langle \nabla f(z_n^j), v_m - z_n^j \rangle) \\ &+ (1 - \alpha_n) (f(v_m) - f(w_n) - \langle \nabla f(w_n), v_m - w_n \rangle) + \beta_n \xi_n, \end{aligned} \quad (6)$$

also

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), v_m - z_n^j \rangle &+ (1 - \alpha_n) \langle \nabla f(w_n), v_m - w_n \rangle - \langle \nabla f(z_{n+1}^j), v_m - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - \alpha_n f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n. \end{aligned} \quad (7)$$

It implies that

$$\begin{aligned} \alpha_n [\langle f(z_n^j), v_m - v \rangle + \langle f(z_n^j), v - z_n^j \rangle] &+ (1 - \alpha_n) [\langle f(w_n), v_m - v \rangle + \langle f(w_n), v - w_n \rangle] \\ &- \langle \nabla f(z_{n+1}^j), v_m - v \rangle - \langle \nabla f(z_{n+1}^j), v - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - \alpha_n f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n. \end{aligned}$$

Taking the advantage of the limit as $m \rightarrow \infty$, we arrive at

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), v - z_n^j \rangle &+ (1 - \alpha_n) \langle \nabla f(w_n), v - w_n \rangle - \langle \nabla f(z_{n+1}^j), v - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - \alpha_n f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n, \forall j = 1, 2, \dots, N. \end{aligned}$$

Implies $v \in C_n^j$ for all $j = 1, 2, \dots, N$. Hence, $v \in C_n$ and so C_n is closed. For any $a_1, a_2 \in C_n$, we obtain that $a_1, a_2 \in C_n^j$ for all $j = 1, 2, \dots, N$. and $a_1, a_2 \in C$. Now since C is convex, $\omega = \lambda a_1 + (1 - \lambda) a_2 \in C$ for $\lambda \in [0, 1]$. Taking the advantage of the definition of C_n , that

$$D_f(a_1, z_{n+1}^j) \leq \alpha_n D_f(a_1, z_n^j) + (1 - \alpha_n) D_f(a_1, w_n) + \beta_n \xi_n, \quad (8)$$

and

$$D_f(a_2, z_{n+1}^j) \leq \alpha_n D_f(a_2, z_n^j) + (1 - \alpha_n) D_f(a_2, w_n) + \beta_n \xi_n. \quad (9)$$

Using (5), (6) and (7), shows that (8) and (9) are equivalent to

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), a_1 - z_n^j \rangle &+ (1 - \alpha_n) \langle \nabla f(w_n), a_1 - w_n \rangle - \langle \nabla f(z_{n+1}^j), a_1 - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n \end{aligned} \quad (10)$$

and

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), a_2 - z_n^j \rangle &+ (1 - \alpha_n) \langle \nabla f(w_n), a_2 - w_n \rangle - \langle \nabla f(z_{n+1}^j), a_2 - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n \end{aligned} \quad (11)$$

By multiplying λ and $(1 - \lambda)$ on both sides of (10) and (11) respectively, we get

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), \lambda a_1 &+ (1 - \lambda) a_2 - z_n^j \rangle + (1 - \alpha_n) \langle \nabla f(w_n), \lambda a_1 + (1 - \lambda) a_2 - w_n \rangle \\ &- \langle \nabla f(z_{n+1}^j), \lambda a_1 + (1 - \lambda) a_2 - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n, \forall j = 1, 2, \dots, N. \end{aligned} \quad (12)$$

Considering the definition of ω and (12), we conclude that

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^j), \omega - z_n^j \rangle &+ (1 - \alpha_n) \langle \nabla f(w_n), \omega - w_n \rangle - \langle \nabla f(z_{n+1}^j), \omega - z_{n+1}^j \rangle \\ &\leq f(z_{n+1}^j) - f(z_n^j) - (1 - \alpha_n) f(w_n) + \beta_n \xi_n, \forall j = 1, 2, \dots, N, \end{aligned}$$

then $\omega \in C_n^j$ for all $j = 1, 2, \dots, N$, therefore $\omega \in C_n$. Implies that C_n is closed and convex for all $n \geq 1$. Hence $C_n \cap Q_n$ is closed and convex for all $n \geq 1$.

Step 3 : We show that $\Omega \subset C_n \cap Q_n$ for all $n \geq 1$. Let $q \in \Omega$, from (2) and by Lemma 2.12, we have

$$\begin{aligned}
 D_f(q, z_{n+1}^j) &= D_n(q, Res_{\Theta, \psi, \varphi}^f(u_n^j)) \\
 &\leq D_f(q, u_n^j) - D_f(Res_{\Theta, \psi, \varphi}^f(u_n^j), u_n^j) \\
 &\leq D_f(q, u_n^j) \\
 &= D_f(q, \nabla f^*(\alpha_n \nabla(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j))) \\
 &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, y_n^j) \\
 &= \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [D_f(q, \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)))] \\
 &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n D_f(q, T_j^n w_n) + (1 - \beta_n) D_f(q, w_n)] \\
 &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n [k_n D_f(q, w_n)] + (1 - \beta_n) D_f(q, w_n)] \\
 &= \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n D_f(q, w_n) + \beta_n (k_n - 1) D_f(q, w_n) \\
 &\quad + (1 - \beta_n) D_f(q, w_n)] \\
 &= \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [D_f(q, w_n) + \beta_n \xi_n] \\
 &= \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, w_n) + \beta_n \xi_n - \alpha_n \beta_n \xi_n \\
 &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, w_n) + \beta_n \xi_n,
 \end{aligned} \tag{13}$$

where $\xi_n = (k_n - 1) D_f(q, w_n)$. So we have that $q \in C_n^j$ for all $j = 1, 2, \dots, N$ and then $q \in C_n = \bigcap_{j=1}^N C_n^j$. Hence $\Omega \subset C_n, \forall n \geq 1$. Next, we prove by induction that $\Omega \subset C_n \cap Q_n, \forall n \geq 1$. It Taking the advantage of the definition of Q_n , that $Q_1 = C$ we arrive at $\Omega \subset C_1 \cap Q_1$. Assume that $\Omega \subset C_k \cap Q_k$ for some $k > 0$. Since $C_k \cap Q_k$ is closed and convex, then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{n+1} = Proj_{C_k \cap Q_k}^f(x_1)$. By using the definition of x_{k+1} , we obtain for all $z \in C_k \cap Q_k$ that

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \geq 0$$

Now, for $\Omega \subset C_k \cap Q_k$, we get that

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - q \rangle \geq 0, \forall q \in \Omega,$$

then $q \in Q_{k+1}$. Hence, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$, now since $\Omega \subset C_n, \forall n \geq 1$. Thus, we conclude that $\Omega \subset C_n \cap Q_n, \forall n \geq 1$ and therefore $x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1)$ is well defined $\forall n \geq 1$. This shows that $\{x_n\}$ is well defined

Step 4 : We show that the sequences $\{x_n\}, \{w_n\}, \{y_n^j\}_{j=1}^\infty, \{u_n^j\}_{j=1}^\infty, \{z_n^j\}_{j=1}^\infty$, and $\{T_j^n w_n\}_{j=1}^\infty, j = 1, 2, \dots, N$ are bounded. Now, by using the definition of Q_n and it follows from Lemma 2.1 that $x_n = Proj_{Q_n}^f(x_1)$, we have

$$\begin{aligned}
 D_f(x_n, x_1) &= D_f(Proj_{Q_n}^f(x_1), x_1) \\
 &\leq D_f(q, x_1) - D_f(q, Proj_{Q_n}^f(x_1)) \\
 &\leq D_f(q, x_1), \forall q \in \Omega \subset Q_n.
 \end{aligned}$$

Therefore $\{D_f(x_n, x_1)\}$ is bounded. Hence by Lemma 2.5, $\{x_n\}$ is bounded. Also, the inequality

$$\begin{aligned}
 D_f(q, x_n) &= D_f(q, Proj_{C_{n-1} \cap Q_{n-1}}^f(x_1)) \\
 &\leq D_f(q, x_1) - D_f(x_n, x_1) \\
 &\leq D_f(q, x_1)
 \end{aligned}$$

implies that $\{D_f(q, x_n)\}$ is bounded. Now by using the fact that $D_f(q, T_j^n x_n) \leq D_f(q, x_n), \forall q \in \Omega, j = 1, 2, \dots, N$ then $\{D_f(q, T_j^n x_n)\}_{n=1}^\infty$ is bounded, for all $j = 1, 2, \dots, N$. Also in view of the that $D_f(q, T_j^n w_n) \leq k_n D_f(q, w_n), \forall q \in \Omega, j = 1, 2, \dots, N$ therefore $\{w_n\}, \{y_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty$, and

$\{T_j^n w_n\}_{n=1}^\infty$, are bounded. Now setting $M = \max\{D_f(q, z_1^j), \sup_n D_f(q, w_n)\}$, then obviously $D_f(q, z_1^j) \leq M$. Let $D_f(q, z_n^j) \leq M$. for some n and $j = 1, 2, \dots, N$ then by (13), we have

$$\begin{aligned} D_f(q, z_{n+1}^j) &\leq \alpha_n M + (1 - \alpha_n)M + \beta_n \xi_n \\ &\leq M + \beta_n \xi_n \end{aligned}$$

Thus, $\{D_f(q, z_{n+1}^j)\}_{j=1}^\infty$ is bounded, and so $\{D_f(q, z_n^j)\}_{j=1}^\infty$. Also by Lemma 2.5 $\{z_n^j\}_{j=1}^\infty$ is also bounded for all $j = 1, 2, \dots, N$.

Step 5 : We show that the sequences $\{x_n\}$, converges strongly to some point $\hat{x}$. Now since $x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1)$ and $x_n = Proj_{Q_n}^f(x_1)$, we get that

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \forall n \geq 1.$$

This implies that $\{D_f(x_n, x_1)\}$ is non decreasing. Since $\{D_f(x_n, x_1)\}$ is bounded, then $\lim_{n \rightarrow \infty} D_f(x_n, x_1)$ exists. On the other hand, we obtain

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, Proj_{Q_n}^f(x_1)) \\ &\leq D_f(x_{n+1}, x_1) - D_f(Proj_{Q_n}^f(x_1), x_1) \\ &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, x_n). \quad (14)$$

Since f is totally convex on bounded sets, f is sequentially consistent, now by Lemma 2.3 and (14), we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (15)$$

Now, since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \rightarrow \infty$ (some point in C). Since C_n and Q_n are closed and convex, then $C_n \cap Q_n$ is closed and convex, implies that C_n is weakly closed and $\hat{x} \in C_n$ for each $n \geq 1$. It follows from $x_{n_k} = Proj_{Q_{n_k}}^f(x_1)$ that

$$D_f(x_{n_k}, x_1) \leq D_f(\hat{x}, x_1), \forall n_k \geq 1.$$

Since f is a lower semi-continuous function on convex set C , then it implies weakly lower semi-continuous on C . Therefore, we have

$$\begin{aligned} \liminf_{n_k \rightarrow \infty} D_f(x_{n_k}, x_1) &= \liminf_{n_k \rightarrow \infty} \{f(x_{n_k}) - f(x_1) - \langle \nabla f(x_1), x_{n_k} - x_1 \rangle\} \\ &\geq f(\hat{x}) - f(x_1) - \langle \nabla f(x_1), \hat{x} - x_1 \rangle \\ &= D_f(\hat{x}, x_1). \end{aligned}$$

Also

$$\begin{aligned} D_f(\hat{x}, x_1) &\leq \liminf_{n_k \rightarrow \infty} D_f(x_{n_k}, x_1) \\ &\leq \limsup_{n_k \rightarrow \infty} D_f(x_{n_k}, x_1) \\ &\leq D_f(\hat{x}, x_1). \end{aligned}$$

It follows that

$$\lim_{n_k \rightarrow \infty} D_f(x_{n_k}, x_1) = D_f(\hat{x}, x_1)$$

and

$$f(x_{n_k}) \longrightarrow f(\hat{x}).$$

Since f is uniformly continuous, we conclude that

$$\lim_{n_k \rightarrow \infty} x_{n_k} = \hat{x}.$$

Also, since $\{D_f(x_n, x_1)\}$ is convergent and $\lim_{n_k \rightarrow \infty} D_f(x_{n_k}, x_1) = D_f(\hat{x}, x_1)$ implies that $\lim_{n \rightarrow \infty} D_f(x_n, x_1) = D_f(\hat{x}, x_1)$. If there exists another subsequence $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_m} \longrightarrow x^*$, now by Lemma 2.1, we obtain

$$\begin{aligned} D_f(\hat{x}, x^*) &= \lim_{n_k, n_m \rightarrow \infty} D_f(x_{n_k}, x_{n_m}) \\ &= \lim_{n_k, n_m \rightarrow \infty} D_f(x_{n_k}, Proj_{Q_{n_m}}^f(x_1)) \\ &\leq \lim_{n_k, n_m \rightarrow \infty} (D_f(x_{n_k}, x_1) - D_f(Proj_{Q_{n_m}}^f(x_1), x_1)) \\ &= \lim_{n_k, n_m \rightarrow \infty} (D_f(x_{n_k}, x_1) - D_f(x_{n_m}, x_1)) \\ &= D_f(\hat{x}, x_1) - D_f(x^*, x_1). \end{aligned}$$

This implies that $\hat{x} = x^*$, therefore

$$\lim_{n \rightarrow \infty} x_n = \hat{x} \quad (16)$$

Step 6 : We show that $\hat{x} \in \Omega$, firstly we show that $\hat{x} \in \cap_{j=1}^N F(T_j)$. Now from the definition ξ_n , we have

$$\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} ((k_n - 1)D_f(q, w_n)) = 0.$$

Also, by definition of w_n , we get that

$$\nabla f(w_n) - \nabla f(x_n) = \theta_n(\nabla f(x_n) - \nabla f(x_{n-1})).$$

Which implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(x_n)\| &= \lim_{n \rightarrow \infty} \|\theta_n \nabla f(x_n - x_{n-1})\| \\ &\leq \|\nabla f(x_n - x_{n-1})\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(x_n)\| = 0.$$

Since f is the legendre function and f^* is uniformly Frechet differentiable on bounded subsets, we have

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (17)$$

Since $\{w_n\}$ is bounded, by Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} D_f(w_n, x_n) = 0.$$

Using (15) and (17), we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0. \quad (18)$$

It follows from Lemma 2.2, that

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, w_n) = 0. \quad (19)$$

By the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n^j) = \langle \nabla f(z_n^j) - \nabla f(x_{n+1}), q - x_{n+1} \rangle + D_f(q, z_n^j) - D_f(q, x_{n+1}).$$

Since f is bounded on bounded subset of E , then ∇f is bounded on bounded subset of E^* and hence by the boundedness of $\{x_n\}$, $\{T_j^n w_n\}_{n=1}^\infty$, and $\{z_n\}_{n=1}^\infty$, the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(T_j^n w_n)\}_{n=1}^\infty$, and $\{\nabla f(z_n)\}_{n=1}^\infty$ are bounded in E^* , which implies that $\{D_f(x_{n+1}, z_n^j)\}_{n=1}^\infty$ is bounded. Now since $x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1) \in C_n$, we get

$$D_f(x_{n+1}, z_{n+1}^j) \leq \alpha_n D_f(x_{n+1}, z_n^j) + (1 - \alpha_n) D_f(x_{n+1}, w_n) + \beta_n \xi_n, \forall j = 1, 2, \dots, N. \quad (20)$$

Using (19), $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \xi_n = 0$ in (20), we have

$$\lim_{n \rightarrow \infty} D_f(x_{n+1}, z_{n+1}^j) = 0, \forall j = 1, 2, \dots, N.$$

Since f is totally convex on bounded sets, it follows from Lemma 2.3 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - z_{n+1}^j\| = 0, \forall j = 1, 2, \dots, N. \quad (21)$$

By triangular inequality, we get

$$\|x_n - z_{n+1}^j\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}^j\|.$$

Using (15) and (21), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - z_{n+1}^j\| = 0. \quad (22)$$

Which implies that $z_{n+1}^j \rightarrow \hat{x}$, since $x_n \rightarrow \hat{x}$. Taking into account that

$$\|w_n - z_{n+1}^j\| \leq \|w_n - x_n\| + \|x_n - z_{n+1}^j\|.$$

Also, by (17) and (22), we have

$$\lim_{n \rightarrow \infty} \|w_n - z_{n+1}^j\| = 0 \quad (23)$$

Now, by Lemma 2.11, f and ∇f are uniformly continuous since f is uniformly Frechet differentiable on bounded subsets. Then using (23), we have

$$\lim_{n \rightarrow \infty} |f(w_n) - f(z_{n+1}^j)| = 0, \forall j = 1, 2, \dots, N. \quad (24)$$

and

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(z_{n+1}^j)\| = 0, \forall j = 1, 2, \dots, N. \quad (25)$$

By definition of Bregman distance, we have

$$\begin{aligned} D_f(q, w_n) &= D_f(q, z_{n+1}^j) = f(q) - f(w_n) - \langle \nabla f(w_n), q - w_n \rangle \\ &= (f(q) - f(z_{n+1}^j) - \langle \nabla f(z_{n+1}^j), q - z_{n+1}^j \rangle) \\ &= f(z_{n+1}^j) - f(w_n) + \langle \nabla f(z_{n+1}^j), q - w_n \rangle \\ &+ \langle \nabla f(z_{n+1}^j), w_n - z_{n+1}^j \rangle - \langle \nabla f(w_n), q - w_n \rangle \\ &= f(z_{n+1}^j) - f(w_n) + \langle \nabla f(z_{n+1}^j) - \nabla f(w_n), q - w_n \rangle \\ &+ \langle \nabla f(z_{n+1}^j), w_n - z_{n+1}^j \rangle \end{aligned} \quad (26)$$

Since $\{z_{n+1}^j\}_{n=1}^\infty$ and $\{\nabla f(z_{n+1}^j)\}_{n=1}^\infty$ are bounded for all $j = 1, 2, \dots, N$, using (23), (24), (25) in (26), we obtain

$$\lim_{n \rightarrow \infty} (D_f(q, w_n) - D_f(q, z_{n+1}^j)) = 0, \quad \forall j = 1, 2, \dots, N. \quad (27)$$

Furthermore, by Lemma 2.12(e) and Lemma 2.10, for each $j = 1, 2, \dots, N$, we get

$$\begin{aligned} D_f(z_{n+1}^j, u_n^j) &\leq D_f(q, u_n^j) - D_f(q, z_{n+1}^j) \\ &= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j))) - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n))) \\ &\quad - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n D_f(q, T_j^n w_n) + (1 - \beta_n) D_f(q, w_n)] \\ &\quad - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n (k_n D_f(q, w_n)) + (1 - \beta_n) D_f(q, w_n)] \\ &\quad - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [\beta_n (1 + (k_n - 1)) D_f(q, w_n) + (1 - \beta_n) D_f(q, w_n)] \\ &\quad - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) [D_f(q, w_n) + \beta_n (k_n - 1) D_f(q, w_n)] \\ &\quad - D_f(q, z_{n+1}^j) \\ &= \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, w_n) + \beta_n \xi_n - \alpha_n \beta_n \xi_n - D_f(q, z_{n+1}^j) \\ &\leq \alpha_n D_f(q, z_n^j) + (1 - \alpha_n) D_f(q, w_n) + \beta_n \xi_n - D_f(q, z_{n+1}^j) \\ &= \alpha_n D_f(q, z_n^j) + D_f(q, w_n) - \alpha_n D_f(q, w_n) + \beta_n \xi_n - D_f(q, z_{n+1}^j) \\ &= \alpha_n (D_f(q, z_n^j) - D_f(q, w_n)) + (D_f(q, w_n) - D_f(q, z_{n+1}^j)) + \beta_n \xi_n. \end{aligned} \quad (28)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \xi_n = 0$ then using (27) in (28), we obtain

$$\lim_{n \rightarrow \infty} D_f(z_{n+1}^j, u_n^j) = 0, \quad \forall j = 1, 2, \dots, N.$$

Since f is totally convex on bounded sets, it follows from Lemma 2.3 that

$$\lim_{n \rightarrow \infty} \|z_{n+1}^j - u_n^j\| = 0, \quad \forall j = 1, 2, \dots, N. \quad (29)$$

By triangular inequality, we get that

$$\|x_n - u_n^j\| \leq \|x_n - z_{n+1}^j\| + \|z_{n+1}^j - u_n^j\|$$

Using (22) and (29), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n^j\| = 0, \quad \forall j = 1, 2, \dots, N. \quad (30)$$

Which implies that $u_n^j \rightarrow \hat{x} \in C$, since $x_n \rightarrow \hat{x} \in C$. Now, taking into account that

$$\|w_n - u_n^j\| \leq \|w_n - x_n\| + \|x_n - u_n^j\|$$

By (17) and (30), we obtain

$$\lim_{n \rightarrow \infty} \|w_n - u_n^j\| = 0 \quad (31)$$

Since f is uniformly Frechet differentiable then by Lemma 2.11 ∇f is uniformly continuous on bounded sets. Using (29) and (31), we conclude that

$$\lim_{n \rightarrow \infty} \|\nabla f(z_{n+1}^j) - \nabla f(u_n^j)\| = 0, \forall j = 1, 2, \dots, N, \quad (32)$$

and

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(u_n^j)\| = 0, \forall j = 1, 2, \dots, N. \quad (33)$$

On the other hand, for each $j = 1, 2, \dots, N$, we have

$$\begin{aligned} \|\nabla f(w_n) - \nabla f(u_n^j)\| &= \|\nabla f(w_n) - \nabla f(\nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j)))\| \\ &= \|\nabla f(w_n) - (\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j))\| \\ &= \|(1 - \alpha_n) \nabla f(w_n) + \alpha_n \nabla f(w_n) - \alpha_n \nabla f(z_n^j) - (1 - \alpha_n) \nabla f(y_n^j)\| \\ &= \|\alpha_n (\nabla f(w_n) - \nabla f(z_n^j)) + (1 - \alpha_n) (\nabla f(w_n) - \nabla f(y_n^j))\| \\ &= \|\alpha_n (\nabla f(w_n) - \nabla f(z_n^j)) + (1 - \alpha_n) [\nabla f(w_n) - \nabla f(\nabla f^*(\beta_n \nabla f(T_j^n w_n) \\ &\quad + (1 - \beta_n) \nabla f(w_n)))]\| \\ &= \|\alpha_n (\nabla f(w_n) - \nabla f(z_n^j)) + (1 - \alpha_n) [\nabla f(w_n) - (\beta_n \nabla f(T_j^n w_n) \\ &\quad + (1 - \beta_n) \nabla f(w_n))]\| \\ &= \|\alpha_n (\nabla f(w_n) - \nabla f(z_n^j)) + (1 - \alpha_n) [\beta_n (\nabla f(w_n) - \nabla f(T_j^n w_n))]\| \\ &\geq (1 - \alpha_n) \beta_n \|\nabla f(w_n) - \nabla f(T_j^n w_n)\| - \alpha_n \|\nabla f(w_n) - \nabla f(z_n^j)\| \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} (1 - \alpha_n) \beta_n \|\nabla f(w_n) - \nabla f(T_j^n w_n)\| &\leq \alpha_n \|\nabla f(w_n) - \nabla f(z_n^j)\| \\ &\quad + \|\nabla f(w_n) - \nabla f(u_n^j)\|, \quad \forall j = 1, 2, \dots, N. \end{aligned} \quad (34)$$

Since $\liminf_{n \rightarrow \infty} (1 - \alpha_n) \beta_n > 0$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, then using (33) in (34), we obtain

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(T_j^n w_n)\| = 0, \quad \forall j = 1, 2, \dots, N. \quad (35)$$

Again for $j = 1, 2, \dots, N$, we have

$$\begin{aligned} \|\nabla f(w_n) - \nabla f(y_n^j)\| &= \|\nabla f(w_n) - \nabla f(\nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)))\| \\ &= \|\nabla f(w_n) - (\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n))\| \\ &= \|\beta_n (\nabla f(w_n) - \nabla f(T_j^n w_n))\| \\ &\leq \beta_n \|\nabla f(w_n) - \nabla f(T_j^n w_n)\| \end{aligned} \quad (36)$$

Using (35) in (36), we obtain

$$\lim_{n \rightarrow \infty} \|\nabla f(w_n) - \nabla f(y_n^j)\| = 0, \quad \forall j = 1, 2, \dots, N.$$

Since f is the legendre function and f^* is uniformly Frenchet differentiable on bounded subsets, we have

$$\lim_{n \rightarrow \infty} \|w_n - y_n^j\| = 0, \quad \forall j = 1, 2, \dots, N. \quad (37)$$

It follows from (16) and (17), that

$$\lim_{n \rightarrow \infty} w_n = \hat{x} \quad (38)$$

Using (37) and (38), we have that $y_n^j \rightarrow \hat{x}$ (as $n \rightarrow \infty$). Since ∇f is uniformly continuous, using (38) we have $\nabla f(w_n) \rightarrow \nabla f(\hat{x})$. Therefore by (35), we obtain that

$$\lim_{n \rightarrow \infty} \nabla f(T_j^n w_n) = \nabla f(\hat{x}).$$

Since ∇f is uniformly continuous, we have

$$\lim_{n \rightarrow \infty} T_j^n w_n = \hat{x}, \quad \forall j = 1, 2, \dots, N. \quad (39)$$

It follows from (18), (29) and (37) that the sequence $\{x_n\}$, $\{y_n^j\}_{n=1}^\infty$, $\{u_n^j\}_{n=1}^\infty$, $\{w_n\}$, and $\{z_n^j\}_{n=1}^\infty$, for all $j = 1, 2, \dots, N$, have the same asymptotic behaviour, therefore by (39), we conclude that

$$\lim_{n \rightarrow \infty} T_j^n x_n = \hat{x}, \quad \forall j = 1, 2, \dots, N. \quad (40)$$

Now, by the assumptions that for all $j = 1, 2, \dots, N$, T_j is uniformly L_j -Lipschitz continuous, we get that

$$\begin{aligned} \|T_j^{n+1} x_n - T_j^n x_n\| &\leq \|T_j^{n+1} x_n - T_j^{n+1} x_{n+1}\| + \|T_j^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T_j^n x_n\| \\ &\leq (L_j + 1) \|x_{n+1} - x_n\| + \|T_j^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - T_j^n x_n\| \end{aligned} \quad (41)$$

By (40) and since $x_n \rightarrow \hat{x}$ (as $n \rightarrow \infty$), we obtain that

$$\lim_{n \rightarrow \infty} \|T_j^{n+1} x_n - T_j^n x_n\| = 0, \quad \forall j = 1, 2, \dots, N,$$

and

$$\lim_{n \rightarrow \infty} T_j^{n+1} x_n = \hat{x}, \quad \forall j = 1, 2, \dots, N,$$

which implies that

$$\lim_{n \rightarrow \infty} T_j T_j^n x_n = \hat{x}, \quad \forall j = 1, 2, \dots, N.$$

It follows from the closedness of T_j that

$$\lim_{n \rightarrow \infty} T_j \hat{x} = \hat{x}, \quad \forall j = 1, 2, \dots, N.$$

Therefore

$$\hat{x} \in \bigcap_{j=1}^N F(T_j).$$

Next, we show that $\hat{x} \in GMEP(\Theta, \psi, \varphi)$. Since $\{x_n\}$ is bounded and so is $\{\nabla f(x_n)\}$, there exists a subsequence $\{\nabla f(x_{n_k})\}$ of $\{\nabla f(x_n)\}$ such that $\nabla f(x_{n_k}) \rightarrow \hat{x} \in C$ as $k \rightarrow \infty$. It follows from (22) and (29) that there exists subsequence $\{\nabla f(z_{n_k}^j)\}$ of $\{\nabla f(z_n^j)\}$ and $\{\nabla f(u_{n_k}^j)\}$ of $\{\nabla f(u_n^j)\}$ such that $\nabla f(z_{n_k}^j) \rightarrow \hat{x} \in C$ and $\nabla f(u_{n_k}^j) \rightarrow \hat{x} \in C$ as $k \rightarrow \infty$, for all $j = 1, 2, \dots, N$ respectively. From $z_{n+1}^j = \text{Res}_{\Theta, \psi, \varphi}^f(u_n^j)$, we have

$$\Gamma(z_{n+1}^j, y) + \langle \nabla f(z_{n+1}^j) - \nabla f(u_n^j), y - z_{n+1}^j \rangle \geq 0, \quad \forall y \in C.$$

Replacing n by n_k and using (A₂), we obtain

$$\langle \nabla f(z_{n_k+1}^j) - \nabla f(u_{n_k}^j), y - z_{n_k+1}^j \rangle \geq -\Gamma(z_{n_k+1}^j, y) \geq \Gamma(y, z_{n_k+1}^j), \quad \forall y \in C, \quad \forall j = 1, 2, \dots, N.$$

Letting $k \rightarrow \infty$, it follows from $\nabla f(z_{n_k}) \rightarrow \hat{x} \in C$ and (A_4) that

$$\Gamma(y, \hat{x}) \leq 0, \forall y \in C,$$

for λ with $0 < \lambda \leq 1$ and $y \in C$, assume that $y_\lambda = \lambda y + (1 - \lambda)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_\lambda \in C$ and $\Gamma(y_\lambda, \hat{x}) \leq 0, \forall y \in C$, so by (A_1) and (A_3) , we obtain

$$\begin{aligned} 0 &= \Gamma(y_\lambda, y_\lambda) \\ &\leq \lambda \Gamma(y_\lambda, y) + (1 - \lambda) \Gamma(y_\lambda, \hat{x}) \\ &\leq \lambda \Gamma(y_\lambda, y). \end{aligned}$$

Dividing by λ , we get

$$\Gamma(y_\lambda, y) \geq 0, \forall y \in C.$$

Letting $\lambda \rightarrow 0$ and by (A_3) , we conclude that

$$\Gamma(\hat{x}, y) \geq 0, \forall y \in C.$$

Thus

$$\hat{x} \in GMEP(\Theta, \psi, \varphi).$$

Therefore

$$\hat{x} \in \Omega := \left[\bigcap_{j=1}^N F(T_j) \right] \cap [GMEP(\Theta, \psi, \varphi)].$$

Step 7 : We show that $\hat{x} = Proj_\Omega^f(x_1)$. Since Ω is a nonempty closed and convex subset of E , $Proj_\Omega^f(x_1)$ is well defined. Suppose that $\hat{u} = Proj_\Omega^f(x_1)$, now by $x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1)$ and $Proj_\Omega^f(x_1) \in \Omega \subseteq C_n \cap Q_n$, we conclude that

$$D_f(x_{n+1}, x_1) \leq D_f(\hat{u}, x_1)$$

Since $\{x_{n_k}\}$ is a weak convergence subsequence of $\{x_n\}$ and by Lemma 2.7, we get that the sequence $\{x_n\}$ converges strongly to $\hat{u}$. It follows from the uniqueness of the limit, we conclude that the sequence $\{x_n\}$ converges strongly to $\hat{x} = Proj_\Omega^f(x_1)$. This completes the proof.

Corollary 3.2. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \rightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be bi functions which satisfying assumptions $(A_1) - (A_4)$, $\psi : C \rightarrow E^*$ be continuous monotone mappings. Let $\{T_j\}_{j=1}^N : E \rightarrow E$ be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each $j \geq 1$, T_j is uniformly L_j - Lipschitz continuous. Assume that $\Omega := \left(\bigcap_{j=1}^N F(T_j) \right) \cap (GMEP(\Theta, \psi)) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:*

$$\left\{ \begin{array}{l} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j)); \\ z_{n+1}^j = Res_{\Theta, \psi}^f(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1 - \alpha_n) D_f(z, w_n) \\ + \beta_n \xi_n^j\}; \\ C_n = \bigcap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \quad \forall n \geq 1, \end{array} \right.$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\liminf_{n \rightarrow \infty} (1 - \alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0, 1)$ and $\xi_n = (k_n - 1)D_f(q, w_n)$. Then, $\{x_n\}$ converges strongly to $Proj_{\Omega}^f x_1$.

Corollary 3.3. Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \rightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be bi functions which satisfying assumptions $(A_1) - (A_4)$. Let $\{T_j\}_{j=1}^N : E \rightarrow E$ be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each $j \geq 1, T_j$ is uniformly L_j - Lipschitz continuous. Assume that $\Omega := (\cap_{j=1}^N F(T_j)) \cap (GMEP(\Theta)) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j)); \\ z_{n+1}^j = Res_{\Theta}^f(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1 - \alpha_n) D_f(z, w_n) \\ + \beta_n \xi_n\}; \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\liminf_{n \rightarrow \infty} (1 - \alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0, 1)$ and $\xi_n = (k_n - 1)D_f(q, w_n)$. Then, $\{x_n\}$ converges strongly to $Proj_{\Omega}^f x_1$.

Corollary 3.4. Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \rightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be bi functions which satisfying assumptions $(A_1) - (A_4)$. Let $\{T_j\}_{j=1}^N : E \rightarrow E$ be a countable family of closed and uniformly Bregman relatively nonexpansive mappings and for each $j \geq 1, T_j$ is uniformly L_j - Lipschitz continuous. Assume that $\Omega := (\cap_{j=1}^N F(T_j)) \cap (GMEP(\Theta)) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1 - \beta_n) \nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1 - \alpha_n) \nabla f(y_n^j)); \\ z_{n+1}^j = Res_{\Theta}^f(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1 - \alpha_n) D_f(z, w_n); \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\liminf_{n \rightarrow \infty} (1 - \alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0, 1)$. Then, $\{x_n\}$ converges strongly to $Proj_{\Omega}^f x_1$.

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Analysis of single and multi-teller models using simulation for reducing waiting time: a case study of Access Bank and Eco Bank, Nig Ltd, in Taraba State Nigeria

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Abstract

The essence of any organization is to solve problem and maximize its services and relevance among its subscribers and customers. In an attempt to actualize these, several practices are put in place depending on the type of services rendered and the type of subscribers. The banking sector tends to be one of the most customer oriented establishments in the private sector and thus infers a lot of carefulness in customer services and customer relations. One of the major factors influencing the success of organizations in today's competitive world is to increase customer satisfaction through the improvement of service quality. In this research paper, it was observed that a customer is expected to come into the bank at every 0.72 minutes at Eco bank while at Access bank at every 0.46 minutes a customer comes in. It was also revealed that if it costs ₦ 400 (₦ 200 happiness cost and ₦ 200 service cost) to attend to a customer the expected hourly cost of facility maintenance in Eco Bank is ₦ 33,200 and for Access Bank is ₦ 52,000. Thus at a single teller access bank spends a minimum cost of ₦ 52, 0000 per hour to attend to 130 customers while at the multi teller, adopting the established average per teller; Eco Bank Spends for 4 tellers, ₦132,000 per hour attending to an average of 83 customers and Access bank spends for 6 tellers, ₦ 312,000 per hour attending to an average of 130 customers. Thus, the cost of facility is dependent on the number of tellers and cost of internet service plus happiness service provided. In addition, the group descriptive statistics, though with varying distribution and sample selection between the banks due to validity of responses collected, there is a mean correlation and deviation in the assessment of empathy and reliability of teller system at both Eco Bank Nigeria and Access Bank Plc. With a related, mean mark above 2.5 and deviation below 0 and significant at 0.125 points to positive autocorrelation. On the other hand, the reliability mean marks are related at 3.148 and 3.528 with a mean deviation of 0.863 and 0.559 respectively. Assurance on its own has a lesser mean mark at 2.815 for Eco bank and 3.111 at Access bank. The deviations are similar at 0.786, which infers no significant difference by mere inspection. Finally, responsiveness as a proxy to measure performance the mean at 2.333 for Eco Bank and Access bank at 2.639 this is quite poor and low thus, we could infer that at both Eco bank and Access bank the responsiveness of the tellers though reliable and assuring is low in responsiveness.

Keywords: empathy; validity; expected; descriptive; customer

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1 Introduction

Banking services is seen and is often a one on one or inter personal relationship due to the confidentiality of financial matters and engagements. In several banks especially at congested or highly populated areas mostly limited in facilities, queuing is common and the major alternative to achieving orderliness and smooth service rendering. In organizations and establishments where services are rendered, it is the paramount desire of the managers to estimate the time and save customers time and maximizes their inflow and opportunities. In the words of [4] as cited in [1], Bank customers have become increasingly demanding, because they require high quality, low priced and immediate service delivery. They want additional improvement of value from their chosen banks. Service delivery in banks is personal, customers are either served immediately or join a queue (waiting line) if the system is busy. A queue occurs where facilities are limited and cannot satisfy demand made against them at a particular period. However, most customers are not comfortable with waiting or queuing. These are common and every day experiences at post offices, ATM, banking halls, fuel stations bet centers and some shops. In the words of [2] as service speeds up, time spent waiting on queue decreases. Service cost however increases as the level of service increases.

The goal of managers is to schedule as few employees as possible while maintaining a minimum customer service level. Managers want queues short enough so that customers do not become dissatisfied and either leaves without transacting their business or transact and never return in the future. However, some waiting can be allowed if the waiting cost is balanced with significant saving in service cost. Thus, it suffices to infer that queuing is a common banking practice which gives the managers especially operations manager concern on how to reduce the queuing or waiting time and increase the quality of service rendered.

In an attempt to solve this conventional challenge and issue, the operations managers have devised means and alternative methods as well as suggested mathematical or statistical operational methods to curb this and maximize customers' service. The provision of the Automated Teller Machine (ATM), is an attempt to render quick and efficient services to customers even outside working hours, internet banking and the use of USSD codes in transactions have been adopted as alternative models to customer service delivery. In the banks, customers form queues and lines to quicken the access to services. The multi teller or single teller model infers a direct human contact between a bank representative or teller and a customer, the multiplicity of this ushers in the multi teller model system. Thus, one could infer that the multi teller system infers the provision of more bank personnel (tellers) who attend to various customers to reduce the queue and the time spent in attending to customers. On the other hand, simulation is a useful technique to analyze complex systems which are expensive to be changed through real experimentation. This approach allows investigation of different service configurations without any physical alteration, leading to the selection of the best solution while spending significantly lower cost.

The banking sector in Nigeria is no exception to this reality and fact. Notable among the banks are the commercial banks that are scattered over the entire 36 states of the federation and Abuja the Capital. In Nigeria there are 21 Commercial banks and each has several departments which one is solely dedicated to customer relations and customer service management. In all attempts banks try as much as possible to satisfy its customers and reduce stress in accessing their services. Each commercial bank is in serious competition with the other considering the open market operation. In all attempt to take charge of market share and maximize shareholders wealth, banks place her customers first and create enabling environment and service accessibilities to

ameliorate the time spent and the stress undertaken in getting served. As the banking activities and services increases, the number of customers increases and each customer desires to be served well, first and possibly treated preferentially. Daily work scheduling is required in many service companies such as Hospitals and Banks. Bad staff-customer scheduling results in long customer waiting times, long queues, and consequently, waiting cost. Bad scheduling can also result in loss of productivity of Tellers due to idle times. On the other hand, good scheduling results in low waiting cost, good Teller utilization, customer satisfaction, and more profit. Operations managers are faced with the problem of recognizing the trade-off that must be taken between providing good service and the cost of waiting for such service.

The main aim of this study is to analyze single and multiple teller models or using simulation for reducing waiting time in banks, this will help us ascertain the average waiting time of customers at the various banking halls of the study areas. It will determine the Multi teller effects on the performance of the banks using simulation models and the facility cost of single and multi-teller system and make relevant recommendations as it relates to effective performance and efficient service delivery.

1.2 Scope of the study

The scope of this research is on the analyses of single and multiple teller models or using simulation for reducing waiting time in banks by customers specifically Eco Bank and Access Bank Nigeria Limited. Other inferences are for reference and purpose of emphases and consolidation.

2 Queuing and waiting

Waiting lines are formed whenever the current demand for a service exceeds the current capacity to provide that service [7]. Because of the difficulty in accurately predicting arrival patterns of customers for service and/or how much time is required to provide service to each customer, accurate decision regarding the capacity to be provided is made quite difficult. Excess service capacity involves excessive costs due to underutilization and insufficient capacity to meet peak loads causes waiting lines to become excessively long and customers may even quit the waiting lines. The ultimate goal of banks is to achieve an economic balance between the cost of service (i.e., cost of idle facility and employee) and costs associated with waiting for the service (i.e., social cost and cost of lost customers). Queuing theory does not directly solve this problem, but tries to provide vital information required for taking decisions.

According to [3], a waiting line (length) system or queuing system is defined by two important elements: the population source of its customers and the process or service system. The customer population can be considered as finite or infinite. However, [5] asserted that, the customer population is finite when the number of customers affects potential new customers for the service system already in the system. When the number of customers waiting in line does not significantly affect the rate at which the population generates new customers, the customer population is considered infinite.

According to [3], in his publication on waiting lines and queuing systems, he stated that Waiting in lines may be due to overcrowded, overfilling or due to congestion.

2.1 Single and multiple teller' characteristics

In queuing system, the terms server or teller and channel are used interchangeably (Wang and Tai, 2000). And is used interchangeably with teller system in banking. Queuing systems are either single

server or teller or multiple server or teller s. Single server or teller examples include gas station, food mart with single checkout counter, a theater with a single person selling tickets and controlling admission into the show. Multiple server or teller examples include gas stations with multiple gas pumps, grocery stores with multiple cashiers, and multiple tellers in a bank [8]. The study however looks at both single channel and multiple channels to make comparative judgment.

In a single-phase system, the service is completed all at once, such as a bank transaction or grocery store checkout counter [6]. In a multiphase system, the service is completed in a series of phases, such as at fast-food restaurant with ordering, pay, and pick-up windows.

As indicated by [6], it was also stated by [9] that, Basic single server or teller model assumes customers are arriving at Poisson arrival rate with exponential service times, and first come, first serviced queue discipline, and infinite queue length, and infinite calling population.

By adding additional resources to single server or teller system either service rate can be increased or arrival rate at each server or teller can be decreased with additional cost overhead. In Single server or teller single-phase system, customer is served once completed. Common examples of single server or teller single-phase are a teller counter in bank, a cashier counter in super market, automated ticketing machine at train station. In single server or teller queuing system wait time or performance of system depends on efficiency of serving person or service machine. Single server or teller single-phase queuing system is most commonly automated system found in our regular life. For example many superstores have replaced manual billing counters with automated machines.

[5] Argued that Single server or teller and multiple-phase system incorporates division of work into phases to keep waiting line moving. Common examples of these systems are automatic or manual car wash, drive through restaurants. Waiting line models are important to a business because they directly affect customer service perception and the costs of providing service. [4] also argued that if system average utilization is low, that suggests the waiting line design is inefficient. Poor system design can result in over staffing. Long waits suggest a lack of concern by the organization or can be view as a perception of poor service quality. Queuing analysis has changed the way businesses use to run and has increased efficiency and profitability of businesses.

3 Data analysis and interpretation

3.1 Waiting time interval from arrival to service delivery 8:15am to 9:51am (74 minutes) Single Teller System (Eco Bank)

Table 3.1 Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
Arrival Interval	54	2.00	6.00	3.8889	.94503	.893
Valid N (listwise)	54					

3.1.1 Duration for Service Delivery: Single teller System (Eco Bank)

Table 3.2 Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
Service Duration	54	2.00	7.00	4.5000	1.17762	1.387
Valid N (listwise)	54					

Waiting time Interval from Arrival to Service Delivery 10:11am - 12:08am (117 minutes)
Single Teller System (Access Bank)

Table 3.3 Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
Waiting Time	54	3.00	7.00	4.9074	1.16988	1.369
Valid N (listwise)	54					

3.1.2 Duration for Service Delivery: Single teller System Access Bank

Table 3.4 Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
Service Delivery Time	54	1.00	10.00	4.6778	2.50785	6.289
Valid N (listwise)	54					

Results of Customer Satisfaction ratings on banking hall queue and Teller systems as Proxy to performance effect

Table 3.5 Group Statistics

	Bank	N	Mean	Std. Deviation	Std. Error Mean
Empathy	ECO BANK	27	3.0370	.85402	.16436
	ACCESS	36	3.5000	.50709	.08452
Reliability	ECO BANK	27	3.1481	.86397	.16627
	ACCESS	36	3.5278	.55990	.09332
Assurance	ECO BANK	27	2.8148	.78628	.15132
	ACCESS	36	3.1111	.74748	.12458
Responsiveness	ECO BANK	27	2.3333	.78446	.15097
	ACCESS	36	2.6389	.76168	.12695

3.2 Facility Cost of Single and Multi-Teller System

Establishing the assumed cost of service and facility cost.

The Average time spent per customer at the bank per hour

Total customers in 1hour 30 minutes Divided by the number of minutes that is the average customer per minute. (Recall, there is already a mean mark of waiting and service time thus, adopting that per customer)

At the single teller Eco Bank = $4.5 + 3.9$ mins is 8 mins 40 seconds.

At the single teller Access Bank = $4.7 + 4.9$ mins is 10 minutes,

where, 3.8 is the mean mark achieved in Table 3.1 (Waiting Time Interval, Eco Bank) and 4.5 is the mean mark achieved in table 3.2 (Duration for Service Delivery, Eco Bank).

4.9 is the mean mark achieved in table 3.3 (Waiting Time Interval, Access Bank) 4.7 is the mean mark achieved in table 3.4 (Duration for Service Delivery, Access Bank).

At each time spent by a customer, there is a cost of happiness and lost opportunity to the bank, and each service provided at the individual teller is charged. Assuming that it cost 200 naira to provide comfort to each customer and extra 200 naira to provide data and internet service per minute in service delivery, the possible cost of each teller facility could be calculated thus:

Adopting the Number of arrival time formulae thus as $\frac{\text{Number of Customers}}{\text{Minutes}}$.

At Eco Bank $\frac{54 \text{ customers}}{74 \text{ minutes}} = 0.72 \text{ minutes}$ (1 hour 14 minutes = 74 minutes).

Thus at Access bank $\frac{54 \text{ customers}}{117 \text{ minutes}} = 0.46 \text{ minutes}$ (1 hour 57 minutes as stated earlier = 117 minutes).

From the above, it was observed that a customer is expected to come into the bank at every 0.72 minutes at Eco bank while at Access bank at every 0.46 minutes a customer comes in. thus, in an hour it is expected that about $\frac{60 \text{ minutes}}{\text{expected minute per arrival}} = \frac{60}{0.72} = 83$ customers at Eco Bank

and $\frac{60 \text{ minutes}}{\text{expected minute per arrival}} = \frac{60}{0.46} = 130$ customers at Access Bank.

Thus, if it costs ₦ 400 (₦ 200 happiness cost and ₦ 200 service cost) to attend to a customer, the expected hourly cost of facility maintenance equals

Eco Bank = ₦ 400 x 83 = ₦ 33,200,
Access Bank = ₦ 400 x 130 = ₦ 52,000.

Thus at a single teller access bank spends a minimum cost of ₦ 52, 0000 per hour to attend to 130 customers, while at the multi teller, adopting the established average per teller;

Eco Bank Spends 4 tellers x ₦ 33,200 = ₦ 132,000 per hour attending to an average of 83 customers.

Access bank spends 6 tellers x ₦ 52,000 = ₦ 312,000 per hour attending to an average of 130 customers.

From the above, the cost of facility is dependent on the number of tellers and cost of internet service plus happiness service provided. The water, snacks and comfort provided at the bank constitute the happiness cost.

4 Discussion and conclusion

In line with the outputs and the analyses interpreted, it becomes pertinent to make the following summary:

1. Commercial banks in contemporary times have adopted the use of multi teller as a means to reduce waiting time and service delivery time though not without extra cost.
2. While it is not uniform in arrival and waiting time among banks and customers, there is an average time which does not exceed certain minutes at teller's points. However, the waiting time of a customer is dependent on the service delivery time of the other customer who

arrived prior to him. That is, from arrival to service delivery of a customer, the time taken by the earlier customer at the teller point is a determinant.

3. In a single teller, the waiting time is more; as many customers are left in the hands of a single teller there by making those who arrived later to wait longer since there is a single queue system.
4. At the multi teller, the maximum waiting time is 15mins with a minimum of 5 mins at the two banks when compared accordingly. More tellers have the likelihood of attending to more customers as seen in the research where Access bank with 6 teller points attended to more customers than Eco bank with 4 teller points.
5. There is an increased cost in the adoption and use of multi teller as more tellers require more service points, human capital, computing items and queue management system. Thus, multi teller adoption and usage requires more cost.
6. Customers were relatively satisfied in the service received in terms of empathy, responsiveness, assurance and reliability in no order. The responses of the customers to the above proxy as a measure of satisfaction were quite similar.

It suffices to conclude from the summary above that; multi teller has the tendency and probability to reduce waiting time in commercial banks thereby improving service delivery but at a cost that is not same with single teller. The waiting times at various commercial banks are dependent on number of teller and the service delivery time from one customer to another. Thus, it suffices to conclude that, while customer satisfaction at the various bank does not differ significantly, the waiting time between a single teller and a multi teller differ. The multi teller experience and waiting time does not differ significantly among commercial banks. The more the teller points, the less the waiting time and better customer service delivery.

It becomes imperative, at this juncture, to make the following recommendations in line with the objective and scope of this study:

1. Commercial banks should improve in their customer service through the adoption of multi teller system.
2. Multi-tellers should be managed to reduce waiting times and improve customer service delivery and in the long run maximize banks corporate objective. Processors at bank tellers should make effort to eliminate biased selection of customers to be attended to as well.
3. Tags and numbers should be given to customers in queues while marks should be made to reduce unfair movement from one teller queue to the other by customers, thereby inculcating equity and fair service delivery as well eliminating dissatisfaction, which is a result of mismanagement of tellers and queue system.
4. The use of First in First Out (FIFO) should be systematically applied and diplomatically adopted in banks to enable customers switch adequately from queue to queue if need be to consolidate waiting time reduction among customers.

Provisions for adequate teller points, ATMs, processors should form part of the banks operational plans and annual operational expenses towards improving customers' satisfaction and service delivery. More computers and service points should be created mostly during periods of festivities and high demand for teller deposit and withdrawal.

As research is contentious, the limitation in scope and objective to this study makes it relevant to recommend the following topics for further studies.

1. A review of Single and Multi-Teller System Management using computerized simulation and manual queuing management System.

2. Alternative to Multi Teller system in a computerized and Robotic era
3. An investigation into the use of ATMs and Human Teller systems for withdrawals and Deposits in a computerized era.
4. Cost Implication and Investigation into the use of ATMs and Human Teller systems for withdrawals and Deposits in a computerized era.
5. The place of banking in the era of Crypto currency.

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High order stiffly stable parameter dependent nested hybrid linear multistep methods for stiff ODEs

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Abstract

In this work, we proposed a High Order Stiffly Stable parameter dependent nested hybrid linear multistep method for stiff initial value problems (IVPs) in ordinary differential equations (ODEs). It forms a super class of the linear multistep methods (LMMs), with properties that embedded the characteristics of LMM and hybrid methods. It is hybrid in nature as a result of the incorporation of one or more off-step points and this is aimed at attaining a better stability properties and higher order than the conventional linear multistep methods (LMMs). The proposed method is A- stable and $A(\alpha)$ –stable for step number $k \leq 6$. A numerical example is given to demonstrate the application of the methods.

Keywords and phrases: nested hybrid multi-step methods; implicit methods; A-stable; $A(\alpha)$ -stability

1 Introduction

The high order stiffly stable parameter dependent nested hybrid linear multistep method form a super class of linear multistep methods (LMMs), with properties that embedded the characteristics of LMM and hybrid methods. This paper gives a continuous modification of [8], the second derivative continuous linear multistep methods for the solution of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The motivation lies in the fact that the new formulation offers the advantage of comprehensive solutions to IVPs unlike the discrete solution generated from [3] and [4] methods. The success of these methods is in their attainable stiff stability characteristics useful for resolving the problem posed by stiffness in IVPs [1].

In this regard, an introduction of high order stiffly stable parameter dependent nested hybrid linear multistep method for stiff ODEs is proposed. It is hybrid in nature as a result of the incorporation of one or more off-step points and this is aimed at attaining a better stability properties and higher order than the conventional linear multistep methods (LMM). This type of methods provides a means of bypassing the Dahlquist order and stability barrier of the conventional LMM [3]. The proposed method is A- stable and $A(\alpha)$ –stable for step number $k \leq 6$. A numerical example is given to demonstrate the application of the methods.

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According to [9], the general form of linear multistep method (LMMs) is

$$Y_0 = \sum_{j=0}^k \alpha_j y_{n+j} + h\beta_k f_{n+k}; \quad Y_0 = y_{n+c_0}. \quad (1)$$

There has been considerable progress in the development of these methods and its hybrid counterpart [8]

$$\begin{aligned} Y_{i+1} &= \sum_{j=0}^k \lambda_j y_{n+j} + h\rho_i f(Y_i); \quad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} &= \sum_{j=0}^{k-1} \ell_j y_{n+j} + h\eta_{s-1} f(Y_{s-1}) + h\phi_k(f_{n+k} + a f_{n+k-1}); \quad c_i = k - \frac{1}{2^{k-i}}, \quad 0 \leq c \leq k. \end{aligned} \quad (2)$$

where Y_{i+1} and y_{n+k} are the hybrid inputs and output.

Hence, the general form of high order stiffly stable parameter dependent nested hybrid multistep methods is

$$\begin{cases} Y_0 = \sum_{j=0}^k \phi_j y_{n+j} + h\lambda_k f_{n+k}; \quad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^k \varphi_j f_{n+j} + h\rho_i f(Y_i); \quad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k(f_{n+k} + a f_{n+k-1}); \quad 0 \leq c \leq k. \end{cases} \quad (3)$$

the parameters ϕ_j , φ_j , δ_j , ρ_i , θ_{s-1} , λ_k and γ_k with $i = 1, \dots, s-1$, $j = 0, 1, \dots, k$ are all real coefficients. The $f_{n+j} = f(x_{n+j}, y_{n+j})$ is the first derivative function, $h = x_{n+1} - x_n$ denotes the mesh size, k and s represents the step number, and the stage respectively, while $Y_i = y(x_n + c_i h) + O(h^{q_{i+1}})$, $i = 0, \dots, s-1$, and y_{n+k} are the stage and the output point of the methods in (3). The $f(Y_i) = \{f(x_n + c_i h, Y_i)\}_{i=0}^{s-1}$ denotes the derivative of the stages. The stability region of the formulae in (3) depends on the choice of the parameter “a”. The $c = [c_0, c_1, \dots, c_s]^T$ is the abscissa vector of the input methods. The elements in c are computed from the abscissa generator: $c_i = \left\{k - \frac{1}{2^{k-i}}\right\}_{i=0}^{s-1}$, $k = 1, 2, \dots, s-1$.

Interestingly, many cases of these nested methods are known to be stiffly stable for the numerical solution of stiff initial value problems in ODEs, [10],

$$y' = f(x, y), \quad x \in [x_0, X], \quad y(x_0) = y_0, \quad f: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m. \quad (4)$$

In ordinary differential equations where $y(x)$ and $f(x, y)$ may be vectors. We seek the numerical solution of this problem using parameter dependent nested hybrid multistep methods (PDNHLMMs) and its hybrid counterpart obtained by the transformation of the continuous linear multistep method of [11]. Such formulae are the adaptive nested Runge - Kutta (ANRK) methods [9], hybrid multistep formulae with nested hybrid predictors [5], second derivative linear multistep with nested predictors [6] which was introduced to overcome the limitation of order reduction that is often encountered during the implementation process in some diagonally and fully implicit Runge-Kutta methods and also to overcome the limitation encountered by nested method in the area of long execution time when applied to large scale initial value problems. These formulae developed did very well in extensive numerical computations of stiff ODEs. The interest of this paper is to introduce an High Stiffly Stable parameter dependent nested hybrid linear multistep method for the solution of the IVPs (1).

2 Order conditions

By Taylor's series technique, the local truncation error of the stage method and the output method in (3) at x_n gives the following order conditions:

$$h^{q_0} : \begin{cases} \sum_{j=0}^k \phi_j = 1, & q_0 = 0, \\ \sum_{j=1}^k j \phi_j + \lambda_k = c_0, & q_0 = 1, \\ \frac{1}{q_0!} \sum_{j=1}^k j^{q_0} \phi_j + \frac{1}{(q_0-1)!} k^{q_0-1} \lambda_k = \frac{c_0^{q_0}}{q_0!}; & q_0 = 2, 3, \dots, \end{cases} \quad (5a)$$

The error constant of the method in (5a) is

$$C_{q_0+1} = \frac{1}{(q_0+1)!} \sum_{j=0}^k j^{q_0+1} \phi_j + \frac{1}{q_0!} k^{q_0} \lambda_k = \frac{c_i^{q_0+1}}{(q_0+1)!} + O(h^{q_0+1}). \quad (5b)$$

Equations (5a) and (5b) are the order conditions and error constants of the first input method in (3) while the order conditions and error constants of the second input method in (3) are

$$h^{q_i} : \begin{cases} \sum_{j=0}^k \varphi_j + \rho_0 = 4 + c_1, & i = 1, 2, \dots, s, \quad i = 0(1)s - 1, \\ \sum_{j=1}^k j \varphi_j + c_0 \rho_i = 8 - \frac{c_i^2}{2!}, & q_i = 1, \\ \frac{1}{q_i!} \sum_{j=1}^k j^{q_i} \varphi_j + \rho_0 c_0^{q_i-1} = -\frac{c_0^{q_i}}{q_i!} + \frac{64}{q_i!}; & q_i = 2, 3, \dots, \end{cases}$$

(6a)

The error constant of the method in (3) is

$$C_{q_i+1} = \frac{1}{(q_i+1)!} \sum_{j=0}^{k-1} j^{q_i+1} \varphi_j + \rho_0 c_0^{q_i} = \frac{c_i^{q_i+1}}{(q_i+1)!} + \frac{64}{(q_i+1)!} O(h^{q_i+1}).$$

(6b)

For the output method in (3), the order conditions and error constants are

$$h^p : \begin{cases} \sum_{j=0}^{k-1} \delta_j + \gamma_k = 1, & p = 0, \\ \sum_{j=1}^{k-1} j \delta_j + \gamma_k (5 + a) = c_4, & p = 1, \\ \frac{1}{p!} \sum_{j=1}^{k-1} j^p \delta_j + \frac{1}{(p-1)!} \theta_{s-1} c_{s-1}^{p-1} + \frac{1}{(p-1)!} (a(-1+k)^{p-1} + k^{p-1}) \gamma_k = \frac{k^p}{p!}; & p = 2, 3, \dots, \end{cases}$$

(7a)

and

$$C_{p+1} = \frac{1}{(p+1)!} \sum_{j=1}^{k-1} j^{p+1} \delta_j + \frac{1}{p!} \theta_{s-1} c_{s-1}^p + \frac{1}{p!} (a(-1+k)^p + k^p) \gamma_k = \frac{k^{p+1}}{(p+1)!} + O(h^{p+1})$$

(7b)

Derivation of the multistep nested hybrid methods

$k=1 \ s=1 \ p=q+1=3 \ c_0=\frac{1}{2} \ a=\frac{1}{2}$			
Y_0	$\frac{1}{4}(y_n+3y_{n+1})-\frac{h}{4}f_{n+1};$	$q_0=2,$	$C_3=\frac{1}{48}$
y_{n+1}	$y_{n+1}=y_n+hf(Y_0)$	$p=2,$	$C_3=\frac{1}{24}$
$k=2 \ s=2 \ p=q_1=q_0+1 \ c_0=\frac{7}{4} \ c_1=\frac{3}{2} \ a=\frac{1}{4}$			
Y_0	$\frac{1}{256}(-3y_n+28y_{n+1}+231y_{n+2})-\frac{21h}{128}f_{n+2};$	$C_4=\frac{7}{2048},$	$q_0=3$
Y_1	$y_{n+1}+\frac{1}{96}(-hf_n+30hf_{n+1}-13hf_{n+2}+32hf(Y_0));$	$C_5=\frac{179}{92160},$	$q_1=4$
y_{n+2}	$y_{n+2}=-\frac{1}{23}y_n+\frac{24}{23}y_{n+1}-\frac{56}{69}hf(Y_1)+\frac{8}{69}h(f_{n+2}+af_{n+1})$	$C_4=\frac{1}{84},$	$p=3.$
$k=3 \ s=3 \ p=q_2=q_1=q_0+1=5 \ c_0=\frac{23}{8} \ c_1=\frac{11}{4} \ c_2=\frac{5}{2} \ a=\frac{1}{2}$			
Y_0	$\frac{1}{49152}(70y_n-48328y_{n+1}+2070y_{n+2}+47495y_{n+3}-4830hf_{n+3}); \ q_0=4,$	$C_5=\frac{161}{262144}$	$q_0=4,$
Y_1	$y_{n+2}+\frac{1143}{10496}hf_n-\frac{1983}{51200}hf_{n+1}+\frac{30549}{71680}hf_{n+2}-\frac{4977}{10240}hf_{n+3}+\frac{3396h}{824025}f(Y_0);$	$C_6=\frac{681}{409600},$	$q_1=5,$
Y_2	$y_{n+2}+\frac{179}{63360}hf_n-\frac{319}{13440}hf_{n+1}+\frac{1979}{5760}hf_{n+2}-\frac{601}{5760}hf_{n+3}+\frac{976h}{3465}f(Y_1);$	$C_6=\frac{41}{46080},$	$q_2=5,$
y_{n+3}	$y_{n+2}-\frac{1}{150}hf_n-\frac{1}{30}hf_{n+1}+\frac{58}{75}hf(Y_2)-\frac{2}{15}h(f_{n+3}+af_{n+2});$	$C_5=\frac{11}{2880},$	$p=4,$

Using the general form of the k -step High Order Stiffly Stable parameter dependent nested hybrid linear multistep methods in (3) above, the method for k_{is} was derived by applying Taylor's series scheme to the expansion, by fixing k_{is} for every $i = 1(n)6$, into (5), (6) and (7) respectively, we have by setting

$k=4 \ s=4 \ p=q_3=q_2=q_1=q_0+1=6 \ c_0=\frac{63}{16} \ c_1=\frac{23}{8} \ c_2=\frac{15}{4} \ c_3=\frac{7}{2} \ a=\frac{1}{2}$			
Y_0	$-\frac{7.3 \times 10^3}{3.4 \times 10^7} y_n + \frac{3.3 \times 10^3}{2.1 \times 10^6} y_{n+1} - \frac{4.4 \times 10^4}{8.4 \times 10^6} y_{n+2} + \frac{3.1 \times 10^4}{2.1 \times 10^6} y_{n+3} + \frac{3.3 \times 10^7}{3.3 \times 10^7} y_{n+4} - \frac{4.6 \times 10^5}{8.4 \times 10^6} hf_{n+4};$	$C_6 = \frac{3.1 \times 10^4}{2.7 \times 10^8}$	$q_0 = 5,$
Y_1	$y_{n+3} - \frac{1.3 \times 10^5}{5.3 \times 10^7} hf_n + \frac{6.7 \times 10^5}{3.4 \times 10^7} hf_{n+1} - \frac{2.3 \times 10^6}{3.0 \times 10^7} hf_{n+2} + \frac{2.7 \times 10^4}{5.5 \times 10^4} hf_{n+3} - \frac{6.8 \times 10^6}{5.8 \times 10^6} hf_{n+4} + \frac{9.4 \times 10^5}{5.9 \times 10^5} hf(Y_0);$	$C_7 = \frac{2.0 \times 10^7}{1.8 \times 10^{10}},$	$q_1 = 6,$
Y_2	$y_{n+3} - \frac{6.8 \times 10^2}{3.2 \times 10^5} hf_n + \frac{3.9 \times 10^3}{2.3 \times 10^5} hf_{n+1} - \frac{6.7 \times 10^2}{1.0 \times 10^4} hf_{n+2} + \frac{3.3 \times 10^4}{7.2 \times 10^4} hf_{n+3} - \frac{5.4 \times 10^2}{1.3 \times 10^3} hf_{n+4} + \frac{1.9 \times 10^4}{2.5 \times 10^4} hf(Y_1);$	$C_7 = -\frac{3.5 \times 10^4}{3.7 \times 10^7},$	$q_2 = 6,$
Y_3	$y_{n+3} - \frac{41}{34560} hf_n + \frac{589}{63360} hf_{n+1} - \frac{131}{3360} hf_{n+2} + \frac{6347}{17280} hf_{n+3} - \frac{997}{11520} hf_{n+4} + \frac{520}{2079} hf(Y_2);$	$C_7 = -\frac{3.5 \times 10^{-4}}{3.7 \times 10^{-7}},$	$q_3 = 6,$
y_{n+4}	$y_{n+3} + \frac{11}{3368} hf_n + \frac{448}{1575} hf_{n+1} + \frac{234}{315} hf_{n+2} + \frac{8328}{11025} hf(Y_3) + \frac{179}{1260} h(f_{n+4} + a f_{n+3})$	$C_6 = \frac{32}{35151},$	$p = 5$
$k=5 \ s=5 \ p=q_4=q_3=q_2=q_1=q_0+1=7 \ c_0=\frac{159}{32} \ c_1=\frac{79}{16} \ c_2=\frac{39}{8} \ c_3=\frac{19}{4} \ c_4=\frac{9}{2} \ a=\frac{1}{2}$			
Y_0	$\frac{1.6 \times 10^6}{4.2 \times 10^{10}} y_n - \frac{9.8 \times 10^6}{3.4 \times 10^{10}} y_{n+1} + \frac{4.4 \times 10^6}{4.3 \times 10^9} y_{n+2} - \frac{2.0 \times 10^7}{8.6 \times 10^9} y_{n+3} + \frac{4.1 \times 10^7}{8.6 \times 10^9} y_{n+4}$ $+ \frac{1.7 \times 10^{11}}{1.7 \times 10^{11}} y_{n+5} - \frac{2.5 \times 10^8}{8.6 \times 10^9} hf_{n+5};$	$C_7 = \frac{1.2 \times 10^7}{5.5 \times 10^{11}}$	$q_0 = 6$
Y_1	$y_{n+4} + \frac{1.5 \times 10^8}{9.9 \times 10^{10}} hf_n - \frac{2.9 \times 10^9}{2.4 \times 10^{11}} hf_{n+1} + \frac{8.3 \times 10^8}{1.8 \times 10^{10}} hf_{n+2} - \frac{2.4 \times 10^9}{2.0 \times 10^{10}} hf_{n+3} + \frac{3.2 \times 10^{10}}{5.8 \times 10^{10}} hf_{n+4}$ $- \frac{4.5 \times 10^9}{1.9 \times 10^9} hf_{n+5} + \frac{1.6 \times 10^9}{5.8 \times 10^8} hf(Y_0);$	$C_7 = \frac{2.2 \times 10^{10}}{2.7 \times 10^{13}}$	$q_1 = 7$

Y_2	$y_{n+4} + \frac{2.1 \times 10^7}{1.5 \times 10^{10}} hf_n - \frac{1.9 \times 10^7}{1.7 \times 10^9} hf_{n+1} + \frac{1.9 \times 10^8}{4.4 \times 10^9} hf_{n+2} - \frac{3.3 \times 10^8}{2.9 \times 10^9} hf_{n+3} + \frac{3.0 \times 10^8}{5.7 \times 10^8} hf_{n+4}$ $- \frac{1.9 \times 10^8}{1.9 \times 10^8} hf_{n+5} + \frac{6.8 \times 10^7}{4.7 \times 10^7} hf(Y_1);$	$C_8 = \frac{2.2 \times 10^8}{2.9 \times 10^{11}}$	$q_2 = 7$
Y_3	$y_{n+4} + \frac{3.5 \times 10^4}{3.0 \times 10^7} hf_n - \frac{6.7 \times 10^5}{7.1 \times 10^7} hf_{n+1} + \frac{9.5 \times 10^5}{2.6 \times 10^7} hf_{n+2} - \frac{2.2 \times 10^4}{2.3 \times 10^5} hf_{n+3} + \frac{8.0 \times 10^6}{1.6 \times 10^7} hf_{n+4}$ $- \frac{8.6 \times 10^5}{2.3 \times 10^6} hf_{n+5} + \frac{1.6 \times 10^6}{2.3 \times 10^6} hf(Y_2);$	$C_8 = \frac{1.2 \times 10^6}{2.1 \times 10^9}$	$q_3 = 7,$
Y_4	$y_{n+4} + \frac{7.6 \times 10^2}{1.2 \times 10^6} hf_n - \frac{4.9 \times 10^3}{9.8 \times 10^5} hf_{n+1} + \frac{3.5 \times 10^4}{1.8 \times 10^6} hf_{n+2} - \frac{6.3 \times 10^4}{1.1 \times 10^6} hf_{n+3}$ $+ \frac{3.7 \times 10^5}{9.7 \times 10^5} hf_{n+4} - \frac{2.4 \times 10^4}{3.2 \times 10^5} hf_{n+5} + \frac{6.3 \times 10^4}{2.8 \times 10^5} hf(Y_3)$	$C_7 = \frac{1.3 \times 10^4}{4.3 \times 10^7}$	$q_4 = 7$
y_{n+5}	$y_{n+4} + \frac{12}{7290} hf_n + \frac{83}{7560} hf_{n+1} + \frac{35}{1080} hf_{n+2} + \left(\frac{589}{1080} - \frac{79}{162}\right) hf_{n+3} + \frac{19032}{25515} hf(Y_4)$ $- \frac{79}{540} h(f_{n+5} + a f_{n+4});$	$C_7 = \frac{373}{362880},$	$p = 6$
$k = 6, s = 6, p = q_5 = q_4 = q_3 = q_2 = q_1 = q_0 + 1 = 8, c_0 = \frac{383}{64}, c_1 = \frac{191}{32}, c_2 = \frac{95}{16}, c_3 = \frac{47}{8}, c_4 = \frac{23}{4}, c_5 = \frac{11}{2}, a = \frac{1}{4},$			
Y_0	$- \frac{9.2 \times 10^8}{1.4 \times 10^{13}} y_n + \frac{9.9 \times 10^9}{1.7 \times 10^{14}} y_{n+1} - \frac{6.2 \times 10^{10}}{2.8 \times 10^{14}} y_{n+2} + \frac{9.2 \times 10^9}{1.8 \times 10^{13}} y_{n+3} - \frac{1.2 \times 10^{11}}{1.4 \times 10^{14}} y_{n+4}$ $+ \frac{5.0 \times 10^{10}}{3.5 \times 10^{13}} y_{n+5} - \frac{1.4 \times 10^{15}}{1.4 \times 10^{15}} y_{n+6} - \frac{1.1 \times 10^{12}}{7.0 \times 10^{13}} hf_{n+6};$	$C_8 = \frac{1.5 \times 10^{11}}{3.6 \times 10^{16}}$	$q_0 = 7$

Y_1	$y_{n+5} - \frac{1.6 \times 10^{14}}{1.6 \times 10^{17}} hf_n + \frac{6.3 \times 10^{13}}{7.4 \times 10^{15}} hf_{n+1} - \frac{1.4 \times 10^{14}}{3.9 \times 10^{15}} hf_{n+2} + \frac{8.6 \times 10^{15}}{9.9 \times 10^{16}} hf_{n+3} - \frac{4.9 \times 10^{15}}{2.9 \times 10^{16}} hf_{n+4}$ $+ \frac{2.3 \times 10^{13}}{3.9 \times 10^{13}} hf_{n+5} - \frac{9.9 \times 10^{15}}{2.1 \times 10^{15}} hf_{n+6} - \frac{1.1 \times 10^{13}}{2.1 \times 10^{12}} hf(Y_0);$	$C_9 = \frac{5.8 \times 10^{17}}{1.0 \times 10^{21}}$	$q_1 = 8$
Y_2	$y_{n+5} - \frac{2.2 \times 10^{10}}{2.3 \times 10^{13}} hf_n + \frac{2.6 \times 10^{10}}{3.2 \times 10^{12}} hf_{n+1} - \frac{5.1 \times 10^{11}}{1.5 \times 10^{13}} hf_{n+2} + \frac{4.8 \times 10^{10}}{5.7 \times 10^{11}} hf_{n+3}$ $- \frac{8.3 \times 10^9}{5.2 \times 10^{10}} hf_{n+4} + \frac{1.1 \times 10^{12}}{1.9 \times 10^{12}} hf_{n+5} - \frac{2.7 \times 10^{11}}{1.2 \times 10^{11}} hf_{n+6} + \frac{4.3 \times 10^{10}}{1.6 \times 10^{10}} hf(Y_1);$	$C_9 = \frac{5.4 \times 10^{11}}{9.9 \times 10^{14}}$	$q_2 = 8$
Y_3	$y_{n+5} - \frac{7.8 \times 10^7}{9.1 \times 10^{10}} hf_n + \frac{6.1 \times 10^8}{8.0 \times 10^{10}} hf_{n+1} - \frac{6.2 \times 10^7}{2.0 \times 10^9} hf_{n+2} + \frac{1.7 \times 10^{10}}{2.1 \times 10^{11}} hf_{n+3} - \frac{9.5 \times 10^9}{6.2 \times 10^{10}} hf_{n+4}$ $+ \frac{2.8 \times 10^9}{5.0 \times 10^9} hf_{n+5} - \frac{1.7 \times 10^{10}}{1.8 \times 10^{10}} hf_{n+6} + \frac{1.2 \times 10^7}{8.6 \times 10^6} hf(Y_2);$	$C_9 = \frac{2.8 \times 10^{11}}{5.6 \times 10^{14}}$	$q_3 = 8$
Y_4	$y_{n+5} - \frac{1.2 \times 10^6}{1.7 \times 10^9} hf_n + \frac{1.5 \times 10^6}{2.4 \times 10^8} hf_{n+1} - \frac{2.8 \times 10^7}{1.1 \times 10^9} hf_{n+2} + \frac{1.3 \times 10^7}{2.1 \times 10^8} hf_{n+3} - \frac{4.7 \times 10^6}{3.7 \times 10^7} hf_{n+4}$ $+ \frac{1.9 \times 10^6}{3.7 \times 10^7} hf_{n+5} - \frac{2.5 \times 10^6}{7.3 \times 10^6} hf_{n+6} + \frac{9.9 \times 10^6}{1.5 \times 10^7} hf(Y_3);$	$C_9 = \frac{3.7 \times 10^7}{9.4 \times 10^{10}}$	$q_4 = 8$
Y_5	$y_{n+5} - \frac{1.3 \times 10^4}{3.6 \times 10^7} hf_n + \frac{5.4 \times 10^3}{1.6 \times 10^6} hf_{n+1} - \frac{1.2 \times 10^4}{8.6 \times 10^5} hf_{n+2} + \frac{6.8 \times 10^4}{1.9 \times 10^6} hf_{n+3} - \frac{6.3 \times 10^4}{8.6 \times 10^5} hf_{n+4} + \frac{1.7 \times 10^5}{4.5 \times 10^5} hf_{n+5}$ $- \frac{5.1 \times 10^5}{7.7 \times 10^6} hf_{n+6} + \frac{6.3 \times 10^4}{2.8 \times 10^5} hf(Y_4)$	$C_9 = \frac{3.0 \times 10^6}{1.5 \times 10^{10}}$	$q_5 = 8$
y_{n+6}	$y_{n+5} - \frac{1279}{850080} hf_n - \frac{23597}{2086560} hf_{n+1} + \frac{815}{90160} hf_{n+2} - \frac{1079}{38640} hf_{n+3} + \left(\frac{24487}{30240} + \frac{3660}{5152} \right) hf_{n+4}$ $+ \frac{3861776}{5020785} hf(Y_5) + \frac{732}{5152} h(f_{n+6} + a f_{n+5});$	$C_8 = \frac{-1177}{947520}$	$P=7$

The stability region of the formula in (3) depends on the choice of the value of the parameter “a” which has the function of smoothening the method for better stability properties. It is obvious that during the application of the methods for the numerical integration of stiff IVPs, errors are bound to occur at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during the computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method [2]. The minimum properties a numerical method could possess are stability properties [14].

Some of the advantages of this work are that It is of high order and stiffly stable and it has a better stability function. The better stability function of the method was made possible because of the present of a parameter “a” which lies between 0 and 1 and it help to obtain a better stability function thereby leading to a smooth stability curve.

Applying the method in (3) to the test equation $y' = \lambda y$, $\text{Re}(\lambda) < 0$, gives the stability polynomial

$$\pi_s(r, z) = w^k - w^{k-1} - z \sum_{j=0}^{k-2} \delta_j w^j - z \theta_{s-1}(R_s(w, z)) - z \gamma_s(w^k + a w^{k-1}), \quad z = \lambda h,$$

(8)

where

$$R_s(w, z) = w^{k-1} + z \sum_{j=0}^k \varphi_j w^j + z \rho_{s-2}(R_{s-1}(w, z)(R_{s-2}(w, z)(R_{s-3}(w, z)(R_{s-4}(w, z)(R_{s-5}(w, z))))))$$

(9)

The behavior of the numerical solution of the High Order stiffly Stable parametric dependent nested hybrid linear multistep methods will depend on the resulting roots

$(w_j(z))$ of $\pi(r, z) = 0$. Adopting the definitions in [13], we have:

$R_0 = \left\{ \pi(w, 0) = 0 \mid \max_{1 \leq j \leq k} |w_j(z)| \leq 1, j = 1, 2, \dots, k \right\}$ is called the zero-stability of the methods in (3).

$R_a = \left\{ \pi(w, z) = 0 \mid \max_{1 \leq j \leq k} |w_j(z)| \leq 1, j = 1, 2, \dots, k \right\}$ is called the absolute stability of the method in

(3).

The methods in (3) with stability polynomial $\pi(w, z)$ is A(α)-stable if $|\pi(w, z)| \leq 1$ for all $z \in S(\alpha)$, $\alpha \in (0, \pi)$, where α is an angle of absolute stability.

$R_A = \{ |\pi(w, z)| \leq 1; R(z) \leq 0 \}$ is called A-stability of the method (3).

The stability plots of the methods in (3) for $k \leq 6$ are given in Figure 1.

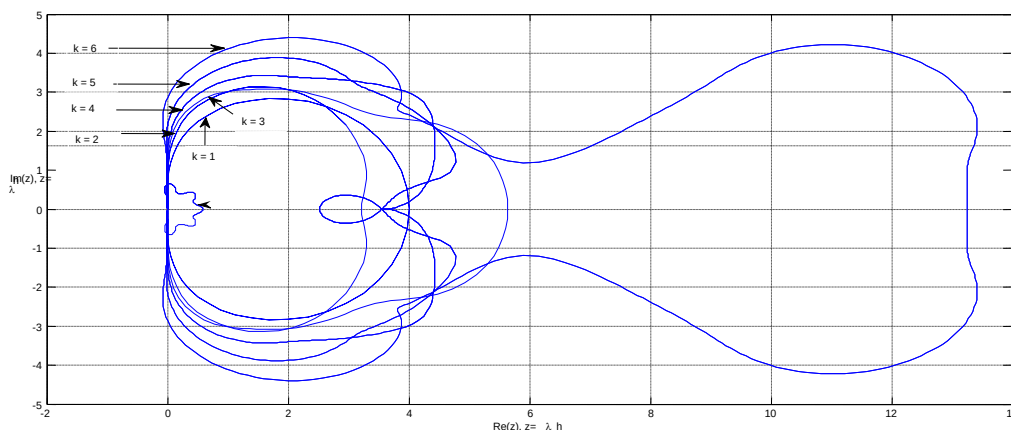


Figure 1: The stability regions (exterior of the closed curves) for the k -step PDNHM method in (2); $k \leq 6$.

3 Numerical experiments

In this section, we consider the application of the proposed method in scheme for $k=1$

$$Y_0 = \frac{1}{4}(y_n + 3y_{n+1}) - \frac{h}{4}f_{n+1}; \quad q_0 = 2, \quad C_3 = \frac{1}{48},$$

$$y_{n+1} = y_n + hf(Y_0); \quad p = 2, \quad C_3 = \frac{1}{24}.$$

(10)

which is of order 3, and also independently derived using Taylor's series expansion on the initial value problem below;

$$y' = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad h = 0.001,$$

$$x = [0, 1], \quad y = [y_1, y_2, y_3, y_4]^T$$

$$\text{Exact solution: } y(x) = [\exp^{-0.1x}, \exp^{-10x}, \exp^{-100x}, \exp^{-1000x}]^T$$

with $x = 0(0.1)2000$. This problem is stiff. Since $S = 10000$, where S is the stiffness ratio.

In solving the IVPs above, the implicitness in the methods has to be resolved using the Newton Raphson iterative scheme below as suggested by [5] and [6],

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - (F'(y_{n+k}^{[s]}))^{-1} F(y_{n+k}^{[s]}), \quad s = 0, 1, \dots$$

(11)

where $F'(y_{n+k}^{[s]})$ is the Jacobian matrix. While an explicit one-step method

$$y_{n+1}^{[0]} = y_n + \frac{h}{2}(f_{n-1} + f_n).$$

(12)

is used to generate the starting value for the schemes in (10).

Table 1: The Numerical Solution of the Problem on PDNHLMM compared with A(α)-Stable LMM

x	PDNHLMM y_n	A(α)-Stable LMM y_n	Exact Solution $y_{(x)}$	PDNHLMM Error = $ y_{(x)} - y_n $	A(α)-Stable LMM Error = $ y_{(x)} - y_n $
0.2	$2.2569438502 \times 10^{-9}$	2.1190578206	$2.2779270412 \times 10^{-9}$	$2.0983190958 \times 10^{-11}$	$1.9920072216 \times 10^{-16}$
0.4	$5.0959769196 \times 10^{-18}$	5.1315793862	$4.6951575726 \times 10^{-18}$	$4.0081934701 \times 10^{-19}$	$5.0021074282 \times 10^{-16}$
0.6		1.1511722549	$9.6774410387 \times 10^{-27}$	$1.8288180836 \times 10^{-27}$	$9.6663711108 \times 10^{-16}$
0.8	$2.5980101770 \times 10^{-35}$	2.5960009977	$1.9946692652 \times 10^{-35}$	$6.0334091172 \times 10^{-36}$	$4.7122272259 \times 10^{-16}$
1.0	$5.8660741151 \times 10^{-44}$	5.8571556981	$4.1113197817 \times 10^{-44}$	$1.7547543333 \times 10^{-44}$	$3.9931006410 \times 10^{-16}$

The graphical representation of the results from the problem is shown in Figure 2 below.

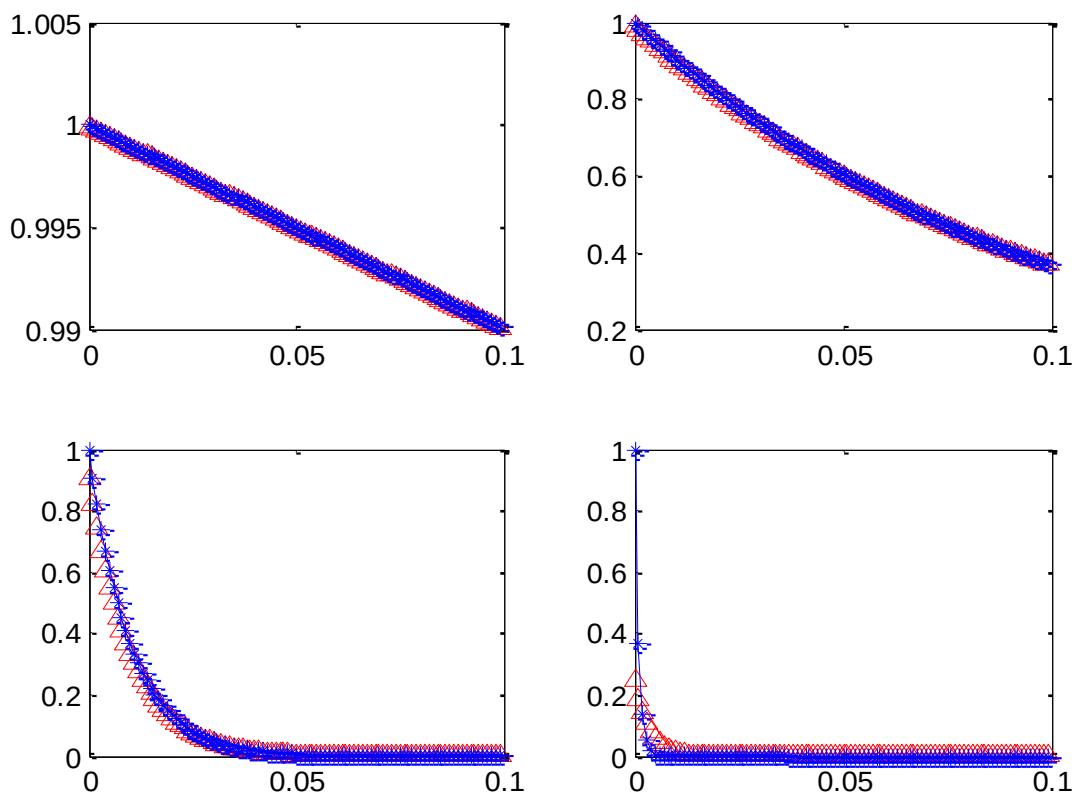


Figure 2

3.1 Discussion of results

Table 1: A careful look at the results from the new methods on the problem solved as compared with the results $A(\alpha)$ -Stable LMM with the step-size $h = 0.001$ used for the problem for $x \in \{0, 1\}$. It reveals that the new methods on the problem solved have a very low error level as compared to the $A(\alpha)$ -Stable LMM. This shows that the new methods (PDNHLMMs) are much more accurate, effective and efficient when solving stiff IVPs in ODEs.

Figure 2: Graphical Representation (Blue Represent Numerical Solutions while Red represents Exact Solutions). Unlike the Euler and RKM, the consistency and convergence of our method can be ascertained during the implementation process. A linear multistep is consistent if the numerical solution converges to that of the exact solution and also, if the method is able to solve exactly two or more ODE problems with different initial conditions. A linear multistep is convergent if and only if it is stable and consistent. A careful look at the graphs of our methods shows that our methods are consistent and convergent as the numerical solutions converge to that of the exact solutions. This is a proof that our methods (PDNHLMMs) compared favorably well with other existing methods.

4 Conclusion

This paper proposes and studies the high Order Stiffly Stable parameter dependent nested hybrid linear multistep methods for the numerical solution of the IVPs (1). We derived the order conditions for methods via Taylor's series expansion; based on the order conditions presented in this paper, hybrid methods for step number $k \leq 6$ were derived. The results of the numerical experiments confirm that the results of our new methods are more accurate than the high quality methods in the scientific literature.

Following this work, there are several interesting modifications to the new methods in (3). For example, a classical modification of the methods can lead to the well-known GLMs [11] of higher order than the existing methods in the literature.

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Re-emergence of Lassa fever disease: the theory of interepidemic effect in action

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Abstract

The Lassa fever virus is one of several viruses that are likely to cause a future epidemic as reported by WHO due to re-emergence. The re-emergence of Lassa fever is due to the phenomenon called interepidemic effect. According to the current status of the interepidemic effect, any public health control measure taken to reduce the basic reproduction number will definitely make the disease to take a very long time to re-emerge. This short expository note establishes the fact that the re-emergence of Lassa fever disease is actually the theory of interepidemic effect in action.

Keywords: Lassa fever; mathematical modeling; interepidemic effect

1 Introduction

Lassa fever (LF) is a serious and potentially deadly viral disease caused by Lassa virus (LASV) that is prevalent in West Africa including Benin, Sierra Leone, Guinea, Liberia, Mali, and Nigeria because LASV reservoir and vector (*Mastomys natalensis*) is abundantly seen in these areas [6]. LASV has 2-21 day or more than that incubation period, thus it increases the risk of its transmission from endemic to non-endemic regions [13]. One of the major concerns with LASV is its high case fatality rate, additionally, the virus can easily spread from human-human contact, which makes it even more dangerous. Another major concern with LASV is the lack of effective vaccines and therapeutics for treating the disease. This is further compounded by the fact that the virus can also spread through aerosol, making it even more difficult to contain [3].

Due to the severe nature of LASV and its potential to spread easily, it is classified as a Biosafety Level 4 agent. This means that it is considered to be one of the most dangerous pathogens known to mankind. Despite this, the overall impact of LASV on the increased threat of viral diseases in West Africa has not been fully understood [2]. However, it is known that LASV is a significant cause of yearly morbidity and mortality in many of Africa's poverty-stricken communities such as during 2022 in Nigeria around 189 deaths reported out of 1067 confirmed cases of LASV. Annually, around 1–3 million infections occur of LF with around 5000 deaths [11].

Lassa virus is a public health security issue. Currently, there is no specific vaccine to prevent from Lassa fever. The antiviral drug ribavirin is used to treat the LF but it is not too much effective. However, researchers are working on developing a vaccine that would provide immunity against the

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virus. One approach being explored is to use a live attenuated version of the virus, which has been genetically modified to be less virulent. Another approach being studied is the use of a subunit vaccine, which consists of pieces of the virus that are able to stimulate an immune response but do not cause illness [7, 2].

Researchers should develop more sensitive and specific diagnostic tools as well as the development of point-of-care diagnostic devices that can be used in resource-limited settings [8]. The development of new antiviral drugs and understanding the epidemiology of Lassa fever is also crucial for controlling and preventing outbreaks. This includes identifying the main vectors and reservoirs of the virus, as well as understanding the risk factors that contribute to the spread of the disease. The Governments of endemic regions should arrange community education and awareness programs to individuals living in or traveling to areas where Lassa fever is endemic should take precautions to avoid contact with infected rats and practice good hygiene to prevent person-to-person transmission of the virus [1].

The incorporation of mathematical and theoretical reasoning into epidemiology, however, has been one of the field's most significant advancements. It is also a superb illustration of how mathematics can be used in practical situations. Lassa fever control is still a problem that needs mathematical answers on a global scale. The current application of mathematical modeling has found interesting new applications in epidemiology [4]. By mathematical model we mean a mathematical representation of a system that can be used to explore its behaviour. A mathematical model is a description of a system using mathematical concepts and language. Mathematical models can help public health epidemiologists understand the spread of disease, and evaluate the potential effectiveness of different control measures to be used based on mathematical analysis [10].

In recent times, several mathematical models have been formulated and analyzed in the literature to study the transmission and spread of Lassa fever. Motivated by the previous studies, [10] presented a novel mathematical model of Lassa fever that takes into account all the transmission modes (rodent-human, human-human and aerosol transmission) of the disease. They proposed that the spread of LF infection should depend on the number of susceptible, exposed, infective and recovered coupled with SIR model for rodent dynamics. Their model accounted for all the known transmission routes of LF disease in humans. They suggested a mass action law for the rate of new infections, and this idea has been basic in compartmental models since late nineteenth century. For the Obasi-Mbah model, it is easy to verify that the basic reproduction number denoted by R_0 , is given as:

$$R_0 = \frac{\beta_2 \kappa \epsilon \Lambda_h}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta)} + \frac{\beta_3 \vartheta \rho \Lambda_r}{\mu_r (\omega + \mu_r)}, \quad (1)$$

accounting for the human transmission and rodent transmission, respectively. The explicit definition of R_0 is that it is the expected number of disease cases produced by a typical infected individual in a wholly susceptible population over the full course of the infectious period. Based on the parameter values obtained by NCDC, the estimated value of R_0 ranges between 1.02 to 13.6 in Nigeria with negligible contribution of aerosol transmission [10]. On average, the geometric mean (GM) between these two extreme numbers is given as:

$$GM = \sqrt{1.02 \times 13.6} \approx 3.7$$

$$\therefore R_0 \approx 3.7 \quad (2)$$

This goes to show that Nigeria is still an endemic region for Lassa fever disease. Re-emerging diseases are diseases that reappear after they have been on a significant decline. Re-emergence may happen because of a breakdown in public health measures for diseases that were once under control. They can also happen when new strains of known disease-causing organisms appear. Human behaviour

affects re-emergence. More importantly, it is believed that re-emergence of infectious diseases is due to the phenomenon called interepidemic effect [10].

2 The theory of interepidemic effect

In 2019, Obasi and Mbah derived the inverse-square relationship between the basic reproduction number and interepidemic period of infectious disease, now referred to as Interepidemic Effect [10]. This is a feature of effective epidemic control measure. The basic reproduction number is used to measure the transmission potential of a disease. By basic reproduction number we mean the expected number of disease cases produced by a typical infected individual in a wholly susceptible population over the full course of the infectious period [5]. The goal of infectious-disease intervention is reducing the basic reproduction number to below one, because such a value means that new infections are in decline and will eventually reach zero. Interepidemic period of disease means occurring between epidemics of disease. In unforced susceptible, exposed, infectious, recovered (SEIR) models of directly transmitted diseases, incidence is predicted to exhibit damped oscillations with an approximate interepidemic period (T) [10]:

$$T = 2\pi\sqrt{AD} \quad (3)$$

where A is the average age of first infection, D is the sum of latent interval and the infectious interval of the disease. Recall that the average age of first infection, A is inversely related with the basic reproduction number, R_0 [9].

$$\begin{aligned} A &= \frac{\kappa_1}{R_0} \\ \Rightarrow \frac{T^2}{4\pi^2} &= AD = \frac{\kappa_1 D}{R_0} \Rightarrow T^2 R_0 = 4\pi^2 \kappa_1 D = \kappa_2 \\ \Rightarrow R_0 &= \frac{\kappa_2}{T^2} \Rightarrow R_0 \propto \frac{1}{T^2} \end{aligned} \quad (4)$$

This indicates that the basic reproduction number is inversely proportional to the square of interepidemic period of an infectious disease. This implies that increase in R_0 would decreased T^2 . This is illustrated in Table 1 as well as in Figure 1 below using the hypothetical data for the basic reproduction number.

Table 1: Estimation of parameters in interepidemic effect

R_0	k_0	T^2	T	$(1/T)^2$
1.02	3	2.941	1.714934	0.34
2.52	3	1.19	1.090871	0.84
4.02	3	0.746	0.863713	1.34
5.52	3	0.543	0.736885	1.842
7.02	3	0.427	0.653452	2.342
8.52	3	0.352	0.593296	2.841
10.02	3	0.299	0.546809	3.344
11.52	3	0.26	0.509902	3.846
13.02	3	0.23	0.479583	4.348
14.52	3	0.207	0.454973	4.831
16.02	3	0.187	0.432435	5.348

Source: Author's computation (2023)

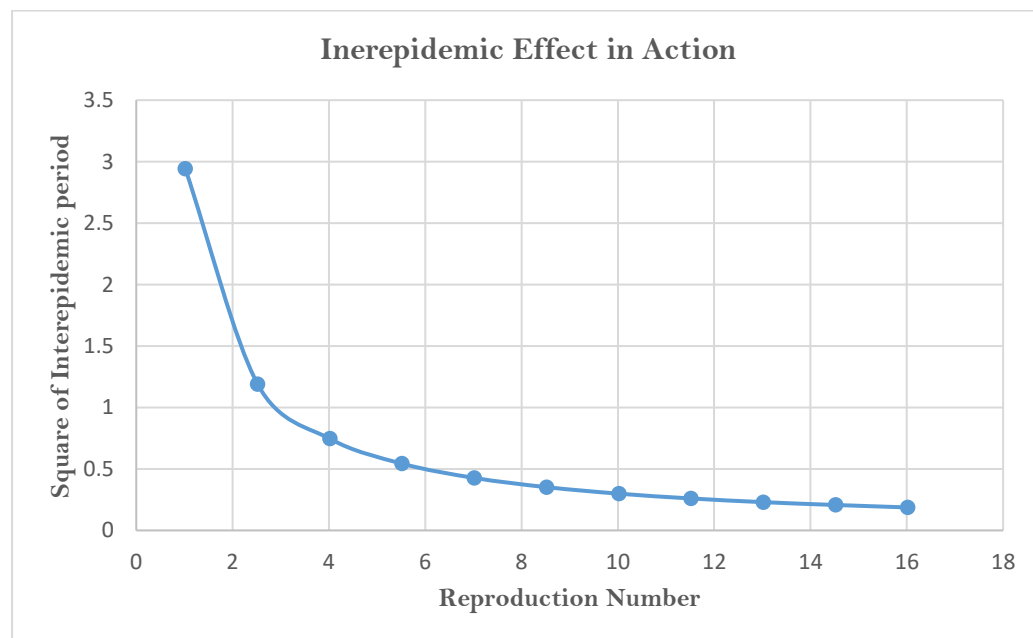


Figure 1: Plot of interepidemic effect in action

Figure 1 shows the relationship describing interepidemic effect. It means that, any public health policy or measures taken to reduce the basic reproduction number will definitely make the disease to take a very long time to re-emerge [10]. This by implication goes to show that introduction of effective control measures will lead to a very long time for the re-emergence of infectious disease. This is a feature of effective epidemic control measure now referred to as interepidemic effect. The interepidemic effect shows that the basic reproduction number in epidemic theory is essentially an important measure used in infectious disease control, immunization and eradication programmes. It should also make you think about what we really mean by the effective control measure. This corroborates the assertion that re-emergence of infectious diseases may happen because of a breakdown in public health measures for diseases that were once under control.

In the interepidemic effect, re-emergence of an infectious disease depends on the effectiveness of the control measures. This goes to show that to achieve sustainable disease control, NCDC is to be aware of the importance of this effect, which in turn helps to achieve Sustainable Development Goals 3 [12]. It is therefore one of the phenomena proving the existence of sustainable disease control. This is a fundamental result in mathematical epidemiology that addresses the issue of re-emergence of infectious diseases. Recall that re-emerging infectious diseases are diseases that once were major health problems globally or in a particular country, and then declined dramatically, but are again becoming health problems for a significant proportion of the population (Lassa fever, malaria and tuberculosis are examples). Among other factors, interepidemic effect has contributed to the understanding of emergence and re-emergence of infectious diseases. This implies that interepidemic effect is used in achieving sustainable control of infectious diseases. Therefore, there should be policy development by NCDC to implement the theory of interepidemic effect. However, this short expository note establishes the fact that the re-emergence of Lassa fever disease is actually the theory of interepidemic effect in action.

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Modeling and numerical analysis of saturation incidence and saturated treatment of dental caries infection

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Abstract

Dental caries are holes in teeth that form when acid in the mouth erodes tooth enamel. In this study, a SEIR model of dental caries infection is constructed with saturation incidence rate and saturated treatment function and we analyze the numerical behavior of the basic reproduction number and stability. Non-standard finite differences (NSFD) scheme was constructed for the SEIR model of dental caries infection and were compared with the Runge Kutta fourth order (RK-4) scheme and Forward Euler (FD) scheme. The NSFD scheme preserves positivity property and also proved to be an effective and efficient scheme for numerical solution of epidemic models. The dental caries (delay) free equilibrium and endemic equilibrium were obtained. Results of the numerical testing reveals that RK-4 scheme and forward Euler (FD) scheme is conditionally convergent and may loss some important properties of continuous system for certain values of step size h . Also, the numerical result further discloses that if medical resources are maximized and treatment not delayed, dental caries infection dynamics will decrease with transient time and the infection will eventually be eradicated from the population.

Keywords: SEIR model; NSFD scheme; Runge Kutta rk-4 scheme; forward Euler (fd) scheme; dental caries infection; positivity

1 Introduction

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Dental caries is a bacterial infection and bacteria must have the proper environment to multiply and thrive. Until teeth are present in a baby's mouth, the bacteria proliferate. Once teeth erupt, however, biofilm forms and adheres to the tooth surface. This matrix (Plaque) increases in amount unless disrupted daily. If allowed to remain, and other cofactors are present, dental disease is likely to result

$$\text{Carbohydrate} + \text{Plaque} = \text{Acid} + \text{Tooth} = \text{Decay}.$$

The bacteria within the plaque are able to ferment the sugars and carbohydrates that they contact. They form lactic acid, which can cause demineralization of the enamel of the tooth. Following exposure to sugar (the breakdown of carbohydrate) for 20 minutes, the enamel of the tooth continues to be attacked by acid. Each repeated snack leads to another 20 minutes of acid formation that can quickly weaken the tooth surface [1]. Once teeth have erupted into the mouth, there are surfaces that are more likely to develop decay at an earlier stage than others. The area along the gum line tends to collect plaque easily. This is where decalcification begins. The pits and fissures on molars tend to have a thinner layer of enamel and are prone to collect food and plaque. These are common sites for decay to occur. Plaque also tends to remain between teeth where the tooth brush cannot reach it. These areas then become susceptible to decay if the plaque is not removed by flossing. Biofilm does not form evenly on all teeth, thus, some teeth are more vulnerable than others to decay [24]. If decay is not treated at an early stage, it is likely that it will extend from the hard outer surface of the tooth to the main living tissue known as the pulp.

The etiological agents of dental caries are many and may be grouped into the exciting or instigating and the predisposing factors. The predisposing factors are diet, general health, oral hygiene, heredity and age, which diet has proven to be the leading contributing factor. The exciting factor is now known to be an acid producing organism [16]. Hahn and Liewehr [14] affirmed that the dental pulp is equipped to mount an innate response to invading caries bacteria which can theoretically slow down bacteria invasion. However the unique location of caries bacteria seems to prevent their being killed or eliminated by phagocytes. Instead, persistent infection leads to the activation of adaptive immunity and overwhelming inflammation and resultant increased edema and intra pulpal pressure, which becomes detrimental to the pulp encase in a low compliance environment. On the other hand, clinical experience indicates that inflamed pulps can recover if the majority of the antigens are removed early enough [16]

Farges et al [11] in their work confirmed that the crowns of the erupted human teeth are covered by symbiotic microbial communities mainly composed of Gram-positive saprophytic bacteria which are normally harmless to the tooth. These communities adhere as biofilms to the highly demineralized enamel that constitutes a barrier which is impermeable to microorganisms and protects the underlying mineralized dentin and the loose connective tissue situated at the centre of the tooth, the dental pulp. However, when placed in a sugar rich environment, specific bacteria populations from these communities release acids that progressively demineralize enamel [15, 10]. This leads to the appearance of a carious lesion characterized by a cavity within which "cariogenic" bacteria proliferate and release additional acids that progressively deepen the lesion. When the enamel barrier is disrupted, dentin becomes degraded by Gram-positive bacteria, including streptococci, lactobacilli and actinomyces that largely dominate the dentin caries microflora [23].

The proliferation and metabolic activity of these microorganisms lead to release of bacterial components into the dentinal tubules and their diffusion towards the peripheral pulp. Dentin

demineralization may also enable the release of bacterial components by host cells at the dentin matrix [6]. Recognition of bacterial components by host cells at the dentin –pulp interface triggers host protective events including antibacterial, immune and inflammatory responses [11]. These events may eliminate early stage bacterial infection and block the route of its progression when accompanied by dentin formation at pulp-dentin interface. Unchecked bacterial invasion results in irreversible chronic pulp inflammation. Subsequently, pulp necrosis, infection of the root canal system and periapical disease may occur [23, 18].

Dental caries management depends on the extent of dental caries and can range from the insertion of a restorative material, endodontic treatment, to tooth extraction [7]. Reversible pulpitis treatment with pulp protection and restorative materials is usually sufficient. Irreversible pulpitis treatment involves a root canal and direct or indirect restorations according to the degree of tooth remnant. There is insufficient evidence to recommend antibiotics [3]. Pulp necrosis is also treated with a root canal or tooth extraction as the last resource. A periapical abscess can complicate pulpitis. A periapical abscess is treatable with endodontic treatment and if they have migrated to the mucosa surface and incision and drainage can also be performed. Periapical abscess complicated by systemic symptoms, cellulitis, or in immunocompromised patients should receive antibiotics in addition to drainage [31, 17].

Gingivitis is a reversible condition that can be treated with professional plaque removal. The use of chlorhexidine rinses is suggested only as an adjuvant to mechanical treatment and for a short period of time given its side effects and complications [12]. Periodontitis may be treated with scaling and may require antibiotics in specific cases but not routinely. Antibiotic therapy for dental infections is necessary for systemic symptoms, fascial space infections, and infections that spread to the bony cortex and surrounding soft tissue [25]. Gram-negative organisms, facultative anaerobes, and strict anaerobes are common organisms found in dental infections, with anaerobes outnumbering aerobic bacteria by a factor of three [8]. Good oral hygiene like regular brushing, flossing and dental cleanings with a dentist can prevent and reverse caries. Using fluoride is a tried and tested way to prevent caries (decay) and cavities. High fluoride tooth paste or direct fluoride application helps remineralize the tooth enamel and prevent caries (decay) and cavities. It usually takes 12 – 24 hours for enough plaque buildup to support bacteria. By brushing and flossing at least once a day, one can remove most of this buildup, with twice a year dental cleaning to remove hard reach plaque that may have been missed.

Ndii et al, [30] constructed nonstandard finite difference scheme (NSFDS) for a water related disease mathematical model. They compared the numerical solutions of NSFDS, Eulers method and MATLAB's ode 45. It is shown numerically that the resulting discrete model preserves essential properties of the continuous model such as positivity and stability. Ashraf and Ahmad [5] in their SEIR Measles model used to study the integrating vaccination as control strategy, constructed an unconditional convergent nonstandard finite difference (NSFD) scheme and it preserve the positivity of all values of h . Also many researchers have compared different numerical schemes with the nonstandard finite difference (NSFD) scheme [21, 4].

The saturated incidence rate tend to a saturated level when I get large and βI measures the infection force when the disease is entering a fully susceptible population while $\frac{1}{1+\alpha I}$ measures the behavioral change of susceptible individuals when their number increased [33, 9] and many other epidemic model have been making use of saturated incidence rate [33, 9, 19, 32, 26, 22, 2]. Abdul – Satar and Naji [1] incorporated saturated recovery in their model while Naji and Abdulateef [29]

stated in their model that the treatment for the infected individual follows the saturated treatment function.

2 Model formulation

Many researchers applied many numerical techniques to solve many mathematical models. In this paper we are interested in the numerical solution of dental caries infection using SEIR epidemic model with saturation incidence rate and saturated treatment function and we shall apply three different numerical schemes such as Runge Kutta RK-4 method, Forward Euler (FD) scheme and Non Standard Finite Difference (NSFD) scheme. The rationale of NSFD scheme is to improve general finite difference scheme by including some of the exact forms and testing would determine which of the scheme is better than the other.

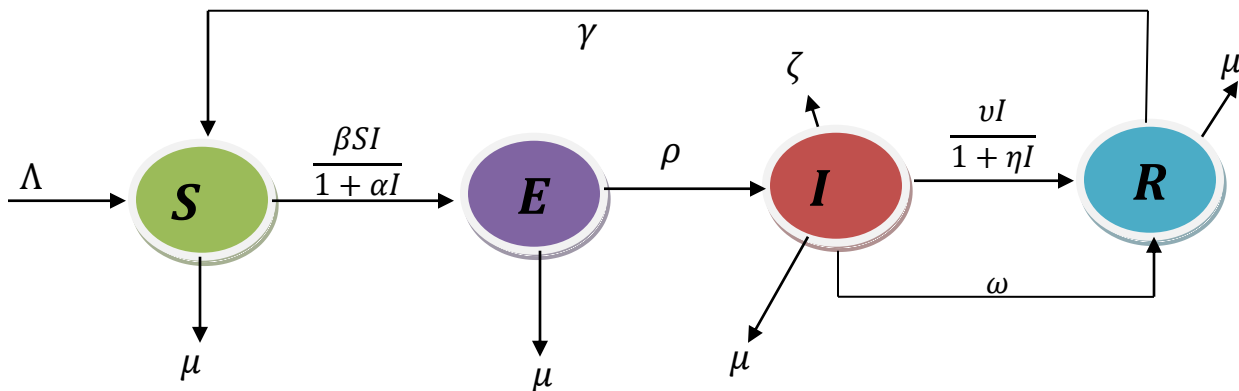


Figure 1: The model flow diagram

$$\left. \begin{aligned} \frac{dS(t)}{dt} &= \Lambda + \gamma R(t) - \left(\mu + \frac{\beta I(t)}{1 + \alpha I(t)} \right) S(t) \\ \frac{dE(t)}{dt} &= \frac{\beta I(t) S(t)}{1 + \alpha I(t)} - (\mu + \rho) E(t) \\ \frac{dI(t)}{dt} &= \rho E(t) - \left(\mu + \zeta + \omega + \frac{v}{1 + \eta I(t)} \right) I(t) \\ \frac{dR(t)}{dt} &= \left(\omega + \frac{v}{1 + \eta I(t)} \right) I(t) - (\mu + \gamma) R(t) \end{aligned} \right\} \quad (1)$$

Description of Variables and Parameters

Variables

$S(t)$ = Susceptible individuals at time t .
 $E(t)$ = Exposed individuals yet to be infectious at time t .
 $I(t)$ = Infectious individuals at time t .
 $R(t)$ = Treated and recovered individuals at time t .

Parameters

β = Disease transmission coefficient (i.e., effective contact rate).
 Λ = Recruitment rate of the population.
 μ = Natural death rate of the population and cut across all the compartments.
 γ = The rate at which treated and recovered individuals move back to susceptible class.
 α = The parameter that measures the inhibitory effects.
 ρ = The rate at which the exposed individuals become infectious.
 ω = recovery rate of infected individuals due to immune response.
 η = The saturated factor that measures the effect of the infected being delayed for treatment.
 v = Maximum medical resources supplied per unit time.
 ζ = The death rate due to the infection.

The most significant and efficient method to prevent and control the spread of dental caries (decay) is through treatment. In conventional epidemic models, the treatment rate of the infection is presumed to be proportional to the number of the infective individuals, which is applicable in dental caries (decay), but in general, the recovery rate depends on the medical resources, such as drugs, hospital personnel's, hospital equipments and efficiency of the treatment. Since most communities or countries have limited capacity for treatment of dental caries (decay), it is necessary to implement a suitable treatment function. Motivated by these facts, and to better understand the effects of the spread of dental caries (decay) with a successful and effective treatment, we proposed a model with saturation incidence rate and saturated treatment function.

The paper is systemized in five sections as follows; in section 2, we have the analysis of the model in which the two equilibria were discussed with the basic reproduction number. In section 3, we solved the proposed model using three (3) different methods. The numerical experiments for the three numerical schemes were performed in section 4 and finally, section was the conclusion.

3 Analysis of the model

From system (1) it follows that

$$N(t) = S(t) + E(t) + I(t) + R(t) \quad (2)$$

$$\begin{aligned} N(t) &\leq \Lambda - \mu(S(t) + E(t) + I(t) + R(t)) \\ &\leq \Lambda - \mu(S + E + I + R). \end{aligned} \quad (3)$$

Then,

$$\limsup_{t \rightarrow \infty} (S + E + I + R) \leq \frac{\Lambda}{\mu}.$$

Thus, the feasible region for system (1) is

$$\Omega = \left\{ (S(t) + E(t) + I(t) + R(t)) : S + E + I + R \leq \frac{\Lambda}{\mu}, S \geq 0, E \geq 0, I \geq 0, R \geq 0 \right\} \quad (4)$$

It can be verified that the region Ω is positively invariant with respect to system (1). Thus we have that the basic reproduction number which is the average number of secondary infection activated by the initial infection is given as

$$R_0 = \frac{\beta \Lambda \rho}{\mu(\mu + \rho)(\mu + \zeta + \omega + v)} \quad (5)$$

The two equilibria of the system are the dental caries (decay) free equilibrium (DCFE) and the dental caries (decay) endemic equilibrium (DCEE) points. The dental caries (decay) free equilibrium (DCFE) point is given by

$$T^0 = (S^0, E^0, I^0, R^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0 \right) \quad (6)$$

while the endemic equilibrium (DCEE) point is given as

$$T^* = (S^*, E^*, I^*, R^*) \quad (7)$$

Where

$$\begin{aligned} S^* &= \frac{[\Lambda(1+\eta I^*)(\mu+\gamma)+\gamma(\omega(1+\eta I^*)+v)I^*](1+\alpha I^*)}{(1+\eta I^*)(\mu+\gamma)[\mu(1+\alpha I^*)+\beta I^*]}, \\ E^* &= \frac{\beta I^*[\Lambda(1+\eta I^*)(\mu+\gamma)+\gamma(\omega(1+\eta I^*)+v)I^*]}{(1+\eta I^*)(\mu+\gamma)[\mu(1+\alpha I^*)+\beta I^*]}, \\ R^* &= \frac{(\omega(1+\eta I^*)+v)I^*}{(1+\eta I^*)(\mu+\gamma)} \\ &= \frac{\rho[\beta I^*[\Lambda(1+\eta I^*)(\mu+\gamma)+\gamma(\omega(1+\eta I^*)+v)I^*]]}{(1+\eta I^*)(\mu+\gamma)[\mu(1+\alpha I^*)+\beta I^*]} = \frac{[(\mu+\zeta+\omega)(1+\eta I^*)+v]I^*}{(1+\eta I^*)} \\ &\Rightarrow \beta \Lambda \rho (1+\eta I^*)(\mu+\gamma) + \beta \rho \gamma \omega (1+\eta I^*) I^* + \beta \rho \gamma v I^* \\ &= [\mu(\mu+\rho)(\mu+\gamma)(1+\alpha I^*) \\ &\quad + \beta(\mu+\rho)(\mu+\gamma)I^*][[(\mu+\zeta+\omega)(1+\eta I^*)+v]] \end{aligned}$$

which after expansion yields

$$\begin{aligned} &[\beta \rho \gamma \omega \eta - \mu \alpha \eta (\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - \eta \beta (\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega)]I^{*2} \\ &\quad + [\beta \rho \gamma \omega + \beta \rho \gamma v + \beta \Lambda \rho \eta (\mu + \gamma) - \mu \eta (\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) \\ &\quad - \mu \alpha (\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - \mu v \alpha (\mu + \rho)(\mu + \gamma) \\ &\quad - \beta (\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - v \beta (\mu + \rho)(\mu + \gamma)]I^* + [(\mu + \gamma)(R_0 - 1)] \\ &= 0 \end{aligned} \quad (8)$$

This can be written as

$$k_2 I^{*2} + k_1 I^* + k_0 = 0 \quad (9)$$

$$\left. \begin{aligned} k_2 &= \beta\rho\gamma\omega\eta - \mu\alpha\eta(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - \eta\beta(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) \\ k_1 &= \beta\rho\gamma\omega + \beta\rho\gamma v + \beta\Lambda\rho\eta(\mu + \gamma) - \mu\eta(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - \mu\alpha(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) \\ &\quad - \mu v\alpha(\mu + \rho)(\mu + \gamma) - \beta(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega) - v\beta(\mu + \rho)(\mu + \gamma) \\ k_0 &= (\mu + \gamma)(R_0 - 1) \end{aligned} \right\} (10)$$

Thus, for the existence of the dental caries (decay) endemic equilibrium, equation (9) must have positive real roots. Therefore, we determine the sign of k_0 by expressing it as

$$k_0 = \beta\Lambda\rho(\mu + \gamma) - \mu(\mu + \rho)(\mu + \gamma)(\mu + \zeta + \omega + v), \quad (11)$$

since

$$k_0 = (\mu + \gamma)(R_0 - 1).$$

We show that at DCEE, $R_0 > 1$, then, we obtain

$$R_0 = \frac{\beta\Lambda\rho}{\mu(\mu + \rho)(\mu + \zeta + \omega + v)} > 1.$$

It can easily be seen that

$$\beta\Lambda\rho > \mu(\mu + \gamma)(\mu + \zeta + \omega + v)$$

Since system (9) is a quadratic equation with respect to I^* , it follows from (11) that k_0 is non positive ($k_0 < 0$) whenever $R_0 < 1$. This proves that $k_0 > 0$ if and only if $R_0 > 1$. Thus, the number of positive real roots depends on the sign of equation (9). This can be analyzed using Descartes rule of sign of equation. Let

$$F(I^*) = k_2 I^{*2} + k_1 I^* + k_0$$

The different possibilities for the roots of $F(I^*)$ are tabulated in Table 1 below.

Table 1: Number of possible positive roots of $F(I^*)$ for $R_0 > 1$.

Cases	k_2	k_1	k_0	R_0	No of sign change	No of positive roots
1	+	+	−	R_0	1	1
2	+	−	−	R_0	1	1

Model system (1) has a unique endemic equilibrium point where one of the cases 1 and 2 in table 1 are satisfied.

4 Numerical modeling

We shall use three finite difference (FD) methods to solve model system (1) in this section. The three methods are Runge Kutta Method of Order 4 (RK4), Forward Euler (FD) Scheme and Non-Standard Finite Difference (NSFD) Method for numerical solution of system (1) and then we will compare our results with a Standard RK4 method.

4.1 Runge- Kutta Method of Order 4

Here, we shall develop the R-K 4 finite difference (FD) scheme for model system (1), thus;

$$\left. \begin{aligned} S_{n+1} - S_n &= h \left[\Lambda + \gamma R_n - \frac{\beta S_n I_n}{1 + \alpha I_n} - \mu S_n \right] \\ E_{n+1} - E_n &= h \left[\frac{\beta S_n I_n}{1 + \alpha I_n} - (\mu + \rho) E_n \right] \\ I_{n+1} - I_n &= h \left[\rho E_n - (\mu + \zeta + \omega) I_n - \frac{v I_n}{1 + \eta I_n} \right] \\ R_{n+1} - R_n &= h \left[\omega I_n + \frac{v I_n}{1 + \eta I_n} - (\mu + \gamma) R_n \right] \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} S_{n+1} &= S_n + h \left[\Lambda + \gamma R_n - \frac{\beta S_n I_n}{1 + \alpha I_n} - \mu S_n \right] \\ E_{n+1} &= E_n + h \left[\frac{\beta S_n I_n}{1 + \alpha I_n} - (\mu + \rho) E_n \right] \\ I_{n+1} &= I_n + h \left[\rho E_n - (\mu + \zeta + \omega) I_n - \frac{v I_n}{1 + \eta I_n} \right] \\ R_{n+1} &= R_n + h \left[\omega I_n + \frac{v I_n}{1 + \eta I_n} - (\mu + \gamma) R_n \right] \end{aligned} \right\}$$

Finding $S_{n+1}, E_{n+1}, I_{n+1}$ and R_{n+1} requires four intermediate calculations such as

Step I

$$\left. \begin{aligned} U_1 &= h \left[\Lambda + \gamma R_n - \frac{\beta S_n I_n}{1 + \alpha I_n} - \mu S_n \right] \\ X_1 &= h \left[\frac{\beta S_n I_n}{1 + \alpha I_n} - (\mu + \rho) E_n \right] \\ Y_1 &= h \left[\rho E_n - (\mu + \zeta + \omega) I_n - \frac{v I_n}{1 + \eta I_n} \right] \\ Z_1 &= h \left[\omega I_n + \frac{v I_n}{1 + \eta I_n} - (\mu + \gamma) R_n \right] \end{aligned} \right\}$$

Step II

$$\left. \begin{aligned} U_2 &= h \left[\Lambda + \gamma(R_n) - \frac{\beta \left(S_n + \frac{U_1}{2} \right) \left(I_n + \frac{Y_1}{2} \right)}{1 + \alpha \left(I_n + \frac{Y_1}{2} \right)} - \mu \left(S_n + \frac{U_1}{2} \right) \right] \\ X_2 &= h \left[\frac{\beta \left(S_n + \frac{U_1}{2} \right) \left(I_n + \frac{Y_1}{2} \right)}{1 + \alpha \left(I_n + \frac{Y_1}{2} \right)} - (\mu + \rho) \left(E_n + \frac{X_1}{2} \right) \right] \\ Y_2 &= h \left[\rho \left(E_n + \frac{X_1}{2} \right) - (\mu + \zeta + \omega) \left(I_n + \frac{Y_1}{2} \right) - \frac{v \left(I_n + \frac{Y_1}{2} \right)}{1 + \eta \left(I_n + \frac{Y_1}{2} \right)} \right] \\ Z_2 &= h \left[\omega \left(I_n + \frac{Y_1}{2} \right) + \frac{v \left(I_n + \frac{Y_1}{2} \right)}{1 + \eta \left(I_n + \frac{Y_1}{2} \right)} - (\mu + \gamma) \left(R_n + \frac{Z_1}{2} \right) \right] \end{aligned} \right\}$$

Step III

$$\left. \begin{aligned} U_3 &= h \left[\Lambda + \gamma(R_n) - \frac{\beta \left(S_n + \frac{U_2}{2} \right) \left(I_n + \frac{Y_2}{2} \right)}{1 + \alpha \left(I_n + \frac{Y_2}{2} \right)} - \mu \left(S_n + \frac{U_2}{2} \right) \right] \\ X_3 &= h \left[\frac{\beta \left(S_n + \frac{U_2}{2} \right) \left(I_n + \frac{Y_2}{2} \right)}{1 + \alpha \left(I_n + \frac{Y_2}{2} \right)} - (\mu + \rho) \left(E_n + \frac{X_2}{2} \right) \right] \\ Y_3 &= h \left[\rho \left(E_n + \frac{X_2}{2} \right) - (\mu + \zeta + \omega) \left(I_n + \frac{Y_2}{2} \right) - \frac{v \left(I_n + \frac{Y_2}{2} \right)}{1 + \eta \left(I_n + \frac{Y_2}{2} \right)} \right] \\ Z_3 &= h \left[\omega \left(I_n + \frac{Y_2}{2} \right) + \frac{v \left(I_n + \frac{Y_2}{2} \right)}{1 + \eta \left(I_n + \frac{Y_2}{2} \right)} - (\mu + \gamma) \left(R_n + \frac{Z_2}{2} \right) \right] \end{aligned} \right\}$$

Step IV

$$\left. \begin{aligned} U_2 &= h \left[\Lambda + \gamma(R_n) - \frac{\beta(S_n + U_3)(I_n + Y_3)}{1 + \alpha(I_n + Y_3)} - \mu(S_n + U_3) \right] \\ X_1 &= h \left[\frac{\beta(S_n + U_3)(I_n + Y_3)}{1 + \alpha(I_n + \frac{Y_1}{2})} - (\mu + \rho)(E_n + X_3) \right] \\ Y_1 &= h \left[\rho(E_n + X_3) - (\mu + \zeta + \omega)(I_n + Y_3) - \frac{v(I_n + Y_3)}{1 + \eta(I_n + Y_3)} \right] \\ Z_1 &= h \left[\omega(I_n + Y_3) + \frac{v(I_n + Y_3)}{1 + \eta(I_n + Y_3)} - (\mu + \gamma)(R_n + Z_3) \right] \end{aligned} \right\}$$

We shall observe that the incremental in the value of $S_{n+1}, E_{n+1}, I_{n+1}$ and R_{n+1} is then

$$\left. \begin{aligned} S_{n+1} &= S_n + \frac{1}{6} [U_1 + 2U_2 + 2U_3 + U_4] \\ E_{n+1} &= E_n + \frac{1}{6} [X_1 + 2X_2 + 2X_3 + X_4] \\ I_{n+1} &= I_n + \frac{1}{6} [Y_1 + 2Y_2 + 2Y_3 + Y_4] \\ R_{n+1} &= R_n + \frac{1}{6} [Z_1 + 2Z_2 + 2Z_3 + Z_4] \end{aligned} \right\} \quad (13)$$

4.2 Forward Euler (FD) Scheme

The Forward Euler Finite Difference Scheme were developed for model system (1) as follows:

$$\left. \begin{aligned} S_{n+1} &= S_n + h \left[\Lambda + \gamma R_n - \frac{\beta S_n I_n}{1 + \alpha I_n} - \mu S_n \right] \\ E_{n+1} &= E_n + h \left[\frac{\beta S_n I_n}{1 + \alpha I_n} - (\mu + \rho) E_n \right] \\ I_{n+1} &= I_n + h \left[\rho E_n - (\mu + \zeta + \omega) I_n - \frac{v I_n}{1 + \eta I_n} \right] \\ R_{n+1} &= R_n + h \left[\omega I_n + \frac{v I_n}{1 + \eta I_n} - (\mu + \gamma) R_n \right] \end{aligned} \right\} \quad (14)$$

4.3 Non Standard Finite Difference Scheme (NSFD)

In this section we design the non standard finite difference (NSFD) scheme that replicates the dynamics of continuous model system (1). Non standard finite difference scheme was proposed by Mickens's in 1989 [21] and now is widely used for system of differential equations that are describing problems in mathematical biology and the other different areas. These Method showed the superiority in preserving the passivity [13, 20, 27, 28] when comparing all others well known numerical method of state variable of the system under study. NSFD scheme maintains dynamic

consistency as well as numerical stability in terms of initial restriction with irregular step length. The model system (1) is developed according to the laws given by Mickens

$$\begin{aligned}
 \frac{S_{n+1} - S_n}{h} &= \Lambda + \gamma R_n - \frac{\beta S_{n+1} I_n}{1 + \alpha I_n} - \mu S_{n+1} \\
 S_{n+1} &= S_n + h\Lambda + h\gamma R_n - \frac{h\beta S_{n+1} I_n}{1 + \alpha I_n} - h\mu S_{n+1} \\
 S_{n+1} + \frac{h\beta S_{n+1} I_n}{1 + \alpha I_n} + h\mu S_{n+1} &= S_n + h\Lambda + h\gamma R_n \\
 S_{n+1} \left[1 + \frac{h\beta I_n}{1 + \alpha I_n} + h\mu \right] &= S_n + h\Lambda + h\gamma R_n \\
 S_{n+1} &= \frac{S_n + h\Lambda + h\gamma R_n}{1 + \frac{h\beta I_n}{1 + \alpha I_n} + h\mu} \\
 \frac{E_{n+1} - E_n}{h} &= \frac{\beta S_n I_n}{1 + \alpha I_n} - (\mu + \rho) E_{n+1} \\
 E_{n+1} + h(\mu + \rho) E_{n+1} &= E_n + \frac{h\beta S_n I_n}{1 + \alpha I_n} \\
 E_{n+1} [1 + h(\mu + \rho)] &= E_n + \frac{h\beta S_n I_n}{1 + \alpha I_n} \\
 E_{n+1} &= \frac{E_n + \frac{h\beta S_n I_n}{1 + \alpha I_n}}{1 + h(\mu + \rho)} \\
 \frac{I_{n+1} - I_n}{h} &= \rho E_n - (\mu + \zeta + \omega) I_{n+1} - \frac{v I_{n+1}}{1 + \eta I_n} \\
 I_{n+1} + h(\mu + \zeta + \omega) I_{n+1} + \frac{h v I_{n+1}}{1 + \eta I_n} &= I_n + h \rho E_n \\
 I_{n+1} &= \left[1 + h(\mu + \zeta + \omega) I_{n+1} + \frac{h v I_{n+1}}{1 + \eta I_n} \right] = I_n + h \rho E_n \\
 I_{n+1} &= \frac{I_n + h \rho E_n}{1 + h(\mu + \zeta + \omega) I_{n+1} + \frac{h v I_{n+1}}{1 + \eta I_n}} \\
 \frac{R_{n+1} - R_n}{h} &= \omega I_n + \frac{v I_n}{1 + \eta I_n} - (\mu + \gamma) R_{n+1} \\
 R_{n+1} + h(\mu + \gamma) R_{n+1} &= R_n + h\omega I_n + \frac{h v I_n}{1 + \eta I_n} \\
 R_{n+1} [1 + h(\mu + \gamma)] &= R_n + h\omega I_n + \frac{h v I_n}{1 + \eta I_n} \\
 R_{n+1} &= \frac{R_n + h\omega I_n + \frac{h v I_n}{1 + \eta I_n}}{1 + h(\mu + \gamma)}
 \end{aligned}$$

$$\left. \begin{aligned} S_{n+1} &= \frac{S_n + h\Lambda + h\gamma R_n}{1 + \frac{h\beta I_n}{1 + \alpha I_n} + h\mu} \\ E_{n+1} &= \frac{E_n + \frac{h\beta S_n I_n}{1 + \alpha I_n}}{1 + h(\mu + \rho)} \\ I_{n+1} &= \frac{I_n + h\rho E_n}{1 + h(\mu + \zeta + \omega)I_{n+1} + \frac{h\nu I_{n+1}}{1 + \eta I_n}} \\ R_{n+1} &= \frac{R_n + h\omega I_n + \frac{h\nu I_n}{1 + \eta I_n}}{1 + h(\mu + \gamma)} \end{aligned} \right\} \quad (15)$$

To obtain success in NSFD system (15) should be sequentially computed since the value of S_{n+1} will be applied in calculating E_{n+1} , E_{n+1} is used to calculate the value of I_{n+1} which in turn is then used in computing R_{n+1} . The process is repeated until we finally approaches the time of interest.

Non-Negativity of the Solution

The model is centered on the transmission dynamics of dental caries in human population, therefore, the proposed numerical scheme cannot produce negative value and we assume that the initial conditions are non negative such that

$$S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0 \quad (16)$$

These variables have approximate values which also are non-negative due to the following

$$S_n \geq 0, E_n \geq 0, I_n \geq 0, R_n \geq 0$$

We shall observe that for time steps greater than zero, that is, $h \geq 0$ and $S_n, E_n, I_n, R_n \geq 0$, numerators and denominators are nonnegative and hence $S_{n+1}, E_{n+1}, I_{n+1}, R_{n+1} \geq 0$.

4.3.1 Convergence analysis of the proposed NSFD scheme

The NSFD scheme converges to the equilibrium point of the absolute of the eigenvalues is less than one. At this point, we shall examine the convergence of the proposed non standard finite difference scheme (NSFD) using

$$\left. \begin{aligned} L &= \frac{S + h\Lambda + h\gamma R}{1 + h\mu + \frac{h\beta I}{1 + \alpha I}} \\ P &= \frac{E + \frac{h\beta SI}{1 + \alpha I}}{1 + h(\mu + \rho)} \\ Q &= \frac{I + h\rho E}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I}} \\ T &= \frac{R + h\left(\omega + \frac{v}{1 + \eta I}\right)I}{1 + h(\mu + \gamma)} \end{aligned} \right\}$$

Then

$$\left. \begin{aligned} L &= \frac{S + h\Lambda + h\gamma R}{1 + h\mu + \frac{h\beta I}{1 + \alpha I}} \\ \frac{\partial L}{\partial S} &= \frac{1}{1 + h\mu + \frac{h\beta I}{1 + \alpha I}} \\ \frac{\partial L}{\partial E} &= 0 \\ \frac{\partial L}{\partial I} &= -\frac{h\beta(S + h\Lambda + h\gamma R)}{(1 + \alpha I)^2 \left(1 + h\mu + \frac{h\beta I}{1 + \alpha I}\right)^2} \\ \frac{\partial L}{\partial R} &= \frac{h\gamma}{1 + h\mu + \frac{h\beta I}{1 + \alpha I}} \end{aligned} \right\}$$

Also,

$$\left. \begin{aligned} P &= \frac{E + \frac{h\beta SI}{1 + \alpha I}}{1 + h(\mu + \rho)} \\ \frac{\partial P}{\partial S} &= \frac{h\beta I}{(1 + \alpha I)(1 + h(\mu + \rho))} \\ \frac{\partial P}{\partial E} &= \frac{1}{1 + h(\mu + \rho)} \\ \frac{\partial P}{\partial I} &= \frac{h\beta S}{(1 + \alpha I)^2(1 + h(\mu + \rho))} \\ \frac{\partial P}{\partial R} &= 0 \end{aligned} \right\}$$

Similarly

$$\left. \begin{aligned} Q &= \frac{I + h\rho E}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I}} \\ \frac{\partial Q}{\partial S} &= 0 \\ \frac{\partial Q}{\partial E} &= \frac{h\rho}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I}} \\ \frac{\partial Q}{\partial I} &= \frac{1}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I}} \left[1 + \frac{hv}{(1 + \eta I)^2 \left(1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I} \right)} \right] [I - h\rho E] \\ \frac{\partial Q}{\partial R} &= 0 \end{aligned} \right\}$$

Finally

$$\left. \begin{aligned} T &= \frac{R + h\left(\omega + \frac{v}{1 + \eta I}\right)I}{1 + h(\mu + \gamma)} \\ \frac{\partial T}{\partial S} &= 0 \\ \frac{\partial T}{\partial E} &= 0 \\ \frac{\partial T}{\partial I} &= \frac{h\omega + \frac{hv}{1 + \eta I}}{1 + h(\mu + \gamma)} \\ \frac{\partial T}{\partial R} &= \frac{1}{1 + h(\mu + \gamma)} \end{aligned} \right\}$$

$$J = \begin{bmatrix} \frac{\partial L}{\partial S} & \frac{\partial L}{\partial E} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial R} \\ \frac{\partial P}{\partial S} & \frac{\partial P}{\partial E} & \frac{\partial P}{\partial I} & \frac{\partial P}{\partial R} \\ \frac{\partial Q}{\partial S} & \frac{\partial Q}{\partial E} & \frac{\partial Q}{\partial I} & \frac{\partial Q}{\partial R} \\ \frac{\partial T}{\partial S} & \frac{\partial T}{\partial E} & \frac{\partial T}{\partial I} & \frac{\partial T}{\partial R} \end{bmatrix} \quad (17)$$

The eigenvalues of the matrix at dental caries free equilibrium $T^0 = (S^0, E^0, I^0, R^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ gives

$$J(T^0) = \begin{bmatrix} \frac{1}{1+h\mu} & 0 & -\frac{h\beta\left(\frac{\Lambda}{\mu} + \Lambda h\right)}{(1+h\mu)^2} & \frac{h\gamma}{1+h\mu} \\ 0 & \frac{1}{1+h(\mu+\rho)} & \frac{h\beta\Lambda}{\mu(1+h(\mu+\rho))} & 0 \\ 0 & \frac{h\rho}{1+h(\mu+\zeta+\omega+v)} & \frac{1}{1+h(\mu+\zeta+\omega+v)} & 0 \\ 0 & 0 & \frac{h(\omega+v)}{1+h(\mu+\gamma)} & \frac{1}{1+h(\mu+\gamma)} \end{bmatrix}$$

It is clear from the above matrix that the two eigenvalues are $\frac{1}{1+h\mu}, \frac{1}{1+h(\mu+\gamma)}$ which are less than unity. The remaining two eigenvalues can be found from the matrix below

$$M(T^0) = \begin{bmatrix} \frac{1}{1+h(\mu+\rho)} & \frac{h\beta\Lambda}{\mu(1+h(\mu+\rho))} \\ \frac{h\rho}{1+h(\mu+\zeta+\omega+v)} & \frac{1}{1+h(\mu+\zeta+\omega+v)} \end{bmatrix}$$

The characteristic determinant is given as $M(T^0) - \lambda I = 0$ and we obtain

$$\begin{aligned} & \begin{vmatrix} \frac{1}{1+h(\mu+\rho)} - \lambda & \frac{h\beta\Lambda}{\mu(1+h(\mu+\rho))} \\ \frac{h\rho}{1+h(\mu+\zeta+\omega+v)} & \frac{1}{1+h(\mu+\zeta+\omega+v)} - \lambda \end{vmatrix} = 0 \\ & \left(\frac{1}{1+h(\mu+\rho)} - \lambda\right)\left(\frac{1}{1+h(\mu+\zeta+\omega+v)} - \lambda\right) \\ & \quad - \left(\frac{h\beta\Lambda}{\mu(1+h(\mu+\rho))}\right)\left(\frac{h\rho}{1+h(\mu+\zeta+\omega+v)}\right) = 0 \\ & \lambda^2 - \frac{\lambda}{1+h(\mu+\rho)} - \frac{\lambda}{1+h(\mu+\zeta+\omega+v)} + \left(\frac{1}{1+h(\mu+\rho)}\right)\left(\frac{1}{1+h(\mu+\zeta+\omega+v)}\right) \\ & \quad - \frac{h^2\beta\rho\Lambda}{\mu(1+h(\mu+\rho))(1+h(\mu+\zeta+\omega+v))} = 0 \\ & \lambda^2 - \left[\frac{2+h(\mu+\rho)+h(\mu+\zeta+\omega+v)}{(1+h(\mu+\rho))(1+h(\mu+\zeta+\omega+v))}\right]\lambda + \frac{\mu-h^2\beta\rho\Lambda}{\mu(1+h(\mu+\rho))(1+h(\mu+\zeta+\omega+v))} \\ & \quad = 0 \quad (18) \end{aligned}$$

Equation (18) can be rewritten as

$$\lambda^2 - \frac{A}{B}\lambda + \frac{C}{\mu B} = 0 \quad (19)$$

where

$$A = 2 + h(\mu + \rho) + h(\mu + \zeta + \omega + v);$$

$$B = (1 + h(\mu + \rho))(1 + h(\mu + \zeta + \omega + v)) \text{ and } C = \mu - h^2\beta\rho\Lambda.$$

It is clear that the absolute of the eigenvalues is less than unity as demonstrated by the spectral radius of the Jacobian matrix $M(T^0)$ for each time step " h " such that $0 < h < 1000$ shown in figure 2 and we may observe from this figure that the spectral radius remains less than unity for each time step. By implication, it means that all the eigenvalues of the Jacobian matrix at dental caries free equilibrium will be within a unit disc and this assures that the proposed NSFD scheme is convergent for each of the selected time step.

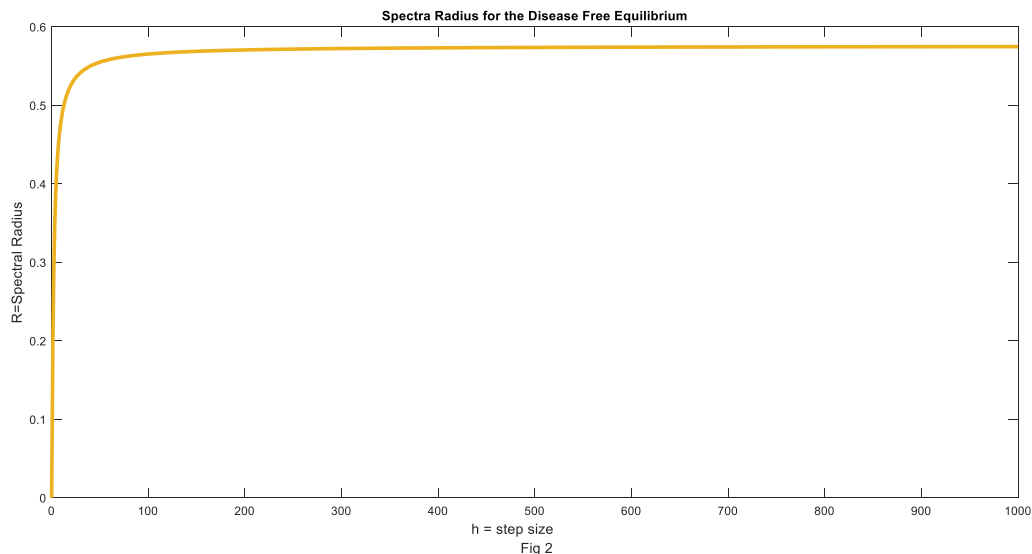


Figure 2: The graph of Spectral radius for the Dental Caries (decay) free equilibrium

Also, the eigenvalues of the Jacobian matrix $J(T^*)$ at dental caries endemic equilibrium $T^* = (S^*, E^*, I^*, R^*)$ will be given as

$$J(T^*) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}$$

$$J_{11} = \frac{1 + \alpha I^*}{(1 + \alpha I^*)(1 + h\mu) + h\beta I^*}; J_{12} = 0; J_{13} = -\frac{h\beta(S^* + h\Lambda + h\gamma R^*)}{(1 + \alpha I^*)^2 \left(1 + h\mu + \frac{h\beta I^*}{1 + \alpha I^*}\right)^2};$$

$$\begin{aligned}
 J_{14} &= \frac{h\gamma(1 + \alpha I^*)}{(1 + \alpha I^*)(1 + h\mu) + h\beta I^*}; J_{21} = \frac{h\beta I^*}{(1 + \alpha I^*)(1 + h(\mu + \rho))}; J_{22} = \frac{1}{1 + h(\mu + \rho)}; \\
 J_{23} &= \frac{h\beta S^*}{(1 + \alpha I^*)^2(1 + h(\mu + \rho))}; J_{24} = 0; J_{31} = 0; J_{32} = \frac{h\rho}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I^*}} \\
 J_{33} &= \frac{1}{1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I^*}} \left[1 + \frac{hv}{(1 + \eta I^*)^2 \left(1 + h(\mu + \zeta + \omega) + \frac{vh}{1 + \eta I^*} \right)} \right] [I^* \\
 &\quad - h\rho E^*]; \\
 J_{34} &= 0; J_{41} = 0; J_{42} = 0; J_{43} = \frac{h\omega(1 + \eta I^*) + hv}{(1 + \eta I^*)(1 + h(\mu + \gamma))}; J_{44} = \frac{1}{1 + h(\mu + \gamma)} \\
 \Rightarrow J(T^*) &= \begin{bmatrix} J_{11} & 0 & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & 0 \\ 0 & J_{32} & J_{33} & 0 \\ 0 & 0 & J_{43} & J_{44} \end{bmatrix}
 \end{aligned}$$

Clearly, we observe from $J(T^*)$ that two eigenvalues are $J_{11} = \frac{1 + \alpha I^*}{(1 + \alpha I^*)(1 + h\mu) + h\beta I^*}$ and $J_{44} = \frac{1}{1 + h(\mu + \gamma)}$ which are both less than unity. The remaining two eigenvalues can be found from the matrix

$$M_1(T^*) = \begin{bmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{bmatrix}.$$

Thus, $M_1(T^*) - \lambda I = 0$ gives

$$\begin{aligned}
 &\begin{vmatrix} J_{22} - \lambda^* & J_{23} \\ J_{32} & J_{33} - \lambda^* \end{vmatrix} = 0 \\
 &(J_{22} - \lambda^*)(J_{33} - \lambda^*) - J_{23}J_{32} = 0 \\
 &\lambda^{*2} - [J_{22} + J_{33}]\lambda^* + J_{22}J_{33} - J_{23}J_{32} = 0,
 \end{aligned}$$

where

$$\begin{aligned}
 J_{22} + J_{33} &= \left[\frac{1}{1 + h(\mu + \rho)} \right] \\
 &\quad + \left[\frac{(1 + \eta I^*)[(1 + \eta I^*) + h(\mu + \zeta + \omega)(1 + \eta I^*) + 2hv]}{[(1 + \eta I^*) + h(\mu + \zeta + \omega)(1 + \eta I^*) + hv]^2} \right] [I^* - h\rho E^*] \\
 J_{22}J_{33} &= \left[\frac{1}{1 + h(\mu + \rho)} \right] \left[\frac{(1 + \eta I^*)[(1 + \eta I^*) + h(\mu + \zeta + \omega)(1 + \eta I^*) + 2hv]}{[(1 + \eta I^*) + h(\mu + \zeta + \omega)(1 + \eta I^*) + hv]^2} \right] [I^* - h\rho E^*]
 \end{aligned}$$

and

$$J_{23}J_{32} = \left[\frac{h\beta S^*}{(1 + \alpha I^*)^2(1 + h(\mu + \rho))} \right] \left[\frac{h\rho(1 + \eta I^*)}{(1 + h(\mu + \zeta + \omega))(1 + \eta I^*) + hv} \right].$$

Observe that the spectral radius of the Jacobian matrix $M_1(T^*)$ for each time step " h " such that $0 < h < 1000$ as shown in figure 3 remains less than unity for each time step. This simply indicates that all the eigenvalues of the Jacobian matrix at the dental caries endemic equilibrium will be within a unit disc which assures that the proposed NSFD scheme is convergent for each of the selected time step.

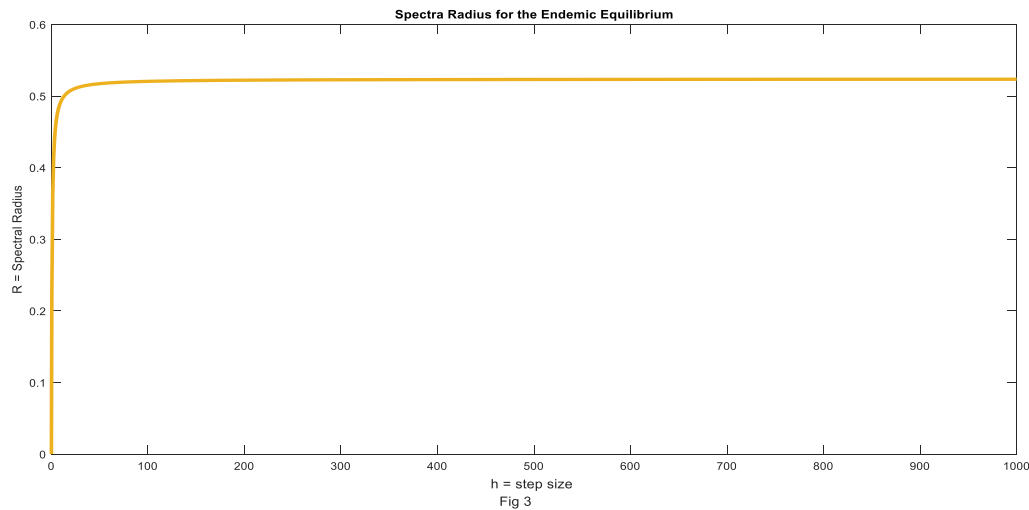


Figure 3: The graph of Spectral radius for the Dental Caries (decay) Endemic equilibrium

5 Numerical Schemes

In this section, we develop three numerical schemes using model system (1) with the help of the parameter values in tables 1 and 2. The numerical schemes that shall be considered in this section are NSFD scheme, R-K 4 and Forward Euler (FD) scheme. Finally, NSFD scheme numerical results will be compared with the other two schemes.

Table 1: Parameter values for the Dental Caries (Decay) Free Equilibrium (DCFE)

Parameter	β	Λ	μ	ζ	ω	α	ρ	η	ν	γ
Value	0.05	0.7	0.13	0.05	0.50	0.0002	0.8	0.000001	0.5	0.000001

Table 2: Parameter values for the Dental Caries (Decay) Endemic Equilibrium (DCFE)

Parameter	β	Λ	μ	ζ	ω	α	ρ	η	ν	γ
Value	0.5	0.7	0.13	0.005	0.30	0.00002	0.7	0.0000001	0.5	0.000001

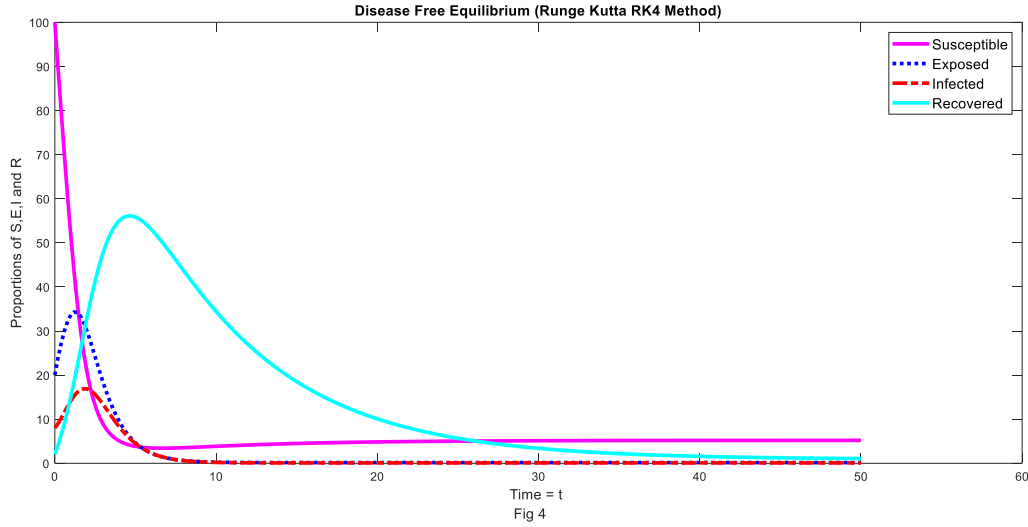


Figure 4: Graph of RungeKutta RK-4 Method for the Dental Caries (Decay) Free Equilibrium with step size $h = 0.2$.

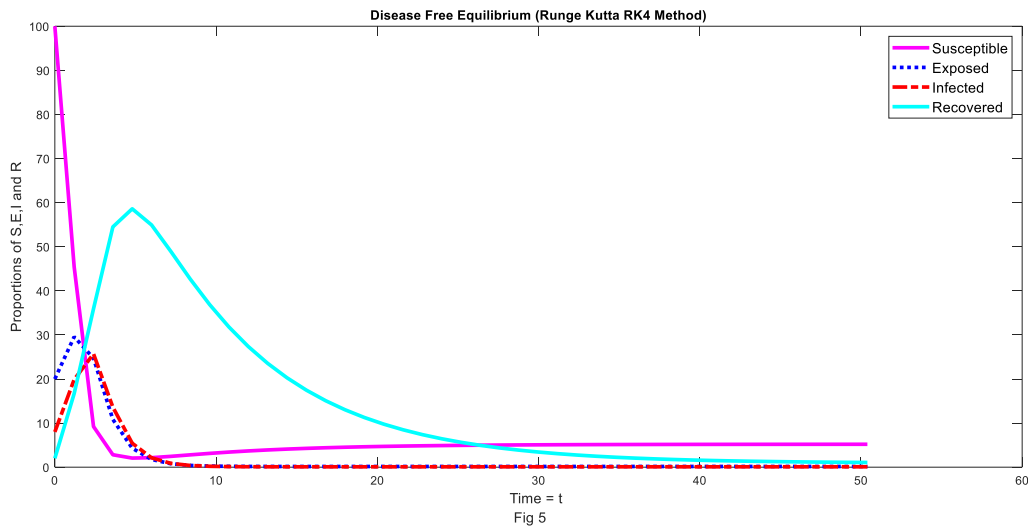


Figure 5: Graph of RungeKutta RK-4 Method for the Dental Caries (Decay) Free Equilibrium with step size $h = 1.2$.

Figures 4 and 5 shows the convergence of the Runge-Kutta fourth order method RK-4 for the dental caries (decay) free equilibrium at $h = 0.2$ and $h = 1.2$ respectively from which we observe the behavior of RK-4 scheme for various values of time step.

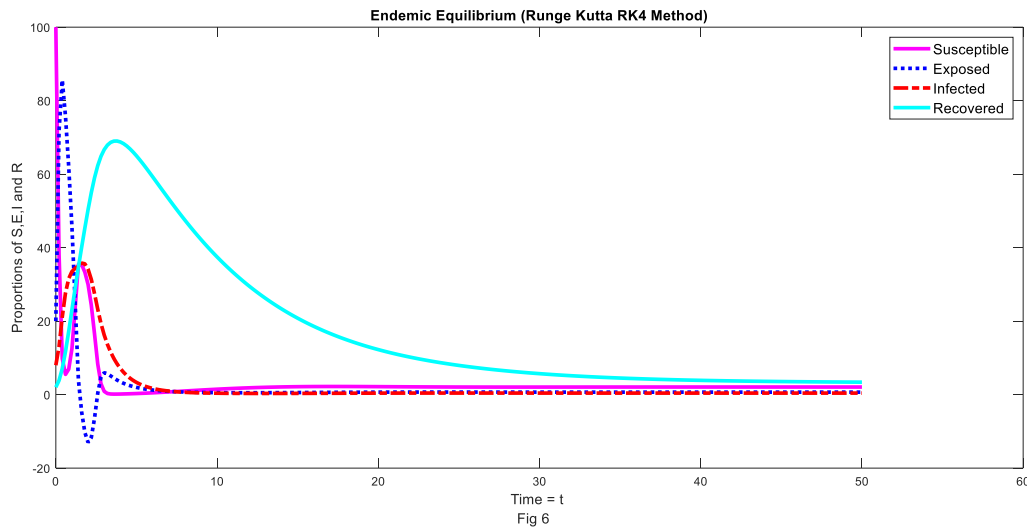


Figure 6: Graph of RungeKutta RK-4 Method for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 0.2$.

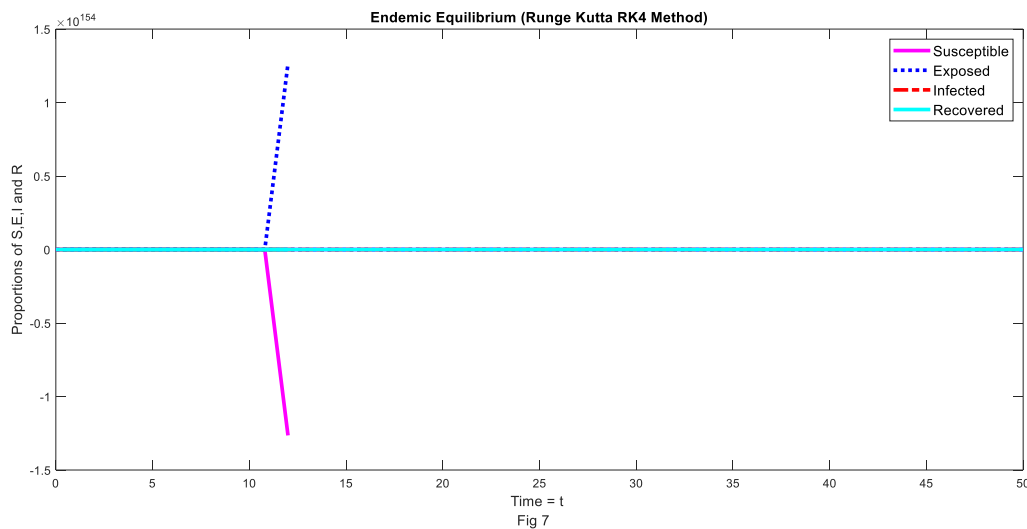


Figure 7: Graph of RungeKutta RK-4 Method for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 1.2$.

Figures 6 and 7 showed that RK-4 scheme loss the positivity property and produce negative values for the compartments at dental caries endemic equilibrium with $h = 0.2$ and $h = 1.2$ respectively. Negative values of the compartments are worthless and they are due to drawback of RK-4 scheme. It shows the non- physical behavior of the compartments which is not consistence with the continuous dynamical system. Actually, figure 7 show the overflow of RK-4 scheme at $h = 1.2$.

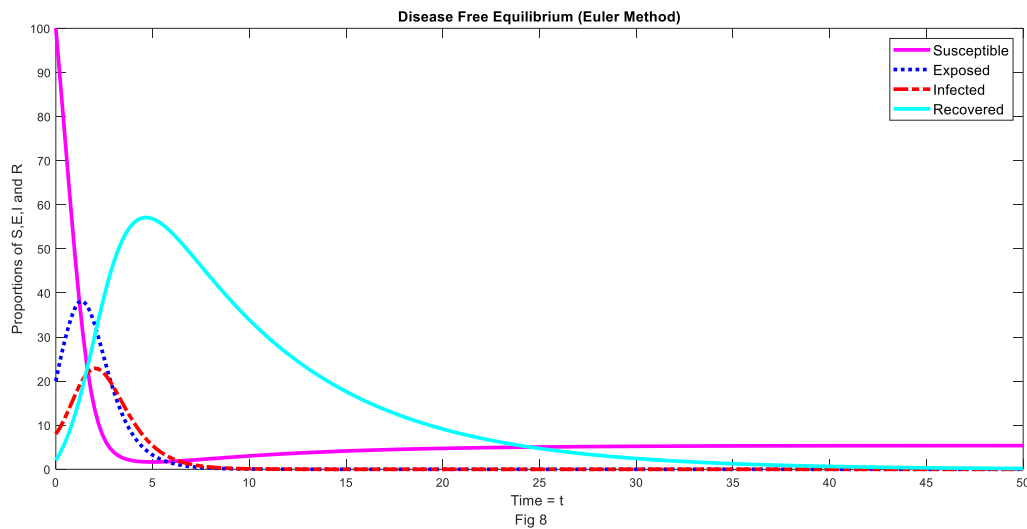


Figure 8: Graph of Forward Euler (FD) Scheme for the Dental Caries (Decay) Free Equilibrium with step size $h = 0.2$.

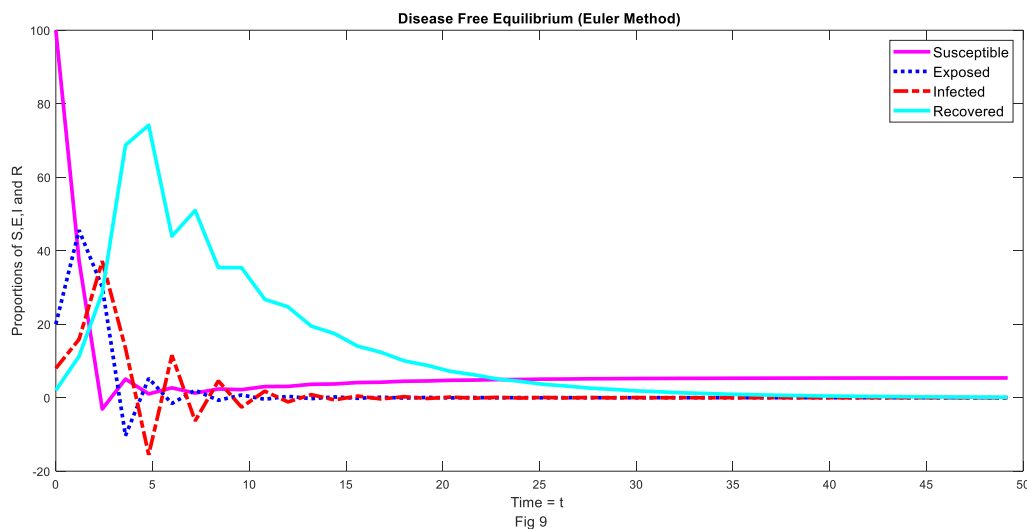


Figure 9: Graph of Forward Euler (FD) Scheme for the Dental Caries (Decay) Free Equilibrium with step size $h = 1.2$.

Figures 8 and 9 showed the convergence of Forward Euler (FD) method for the dental caries (decay) free equilibrium at $h = 0.2$ and $h = 1.2$ respectively. We observe that the scheme lose the positivity property and produce negative values of the compartments at $h = 1.2$. Therefore, we should note that the negative values are due to the drawback of the Forward Euler (FD) scheme. Also, figure 9 shows the overflow of the Forward Euler (FD) scheme.

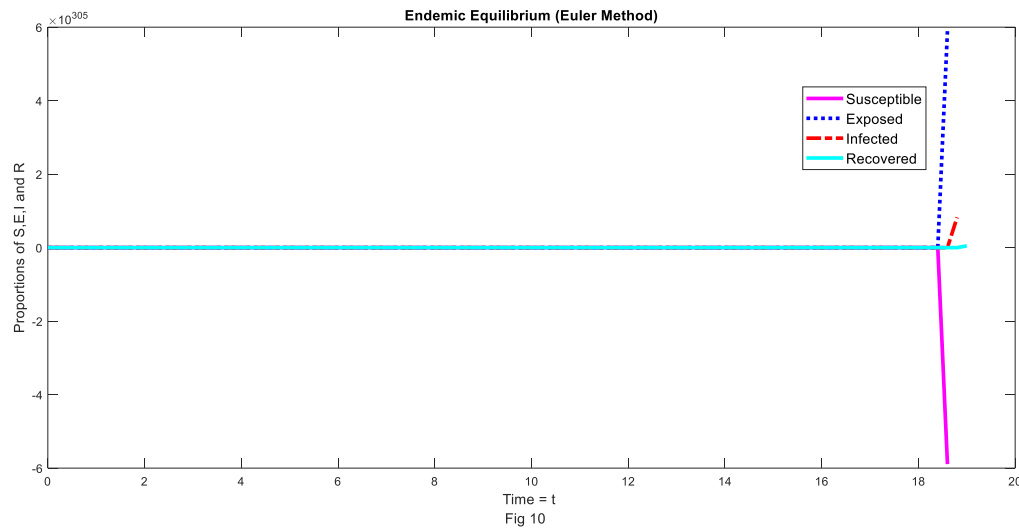


Figure 10: Graph of Forward Euler (FD) Scheme for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 0.2$.

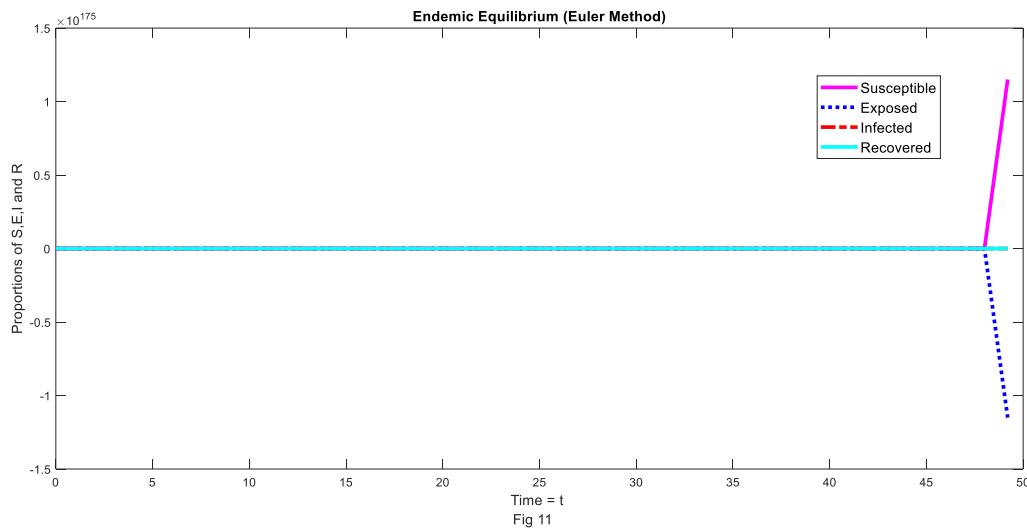


Figure 11: Graph of Forward Euler (FD) Scheme for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 1.2$.

Figures 10 and 11 represents the behavior of both the susceptible, exposed, infected and recovered individuals using Forward Euler (FD) scheme for the dental caries endemic equilibrium at $h = 0.2$ and $h = 1.2$ respectively. We shall observe that the scheme at the endemic equilibrium produce negative value for all the compartments. This is worthless and is due to the drawback of the Forward Euler (FD) scheme.

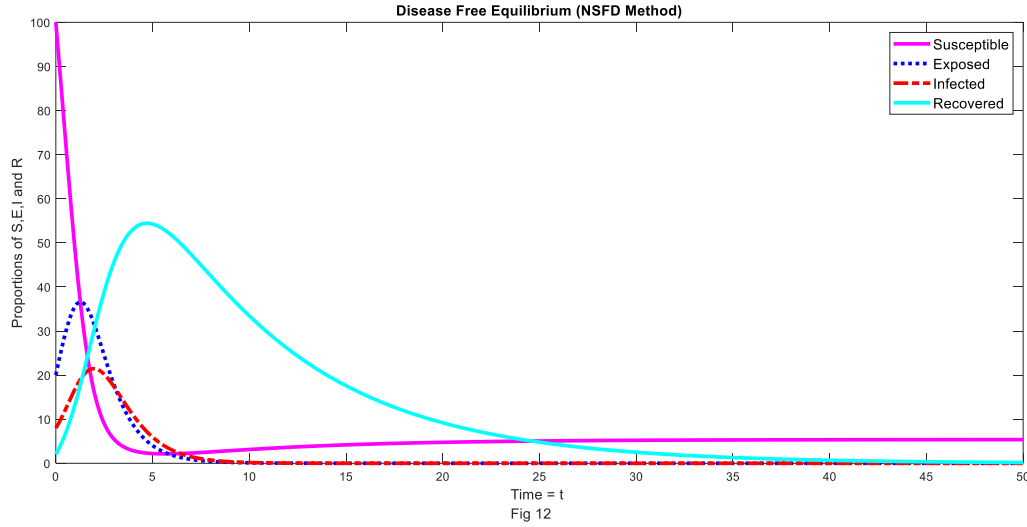


Figure 12: Graph of Non Standard Finite Difference (NSFD) Scheme for the Dental Caries (Decay) Free Equilibrium with step size $h = 0.2$.

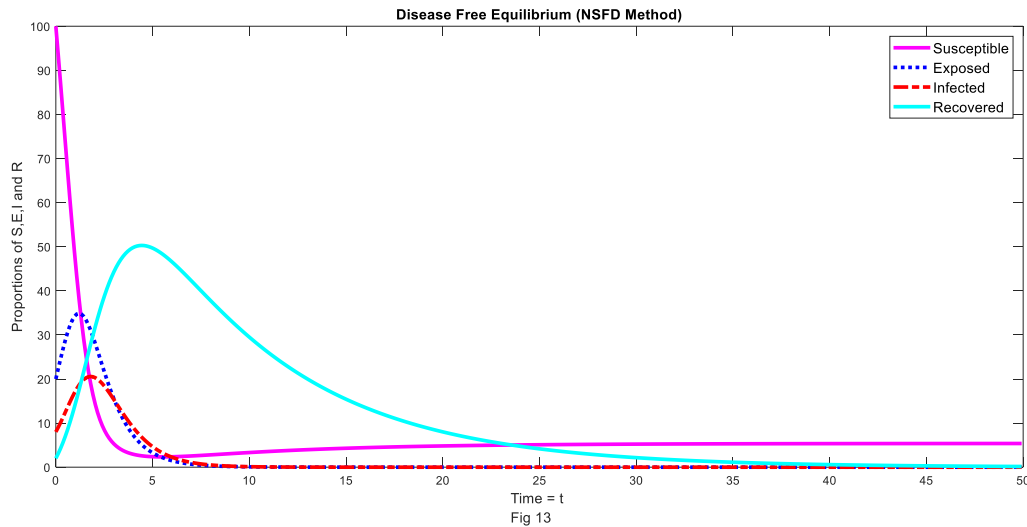


Figure 13: Graph of Non Standard Finite Difference (NSFD) Scheme for the Dental Caries (Decay) Free Equilibrium with step size $h = 1.2$.

Figures 12 and 13 showed the graphs of susceptible, exposed, infected and recovered populations using non-standard finite difference (NSFD) scheme with $h = 0.2$ and $h = 1.2$ respectively at the dental caries free equilibrium. These graphs of the proposed NSFD scheme preserves positivity property and converges to the true steady state of the continuous model for each values of the time step h .

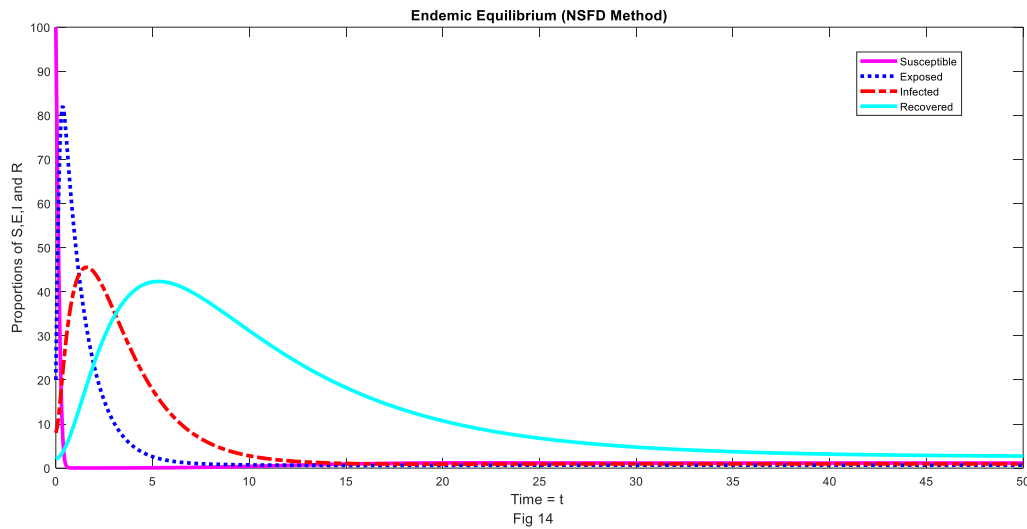


Figure 14: Graph of Non Standard Finite Difference (NSFD) Scheme for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 0.2$.

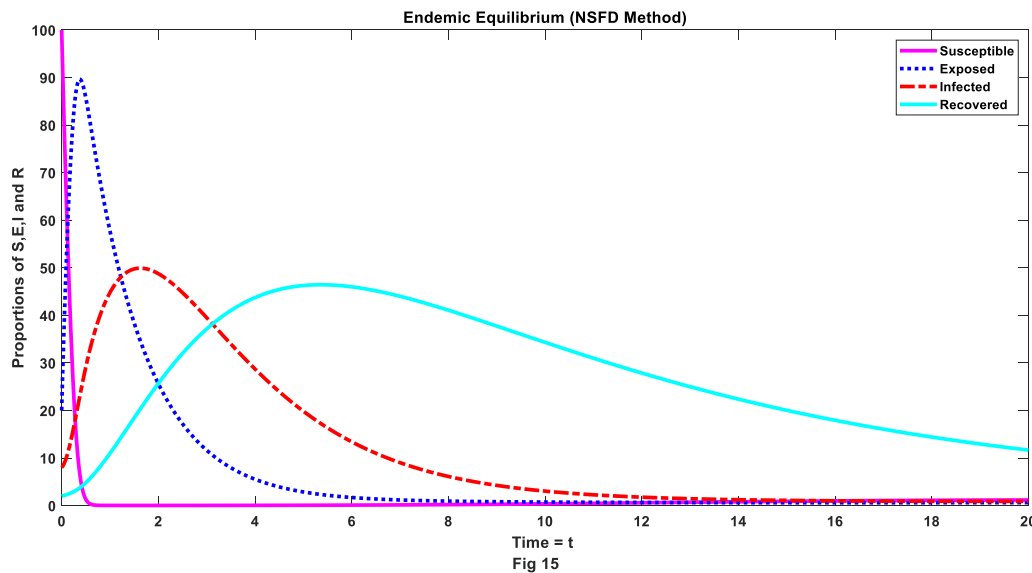


Figure 15: Graph of Non Standard Finite Difference (NSFD) Scheme for the Dental Caries (Decay) Endemic Equilibrium with step size $h = 1.2$.

Figure 14 and 15 showed the dental caries endemic equilibrium for the different compartments of the model system (1) using NSFD scheme at $h = 0.2$ and $h = 1.2$ respectively. The graphs indicated that NSFD scheme is unconditionally convergent for all time steps and preserves positivity property.

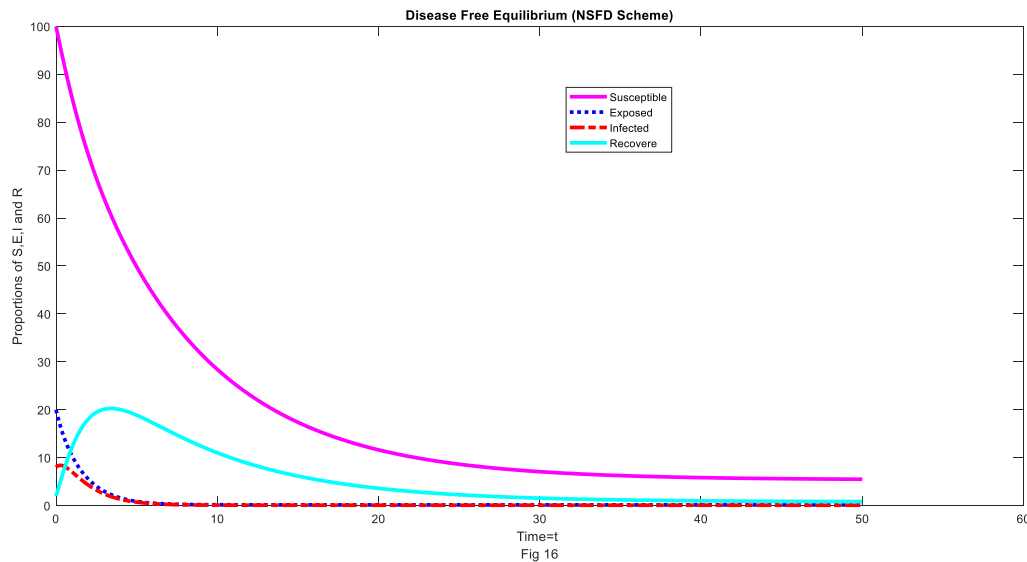


Figure 16: Graph of Non Standard Finite Difference (NSFD) Scheme for the Dental Caries (Decay) infection with step size $h = 0.2$, $\eta = 0.001$ and $\nu = 0.9$.

Figure 16 displayed the effect of maximum medical resources, saturated factors that measure the effect of the infected being delayed for treatment and saturated factor that measures the inhibitory effects, that is, prevention from being exposed. It is observed that with, with adequate prevention from being exposed, maximum medical resources and treatment not delayed, both the exposed and infected individuals dynamics will automatically reduce in the population thereby controlling dental caries (decay) infection in the population.

6 Conclusion

Many researchers applied many numerical techniques to solve many mathematical models. To analyze the dental caries (decay) SEIR epidemic model, we used three different numerical schemes such as; Runge Kutta RK-4 method, Forward Euler (FD) scheme and Non-Standard Finite Difference (NSFD) scheme. In this paper we are interested in the numerical solution dental caries infection using SEIR epidemic model with saturation incidence rate and saturated treatment function. In this research work, we developed Runge Kutta RK-4 method, Forward Euler (FD) scheme and Non Standard Finite Difference (NSFD) scheme for our SEIR epidemic model.

The basic reproduction number and stability analysis for dental caries (decay) free equilibrium and endemic equilibrium points are evaluated. In the three numerical schemes applied in this work, the Non-standard finite difference scheme was found to preserve the essential properties such as positivity, unconditional convergence, and stability. We analyze the convergence of the three numerical schemes of the epidemic model of dental caries (decay) and found from the result that NSFD scheme converges unconditionally for each step size, while Runge-Kutta RK-4 method and Forward Euler (FD) scheme fail to converge at the endemic equilibrium even with the small step size $h = 0.2$. The result also showed that only NSFD scheme preserve positivity property where as Runge Kutta RK-4 method and Forward Euler (FD) scheme fail preserve positivity property.

The effect of maximum medical resources and saturated factors that measure the effect of delaying the treatment of infected dental caries individuals were discussed graphically in figure 16. It was found that if medical resources are maximized and treatment of dental caries infected individuals not delayed, the exposed and infected individuals in the population will be decreased thereby reducing the effective contact rate. In this regard, dental caries (decay) is effectively and efficiently controlled in the population.

From the numerical analysis we conclude that the non-standard finite difference (NSFD) scheme performed better than Runge Kutta RK-4 method and Forward Euler (FD) scheme when considered especially in obtaining solution curves that are consistent with analytic solution. Also, that if maximum medical resources are made available with timely treatment of the infected dental caries individuals, dental caries infection will be eradicated in the population.

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An application of Homotopy Perturbation Method (HPM) for solving Influenza virus model in a population

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Abstract

This article examined application of Homotopy Perturbation Method (HPM) for solving influenza virus model. Some concepts and assumptions of homotopy perturbation method are discussed and homotopy perturbation method is used to solve differential equations with boundary conditions arising from influenza virus model. The total population is subdivided into eight epidemiological compartments in a homogenous and heterogeneous population. The obtained results as compared to those in existing literature is accurate. Analysis of the homotopy perturbation method showed that the method is flexible and very easy to understand.

Keywords and phrases: Homotopy Perturbation Method; infectious disease model; differential equation; convergent; series solution; Influenza virus model

1 Introduction

The homotopy perturbation method is a very unique tool and very effective technique for evaluating solutions of nonlinear equations without requiring process of linearization. This system was introduced by [4]. HPM is a combination of the perturbation and homotopy methods. The homotopy perturbation method yields a very rapid convergence series in most cases, usually only a few iterations leading to accurate solutions [16]. It is a universal one which can solve various kinds of linear and nonlinear equations. Benefits of HPM which distinguished it from other analytical methods include, HPM is a series expansion method that indirectly dependent on small or large physical parameters [2, 5], Therefore, it can be used to solve both varieties problems. It gives convergent series solutions and the perturbation equation is easy to construct. He used HPM to solve Schrodinger equation initial value problems [3]. Many researchers have used mathematical modelling to analyze infectious diseases and Homotopy Perturbation Method, see [1,6,8,9,10,11, 13,14, 15]. [1] developed basic reproduction numbers of infectious disease models and analysed basic reproduction numbers of many infectious diseases.

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2 The model equations

The model is given below:

$$\begin{aligned}
 \frac{ds}{dt} &= \Lambda - \beta_1 SI_1 - \beta_2 SI_2 - \beta_3 SI_3 - \mu S + \omega R, \\
 \frac{dI_1}{dt} &= \beta_1 SI_1 - (\sigma_1 + \gamma_1 + \rho_1 + \mu)I_1, \\
 \frac{dT_1}{dt} &= \sigma_1 I_1 - (\delta_1 + \varepsilon_1 + \mu)T_1, \\
 \frac{dI_2}{dt} &= \beta_2 SI_2 - (\sigma_2 + \gamma_2 + \rho_2 + \mu)I_2, \\
 \frac{dT_2}{dt} &= \sigma_2 I_2 - (\delta_2 + \varepsilon_2 + \mu)T_2, \\
 \frac{dI_3}{dt} &= \beta_3 SI_3 - (\sigma_3 + \gamma_3 + \rho_3 + \mu)I_3, \\
 \frac{dT_3}{dt} &= \sigma_3 I_3 - (\delta_3 + \varepsilon_3 + \mu)T_3, \\
 \frac{dR}{dt} &= \delta_1 T_1 + \gamma_1 I_1 + \delta_2 T_2 + \gamma_2 I_2 + \delta_3 T_3 + \gamma_3 I_3 - \mu R - \omega R,
 \end{aligned} \tag{1}$$

where

$$\begin{aligned}
 k_1 &= \sigma_1 + \gamma_1 + \rho_1 + \mu \\
 k_2 &= \sigma_2 + \gamma_2 + \rho_2 + \mu \\
 \alpha_1 &= \delta_1 + \varepsilon_1 + \mu \\
 \alpha_2 &= \delta_2 + \varepsilon_2 + \mu \\
 \alpha_3 &= \delta_3 + \varepsilon_3 + \mu
 \end{aligned}$$

P is an embedding parameter in $[0,1]$. Then we construct the Homotopy:

$$\begin{aligned}
 (1 - \rho) \frac{ds}{dt} + \rho \left[\frac{ds}{dt} - \Lambda + \beta_1 SI_1 + \beta_2 SI_2 + \beta_3 SI_3 + \mu S - \omega R \right] &= 0 \\
 (1 - \rho) \frac{dI_1}{dt} + \rho \left[\frac{dI_1}{dt} - \beta_1 SI_1 + ((\sigma_1 + \gamma_1 + \rho_1 + \mu)I_1) \right] &= 0, \\
 (1 - \rho) \frac{dT_1}{dt} + \rho \left[\frac{dT_1}{dt} - \sigma_1 I_1 + (\delta_2 + \varepsilon_2 + \mu)T_1 \right] &= 0, \\
 (1 - \rho) \frac{dI_2}{dt} + \rho \left[\frac{dI_2}{dt} - \beta_2 SI_2 + (\sigma_2 + \gamma_2 + \rho_2 + \mu)T_2 \right] &= 0, \\
 (1 - \rho) \frac{dT_2}{dt} + \rho \left[\frac{dT_2}{dt} - \sigma_2 I_2 + (\delta_2 + \varepsilon_2 + \mu)I_2 \right] &= 0 \\
 (1 - \rho) \frac{dI_3}{dt} + \rho \left[\frac{dI_3}{dt} - \beta_3 SI_3 + (\sigma_3 + \gamma_3 + \rho_3 + \mu)I_3 \right] &= 0 \\
 (1 - \rho) \frac{dT_3}{dt} + \rho \left[\frac{dT_3}{dt} - \sigma_3 I_3 + (\delta_3 + \varepsilon_3 + \mu)T_3 \right] &= 0 \\
 \frac{dR}{dt} + \rho [-\delta_1 T_1 - \gamma_1 I_1 - \delta_2 T_2 + \gamma_2 I_2 + \delta_3 T_3 + \gamma_3 I_3 + (\mu + \omega)R] &= 0.
 \end{aligned}$$

This implies

$$\left. \begin{aligned} \frac{ds}{dt} &= \rho \left[\frac{ds}{dt} + \Lambda - \beta_1 SI_1 - \beta_2 SI_2 - \beta_3 SI_3 - \mu + \omega R \right] \\ \frac{dI_1}{dt} &= \rho \left[\frac{dI_1}{dt} + \beta_1 SI_1 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) I_1 - \frac{dI_1}{dt} \right] \\ \frac{dT_1}{dt} &= \rho \left[\frac{dT_1}{dt} + \sigma_1 I_1 - (\delta_1 + \varepsilon_1 + \mu) T_1 - \frac{dT_1}{dt} \right] \\ \frac{dI_2}{dt} &= \rho \left[\frac{dI_2}{dt} + \beta_2 SI_2 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) I_2 - \frac{dI_2}{dt} \right] \\ \frac{dI_3}{dt} &= \rho \left[\frac{dI_3}{dt} + \beta_3 SI_3 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) I_3 - \frac{dI_3}{dt} \right] \\ \frac{dR}{dt} &= \rho \left[\frac{dR}{dt} + \delta_1 T_1 + \gamma_1 I_1 + \delta_2 T_2 + \gamma_2 I_2 + \delta_3 T_3 + \gamma_3 I_3 + (\mu + \omega) R - \frac{dR}{dt} \right] \end{aligned} \right\} \quad (2)$$

$$\begin{aligned} S_{(T)} &= S_0 + \rho S_1 + \rho^2 S_2 + \dots \\ I_{1(t)} &= I_{1(0)} + \rho I_{1(1)} + \rho^2 I_{1(2)} + \dots \\ T_{1(t)} &= T_{1(0)} + \rho I_{1(1)} + \rho^2 T_{1(2)} + \dots \\ I_{2(t)} &= I_{2(0)} + \rho I_{2(1)} + \rho^2 I_{2(2)} + \dots \\ T_{2(t)} &= T_{2(0)} + \rho I_{2(1)} + \rho^2 T_{2(2)} + \dots \\ I_{3(t)} &= I_{3(0)} + \rho I_{3(1)} + \rho^2 I_{3(2)} + \dots \\ T_{3(t)} &= T_{3(0)} + \rho I_{3(1)} + \rho^2 T_{3(2)} + \dots \\ R_{(t)} &= R_0 + \rho R_1 + \rho^2 R_2 + \dots \end{aligned}$$

The initial condition are as follows:

$$\begin{aligned} S_{(0)} &= u_1, I_1(0) = u_2, T_1(0) = u_3, I_2(0) = u_4, T_2(0) = u_5, \\ I_3(0) &= u_6, T_3(0) = u_7 \text{ and } R(0) = u_8. \end{aligned}$$

Carrying out the necessary substitutions, we have that,

$$\begin{aligned} \frac{d}{dt} (S_0 + \rho S_1 + \rho^2 S_2 + \dots) \\ &= \rho \left[\begin{aligned} &\Lambda - \beta_1 (S_0 + \rho S_1 + \rho^2 S_2 + \dots) (I_{1(0)} + \rho I_{1(1)} + \rho^2 I_{1(2)} + \dots) \\ &- \beta_2 (S_0 + \rho S_1 + \rho^2 S_2 + \dots) (I_{2(0)} + \rho I_{2(1)} + \rho^2 I_{2(2)} + \dots) \\ &- \beta_3 (S_0 + \rho S_1 + \rho^2 S_2 + \dots) (I_{3(0)} + \rho I_{3(1)} + \rho^2 I_{3(2)} + \dots) \\ &- \mu (S_0 + \rho S_1 + \rho^2 S_2 + \dots) + \omega (R_0 + \rho R_1 + \rho^2 R_2 + \dots) \end{aligned} \right] \\ \frac{d}{dt} (I_{1(0)} + \rho I_{1(1)} + \rho^2 I_{1(2)} + \dots) \\ &= \rho \left[\begin{aligned} &\beta_1 (S_0 + \rho S_1 + \rho^2 S_2 + \dots) (I_{1(0)} + \rho I_{1(1)} + \rho^2 I_{1(2)} + \dots) \\ &- (\sigma_1 + \gamma_1 + \rho_1 + \mu) (I_{1(0)} + \rho I_{1(1)} + \rho^2 I_{1(2)} + \dots) \end{aligned} \right] \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt}(T_{1(0)} + PT_{1(1)} + P^2T_{1(2)} + \dots) &= P \left[\begin{array}{c} \sigma_1(I_{1(0)} + PI_{1(1)} + P^2I_{1(2)} + \dots) \\ (\delta_1 + \varepsilon_1 + \mu)(T_{1(0)} + PT_{1(1)} + P^2T_{1(2)} + \dots) \end{array} \right] \\
 \frac{d}{dt}(I_{1(0)} + PT_{2(1)} + P^2I_{2(2)} + \dots) &= P \left[\begin{array}{c} \beta_2(S_0 + PS_1 + P^2S_2 + \dots)(I_{2(0)} + PI_{2(1)} + P^2I_{2(2)} + \dots) \\ -(\sigma_2 + \gamma_1 + \rho_1 + \mu)(I_{2(0)} + PI_{2(1)} + P^2I_{2(2)} + \dots) \end{array} \right] \\
 \frac{d}{dt}(T_{2(0)} + PT_{2(1)} + P^2T_{2(2)} + \dots) &= P \left[\begin{array}{c} \sigma_2(I_{2(0)} + PI_{2(1)} + P^2I_{2(2)} + \dots) \\ (\delta_2 + \varepsilon_2 + \mu)(T_{2(0)} + PT_{2(1)} + P^2T_{2(2)} + \dots) \end{array} \right] \\
 \frac{d}{dt}(I_{3(0)} + PI_{3(1)} + P^2I_{3(2)} + \dots) &= P \left[\begin{array}{c} \beta_3(S_0 + PS_1 + P^2S_2 + \dots)(I_{3(0)} + PI_{3(1)} + P^2I_{3(2)} + \dots) \\ -(\sigma_3 + \gamma_3 + \rho_3 + \mu)(I_{3(0)} + PI_{3(1)} + P^2I_{3(2)} + \dots) \end{array} \right] \\
 \frac{d}{dt}(T_{3(0)} + PT_{3(1)} + P^2T_{3(2)} + \dots) &= P \left[\begin{array}{c} \sigma_3(I_{3(0)} + PI_{3(1)} + P^2I_{3(2)} + \dots) \\ (\delta_3 + \varepsilon_3 + \mu)(T_{3(0)} + PT_{3(1)} + P^2T_{3(2)} + \dots) \end{array} \right] \\
 \frac{d}{dt}(R_0 + PR_1 + P^2R_2 + \dots) &= P \left[\begin{array}{c} \delta_1(T_{1(0)} + PT_{1(1)} + P^2T_{1(2)} + \dots) + \\ \gamma_1(I_{1(0)} + PI_{1(1)} + P^2I_{1(2)} + \dots) + \\ \delta_1(T_{2(0)} + PT_{2(1)} + P^2T_{2(2)} + \dots) + \\ \gamma_2(I_{2(0)} + PI_{2(1)} + P^2I_{2(2)} + \dots) + \\ \delta_1(T_{3(0)} + PT_{3(1)} + P^2T_{3(2)} + \dots) + \\ \gamma_3(I_{3(0)} + PI_{3(1)} + P^2I_{3(2)} + \dots) + \\ (\mu - \omega)(R_0 + PR_1 + P^2R_2 + \dots) \end{array} \right]
 \end{aligned}$$

Comparing them according to their identical powers:

$$\begin{aligned}
 P^0: \quad \frac{ds}{dt} &= 0 \\
 \frac{dI_1}{dt} &= 0 \\
 \frac{dT_1}{dt} &= 0 \\
 \frac{dI_2}{dt} &= 0 \\
 \frac{dT_2}{dt} &= 0 \\
 \frac{dI_3}{dt} &= 0 \\
 \frac{dT_3}{dt} &= 0
 \end{aligned}$$

$$\frac{dR}{dt} = 0.$$

$$\begin{aligned} P^1: \quad & \frac{dS_1}{dt} = \Lambda - \beta_1 S_0 I_{1(0)} - \beta_2 S_0 I_{2(0)} - \beta_3 S_0 I_{3(0)} - \mu S_0 + \omega R_0 \\ & \frac{dI_1}{dt} = \beta_1 S_0 T_{1(0)} - (\sigma_1 + \gamma_1 + \rho_1 + \mu) I_{1(0)} \\ & \frac{dT_1}{dt} = \sigma_1 I_{1(0)} - (\delta_1 + \varepsilon_1 + \mu) T_{1(0)} \\ & \frac{dI_2}{dt} = \beta_2 S_0 I_{2(0)} - (\sigma_2 + \gamma_2 + \rho_2 + \mu) I_{2(0)} \\ & \frac{dT_2}{dt} = \sigma_2 I_{2(0)} - (\delta_2 + \varepsilon_2 + \mu) T_{2(0)} \\ & \frac{dI_3}{dt} = \beta_3 S_0 I_{3(0)} - (\sigma_3 + \gamma_3 + \rho_3 + \mu) I_{3(0)} \\ & \frac{dT_3}{dt} = \sigma_3 I_{3(0)} - (\delta_3 + \varepsilon_3 + \mu) T_{3(0)} \\ & \frac{dR}{dt} = \delta_1 T_{1(0)} + \gamma_1 I_{1(0)} + \delta_2 T_{2(0)} + \gamma_2 I_{2(0)} + \delta_3 T_{3(0)} + \gamma_3 I_{3(0)} + (\mu - \omega). \end{aligned}$$

$$\begin{aligned} P^2: \quad & \frac{dS_2}{dt} = \Lambda - \beta_1 S_0 I_{1(0)} - \beta_2 S_0 I_{2(0)} - \beta_3 S_0 I_{3(1)} - \mu S_1 + \omega R_1 \\ & \frac{dI_1}{dt} = \beta_1 S_0 T_{1(0)} - (\sigma_1 + \gamma_1 + \rho_1 + \mu) I_{1(1)} \\ & \frac{dT_1}{dt} = \sigma_1 I_{1(0)} - (\delta_1 + \varepsilon_1 + \mu) T_{1(1)} \\ & \frac{dI_2}{dt} = \beta_2 S_1 I_{2(1)} - (\sigma_2 + \gamma_2 + \rho_2 + \mu) I_{2(1)} \\ & \frac{dT_2}{dt} = \sigma_2 I_{2(1)} - (\delta_2 + \varepsilon_2 + \mu) T_{2(1)} \\ & \frac{dI_3}{dt} = \beta_3 S_1 I_{3(1)} - (\sigma_3 + \gamma_3 + \rho_3 + \mu) I_{3(1)} \\ & \frac{dT_3}{dt} = \sigma_3 I_{3(1)} - (\delta_3 + \varepsilon_3 + \mu) T_{3(1)} \\ & \frac{dR}{dt} = \delta_1 T_{1(1)} + \gamma_1 I_{1(1)} + \delta_2 T_{2(1)} + \gamma_2 I_{2(1)} + \delta_3 T_{3(1)} + \gamma_3 I_{3(1)} + (\mu - \omega) R_1. \end{aligned}$$

For powers of P^0 , we have that

$$\begin{aligned} S_0 = u_1, I_{1(0)} = u_2, T_{1(0)} = u_3, I_{2(0)} = u_4, T_{2(0)} = u_1, T_{2(0)} = u_5 \\ I_{2(0)} = u_5, T_{3(0)} = u_7, R_0 = U_8 \end{aligned}$$

For POWER s of p^1 , we have

$$\begin{aligned}\frac{ds}{dt} &= \Lambda - \beta_1 u_1 u_2 - \beta_2 u_1 u_4 - \beta_3 u_1 u_6 - \mu u_1 \omega u_8 \\ \frac{dI_1}{dt} &= \beta_1 u_1 u_2 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) u_2 \\ \frac{dT_1}{dt} &= \sigma_1 u_2 - (\delta_1 + \varepsilon_1 + \mu) u_3 \\ \frac{dI_2}{dt} &= \beta_2 u_1 u_4 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) u_4 \\ \frac{dT_2}{dt} &= \sigma_2 u_4 - (\delta_2 + \varepsilon_2 + \mu) u_5 \\ \frac{dI_3}{dt} &= \beta_3 u_1 u_6 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) u_6 \\ \frac{dT_3}{dt} &= \sigma_3 u_6 - (\delta_3 + \varepsilon_3 + \mu) u_7 \\ \frac{dR}{dt} &= \delta_1 u_3 + \gamma_1 u_2 + \delta_2 u_5 + \gamma_2 u_4 + \delta_3 u_7 + \gamma_3 u_6 + (\mu - \omega) u_8.\end{aligned}$$

Now let $u_1 u_2 = u_9, u_1 u_4 = u_{10}, u_1 u_6 = u_{11}$. Hence, integrating with respect to t, we have

$$\begin{aligned}S &= (\Lambda - \beta_1 u_9 - \beta_2 u_{10} - \beta_3 u_{11} - \mu u_1 - \omega u_8) t \\ S_{(1)} &= D_1 t \text{ where } D_1 = (\Lambda - \beta_1 u_9 - \beta_2 u_{10} - \mu u_1 - \omega u_8) \\ I_{1(1)} &= [\beta_1 u_9 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) u_2] t \\ I_{1(1)} &= D_2 t \text{ where } D_2 = [\beta_1 u_9 - (\sigma_1 + \gamma_1 + \rho_1 + \mu)] \\ T_{1(1)} &= [\beta_1 u_2 - (\delta_1 + \varepsilon_1 + \mu_3)] t \\ I_{2(1)} &= [\beta_2 u_{10} - (\sigma_2 + \gamma_2 + \rho_2 + \mu) u_4] t \\ T_{2(1)} &= D_4 t \text{ where } D_3 = [\beta_2 u_{10} - (\sigma_2 + \gamma_2 + \rho_2 + \mu)] u_4 \\ T_{2(1)} &= [\sigma_2 u_4 - (\delta_2 + \varepsilon_2 + \mu_3) u_5] t \\ T_{2(1)} &= D_5 t \text{ where } D_5 = [\sigma_2 u_4 - (\delta_2 + \varepsilon_2 + \mu) u_5] \\ I_{3(1)} &= [\beta_3 u_{11} - (\sigma_3 + \gamma_3 + \rho_3 + \mu) u_6] t \\ T_{3(1)} &= D_6 t \text{ where } D_6 = [\beta_3 u_{11} - (\sigma_3 + \gamma_3 + \rho_3 + \mu)] \\ T_{3(1)} &= [\sigma_3 u_6 - (\delta_3 + \varepsilon_3 + \mu) u_7] t \\ T_{3(1)} &= D_7 t \text{ where } D_7 = [\sigma_3 u_6 - (\delta_3 + \varepsilon_3 + \mu)] \\ R_{(1)} &= [\gamma_1 u_2 + \delta_1 u_3 + \gamma_2 u_4 + \delta_2 u_5 + \gamma_3 u_6 + \delta_3 u_7 + (\mu - \omega)] t \\ R_{(1)} &= D_8 t \text{ where } D_8 = [\gamma_1 u_2 + \delta_1 u_3 + \gamma_2 u_4 + \delta_2 u_5 + \gamma_3 u_6 + \delta_3 u_7 + (\mu - \omega)] u_8.\end{aligned}$$

For the power of P^2 , integrating we have,

$$\begin{aligned}S_2 &= (-\beta_1 D_1 D_2 - \beta_2 D_1 D_4 - \beta_3 D_1 D_2 - \mu D_1 + \omega D_8) \frac{t^2}{2} \\ I_{1(2)} &= (\beta_1 D_1 D_2 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) D_2) \frac{t^2}{2}\end{aligned}$$

$$\begin{aligned}T_{1(2)} &= (\sigma_1 D_1 - (\delta_1 + \varepsilon_1 + \mu) D_3) \frac{t^2}{2} \\I_{2(2)} &= (\beta_2 D_1 D_4 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) D_2) \frac{t^2}{2} \\T_{2(2)} &= (\sigma_2 D_4 - (\delta_2 + \varepsilon_2 + \mu) D_5) \frac{t^2}{2} \\I_{3(2)} &= (\beta_3 D_1 D_6 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) D_6) \frac{t^2}{2} \\T_{3(2)} &= (\sigma_3 D_6 - (\delta_3 + \varepsilon_3 + \mu) D_7) \frac{t^2}{2} \\R_{(2)} &= (\gamma_1 D_2 + \delta_1 D_3 + \gamma_2 D_4 + \delta_2 D_5 + \gamma_3 D_6 + \delta_3 D_7 + (\mu - \omega) D_8) \frac{t^2}{2}.\end{aligned}$$

Let the $0(\rho^3)$ be very small such that by asymptotic series approximation:

$$\begin{aligned}S(t) &= u_1 + \rho D_1 t + \rho^2 (-\beta_1 D_1 D_2 - \beta_2 D_1 D_4 - \beta_3 D_1 D_6 - \mu D_1 + \omega D_8) \frac{t^2}{2} + o(\rho^3) \\I_1(t) &= u_2 + \rho D_2 t + \rho^2 (-\beta_1 D_1 D_2 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) D_2) \frac{t^2}{2} + o(\rho^3) \\T_1(t) &= u_3 + \rho D_3 t + \rho^2 (\sigma_1 D_1 - (\delta_1 + \varepsilon_1 + \mu)) \frac{t^2}{2} + o(\rho^3) \\I_2(t) &= u_4 + \rho D_4 t + \rho^2 (\beta_2 D_1 D_2 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) D_2) \frac{t^2}{2} + o(\rho^3) \\T_2(t) &= u_5 + \rho D_5 t + \rho^2 (\sigma_2 D_1 - (\delta_2 + \gamma_2 + \mu) D_3) \frac{t^2}{2} + o(\rho^3) \\I_3(t) &= u_6 + \rho D_6 t + \rho^2 (\beta_3 D_1 D_6 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) D_6) \frac{t^2}{2} + o(\rho^3) \\T_3(t) &= u_7 + \rho^2 (\sigma_3 D_6 - (\delta_3 + \varepsilon_3 + \mu) D_5) \frac{t^2}{2} + o(\rho^3) \\R(t) &= u_8 + \rho D_8 t + \rho (\gamma_1 D_2 + \delta_1 D_3 + \gamma_2 D_4 + \delta_2 D_5 + \gamma_3 D_6 + \delta_3 D_7 + (\mu - \omega) D_8) \frac{t^2}{2}.\end{aligned}$$

We have

$$\begin{aligned}S(t) &= u_1 + \rho D_1 t + \rho^2 (-\beta_1 D_1 D_2 - \beta_2 D_1 D_4 - \beta_3 D_1 D_6 - \mu D_1 + \omega D_8) \frac{t^2}{2} < o(\rho^3), \\I_1(t) &= u_2 + \rho D_2 t + \rho^2 (\beta_1 D_1 D_2 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) D_2) \frac{t^2}{2} < o(\rho^3), \\T_1(t) &= u_3 + \rho D_3 t + \rho^2 (\sigma_1 D_1 - (\delta_1 + \varepsilon_1 + \mu) D_3) \frac{t^2}{2} < o(\rho^3) \\I_2(t) &= u_4 + \rho D_4 t + \rho^2 (\beta_2 D_1 D_2 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) D_2) \frac{t^2}{2} < o(\rho^3) \\T_2(t) &= u_5 + \rho D_5 t + \rho^2 (\sigma_2 D_1 - (\delta_2 + \gamma_2 + \mu) D_3) \frac{t^2}{2} < o(\rho^3)\end{aligned}$$

$$I_3(t) = u_6 + \rho D_6 t + \rho^2(\beta_3 D_1 D_6 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) D_6) \frac{t^2}{2} < o(\rho^3)$$

$$T_3(t) = u_7 + \rho^2 D_7 - \rho^2(\sigma_3 D_6 - (\delta_3 + \varepsilon_3 + \mu) D_5) \frac{t^2}{2} < o(\rho^3)$$

$$R(t) = u_8 + \rho D_8 t + \rho^2(\gamma_1 D_2 + \delta_1 D_3 + \gamma_2 D_4 + \delta_2 D_5 + \gamma_3 D_6 + \delta_3 D_7 + (\mu - \omega) D_8) < o(\rho^3).$$

Thus,

$$S(t) \approx u_1 + \rho D_1 t + \rho^2(-\beta_1 D_1 D_2 - \beta_2 D_1 D_4 - \beta_3 D_1 D_6 - \mu D_1 + \omega D_8) \frac{t^2}{2}$$

$$I_1(t) \approx u_2 + \rho D_2 t + \rho^2(-\beta_1 D_1 D_2 - (\sigma_1 + \gamma_1 + \rho_1 + \mu) D_2) \frac{t^2}{2}$$

$$T_1(t) \approx u_3 + \rho D_3 t + \rho^2(\sigma_1 D_1 - (\delta_1 + \varepsilon_1 + \mu)) \frac{t^2}{2}$$

$$I_2(t) \approx u_4 + \rho D_4 t + \rho^2(\beta_2 D_1 D_2 - (\sigma_2 + \gamma_2 + \rho_2 + \mu) D_2) \frac{t^2}{2}$$

$$T_2(t) \approx u_5 + \rho D_5 t + \rho^2(\sigma_2 D_1 - (\delta_2 + \gamma_2 + \mu) D_3) \frac{t^2}{2}$$

$$I_3(t) \approx u_6 + \rho D_6 t + \rho^2(\beta_3 D_1 D_6 - (\sigma_3 + \gamma_3 + \rho_3 + \mu) D_6) \frac{t^2}{2}$$

$$T_3(t) \approx u_7 + \rho^2(\sigma_3 D_6 - (\delta_3 + \varepsilon_3 + \mu) D_5) \frac{t^2}{2}$$

$$R(t) \approx u_8 + \rho D_8 t + \rho(\gamma_1 D_2 + \delta_1 D_3 + \gamma_2 D_4 + \delta_2 D_5 + \gamma_3 D_6 + \delta_3 D_7 + (\mu - \omega) D_8).$$

3 Conclusion

In this article, application of Homotopy Perturbation Method (HPM) for solving an influenza virus model was investigated, with some concepts of homotopy perturbation method and steps involved in HPM presented. It is observed that homotopy perturbation method gives a series solution that is mostly convergent for both linear and nonlinear differential equations [10]. From analysis of the model using HPM, it is discovered that the method is a sound analytical technique which is easy to apply in solving physical problems

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Solution of fractional order integro-differential equations by two semi-analytical methods

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Abstract

This paper is concerned with the solution of fractional order integro-differential equations where the fraction order, η is in the interval $(3, 4]$. Two semi-analytical methods; Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) are implemented for the solution of the class of problems considered. Few examples are solved and the results obtained compared favorably with the existing results in the literature. All codes used in solving the problems are written in Mathematica 11.3

Keywords and phrases: integro-differential equation; semi-analytical method; fractional order; caputo derivative; correction functional

1 Introduction

Recently, serious interest has been developed in fractional differential equation due to their many applications in physics and engineering. Absence of analytical solutions in most differential equations lead to application of approximation and numerical methods. A Chinese mathematician was the first to propose the Variational Iteration Method (VIM) [9] and Homotopy Perturbation Method (HPM) [10] to solve the linear and non-linear problems. Variational iteration method is based on Lagrange multiplier and homotopy perturbation method is a coupling of traditional perturbation method and homotopy in topology. In ascertaining the solutions of functional equations, which occur in engineering and scientific problems, authors apply these methods successfully.

The Adomain decomposition method presents solutions of functional equations, but some difficulties arise during the computation of Adomian polynomials and variational iteration method (VIM) and homotopy perturbation method (HPM) overcome the difficulties [3,14]. The Integro-differential Equation (IDE) is an equation that involves both integral and derivatives of an unknown function [1]. In recent years, several numerical methods such as spline functions, collocation method, etc., have been used to solve integro-differential equations. The purpose of this research is to apply both variational iteration method (VIM) [9,14,18] and homotopy perturbation method (HPM) [7,8,10] to solve fractional order integro-differential equations.

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2 Preliminaries

2.1 Fractional order derivative

Riemann – Liouville gave a very good definition of fractional order derivative, but with a draw back. The latter definition given by Caputo in 1967 accounted for the draw back and offered a more reliable definition that does not depend on the availability of initial conditions of the model being proffered solution to [2,15,18,19,20]. The superiority of Caputo derivative was established by Ford and Simpson in 2001.

2.2 Caputo Derivatives

Definition: Let $\beta \in \mathbb{R}_+$ and $\eta = [\beta]$. The operator D_{*a}^β defined by

$$D_{*a}^\beta f(t) = J_a^{\eta-\beta} D^\eta f(t) = \frac{1}{\Gamma(\eta-\beta)} \int_a^t (t-\tau)^{\eta-\beta-1} \left(\frac{d}{d\tau}\right)^\eta f(\tau) d\tau, \quad (1)$$

for $a \leq t \leq b$, is called the Caputo differential operator of order β . The definition can equally be expressed as

$$D^\beta t^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\beta+1)} t^{\eta-\beta}. \quad (2)$$

2.3 Statement of problem

$$D^\eta u(x) = f(x) + \mu u(x) + \int_0^x [g(t)u(t) + h(t)F(u(t))] dt, \quad (3)$$

$$0 < x < b, \quad 3 < \eta \leq 4,$$

subject to the following boundary conditions

$$u(0) = \mu_0, \quad u''(0) = \mu_2$$

$$u(b) = \beta_0, \quad u''(b) = \beta_2.$$

2.4 Implementation of Variational Iteration Method Algorithm

To illustrate (3) using VIM, we construct the correction functional as

$$u_{k+1}(x) = u_k(x) + J^\eta [D^\eta u_k(x) - f(x) - \mu u_k - \int_0^x g(t)u_k(t) + h(t)F(u_k(t))dt],$$

where J^β is the Reimann-Liouville fractional integral operator of order $\beta = \eta + 1 - m$, λ is a general lagrange multiplier and u_k denotes restricted variation, $\lambda(s) = \frac{(-1)^m}{(m-1)!} (s-x)^{m-1}$, where m is the order of the differential equation.

If $m = 4$

$$\begin{aligned} \lambda(s) &= \frac{1}{3!} (s-x)^3 \\ &= \frac{1}{6} (s-x)^3 \end{aligned}$$

$$J^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(t) dt, \quad \eta > 0$$

$$u_{k+1}(x) = u_k(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \lambda(s) (D^\eta u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t)u_k(t) + h(t)F(u_k(t))dt).$$

But $\beta = \eta + 1 - m$, if $m = 4$,

$$\begin{aligned} \beta &= \eta + 1 - 4 \\ \beta &= \eta - 1. \end{aligned}$$

Also,

$$\begin{aligned} \beta - 1 &= \eta - 3 - 1 \\ \beta - 1 &= \eta - 4 \end{aligned}$$

$$u_{k+1}(x) = u_k(x) + \frac{1}{6\Gamma(\eta-3)} \int_0^x (x-s)^{\eta-4} (s-x)^3 (D^\eta u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t)u_k(t) + h(t)F(u_k(t))dt).$$

But

$$\begin{aligned} (x-s)^{\eta-4} (s-x)^3 &= -(x-s)^3 (x-s)^{\eta-4} \\ &= -(x-s)^{\eta-1}. \end{aligned}$$

Also,

$$\frac{1}{6\Gamma(\eta-3)} = \frac{(\eta-1)(\eta-2)(\eta-3)}{6\Gamma(\eta)}.$$

So,

$$u_{k+1}(x) = u_k(x) + \frac{(\eta-1)(\eta-2)(\eta-3)}{6\Gamma(\eta)} \int_0^x (-(x-s)^{\eta-1}) (D^\eta u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t)u_k(t) + h(t)F(u_k(t))dt)$$

$$u_0(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0),$$

since $u_0 = \mu_0, u''(0) = \mu_2$. Then,

$$u_0(x) = \mu_0 + \mu_1 x + \frac{x^2}{2}\mu_2 + \frac{x^3}{6}\mu_3,$$

where $u'(0) = \mu_1$ and $u'''(0) = \mu_3$.

When $k = 0$:

$$u_1(x) = u_0(x) + \frac{-(\eta-3)(\eta-2)(\eta-1)}{6\Gamma(\eta)} \int_0^x (-(x-s)^{\eta-1}) (D^\eta u_0(x) - f(x) - \mu u_0(x) - \int_0^x g(t)u_0(t) + h(t)F(u_0(t))dt).$$

But $J^\eta = \frac{1}{\Gamma(\eta)} \int_0^x (x-s)^{\eta-1} ds$

$$u_1(x) = u_0(x) + \frac{-(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta u_0(x) - f(x) - \mu u_0(x) - \int_0^x g(t)u_0(t) + h(t)F(u_0(t))dt).$$

3 Numerical examples on fractional order integro-differential equation using VIM

Solutions of fractional order integro-differential equation by using VIM are considered here. The following problems are solved using VIM

Problem 1:

Consider the following non-linear integro-differential equation.

$$D^\eta(x) = e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x u(t)u''(t)dt, \quad 0 < x < 1, \quad 3 < \eta \leq 4,$$

with the boundary conditions

$$\begin{aligned} u(0) &= 1, \quad u'(0) = 1 \\ u(1) &= e, \quad u''(0) = 1. \end{aligned}$$

The correction functional is

$$u_{k+1}(x) = u_k(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta u_k - e^x + \frac{1}{2}e^{2x} - \frac{1}{2} - \int_0^x u_k(t)u_k''(t)dt).$$

We take the truncated Taylor expansion for e^x and e^{2x} as $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ And $e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3}$.

Also,

$$\begin{aligned} u_0(x) &= u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0) \\ u(0) &= 1, \quad u'(0) = 1, \quad u''(0) = 1 \\ \therefore u_0(x) &= 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} \end{aligned}$$

where $A = u'''(0)$.

When $k = 0$,

$$u_1(x) = u_0(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta u_0(x) - e^x + \frac{1}{2}e^{2x} - \frac{1}{2} - \int_0^x u_0(t)u_0''(t)dt)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(D^\eta \left(1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} \right) - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}) + \frac{1}{2} (1 + 2x + 2x^2 + \frac{4x^3}{3}) - \frac{1}{2} - \int_0^x (1 + t + \frac{t^2}{2} + \frac{At^3}{6}) (2 + At) dt \right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(D^\eta \left(1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} \right) - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{2} + x + x^2 + \frac{2x^3}{3} - \frac{1}{2} + 2x + x^2 + \frac{Ax^2}{2} + \frac{x^3}{6} + \frac{Ax^3}{3} + \frac{7Ax^4}{48} + \frac{A^2x^5}{30} \right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(0 - 1 + 2x + \frac{1}{2} (3 + A)x^2 + \frac{1}{3} (2 + A)x^3 + \frac{7Ax^4}{48} + \frac{A^2x^5}{30} \right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(-1 + 2x + \frac{1}{2} (3 + A)x^2 + \frac{1}{3} (2 + A)x^3 + \frac{7Ax^4}{48} + \frac{A^2x^5}{30} \right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{x^\eta}{\Gamma(1+\eta)} + \frac{2x^{\eta+1}}{\Gamma(2+\eta)} + \frac{(3+A)x^{\eta+2}}{\Gamma(3+\eta)} + \frac{(4+A)x^{\eta+3}}{\Gamma(4+\eta)} + \frac{7Ax^{\eta+4}}{2\Gamma(5+\eta)} + \frac{4A^2x^{\eta+5}}{\Gamma(6+\eta)} \right).$$

Imposing the boundary condition $u(1) = 3$,

$$u_1(x) = \frac{5}{2} + \frac{A}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{1}{\Gamma(1+\eta)} + \frac{2}{\Gamma(2+\eta)} + \frac{(3+A)}{\Gamma(3+\eta)} + \frac{(4+A)}{\Gamma(4+\eta)} + \frac{7A}{2\Gamma(5+\eta)} + \frac{4A^2}{\Gamma(6+\eta)} \right) = e.$$

Solving for A in $u(1)$, we have:

At $\eta = 4.00$, $A = 1.20370$

At $\eta = 3.75$, $A = 1.22641$

At $\eta = 3.50$, $A = 1.25565$

At $\eta = 3.25$, $A = 1.28589$.

Substituting the values of A in $u_1(x)$, we have:

at $\eta = 4.00$, $A = 1.203699334$

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.200617x^3 + 0.041667x^4 - 0.166667x^5 - 0.00583847x^6 \\ - 0.00127131x^7 - 0.000208976x^8 - 0.000015971x^9$$

at $\eta = 3.75$, $A = 1.22641$

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.204402x^3 + 0.0362688x^{3.75} - 0.015297x^{4.75} - 0.0056123x^{5.75} \\ - 0.00126946x^{6.75} - 0.000217922x^{7.75} - 0.0000174538x^{8.75}$$

at $\eta = 3.50$, $A = 1.25565$

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.209275x^3 + 0.0268662x^{3.5} - 0.0119405x^{4.5} - 0.00461952x^{5.5} \\ - 0.00108739x^{6.5} - 0.000195714x^{7.5} - 0.0000165209x^{8.5}$$

at $\eta = 3.25$, $A = 1.28589$

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.214315x^3 + 0.014144x^{3.25} - 0.00665618x^{4.25} \\ - 0.00271692x^{5.25} - 0.00066656x^{6.25} - 0.00125927x^{7.25} \\ - 0.00001121x^{7.25}$$

Table 1: Varying the value of x from 0 to 1, at $\eta = 4$, $u(x)$

x	Exact Solution	VIM	Error
0.0	1.0000000	1.0000000	0.0000000
0.1	1.1051709	1.1052046	0.0000337
0.2	1.2214028	1.2216659	0.0002631
0.3	1.3498588	1.3507091	0.0008503
0.4	1.4918247	1.4937093	0.0018846
0.5	1.6487213	1.6520584	0.0033371
0.6	1.8221188	1.8271257	0.0050069
0.7	2.0137527	2.0202108	0.0064581
0.8	2.2255409	2.2324886	0.0069477
0.9	2.2324886	3.4596031	0.2271151
1.0	2.7182818	2.7182818	0.0000000

Problem 2:

Consider the following non-linear integro-differential equation.

$$D^\eta u(x) = e^{-x} + e^{-3x} - 1 + 3 \int_0^x u^3(t)dt, \quad (4)$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = -1 \\ u''(0) = 1, \quad u'''(0) = 1.$$

The correction functional is

$$u_{k+1}(x) = u_k(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta u_k(x) - e^{-x} - e^{-3x} + 1 - 3 \int_0^x u_k^3(t) u_k''(t) dt).$$

We take the truncated Taylor expansion for e^{-x} and e^{-3x} as $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ and $e^{-3x} = 1 - 3x + \frac{9x^2}{2} + \frac{9x^3}{2}$.

Also,

$$\begin{aligned} u_0(x) &= u(0) + xu'(0) + \frac{x^2}{2!} u''(0) + \frac{x^3}{3!} u'''(0) \\ u(0) &= 1, \quad u'(0) = -1, \quad u''(0) = 1, \quad u'''(0) = 1 \\ \therefore u_0(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6}. \end{aligned}$$

When $k = 0$,

$$\begin{aligned} u_1(x) &= u_0(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta u_0(x) - e^{-x} - e^{-3x} + 1 - 3 \int_0^x u_0^3(t) dt) \\ u_1(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} \\ &\quad - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(D^\eta \left(1 - x + \frac{x^2}{2} + \frac{x^3}{6} \right) - \left(1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} \right) \right. \\ &\quad \left. - \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} \right) + 1 - 3 \int_0^x \left(1 - t + \frac{t^2}{2} + \frac{t^3}{6} \right)^3 dt \right) \\ u_1(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} \\ &\quad - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(0 - 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - 1 + 3x - \frac{9x^2}{2} + \frac{9x^3}{2} + 1 \right. \\ &\quad \left. - 3 \int_0^x \left(1 - t + \frac{t^2}{2} + \frac{t^3}{6} \right)^3 dt \right) \\ u_1(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} \\ &\quad - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(-1 + x - \frac{x^2}{2} + \frac{13x^3}{6} + \frac{21x^4}{8} - \frac{3x^5}{4} - \frac{x^6}{8} + \frac{x^7}{8} \right. \\ &\quad \left. - \frac{x^8}{64} - \frac{x^9}{72} - \frac{x^{10}}{720} \right) \\ \therefore u_1(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{x^\eta}{\Gamma(1+\eta)} + \frac{x^{\eta+1}}{\Gamma(2+\eta)} - \frac{x^{\eta+2}}{\Gamma(3+\eta)} - \frac{13x^{\eta+3}}{\Gamma(4+\eta)} + \frac{63x^{\eta+4}}{\Gamma(5+\eta)} - \right. \\ &\quad \left. \frac{90x^{\eta+5}}{\Gamma(6+\eta)} - \frac{90x^{\eta+6}}{\Gamma(7+\eta)} + \frac{630x^{\eta+7}}{\Gamma(8+\eta)} - \frac{630x^{\eta+8}}{\Gamma(9+\eta)} + \frac{630x^{\eta+9}}{\Gamma(10+\eta)} - \frac{5040x^{\eta+10}}{\Gamma(11+\eta)} \right), \end{aligned}$$

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{1}{\Gamma(1+\eta)} + \frac{1}{\Gamma(2+\eta)} - \frac{1}{\Gamma(3+\eta)} - \frac{13}{\Gamma(4+\eta)} + \frac{63}{\Gamma(5+\eta)} - \frac{90}{\Gamma(6+\eta)} - \frac{90}{\Gamma(7+\eta)} + \frac{630}{\Gamma(8+\eta)} - \frac{630}{\Gamma(9+\eta)} + \frac{630}{\Gamma(10+\eta)} - \frac{5040}{\Gamma(11+\eta)} \right).$$

Varying the values of η in $u_1(x)$ we have:

at $\eta = 4.00$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \frac{13x^7}{5040} - \frac{x^8}{640} + \frac{x^9}{4032} + \frac{x^{10}}{40320} - \frac{x^{11}}{63360} + \frac{x^{12}}{760320} + \frac{x^{13}}{9884160} + \frac{x^{14}}{17297280}$$

At $\eta = 3.75$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0362688x^{3.75} - 0.00763555x^{4.75} + 0.00132792x^{5.75} + 0.00255748x^{6.75} - 0.00159922x^{7.75} + 0.000261097x^{8.75} + 0.0000267791x^{9.75} - 0.0000174376x^{10.75} + 1.48405 \times 10^{-6}x^{11.75} + 1.16396 \times 10^{-7}x^{12.75} + 6.77213 \times 10^{-8}x^{13.75}$$

At $\eta = 3.50$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0268662x^{3.5} - 0.00597026x^{4.5} + 0.0010855x^{5.5} + 0.002171x^{6.5} - 0.0014028x^{7.5} + 0.000235765x^{8.5} + 0.0000248174x^{9.5} - 0.000016544x^{10.5} + 1.43869 \times 10^{-6}x^{11.5} + 1.15095 \times 10^{-7}x^{12.5} + 6.82045 \times 10^{-8}x^{13.5}$$

At $\eta = 3.25$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.014144x^{3.25} - 0.00332809x^{4.25} + 0.000633922x^{5.25} + 0.00131856x^{6.25} - 0.000881371x^{7.25} + 0.000152618x^{8.25} + 0.0000164998x^{9.25} - 0.0000112678x^{10.25} + 1.00158 \times 10^{-6}x^{11.5} + 8.17618 \times 10^{-7}x^{12.5} + 4.93656 \times 10^{-8}x^{13.5}.$$

Table 2: Varying the values of $x = 0(0.04)0.36$, at $\eta = 4$, $u(x)$

x	Exact Solution	VIM	Error
0.00	1.0000000	1.0000001	0.0000000
0.04	0.9607894	0.9608108	0.0000214
0.08	0.9231163	0.9232870	0.0001707
0.12	0.8869204	0.8874964	0.0005760
0.16	0.8521438	0.8535091	0.0013653
0.20	0.8187308	0.8213975	0.0026667
0.24	0.7866279	0.7912360	0.0004608
0.28	0.7557837	0.7631014	0.0073177
0.32	0.7261490	0.7370725	0.0109235
0.36	0.6976763	0.7132301	0.0155538

Statement of Problem

$$D^\eta u(x) = f(x) + \mu u(x) + \int_0^x [g(t)u(t) + h(t)F(u(t))]dt \quad (5)$$

$$0 < x < b, \quad 3 < \eta \leq 4,$$

subject to the following boundary conditions

$$\begin{aligned} u(0) &= \mu_0, & u''(0) &= \mu_2 \\ u(b) &= \beta_0, & u''(b) &= \beta_2. \end{aligned}$$

4 Implementation of Homotopy Perturbation Method Algorithm

To illustrate equation (5) using HPM, we construct the homotopy equation

$$(1-p)D^\eta u(x) = ph(D^\eta u(x) - f(x) - \mu u(x) - \int_0^x [g(t)u(t) + h(t)F(u(t))]dt).$$

Putting $h = -1$,

$$D^\eta u(x) - pD^\eta u(x) = -(pD^\eta u(x) - pf(x) - p\mu u(x) - p \int_0^x [g(t)u(t) + h(t)F(u(t))]dt)$$

$$D^\eta u(x) - pf(x) - p\mu u(x) + \int_0^x [g(t)u(t) + h(t)F(u(t))]dt = 0$$

$$D^\eta u(x) = pf(x) - p\mu u(x) + \int_0^x [g(t)u(t) + h(t)F(u(t))]dt$$

$$u(x) = \sum_{n=0}^{\infty} y_n(x)p^n$$

$$D^\eta \sum_{n=0}^{\infty} y_n(x)p^n$$

$$= pf(x) - p\mu \sum_{n=0}^{\infty} y_n(x)p^n$$

$$+ \int_0^x \left[g(t) \sum_{n=0}^{\infty} y_n(x)p^n + h(t)F\left(u(t) \sum_{n=0}^{\infty} y_n(x)p^n\right) \right] dt = 0$$

$$\begin{aligned} D^\eta(u_0(x) + u_1(x)p + u_2(x)p^2 + u_3(x)p^3 + \dots) \\ = p(f(x) + \mu u_0(x) + \mu u_1(x)p + \mu u_2(x)p^2 + \mu u_3(x)p^3 + \dots \\ + \int_0^x [g(t)u_0(t) + g(t)y_1(t)p + g(t)y_2(t)p^2 + \dots h(t)F(u_0(t) \\ + h(t)F(y_1(t)p) + h(t)F(y_2(t)p^2) + h(t)F(y_3(t)p^3) \dots] dt \end{aligned}$$

$$P^0: D^\eta(u_0(x)) = 0$$

$$P^1: D^\eta(u_1(x)) = f(x) + \mu u_0(x) + \int_0^x [g(t)u_0(t) + h(t)F(u_0(t))]dt$$

$$P^2: D^\eta(u_2(x)) = \mu u_1(x) + \int_0^x [g(t)y_1(t) + h(t)F(y_1(t))]dt$$

$$P^3: D^\eta(u_3(x)) = \mu u_2(x) + \int_0^x [g(t)y_2(t) + h(t)F(y_2(t))]dt.$$

5 Numerical examples on fractional order integro-differential using HPM

Solutions of fractional order integro-differential equation by using HPM are considered here. The following problems are solved using HPM.

Problem 1:

Consider the following non-linear integro-differential equation:

$$D^\eta u(x) = e^x - \frac{1}{2}e^{2x} \frac{1}{2} - \int_0^x u(t)u''(t)dt, \quad 0 < x < 1, \quad 3 < \eta \leq 4$$

with the boundary conditions

$$\begin{aligned} u(0) &= 1, & u(1) &= e \\ u'(0) &= 1, & u''(0) &= 1. \end{aligned}$$

The homotopy equation is

$$D^\eta \sum_{k=0}^{\infty} u_k(x)P^k = P \left\{ e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x \left(\sum_{k=0}^{\infty} u_k(x)P^k \right) \left(\sum_{k=0}^{\infty} u_k''(x)P^k \right) dt \right\}$$

$$\begin{aligned} D^\eta(u_0(x) + Pu_1(x) + P^2u_2(x) + \dots) \\ = P \left\{ e^x - \frac{1}{2}e^{2x} + \frac{1}{2} \right. \\ \left. - \int_0^x (u_0(x) + Pu_1(x) + P^2u_2(x) + \dots)(u_0''(x) + Pu_1''(x) + P^2u_2''(x) + \dots) \right\} \end{aligned}$$

$$P^0: D^\eta u_0(x) = 0$$

$$u_0(x) = 0$$

$$P^1: D^\eta u_1(x) = e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x u_0(t)u_0''(t)dt$$

$$P^2: D^\eta u_2(x) = - \int_0^x u_1(t)u_1''(t)dt$$

$$P^3: D^\eta u_3(x) = - \int_0^x u_2(t)u_2''(t)dt$$

$$u_0(x) = u_0 + xu_0' + \frac{x^2}{2!}u_0'' + A \frac{x^3}{3!}u_0''' \Rightarrow 1 + x + \frac{x^2}{2} + \frac{Ax^3}{5}$$

$$u_0(x) = 1$$

$$u_1 = D^{-\eta} \left[e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x (u_0(t))(u_0''(t)) \right] dt.$$

Let the truncated e^x and e^{2x} be

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \text{ and } e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3},$$

$$\text{also } D^{-\eta} = J^\eta$$

$$u_1(x) = J^\eta \left\{ 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} - \frac{1}{2} \left(1 + 2x + 2x^2 + \frac{4x^3}{3} \right) + \frac{1}{2} - \int_0^x (1)(0) \right\}$$

$$u_1(x) = J^\eta \left\{ 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} - \frac{1}{2} - x - x^2 + \frac{2x^3}{3} + \frac{1}{2} \right\}$$

$$u_1(x) = J^\eta \left\{ 1 - \frac{x^2}{2} - \frac{x^3}{2} \right\}$$

$$u(x) = u_0(x) + u_1(x)$$

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} + \frac{x^\eta}{\Gamma(\eta+1)} - \frac{x^{\eta+2}}{\Gamma(\eta+3)} - \frac{2x^{\eta+3}}{\Gamma(\eta+4)}$$

$$u(1) = \frac{5}{2} + \frac{A}{6} + \frac{1}{\Gamma(\eta+1)} - \frac{1}{\Gamma(\eta+3)} - \frac{2}{\Gamma(\eta+4)} = e.$$

Solving for A in $u(1)$, we have:

$$\text{at } \eta = 4.00$$

$$A = 1.070405256,$$

$$\text{at } \eta = 3.75$$

$$A = 0.965114,$$

$$\text{at } \eta = 3.50$$

$$A = 0.821115,$$

$$\text{at } \eta = 3.25$$

$$A = 0.628341.$$

Substituting the values of A in $u(x)$ we have:

$$\text{at } \eta = 4.00, A = 1.070405256$$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.178401x^3 + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^7}{2520}$$

$$\text{at } \eta = 3.75, A = 0.965114$$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.160852x^3 + 0.0602911x^{3.75} - 0.00220745x^{5.75} - 0.00065406x^{6.75}$$

$$\text{at } \eta = 3.50, A = 0.821115$$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.136852x^3 + 0.0859717x^{3.5} - 0.00347361x^{5.5} - 0.0010688x^{6.5}$$

$$\text{at } \eta = 3.25, A = 0.628341$$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.104724x^3 + 0.120699x^{3.25} - 0.00540947x^{5.25} - 0.00173103x^{6.25}.$$

Table 3: Varying the values of x from 0 to 1, at $\eta = 4$, $u(x)$

x	Exact Solution	HPM	Error
0.0	1.0010000	1.0000000	0.0000000
0.1	1.1051709	1.1051826	0.0000117
0.2	1.2214028	1.2214938	0.0000910
0.3	1.3498588	1.3501532	0.0002944
0.4	1.4918247	1.4924710	0.0006463
0.5	1.6487213	1.6498795	0.001158
0.6	1.8221188	1.8238587	0.0017382
0.7	2.0137527	2.0159996	0.0022469
0.8	2.2255409	2.2279606	0.0024197
0.9	2.4596031	2.4614638	0.0018607
1.0	2.7182818	2.7182818	0.0000000

Problem 2:

Consider the following non-linear integro-differential equation:

$$D^\eta u(x) = e^{-x} + e^{-3x} - 1 + 3 \int_0^x u^3(t) dt$$

$$0 < x < 1, \quad 3 < \eta \leq 4$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = -1$$

$$u''(0) = 1, \quad u'''(0) = 1.$$

The homotopy equation is

$$D^\eta \sum_{k=0}^{\infty} u_k(x) P^k = P \{ e^{-x} + e^{-3x} - 1 + 3 \int_0^x \sum_{k=0}^{\infty} u_k^3(t) P^k dt \}$$

$$D^\eta (u_0(x) + Pu_1(x) + P^2u_2(x) + \dots)$$

$$= P \left\{ e^{-x} + e^{-3x} - 1 + 3 \int_0^x (u_0(t) + Pu_1(t) + P^2u_2(t) + \dots)^3 dt \right\}$$

$$D^\eta (u_0(x) + Pu_1(x) + P^2u_2(x) + \dots)$$

$$= P \left\{ e^{-x} + e^{-3x} - 1 \right.$$

$$+ 3 \int_0^x (u_0^3(t) + 3Pu_1(t)u_0^2(t) + 3P^2u_0^2(t)u_2(t) + u_1^2(t)u_0(t))$$

$$+ P^3(6u_0(t)u_1(t)u_2(t) + u_1^3(t) + \dots) dt \left. \right\}$$

$$P^0: D^\eta u_0(x) = 0$$

$$P^1: D^\eta u_1(x) = e^{-x} - e^{-3x} - 1 + 3 \int_0^x (u_0^3(t)) dt$$

$$P^2: D^\eta u_2(x) = 3 \int_0^x (3u_1(t)u_0^2(t)) dt$$

$$\begin{aligned}
 &= 9 \int_0^x (u_1(t)u_0^2(t))dt \\
 P^3: D^\eta u_3(x) &= - \int_0^x 3(u_0^2(t)u_2(t) + u_1^2(t)u_0(t))dt \\
 &= 9 \int_0^x ((u_0^2(t)u_2(t) + u_1^2(t)u_0(t))) dt.
 \end{aligned}$$

We take the truncated Taylor expression for e^{-x} and e^{-3x} as

$$\begin{aligned}
 e^{-x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{6!} \\
 e^{-3x} &= 1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} \\
 u_0(x) &= u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0) \\
 u_0(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} \\
 u_0(x) &= 1 \\
 u_1(x) &= D^{-\eta} \left\{ e^{-x} + e^{-3x} - 1 + 3 \int_0^x (u_0^3(t))dt \right\} \\
 D^{-\eta} &= J^\eta \\
 u_1(x) &= J^\eta \left\{ 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} - 1 + 3 \int_0^x (1)^2 \right\} \\
 u_1(x) &= J^\eta \left\{ 1 - x + 5x^2 - \frac{14x^3}{3} \right\} \\
 u_1(x) &= -x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^\eta}{\Gamma(\eta+1)} - \frac{x^{\eta+1}}{\Gamma(\eta+2)} + \frac{10x^{\eta+2}}{\Gamma(\eta+3)} - \frac{28x^{\eta+3}}{\Gamma(\eta+4)} \\
 u_2(x) &= D^\eta \left\{ 9 \int_0^x (u_1(t)u_0^2(t))dt \right\} \\
 D^{-\eta} &= J^\eta \\
 u_2(x) &= J^\eta \left\{ 9 \int_0^x \left(-t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^\eta}{\Gamma(\eta+1)} - \frac{t^{\eta+1}}{\Gamma(\eta+2)} + \frac{10t^{\eta+2}}{\Gamma(\eta+3)} - \frac{28t^{\eta+3}}{\Gamma(\eta+4)} \right) (1)^2 dt \right\} \\
 u_2(x) &= J^\eta \left\{ \frac{-9x^2}{2} + \frac{3x^2}{2} + \frac{3x^4}{8} + \frac{216x^{\eta+1}}{\Gamma(\eta+5)} - \frac{108x^{\eta+2}}{\Gamma(\eta+5)} + \dots \right\} \\
 u(x) &= \frac{-9x^{\eta+2}}{\Gamma(\eta+3)} + \frac{9x^{\eta+3}}{\Gamma(\eta+4)} + \frac{9x^{\eta+4}}{\Gamma(\eta+5)} + \dots \\
 u(x) &= u_0(x) + u_1(x) + u_2(x) \\
 u(x) &= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^\eta}{\Gamma(\eta+1)} - \frac{x^{\eta+1}}{\Gamma(\eta+2)} + \frac{10x^{\eta+2}}{\Gamma(\eta+3)} - \frac{28x^{\eta+3}}{\Gamma(\eta+4)} - \frac{9x^{\eta+2}}{\Gamma(\eta+3)} \\
 &\quad + \frac{9x^{\eta+3}}{\Gamma(\eta+4)} + \frac{9x^{\eta+4}}{\Gamma(\eta+5)} + \dots
 \end{aligned}$$

Varying the values of η in $u(x)$, we have:

At $\eta = 4.00$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{1008} + \frac{x^8}{4480} + \dots$$

At $\eta = 3.75$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0602911x^{3.75} - 0.0126929x^{4.75} + 0.00220745x^{5.75} - 0.00163515x^{6.75} + 0.000379777x^{7.75}$$

At $\eta = 3.50$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0859717x^{3.5} - 0.0191048x^{4.5} + 0.00347361x^{5.5} - 0.002672x^{6.5} + 0.000641281x^{7.5}$$

At $\eta = 3.25$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.120699x^{3.25} - 0.0283997x^{4.25} + 0.00540947x^{5.25} - 0.00432758x^{6.25} + 0.00107443x^{7.25}$$

Table 4: Varying the values of $x = 0(0.04)0.36$, at $\eta = 4$, $u(x)$

x	Exact Solution	HPM	Absolute Error
0.0	1.0010000	1.0000000	0.0000000
0.04	0.9607894	0.9608108	0.0000214
0.08	0.9231163	0.9232870	0.0001707
0.12	0.8869204	0.8874964	0.0005762
0.16	0.8521438	0.8535091	0.0013653
0.2	0.8187308	0.8213974	0.0026666
0.24	0.7866279	0.7912358	0.0046079
0.28	0.75578374	0.7631010	0.0187122
0.32	0.7261490	0.7370715	0.0109225
0.36	0.6976763	0.7132278	0.0155515

6 Discussion

Tables 1 to 4 present results of problems 1 to 4 where the first two are for the implementation VIM, while HPM is implemented for the results in the last two tables. In results of VIM and HPM are evaluated for $\eta = 4$ at different step-sizes. It is obvious that the results for both VIM and HPM improve with reduction in step length.

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Analysis of heat and mass transfer on magnetohydrodynamics (MHD) natural convective fluid flow over vertical and inclined porous media

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Abstract

Heat and mass transfer effects on MHD natural convective fluid flow over vertical and inclined porous media are analyzed in this paper, unsteadily. Non-dimensional differential equations are derived from the governing equations and are solved numerically using the fourth-order classical Runge-kutta method. First, we did a first-order reduction of the coupled ordinary differential equations. Furthermore, MATLAB was used to obtain and evaluate numerical expressions for the distributions of fluid velocities, temperatures, and concentrations. The Soret and dufour effects, or the impact of an inclination angle on the flow field, were the subject of an attempt at explanation. The equations for momentum, energy, and concentration are shown to be solvable as coupled second-order partial differential equations. The velocity, temperature, and concentration profiles in the flow domain are displayed graphically for a range of issue parameters. When the fluid's temperature, concentration, and velocity all rise due to the Grashof number (Gr), modified Grashof number (Gc), the Soret number (Sr), angle of inclination (α), Dufour number (Df), magnetic parameter (M) Increase and thus thickening the thermal boundary layer across the channel. There is a negative correlation between the Grashof number (Gr) and fluid velocity, temperature, and concentration, However, the magnetic parameter (M), chemical reaction parameter (Kr), Schmidt number (Sc), Prandtl number (Pr), and the angle of inclination parameter (I) are all directly related to one another. Heat transfer at the wall is improved by a rise in the Prandtl number (Pr), radiation parameter (R), and Grashof number (Gr), whereas mass transfer is sped up by an increase in the chemical reaction parameter (Q), the Schmidt number (Sc), the Grashof number (Gr), and the heat source parameter (Q). While we used mat lab, an order-based computer program, and a traditional fourth-order Range-Kutta iteration technique to calculate the table, interested researchers are free to try out other order-based computer programs, such as maple, cubic, etc., to provide alternative solutions.

Keywords and phrases: MHD; heat source; heat transfer; mass transfer; porous medium; inclined angle; thermal radiation

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1 Introduction

As a result of its ubiquitous presence in many scientific and engineering disciplines, Magnetohydrodynamic (MHD) research in heat and mass movement has significant real-world applications for engineers and scientists. Unsteady MHD free convective flow over a vertical porous plate has been studied extensively for their effects on heat mass transport. Unsteady temperature with heat and mass change flows across a proliferating material at the control of a magnetic zone/edge with chemical reactions has attracted a growing number of researchers in recent years. Geothermal energy usage, cooling of reentry vehicles and rocket boosters, thermal energy storage, etc. are only a few examples of how this concept has been put into practice in other scientific and technological fields. [29] Studied the effects of unsteady MHD fluid flow over a vertically inclined porous plate on chemical reactions, aligned magnetic fields, radiation, and soot. [14] Investigate the convective heating of a vertical channel using a porous material and the resulting unsteady MHD free convection flow of an exothermic fluid. [17] Studied a generic solution for an unstable MHD natural convection flow involving an infinite vertical plate immersed in a porous medium and subject to random motion.

Heat and mass transfer effects on MHD flow over an inclined porous plate were investigated by [36] in the presence of a chemical reaction. [30] Investigated heat and mass transport in an unsteady MHD flow between two vertical porous plates. [41] Investigate Heat and mass transfer effects on unsteady MHD flow past an inclined plate embedded in porous medium in the presence of hall current and viscous dissipation. [35] Studied Steady on MHD Heat and mass transfer flow of an inclined porous plate in the presence of Radiation and Chemical Reaction. [39] Investigate unsteady MHD Slip flow past a inclined porous plate with hall current and temperature gradient dependent heat source with fluctuating temperature and concentration in presence of radiation and Soret effect in a rotating system.

The porous medium in a horizontal channel with an inclination magnetic field is studied by [25] for its potential to facilitate both unsteady fluid flow and heat transfer. Using a periodic magnetic field and a constant pressure gradient, [1] investigated unsteady MHD flow and heat transfer through a porous medium between parallel plates. The regulation of micropolar fluid flow and heat transfer via porous media in a horizontal channel was investigated by [23]. Steady Magnetohydrodynamic Micropolar Fluid Flow and Heat and Mass Transfer in Permeable Channel with Thermal Radiation were studied by [3]. Researchers [15] looked at the effects of a chemical reaction on unsteady free convection MHD flow across a vertical cone in porous media.

The work in [27] investigated the free convection flow of magnetohydrodynamics (MHD) past an infinite sloping plate. Non-Newtonian fluid flow with heat transfer over a porous stretching/shrinking sheet is described by [46]. [28] Have taken into account the magnetohydrodynamic (MHD) flow and heat transmission of a viscous incompressible fluid via a porous material. In the presence of a heat source, the Soret effect, and a chemical reaction influenced by viscous dissipation ohmic heating, [45] investigated MHD free convection flow across an inclined porous plate. Heat and mass transport of MHD for a non-stationary viscous oscillatory flow. Magnetohydrodynamic (MHD) study into the fundamentals of chemical reactions, such as heat and mass transfer, and unsteady viscous oscillatory flow, was given by [2].

Suction, viscous dissipation and thermal radiation were all factors in the transient magnetohydrodynamic natural convection boundary layer flow that [47] examined. (i) Ground water flow through soil and rocks, which is crucial for agriculture and pollution control, is an example of MHD fluid in a porous medium. (ii) Removing rock's natural gas and soil from it. Bone, cartilage, and muscle have all been established by [4] as functional tissues thanks to the steady stream of blood and nutrients that feed them. [5] Studied MHD heat and mass transfer in porous medium, with an emphasis on the quality of environmental cleanup and restoration. Heat and mass transfer in a permeable channel subjected to thermal radiation was investigated by [33]. Results of soot and

douour analysis of Revlin-Erickson fluid's MHD free convective drift through a semi-endless vertical plate are reported. Using chemical reaction, an arbitrary shear stress, and exponential heating. [16] Investigated convection heat mass transfer and MHD flow over a vertical plate. Thermal radiation heat transfer has become so important at greater temperatures, as indicated by [20]. The significance of heat transmission via radiation has to be taken into account. With the addition of thermal radiation effects, the importance of MHD flow, mass, and heat transfer in industrial settings grows. High temperatures are required for many scientific and technical processes. For the purpose of modeling the appropriate instruments, the idea of heat transmission via thermal radiation stands out as particularly salient. The quality of the finished product is highly sensitive to heat and other regulating elements. Using radiative heat transfer in the process could lead to the desired end result.

Heat and mass transfer via MHD free convection with Dufour and chemical reaction effects were investigated by [12]. [38] Studied the dynamics of MHD and convective heat transfer over a vertical plate. The importance of mass transfer in the context of multi-phase, two-dimensional, oscillatory, suction-driven, and thermally-driven flows through a vertical porous plate in a porous medium.

Chaotic MHD In a rotary system featuring a hall current and temperature gradient dependent heat source with temperature and concentration changes, [39] Studied slip flow. [6] Studied the heat and mass transport in a porous medium between two vertical plates undergoing a chemical reaction in an unstable MHD convective flow, on a vertically stretched, permeable surface. [24] Studied the impact of heat production/absorption, thermal radiation, and chemical reaction on steady MHD flow and heat transfer.

The effect of chemical reactions and radiation absorption on convective flows in a porous vertical wavy channel transporting traveling thermal waves was studied by [22]. Unsteady MHD boundary layer flows with slip along an accelerating infinitized vertical plate with a ramping plate temperature were studied by [9]. Recent years have seen substantial progress in study into heat and mass transfer in magneto hydrodynamic flows due to their extensive application in devices like the MHD power generator and the Hall accelerator. Additionally applicable to studies of planetary atmospheres is the effect of a magnetic field on the motion of an electrically conducting viscous fluid undergoing mass transfer and radiation absorption. MHD free convection was investigated by [10] using a semi-infinite vertical porous plate and the diffusion-thermo effect. Unsteady MHD free convective mass transfer is investigated by passing a semi-infinite vertical porous plate through a fluid that is viscous, incompressible, and optically thin. [40] Examined the effects of heat and mass transfer on the flow of a chemically reactive and radiating fluid through a porous medium, through a vertically moving plate with an arbitrarily ramped temperature, in an unsteady MHD natural convection scenario. [32] Study Radiation and heat absorption effects on an unsteady MHD boundary layer flow along an accelerated infinite vertical porous plate with ramped plate temperature in existence of slip condition.

The work in [8] looked into the impact of radiation and chemical reactions on the dynamics of viscoelastic fluid flows in a porous medium. [43] Study effect of thermal radiation on MHD free convective flow via a vertical porous plate in the presence of chemical reaction-FEM, these studies investigate the impact of thermal radiation on the flow of fluids through such plate. [26] Analyzed the effects of thermal radiation on heat and mass transmission in a porous media. In a study published in [11]. looked at the effects of a chemical reaction on free convective MHD radiation flow through a porous vertical plate in a porous median. [44] Studied the impact of radiation and chemical reaction on MHD mixed convective flow and mass transfer using a vertical surface, ohmic heating, and viscous dissipation. Researchers [34] looked at how chemical reactions affected heat and mass transport in a magnetohydrodynamic (MHD) flow over a porous, inclined plate. Soret and Dufort effects on the MHD free convective flow behind an impulsively moving vertical plate in an

oblique magnetic field were also investigated [42]. The effects of radiation and chemical reactions on the MHD free convective heat and mass transfer flow of a viscoelastic fluid across a porous heat source (sink) were studied by [37]. The effect of thermal radiation on unsteady magneto-convective heat and mass transport through a vertical permeable sheet with a chemical reaction was studied by [19]. The slip MHD micro fluid flow on a stretching sheet was analyzed by [13]. [21] Study Radiation effect on chemically reacting magnetohydrodynamics (MHD) boundary layer flow of heat and mass transfer through a porous vertical flat plate. [7] investigated Effects of variable transport properties on heat and mass transfer in MHD bioconvective nanofluid rheology with gyrotactic microorganisms. Convection heat mass transfer and MHD flow over a vertical plate with chemical reaction, arbitrary shear stress and exponential heating was study by [18]. Investigation of Steady MHD Natural Convection Heat and Mass Transfer Flow above a Vertical Porous Surface with Thermal Diffusion and Inclined Magnetic Field was done by [31].

In this work, we to determine the heat and mass transfer effects on unsteady MHD natural convective fluid flow over a vertical and inclined porous media. The numerical method is used to achieve our aim. We considered unsteady MHD flow of a viscous incompressible fluid which has the properties electrically conducting, chemically reacting and thermally radiating on limitless vertical porous media with periodical temperature with concentration in flow scheme. The Suction velocity v is of the form

$$v = -v_o(1 + \varepsilon A e^{i\omega t}) \quad (1)$$

where, v_o = Suction velocity which is non-zero positive Constance. The suction is directed toward the plate, as shown by the negative sign; ε = Small such that $A\varepsilon \ll 1$, ω = Frequency of oscillation, t = time and A = Positive real constant

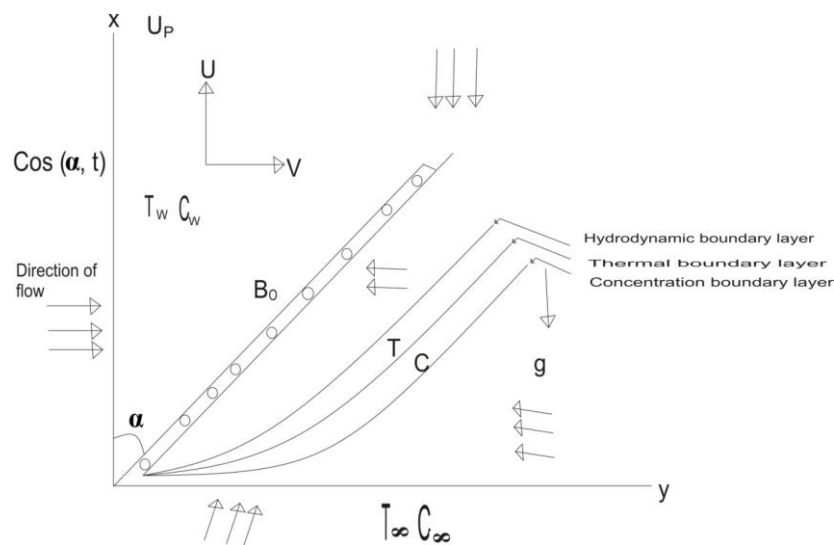


Figure 1: Physical Model and coordinate system of the problem.

The x-axis takes upward vertically alongside vertical porous plate and y-axis forming right angle to that. A consistent magnetic force B_0 is used perpendicular to the surface of the plate. Consider an infinite flat is in the x-direction so all value are independent of x. Assume the variables are dependent on y and t most effective.

The radiative heat flux is described using Rosseland's approximation, which assumes that the fluid is a gray absorbing emitting radiation but non scattering medium. When contrasted to the y-direction, the x-direction component is deemed to be insignificant. A homogeneous magnetic fluid was applied perpendicular to the surface of stretching. Since it was believed that both the magnetic Reynolds number and the transverse applied magnetic field were quite small, the x-axis is the direction along the stretching surface and indicates the direction of motion. The slant or inclined angle α and y-axis is perpendicular to it. At time $t > 0$ the plate is suddenly accelerated in the direction of flow, against gravity, along the x-axis with constant velocity U_p . Furthermore, it is believed that the suction velocity, wall temperature, and concentration all change exponentially with time, where u Is an inlet velocity component in x-direction, v Is an inlet velocity component in y-direction and w Is an inlet velocity component in z-direction. Therefore, the fluid element of length dx, dy, dz in the direction of x, y, z.

In this work, we assumed that: Induced magnetic is negligible because magnetic fields Reynolds number will be small. In between fluid and species concentration a first order chemical reaction exists. Heat generation throughout the chemical reaction negligible because the extend of species concentration could be very low. Liquid is grey and it absorbs and emits radiation in a non-scattering medium. A small velocity encountered in the free convective flow, so viscous dissipation and joule heating phrases will be neglected. No outside electric force will be carried out, thus electric field will be assumed to be zero. At constant temperature (T_w) the wall is maintained and concentration (C_w) higher than the ambient temperature (T_∞) and concentration (C_∞) respectively. A uniform magnetic field acts perpendicular to the porous plate, which absorbs the fluid with a suction velocity varying with time. The porous plate assumed to move with a constant velocity in the direction of the fluid flow.

Under the above assumption, the governing equations for the situation of the problem that describe the physical situation of the new equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - v_0(1 + \varepsilon A e^{i\omega t}) \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta_T(T - T_\infty) \cos \alpha + g\beta_C(C - C_\infty) \cos \alpha - \left(\frac{\sigma\beta_0^2}{\rho} + \frac{\nu}{K} \right) u \quad (3)$$

$$\rho C_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} - v_0(1 + \varepsilon A e^{i\omega t}) \frac{\partial T}{\partial y} \right] = K \frac{\partial^2 T}{\partial y^2} + Q_0(T - T_\infty) + \nu \left(\frac{\partial u}{\partial y} \right)^2 - \frac{\partial q_r}{\partial y} + \frac{D_m K_T \partial^2 T}{T_m \partial y^2} \quad (4)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} - v_0(1 + \varepsilon A e^{i\omega t}) \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - K_r(C - C_\infty) + \frac{DK_r \partial^2 T}{C_m \partial y^2} \quad (5)$$

The 2nd and 3rd terms on the right hand side of equation (3) denote thermal and concentration buoyancy effects respectively. Also the 4th and 5th terms on the right side of the equation (4) represents the radiative heat flux and the Dufour and Soret effects, with the following initial and boundary conditions;

$$t \leq 0; u = 0, T = T_{\infty}, C = C_{\infty}, \text{ for all } y$$

$$\begin{aligned} t > 0 \{ u = u_p, T = T_{\infty} + \varepsilon(T_w - T_{\infty})e^{i\alpha t}, C = C_{\infty} + \varepsilon(C_w - C_{\infty})e^{i\alpha t}, at y = 0 \} \\ u \rightarrow 0, T \rightarrow 0, C \rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned} \quad (6)$$

where T_w is the Temperature near to the wall, T_{∞} is the Measures of temperature a long way from the wall or ambient temperature, C_w is the Species density near to the wall, C_{∞} is the Species density a long way from the wall or ambient Species density, u is the velocity component in x-direction, v is the velocity component in y-direction, B_0 is the magnetic field strength, g is the acceleration due to gravity, D is the Chemical Molecular dispense, C is the Species density, T is the Temperature of fluid σ is the Electrical Conductibility, K is the Thermal Conductibility, C_p is the Specific Warmness at the consistent strain, ν is the Kinematical Viscosity, β_T is the Constant of the volume elaboration for heat transfer, β_C is the Volumetric Constant of elaboration for mass transfer, K_r is the Chemical Reaction parameter, D_m is the Mass diffusivity q_r is the Radiative heat flow, D_k is the Darcy porosity parameter. Q is the Heat sources parameter, ρ is the Density of the fluid, α is the Inclined angle, U_p is the Plate velocity, ε is a positive constant, v_0 is the Suction velocity, K_T is the thermal diffusion effect, T_m is the mean temperature of the fluid, β_T and β_C are the thermal and mass expansion confidents respectively, Gr is Grashof number for heat transfer, Gc is Grahof number for mass transfer.

$$\text{Using Rosseland diffusion approximation for radiation heat flux } q_r = -\frac{4\sigma}{3\delta k} \frac{\partial T^4}{\partial y^2} \quad \text{Using}$$

Taylor series to expand T^4 and the free stream T_{∞} and neglecting higher order terms, this gives the approximation

$$T_{\infty}^4 \cong 4T_{\infty}^3 T - 3\delta k T_{\infty}^4 \quad (7)$$

Introducing the stream function (x,y) such that:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (8)$$

Using equation (8) into equation (2),(3),(4) and (5), Equation (2) its satisfiers, therefore (3) (4) and (5) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (9)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u - \frac{\nu}{K} u + g\beta_T (T - T_\infty) \cos \alpha + g\beta_C (C - C_\infty) \cos \alpha \quad (10)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{K}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho C_p} (T - T_\infty) - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} + \frac{D_\kappa K_\tau}{T_\tau} \frac{\partial^2 T}{\partial y^2} \quad (11)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - K_r (C - C_\infty) + \frac{DK_\Gamma}{C_m} \frac{\partial^2 T}{\partial y^2} \quad (12)$$

Subject to the boundary condition below:

$$\left\{ \begin{array}{l} u = U_0 \ell^{x/L}, v = -U_0 \ell^{y/L}, C = C_\infty \quad @ y = 0 \\ y \rightarrow \infty, \quad u \rightarrow 0, C \rightarrow C_\infty \quad @ y = 0 \end{array} \right\} \quad (13)$$

In order to solve equation (2) to (5) we introduce the following similarity transformations and dimensionless concentration and temperature.

$$\psi(x, y) = (2t \text{Re})^{1/2} \nu \ell^{y/2L} f(\eta) \quad (14)$$

$$\eta = \frac{y}{L} \left(\frac{\text{Re}}{2t} \right)^{1/2} \ell^{y/2L} \quad (15)$$

$$\phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} = C = C_\infty + C_0 \ell^{y/2L} \phi(\eta) \quad (16)$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} = T = T_\infty + T_0 \ell^{y/2L} \theta(\eta) \quad (17)$$

$$T_0 = T_w - T_\infty, C_0 = C_w - C_\infty$$

where η is the similarity variable, $\psi(x, y)$ is the dimensionless stream function $\phi(\eta)$ is the dimensionless concentration and $\theta(\eta)$ is the dimensionless temperature. The equations are solving by their steps, by inserting (8) into (2) we have

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(- \frac{\partial \psi}{\partial x} \right) &= 0 \\ \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} &= 0 \end{aligned} \quad (18)$$

It's satisfied the Cauchy-Riemann equation. Using (8) into (3), (4), and (5), we have

$$\frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial y} \right) + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^2}{\partial y^2} \left(\frac{\partial \psi}{\partial y} \right) - \frac{\sigma \beta_0^2}{\rho} \frac{\partial \psi}{\partial y} - \frac{\nu}{k} \frac{\partial \psi}{\partial y} + g \beta_T (T - T_\infty) \cos \alpha + g \beta_c (C - C_\infty) \cos \alpha \quad (19)$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{K}{\rho C_p} (T - T_\infty) - \frac{\partial q_r}{\partial y} \frac{1}{\rho C_p} + \frac{D_\kappa K_T}{T_\tau} \frac{\partial^2 T}{\partial y^2} \quad (20)$$

$$\frac{\partial \psi}{\partial y} \frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial C}{\partial y} = D \frac{\partial^2 T}{\partial y^2} + K_r (C - C_\infty) + \frac{DK_\Gamma}{C_m} \frac{\partial^2 T}{\partial y^2} \quad (21)$$

Non- dimensional quantities are defined by

$$M = \frac{4tL^3 \sigma B_0^2}{\text{Re } \nu} \text{Re } \ell^{-\frac{x}{L}}, \quad Gr = \frac{4tL^3 g \beta_T (T_0)}{(\text{Re } \nu)^2} \ell^{-3\frac{x}{2L}}, \quad Gc = \frac{4tL^3 g \beta_c (C_0)}{(\text{Re } \nu)^2} \ell^{-3\frac{x}{2L}}, \quad D_k = \frac{2L^2}{\text{Re } K \nu} \text{Re } \ell^{-\frac{x}{L}}$$

$$R = \frac{4\sigma T_\infty^3}{\partial k}, \quad Q = \frac{Q_0 2L^2 \nu}{K \text{Re}} \ell^{-\frac{x}{L}}, \quad P_r = \frac{\nu \rho C_p}{K}, \quad D_f = \frac{D_m K_T}{T_m} \frac{\nu \rho C_p}{K} \ell^{-3\frac{x}{2L}}, \quad \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad (22)$$

$$S_c = \frac{\nu}{D}, \quad K_r = \frac{K_r}{\text{Re } \nu} C_0 \ell^{-3\frac{x}{2L}}, \quad S_r = \frac{DK_\Gamma}{C_m} C_0 \text{Re } \ell^{\frac{x}{L}}$$

where Gr is the Grashof number, Gc is the modified Grashof number, M is the magnetic parameter, Kr is the Chemical Reaction parameter, Pr is the Prandtl number, Sc is the Schmidt number, D_k is the Darcy porosity parameter, Q is the Heat sources parameter, R is the radiation parameter, Sr is the Soret number, Df is the Dufour number respectively.

From equation (9) to (12), the following nonlinear equations (19) to (21) were obtained:

$$f''' + 2tff'' + 2\eta f'' - 4tf'^2 - (M + D_k)f' + Gr\theta \cos \alpha + Gc\phi \cos \alpha = 0 \quad (23)$$

$$\left(1 + \frac{4}{3}R \right) \theta'' + P_r \left(f\theta' + \frac{\eta}{t} \theta' + Q\theta + D_f \theta'' \right) = 0 \quad (24)$$

$$\phi'' + S_c \left(f\phi' + \frac{\eta}{2t} \phi' - K_r \phi + S_r \phi'' \right) = 0 \quad (25)$$

And the transform boundary conditions

$$\begin{aligned} f &= f_w, \quad f'(0) = 0, \quad \theta(0) = 1, \quad \phi'(1) = 1 \quad \text{at } \eta = 0 \\ f &= 0, \quad f'(1) = 0, \quad \theta'(0) = 1, \quad \phi(0) = 0 \quad \text{at } \eta = 1 \end{aligned} \quad (26)$$

2 Method of solution

Non-linear ordinary differential equations are utilized for controlling the convection of heat and other physical substances via porous media. In order to control the convective motion of MHD fluids, these equations are used. With the shooting approach and a fourth-order Runge-Kutta

scheme, non-linear ODE systems with boundary constraints can be numerically solved. The Runge-Kutta technique works well for such purposes. The method has been widely employed due to its efficacy in solving boundary layer equations, which appears to yield correct results (Ibrahim and Makinde, 2011). It seems to be the most flexible of the common methods. The scheme is also applicable to various types of boundary layer flow problems including the free and mixed connection flows. In the numerical system employed in solving the equation or models, that is the (BVP) boundary value problem. Shooting method reformulates the BVP to IVP by adding sufficient number of conditions at one end and adjusts these conditions until the giving conditions are satisfied at the other end while Runge-Kutta method solves the initial value problems. Here 4th order Runge-Kutta techniques where used to find the root and the scheme which is given below:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (27)$$

$$k_1 = hf(x_n, y_n) \quad (28)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \quad (29)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \quad (30)$$

$$k_4 = hf(x_n + h, y_n + hk_3) \quad (31)$$

$n = 0, 1, 2, \dots$

To apply the fourth order classical Runge-kutta method, for unsteady we first reduce the coupled ordinary differential equations in equation of order one

$$\begin{aligned} x_1 &= f, x_2 = f', x_3 = f'', x_4 = \theta, x_5 = \theta', x_6 = \phi, x_7 = \phi' \\ x_1' &= f' = x_2, x_2' = \theta' = x_5, x_3' = \phi' = x_7 \\ x_2' &= f'' = x_3, x_5' = \theta'' = x_6, x_7' = \phi'' \\ x_3' &= f''' \end{aligned} \quad (32)$$

$$x_3' = 4tx_2^2 + (M + D_k)x_2 - (2tx_1 + 2\eta)x_3 - G_r x_4 \cos \alpha - G_c x_6 \cos \alpha \quad (33)$$

$$x_5' = \frac{-P_r x_1 x_5 - P_r \frac{\eta}{t} x_5 - P_r Q x_4}{\left(1 + \frac{4}{3}R + P_r D_f\right)} \quad (34)$$

$$x_7' = \frac{S_c K_r x_6 - S_c x_1 x_6 - S_c \frac{\eta}{2t} x_7}{(1 + S_c S_r)} \quad (35)$$

3 Result and discussions

For the case of different values of the thermal Grashof number (Gr), the velocity profiles on the boundary layer are shown in figures 2, 3 and 4, it is observed that an increase in the Grashof number (Gr) the peak values of the velocity increase rapidly near the porous medium and then decay smoothly to the free stream velocity, temperatures and concentration.

The effect of the heat source (Q) on the velocity, temperature and concentration fields, it is visible that for the increase of the heat source parameter (Q), the velocity decreases but temperatures and concentration are both increasing for unsteady case. The profiles are presented in figures 5, 6 and 7.

Also the effect of the magnetic field on the velocity, temperature and concentration profiles. It is observed that the velocity, temperature and concentration decreases with an increasing magnetic parameter (M). This is because of the Lorentz force acting on the fluid flow. This type of resisting force slows down the fluid velocity as it is in figure. The effect of the permeability parameter in the velocity, temperature and concentration profile is depicted in figures 8, 9, 10. The effects of the modified Grashof number (Gc) on the velocity, temperatures and concentration profile. It is observed that as the modified Grashof number increases the velocity, temperatures and concentration profiles increases. The profiles are presented in figures 11, 12 and 13. The profiles are presented in figures. 14, 15, and 16 displayed the effect of various values of (Sr) on the velocity, temperature and concentration distribution for fixed values of Prandtl number Pr , Gr , M , and Q . It was observed, from the graph that increasing the (Sr) leads to increase in the velocity but has no effect on the temperature and the concentration distribution across the channel.

The effect of the Schmidt number (Sc) on the velocity, temperatures and concentration profiles, it is observed that as the Schmidt number increases the velocity, temperatures and concentration profiles also increases, due to the molecular diffusivity and boundary layer thickness. The profiles are presented in figures 16, 17, and 18. The influence of the radiation parameter (R) on the the velocity, temperatures and concentration profiles, it is observed that as the thermal radiation increases, the velocity, temperatures and concentration increases but without much effect on the profile due to the boundary layer getting thicker. The profile are presented in figures 19, 20, and 21. Figures 22, 23, and 24 represent the velocity, temperature and concentration profiles for different values of the Prandtl number (Pr). It is observed that the Prandtl number (Pr) increases the temperatures profiles increases but have no effect on velocity and concentration profiles due to the thermal boundary layer thickness. Because of the smaller values of the Prandtl numbers, the rate of heat transfer is reduced and the boundary layer is thicker. The effects of chemical reaction parameters and the velocity, temperatures and concentration profiles are presented in figures 25, 26, and 27 respectively. It is observed that on the velocity, temperatures and concentration profiles are decaying functions of amassed values of the chemical reaction parameter (Kr). Increased in values of chemical reaction parameter velocity and temperatures profiles increases but have no effect on concentration profile.

The effect of D_k the Darcy porosity parameter (D_k) on the velocity, temperatures and concentration profiles, it is observed that as the Darcy porosity parameter increases the velocity, temperatures and concentration profiles has no effect for. The profiles are presented in figures 28, 29 and 30. Figures 31, 32, and 33 displayed the effect of various values of (D_f) on the velocity, temperature and concentration distribution for fixed values of Prandtl number Pr , Gr , M , and Q . It was observed, from the graph that increasing the (D_f) leads to increase in the velocity, temperature

and concentration distribution across the channel. Increase in Prandtl number (Pr), radiation parameter (R), Grashof number (Gr), Increase the heat transfer at the wall and under the effect of the chemical reaction parameter(Q), Schmidt number (Sc), Grashof number (Gr), and heat source parameter (Q), enhance the rate of mass transfer. Increase magnetic parameter (M), angle of inclination, decreases in velocity, temperature and concentration but has not much effect on Grashof number (Gr).

Table 1: Numerical Results for different Parameter Values for unsteady

Parameters	Values	$f''(0)$	$-\theta'(0)$	$-\phi'(0)$
Gr	5.0	1.1823	1.7538	8.6844
	10.0	1.2773	1.7596	3.6852
	15.0	1.4447	1.7753	8.7298
	20.0	9.6917	1.7537	8.8700
Q	0.1	1.9892	5.6279	3.2117
	0.3	7.7893	5.6548	3.2144
	0.5	-2.5041	5.6563	3.2327
	0.7	-4.1751	5.6533	3.2419
M	1.0	5.6836	5.4372	2.3821
	3.0	7.4730	5.4571	2.4304
	5.0	-2.2493	5.3719	2.4516
	7.0	-8.0034	5.3738	2.3627
Gc	5.0	3.4097	-3.4713	-5.5462
	10.0	2.9211	-3.1122	-4.5580
	15.0	2.2377	-2.4820	-3.6644
	20.0	1.8517	-1.4852	-1.3344

Sr	0.2	2.5968	1.1017	1.7097
	0.4	3.1234	2.1018	2.7095
	0.6	4.5950	3.1234	4.2345
	0.8	5.2223	4.1011	4.7788
Sc	5.0	1.1823	1.7538	8.6844
	10.0	1.2773	1.7596	3.6852
	15.0	1.4447	1.7753	8.7298
	20.0	9.6917	1.7537	8.8700
R	1.0	5.6836	5.4372	2.3821
	3.0	7.4730	5.4571	2.4304
	5.0	-2.2493	5.3719	2.4516
	7.0	-8.0034	5.3738	2.3627
Pr	0.2	2.5968	1.1017	1.7097
	0.4	3.1234	2.1018	2.7095
	0.6	4.5950	3.1234	4.2345
	0.8	5.2223	4.1011	4.7788
Kr	0.1	1.9892	5.6279	3.2117
	0.3	7.7893	5.6548	3.2144
	0.5	-2.5041	5.6563	3.2327

	0.7	-4.1751	5.6533	3.2419
Dk	5.0	3.4097	-3.4713	-5.5462
	10.0	2.9211	-3.1122	-4.5580
	15.0	2.2377	-2.4820	-3.6644
	20.0	1.8517	-1.4852	-1.3344
Df	5.0	3.4097	-3.4713	-5.5462
	10.0	2.9211	-3.1122	-4.5580
	15.0	2.2377	-2.4820	-3.6644
	20.0	1.8517	-1.4852	-1.3344

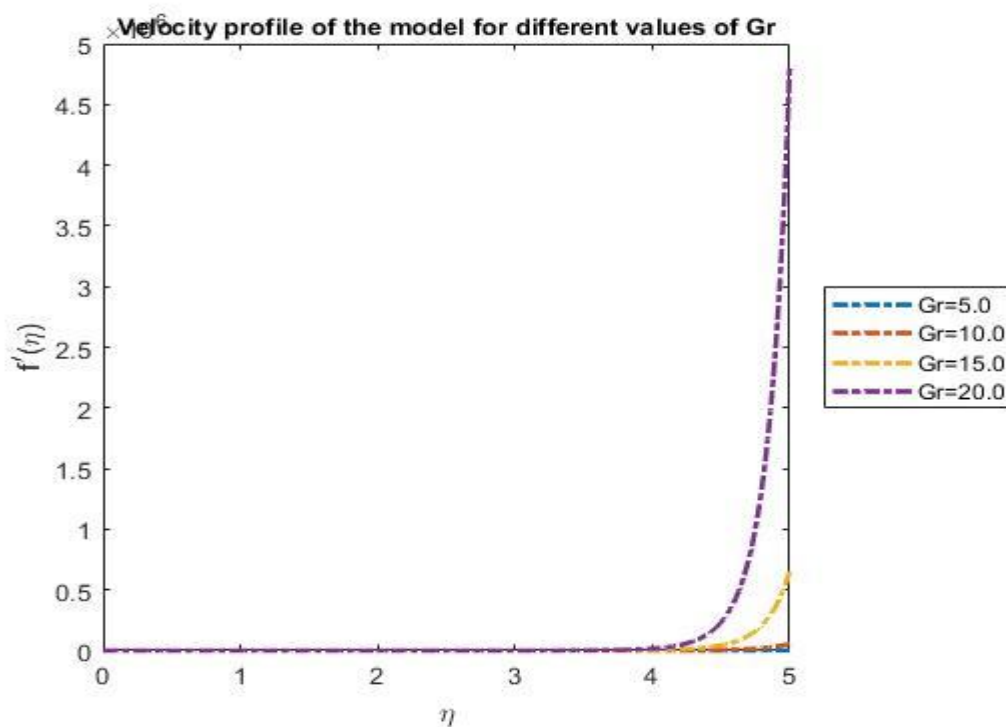


Figure 2: Velocity profile of the model for different values of Gr Number

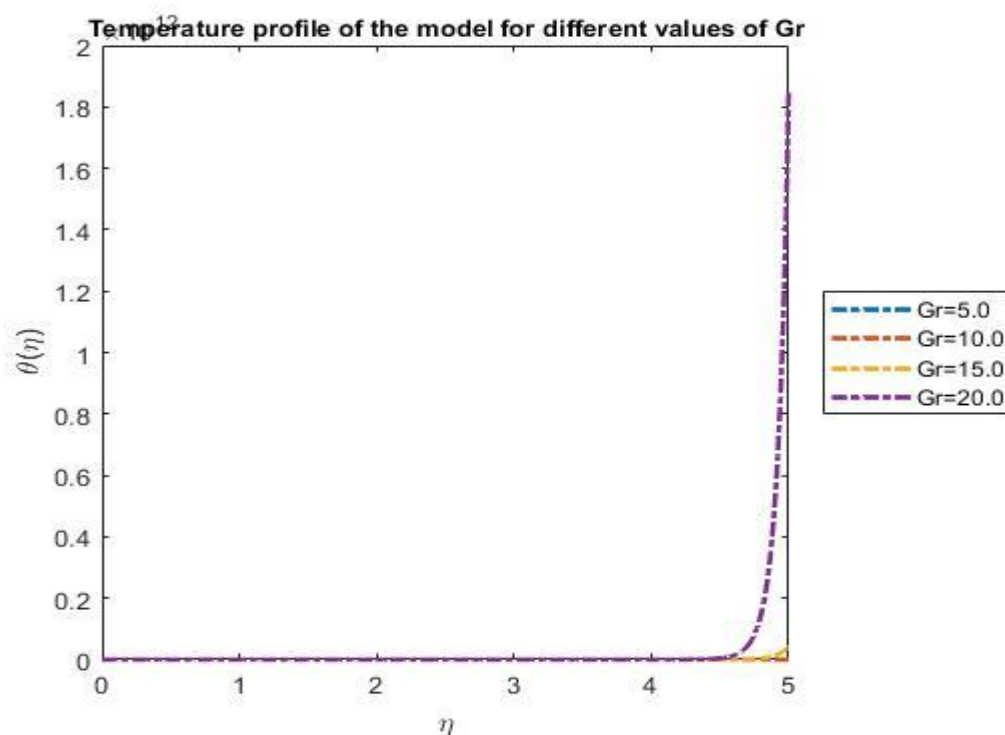


Figure 3: Temperature profile of the model for different values of Gr Number

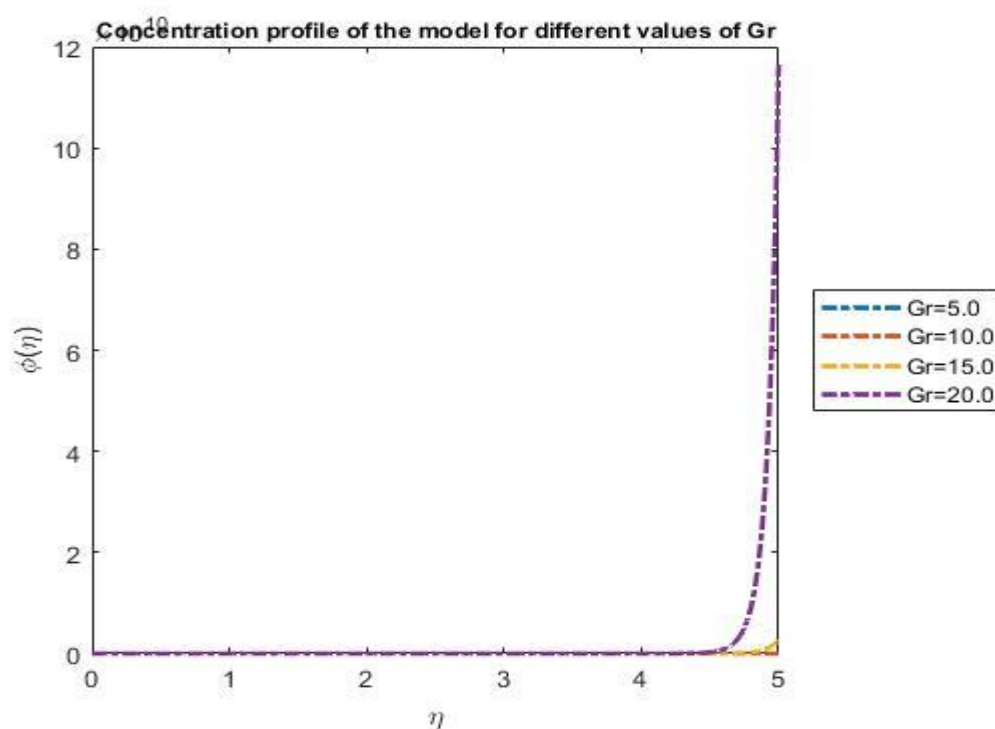


Figure 4: Concentration profile of the model for different values of Gr Number

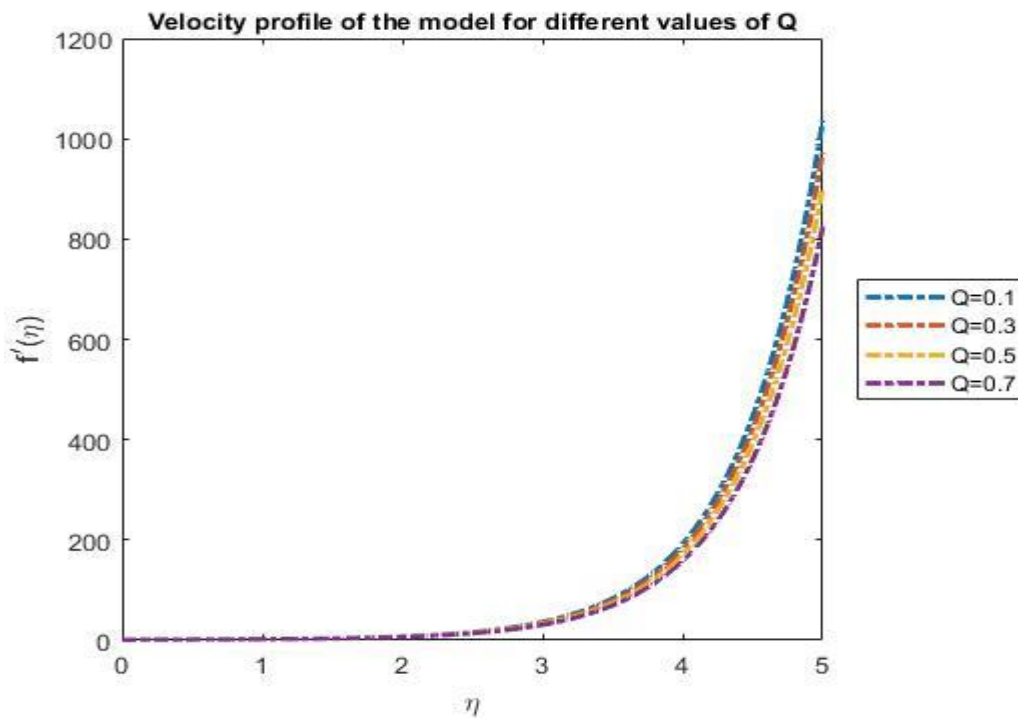


Figure 5: Velocity profile of the model for different values of Q Number

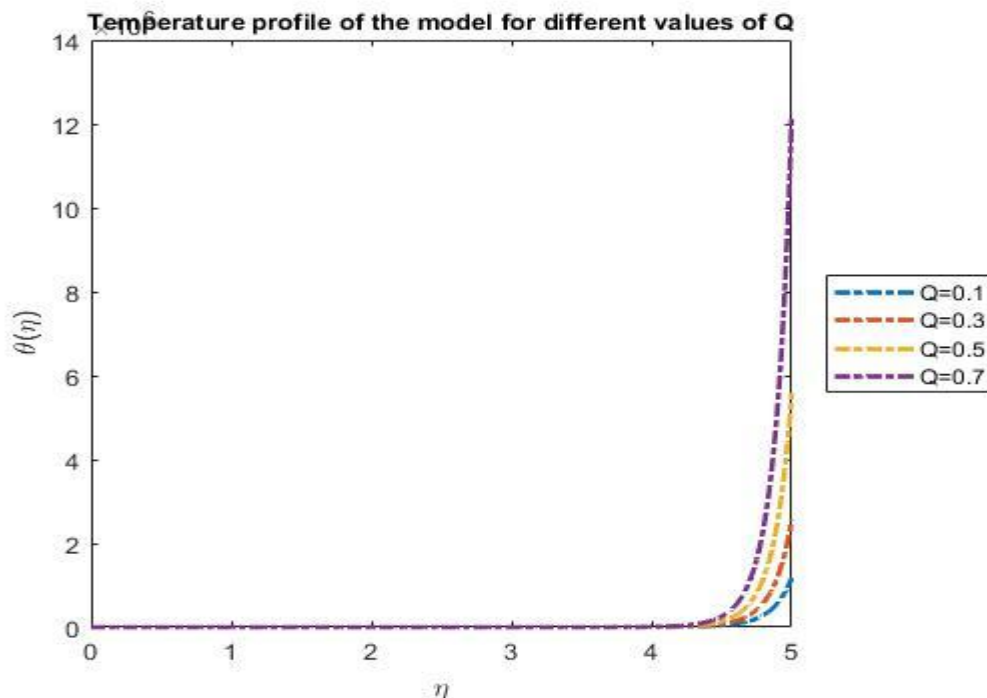


Figure 6: Temperature profile of the model for different values of Q Number

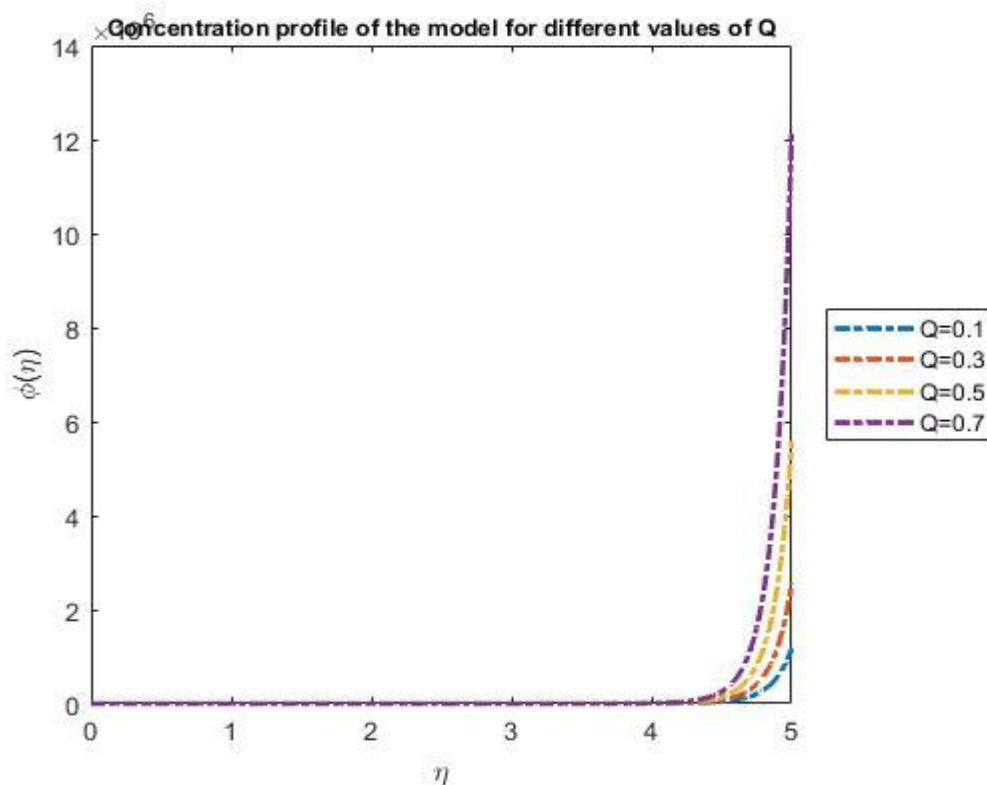


Figure 7: Concentration profile of the model for different values of Q Number

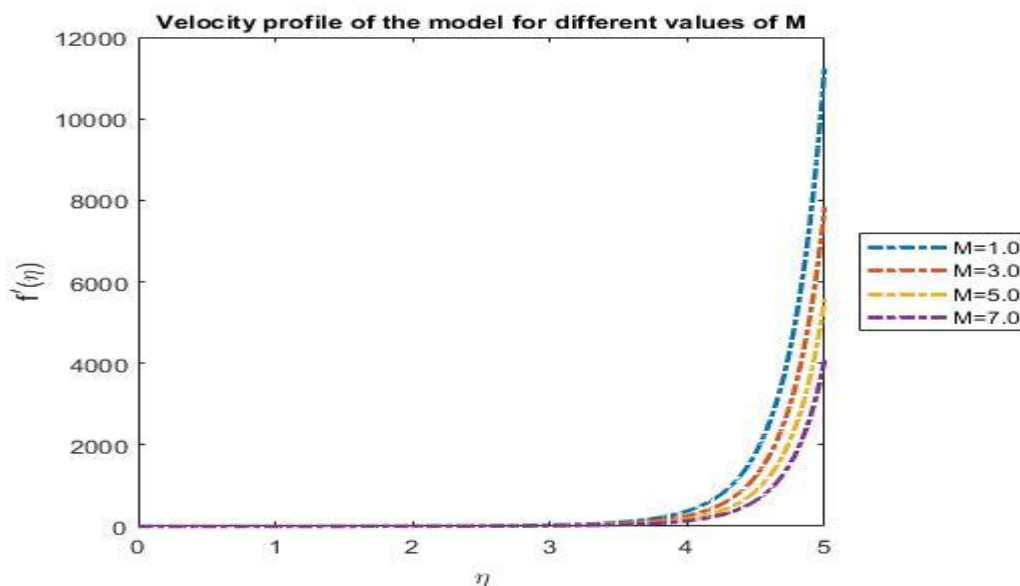


Figure 8: Velocity profile of the model for different values of M Number

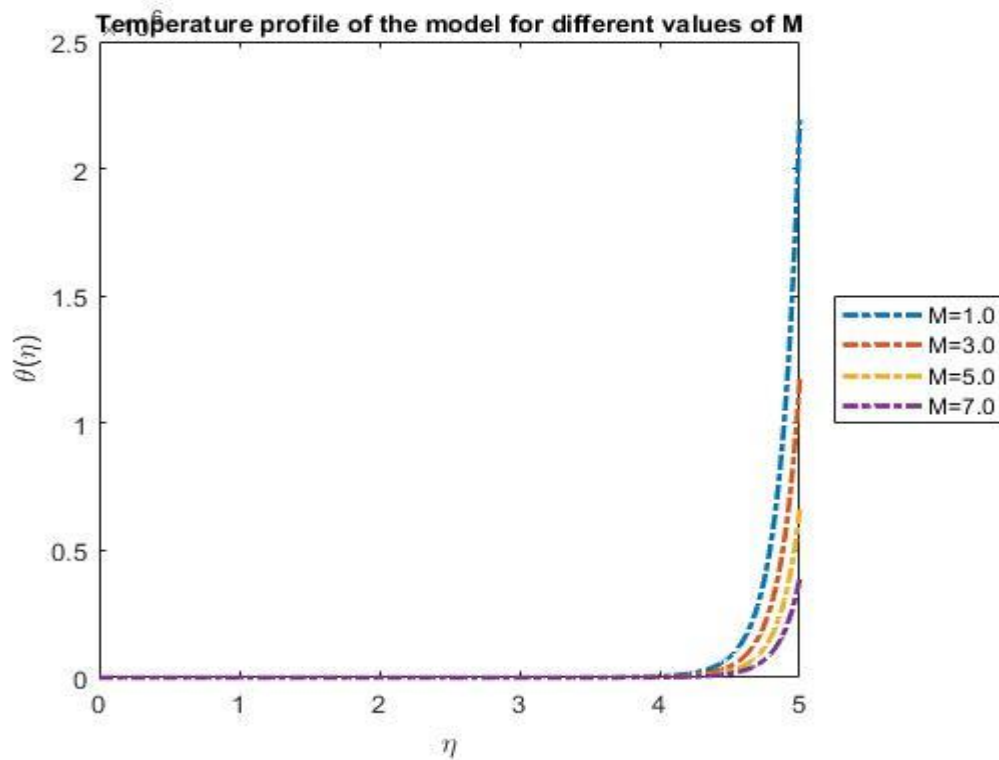


Figure 9: Temperature profile of the model for different values of M Number

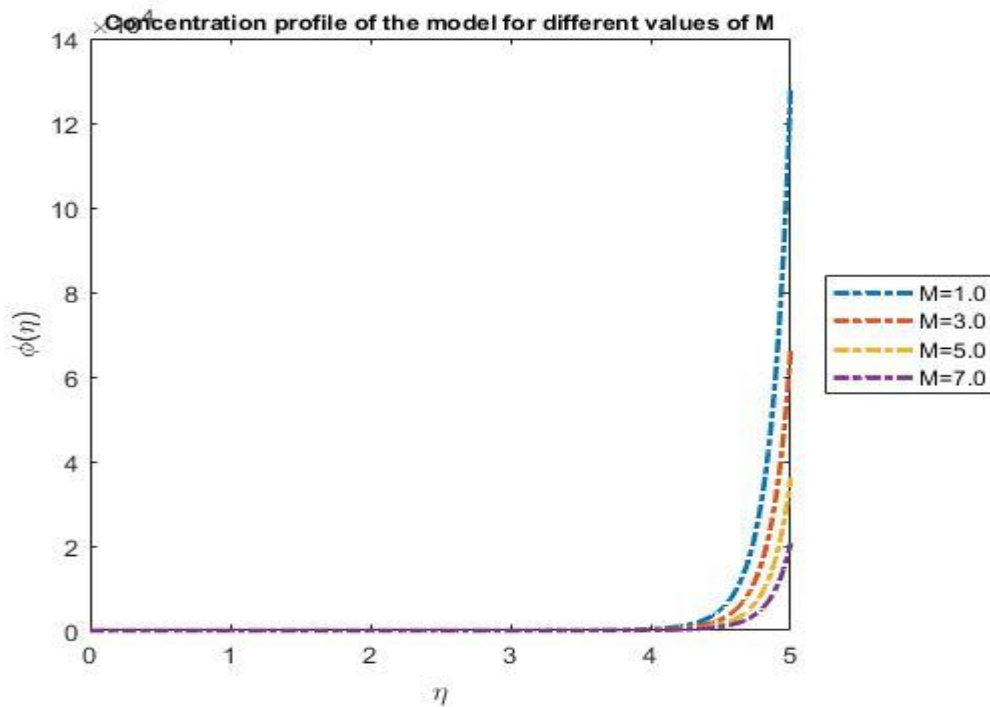


Figure 10: Concentration profile of the model for different values of M Number

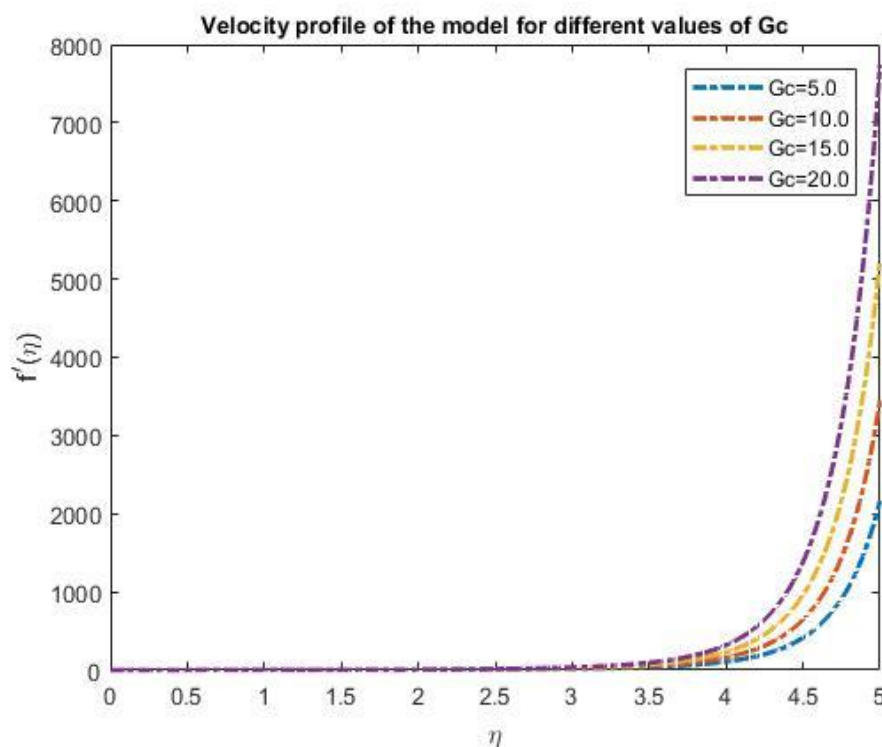


Figure 11: Velocity profile of the model for different values of Gc Number

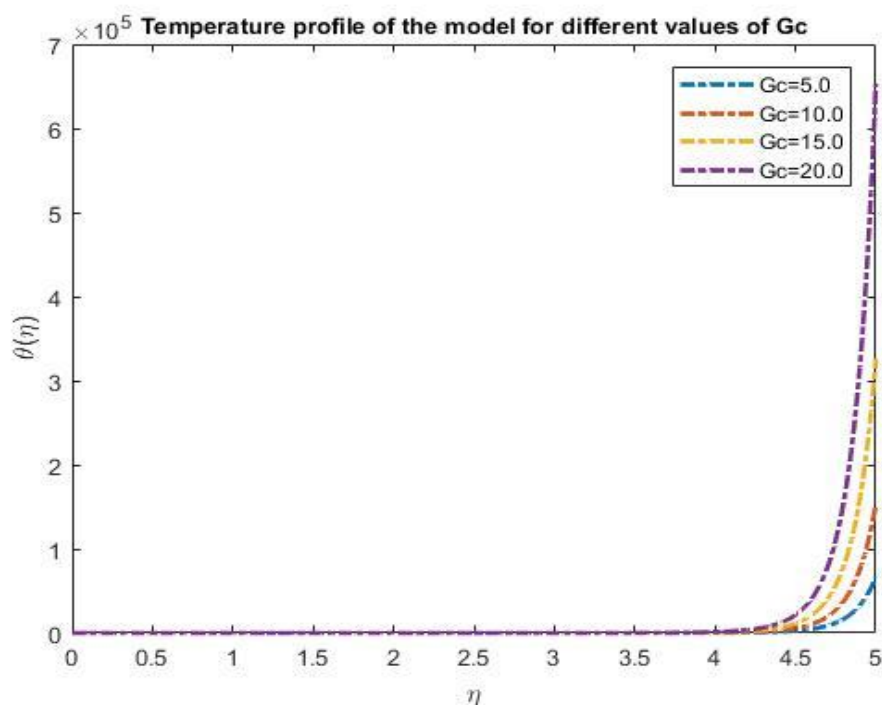


Figure 12: Temperature profile of the model for different values of Gc Number

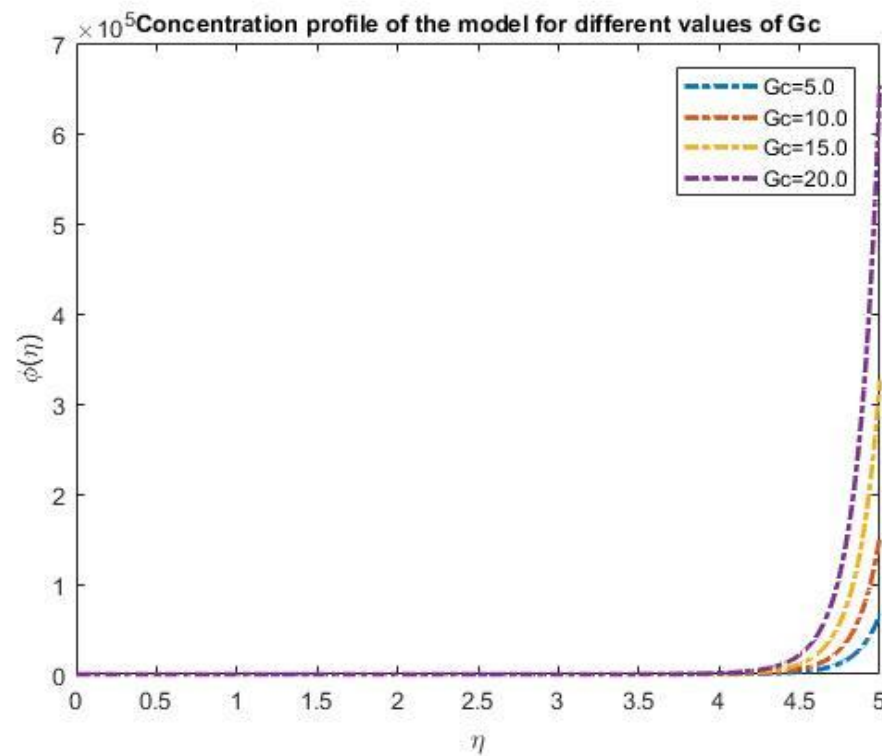


Figure 13: Concentration profile of the model for different values of Gc Number

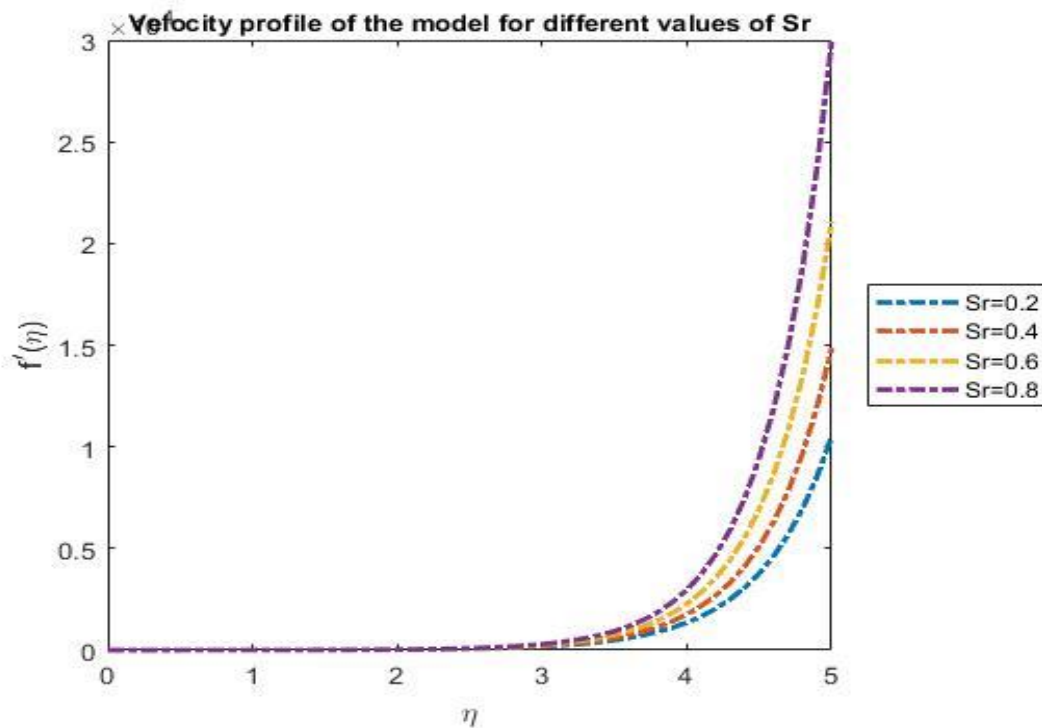


Figure 14: Velocity profile of the model for different values of Sr Number

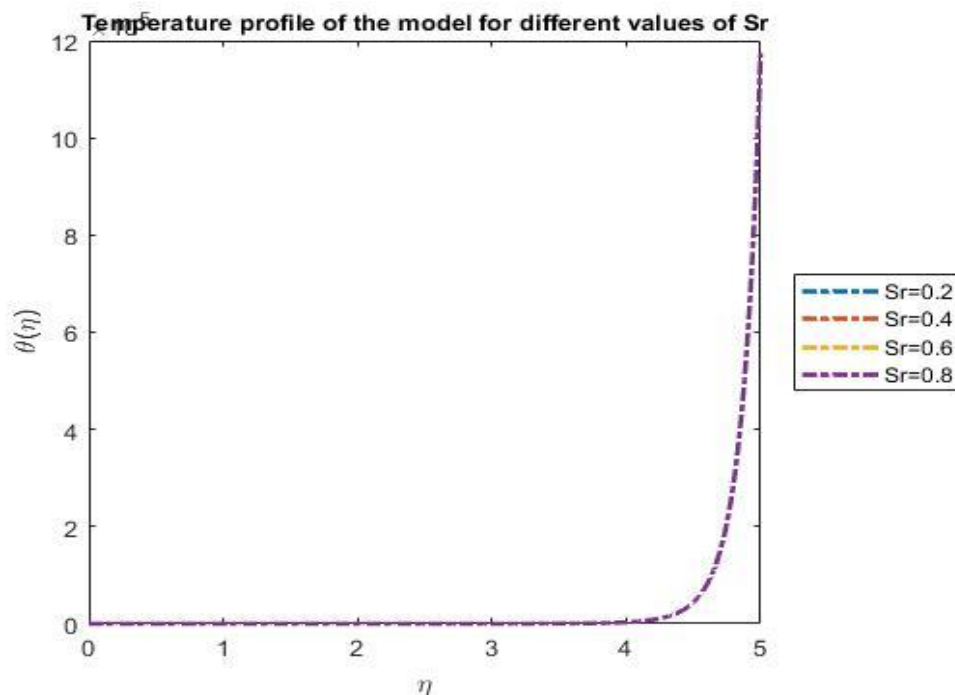


Figure 15: Temperature profile of the model for different values of Sr Number

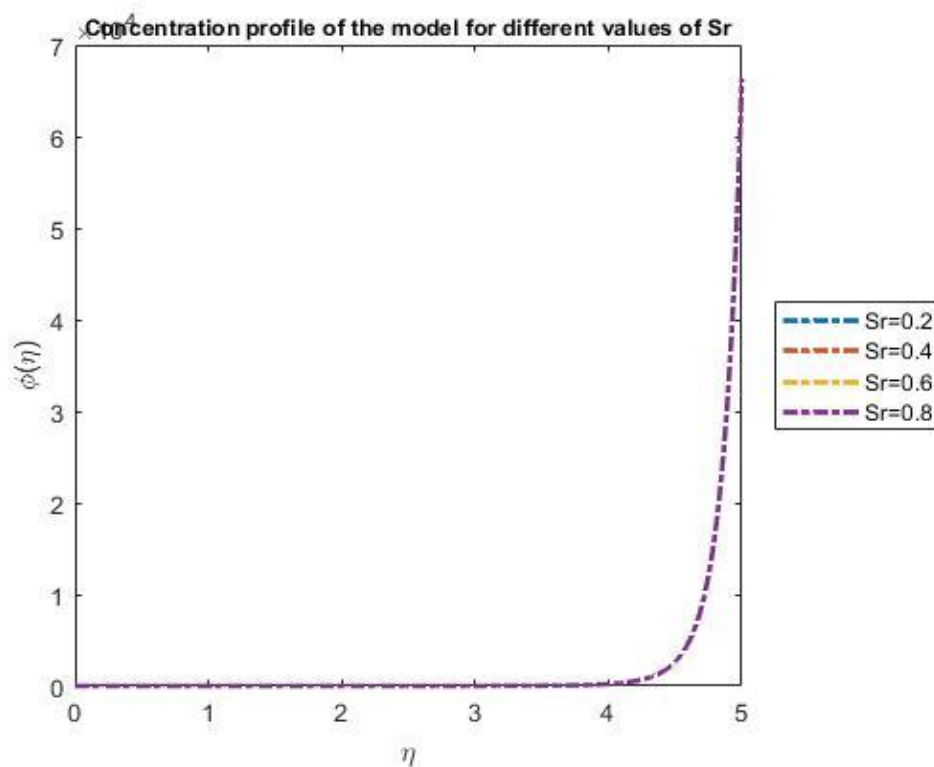


Figure 16: Concentration profile of the model for different values of Sr Number

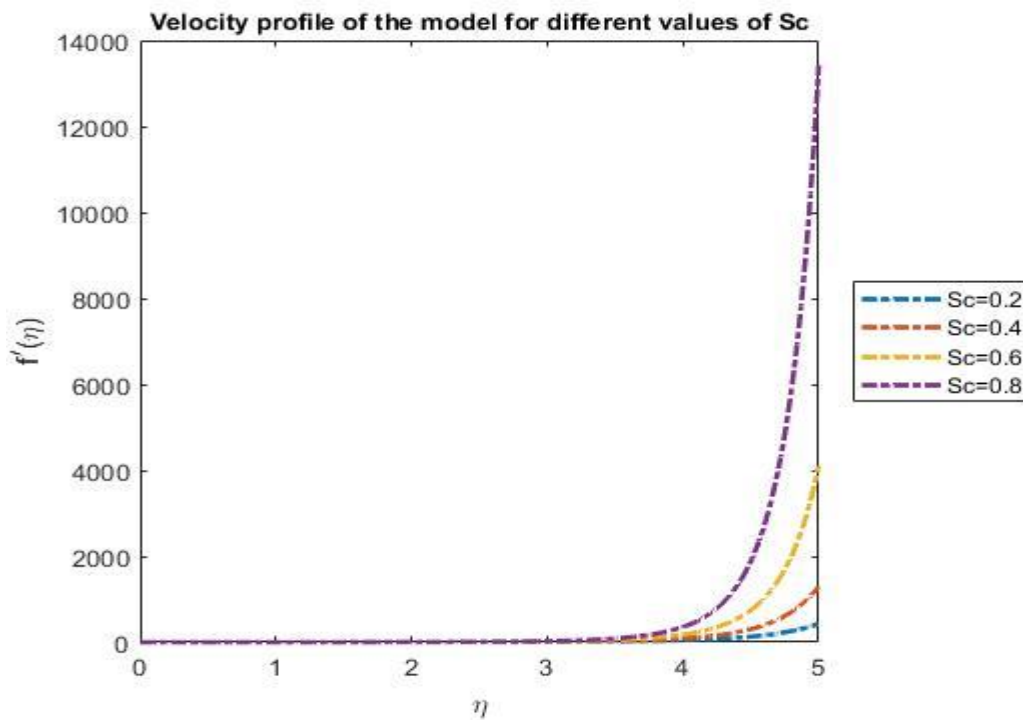


Figure 17: Velocity profile of the model for different values of Sc Number

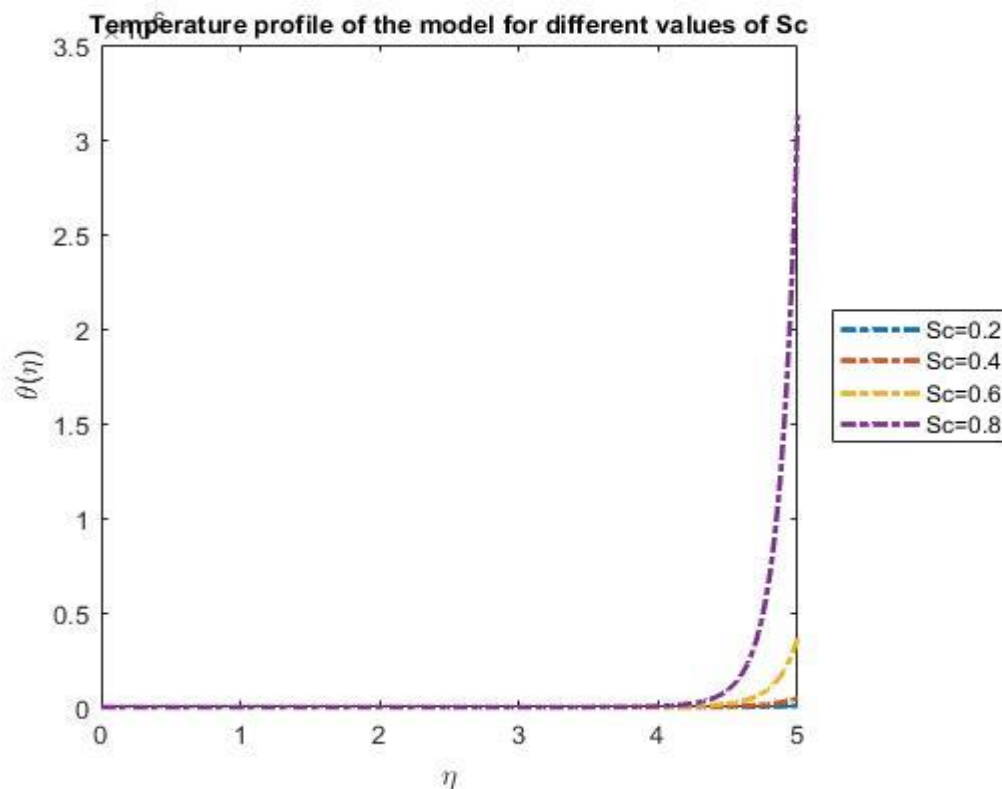


Figure 18: Temperature profile of the model for different values of Sc Number

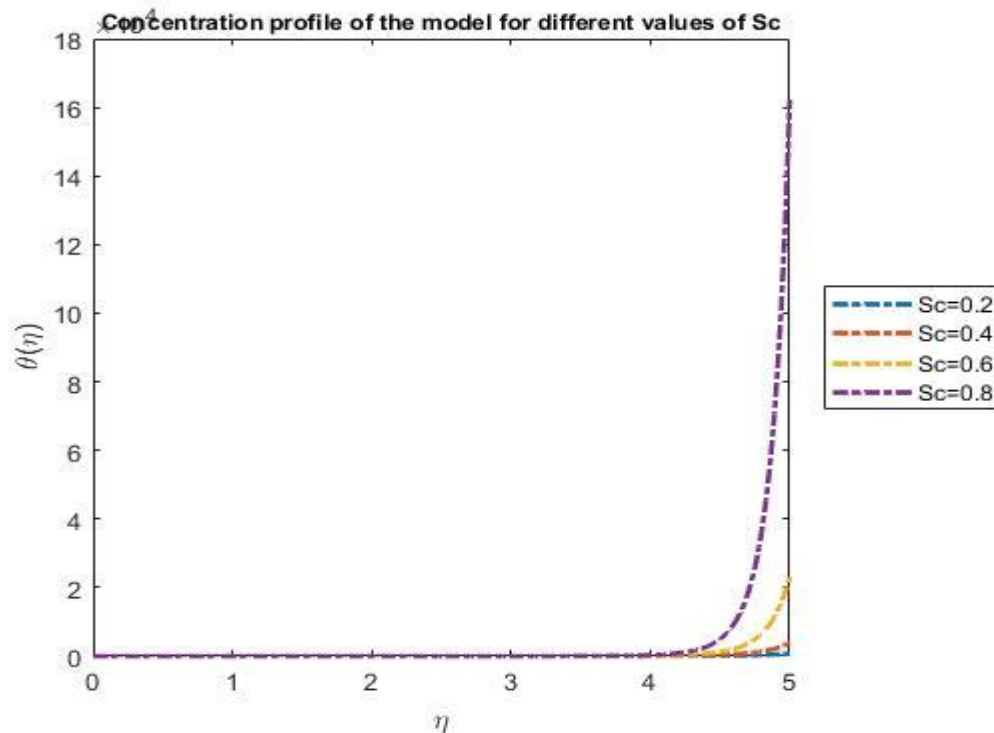


Figure 19: Concentration profile of the model for different values of Sc Number

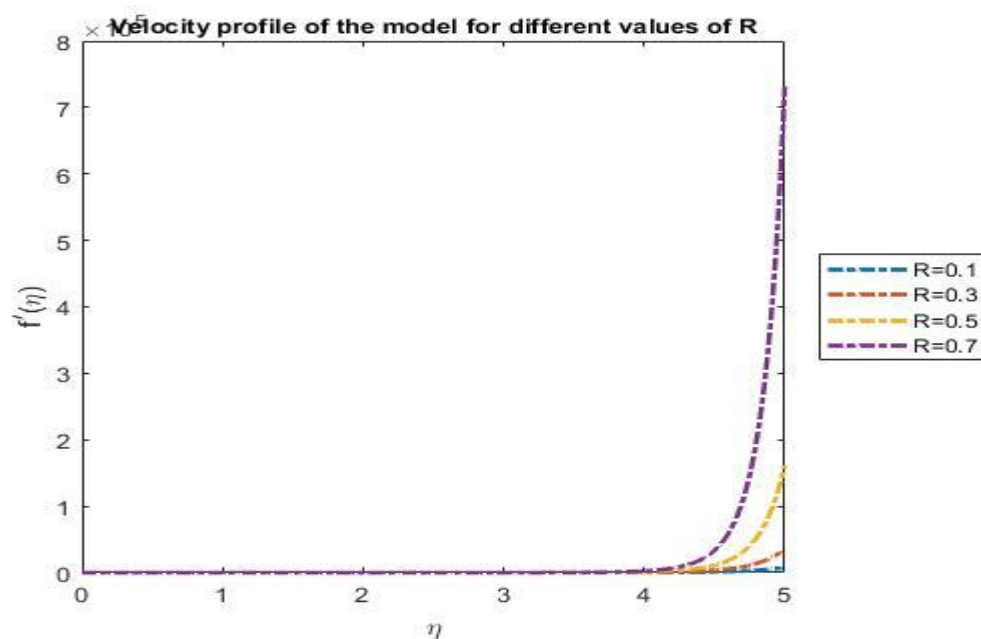


Figure 20: Velocity profile of the model for different values of R Number

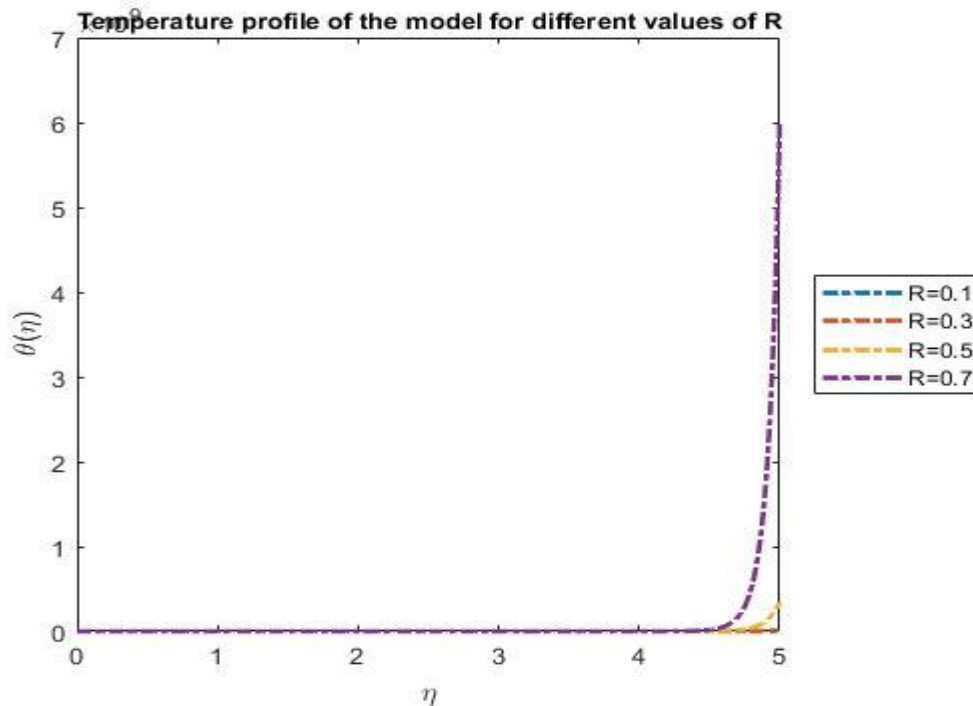


Figure 21: Temperature profile of the model for different values of R Number

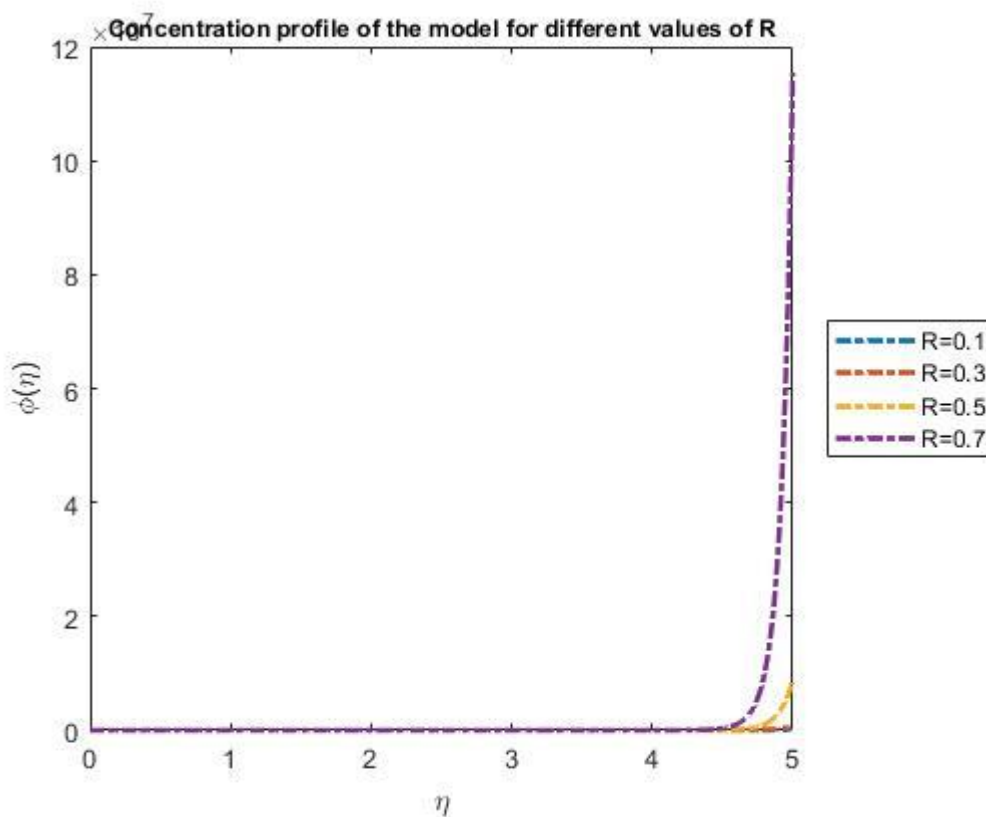


Figure 22: Concentration profile of the model for different values of R Number

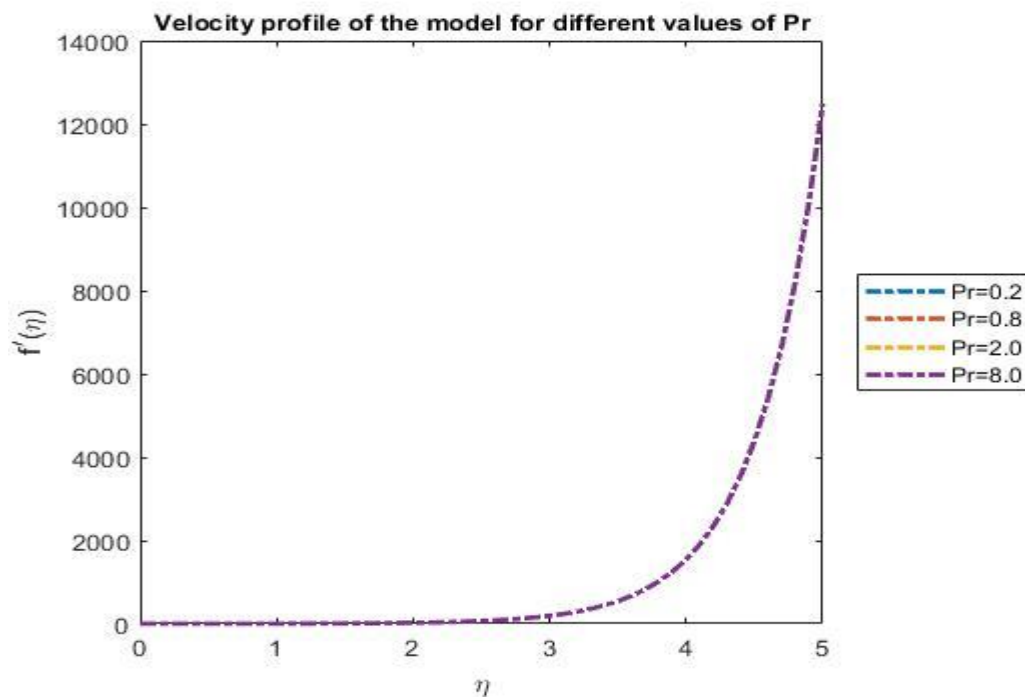


Figure 23: Velocity profile of the model for different values of Pr Number

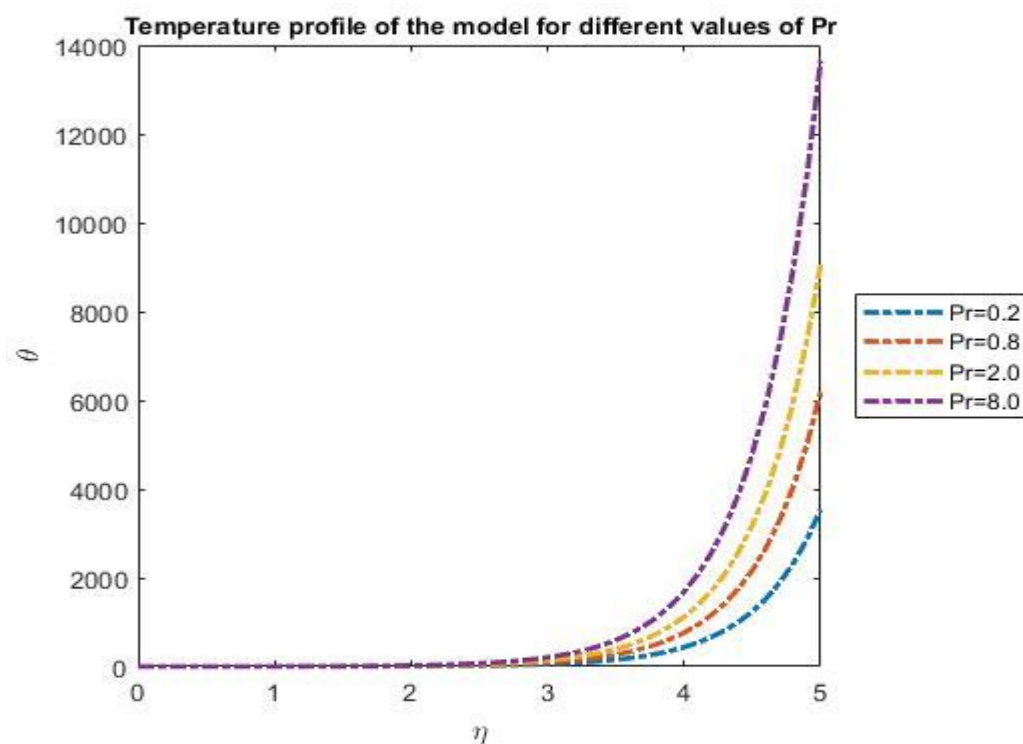


Figure 24: Temperature profile of the model for different values of Pr Number

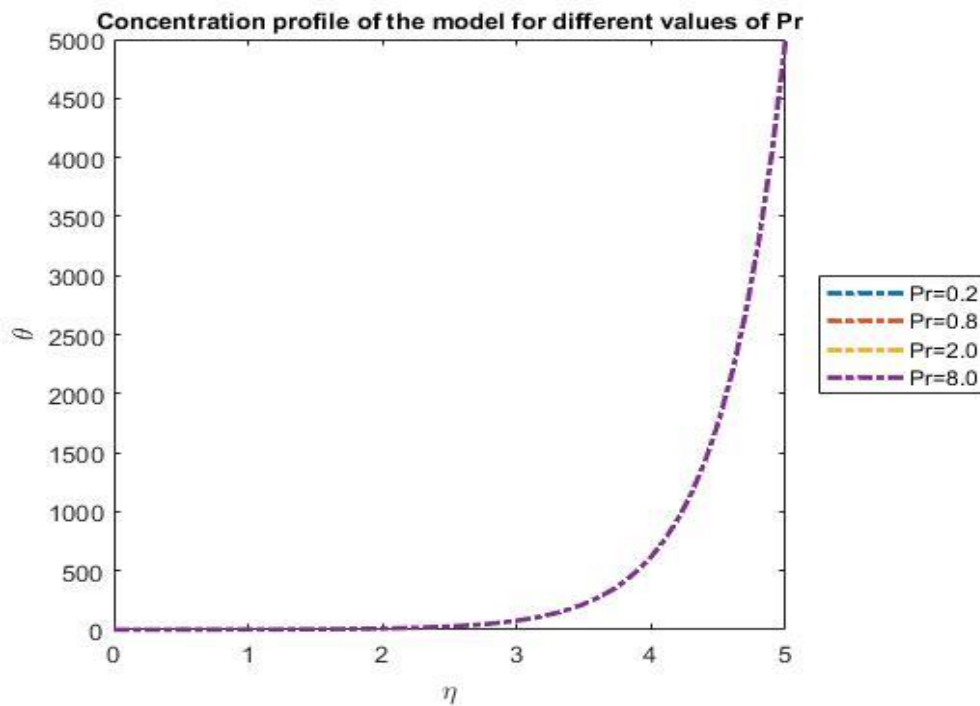


Figure 25: Concentration profile of the model for different values of Pr Number

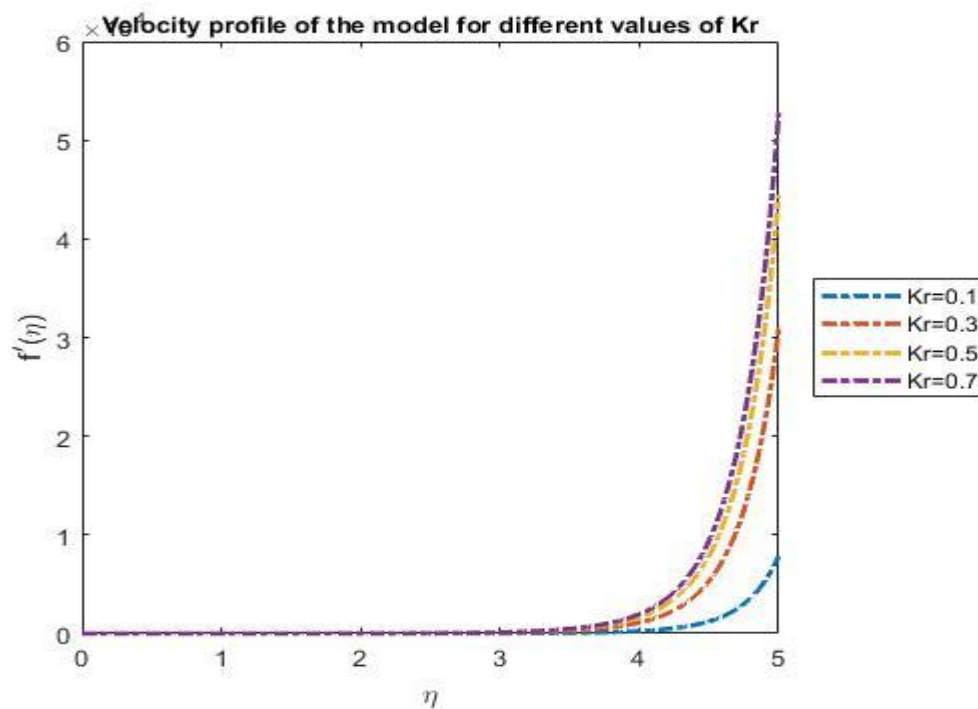


Figure 26: Velocity profile of the model for different values of Kr Number

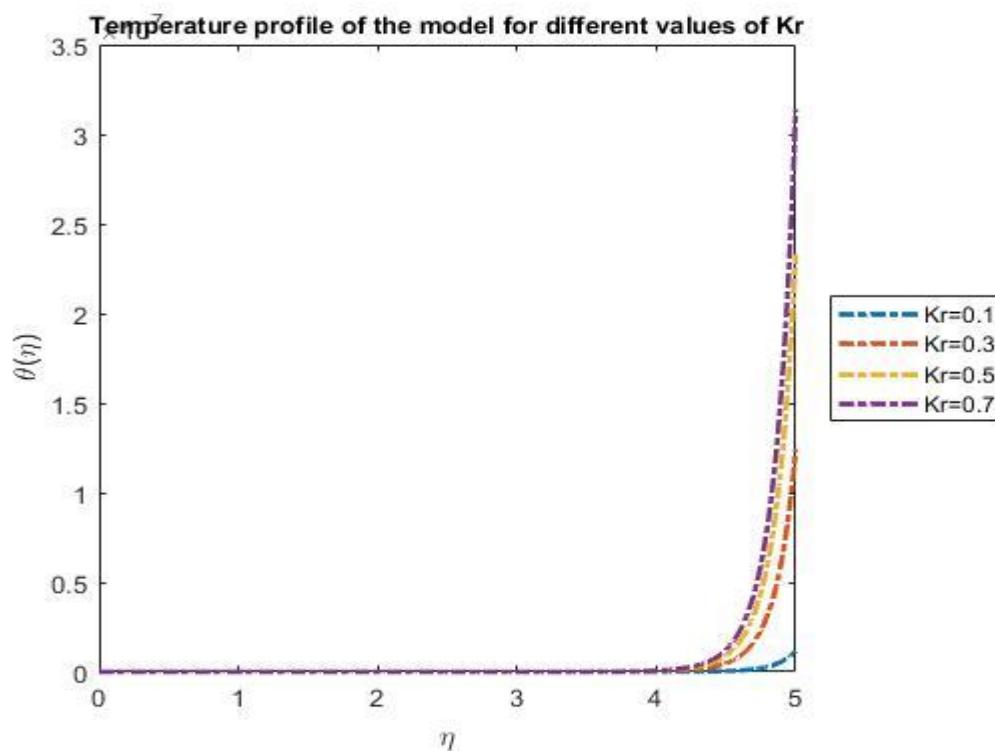


Figure 27: Temperature profile of the model for different values of Kr Number

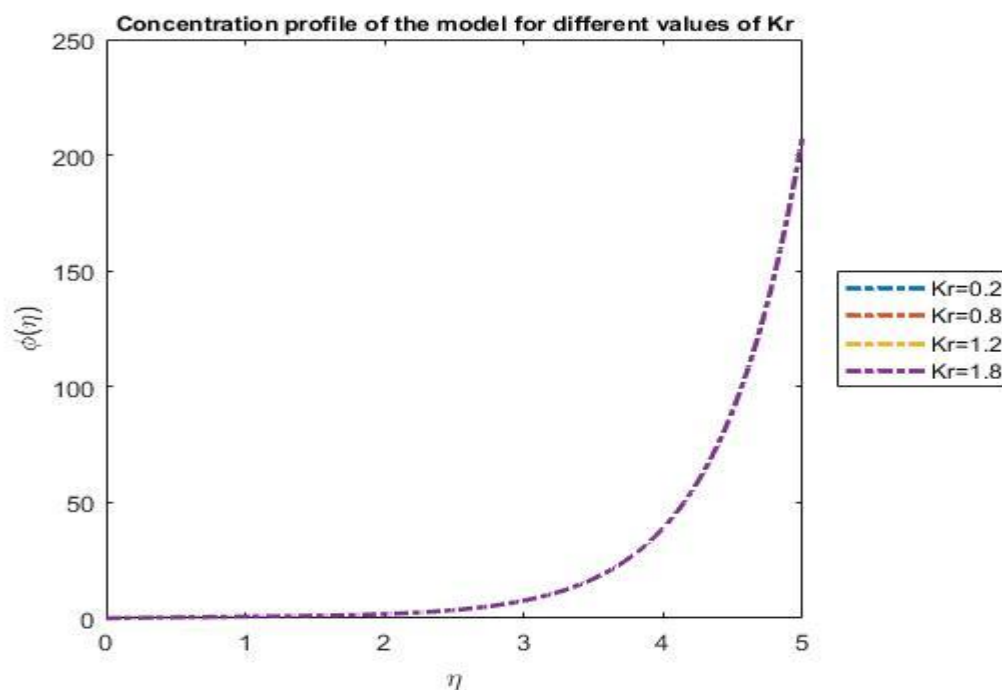


Figure 28: Concentration profile of the model for different values of Kr Number

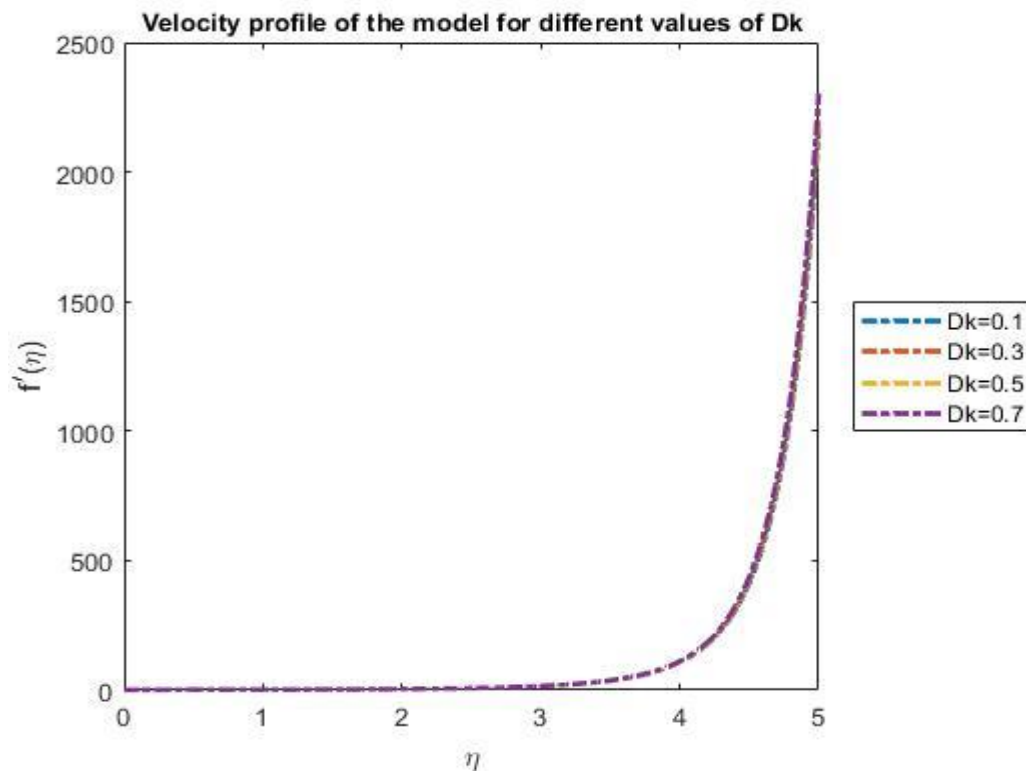


Figure 29: Velocity profile of the model for different values of Dk Number

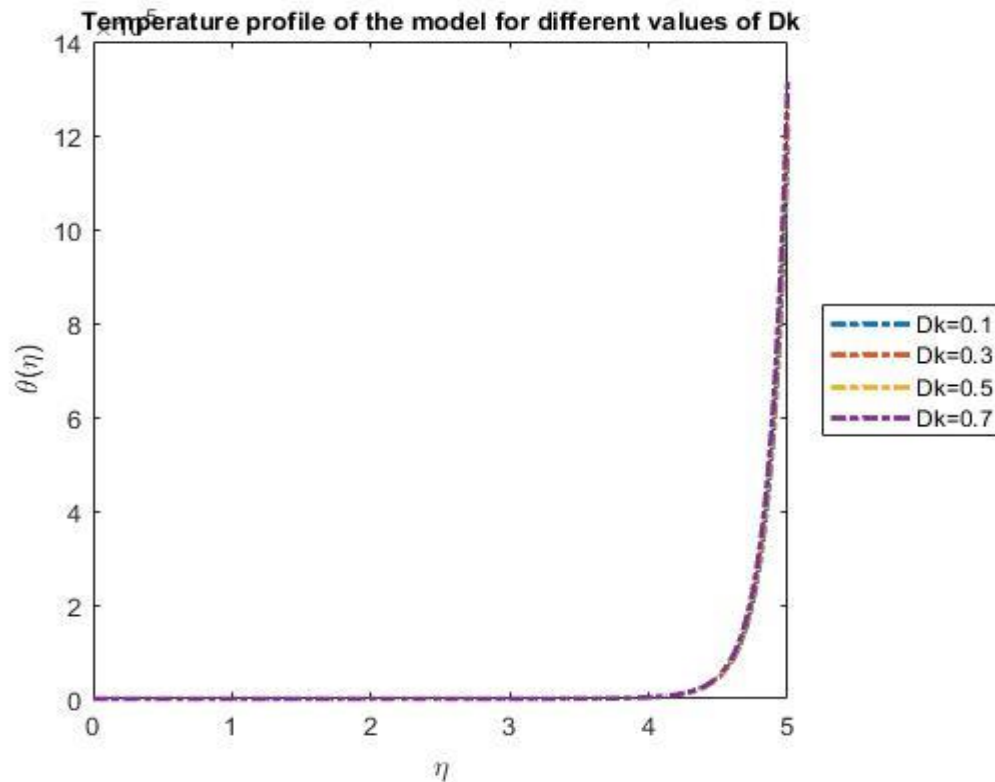


Figure 30: Temperature profile of the model for different values of Dk Number

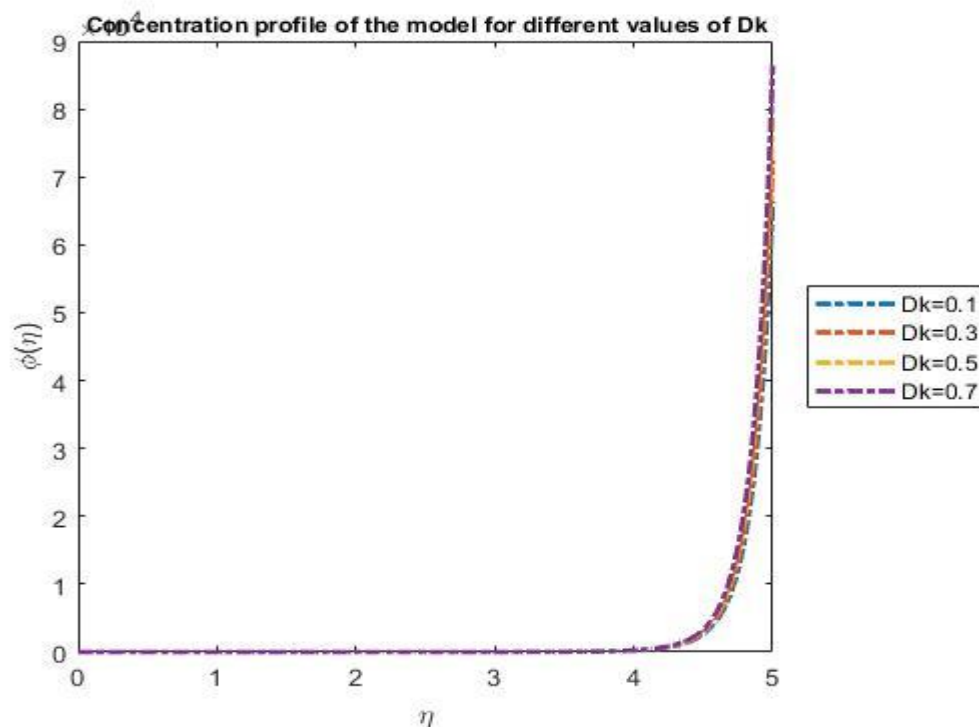


Figure 31: Concentration profile of the model for different values of Dk Number

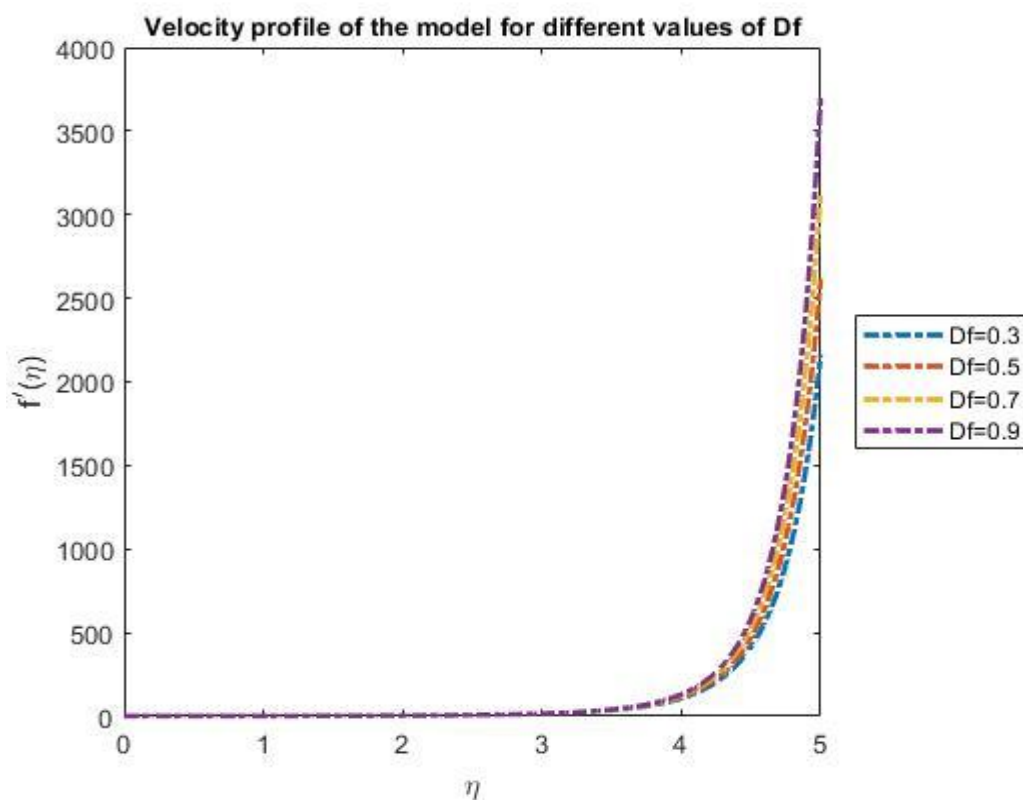


Figure 32: Velocity profile of the model for different values of Df Number

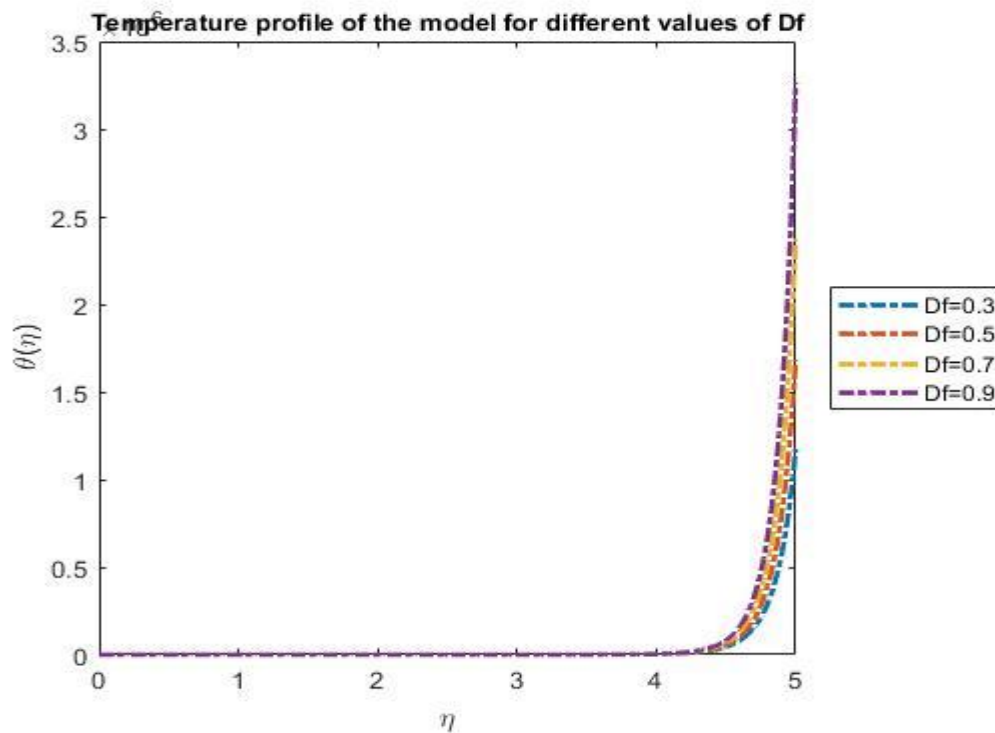


Figure 33: Temperature profile of the model for different values of Df Number

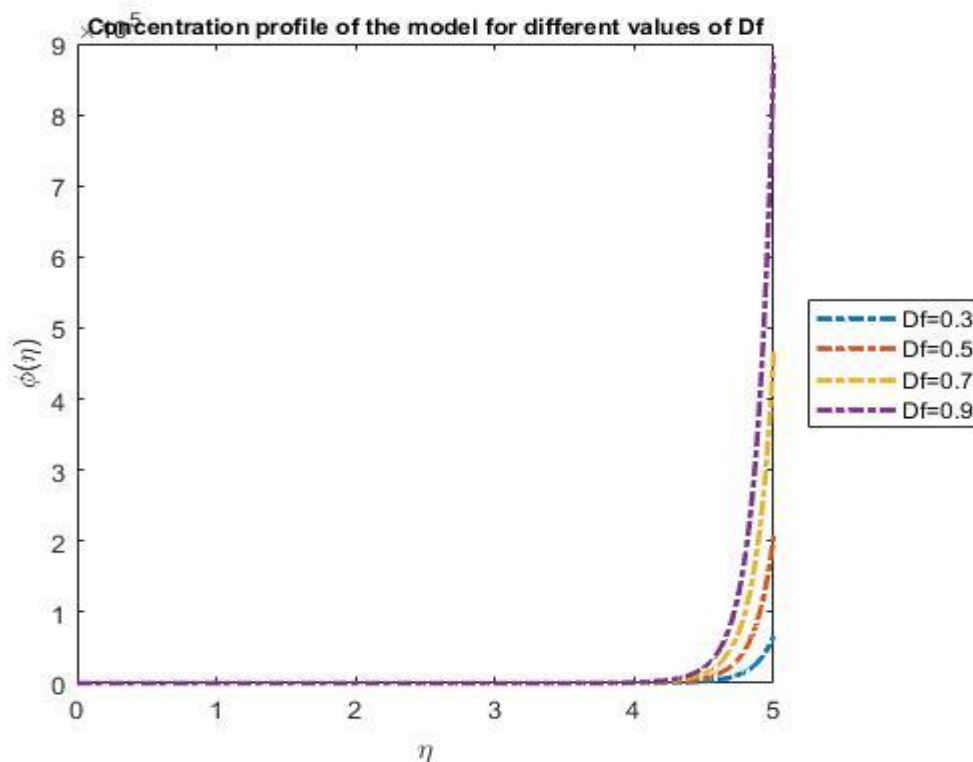


Figure 34: Concentration profile of the model for different values of Df Number

4 Conclusion

This Study Analyses heat and mass transfer effects on MHD natural convective fluid flow over a vertical and inclined porous media. Effects of various values of Grashof number Gr on the thermal boundary layer for fixed values of M , Q , G_c , Pr . From the graph, it was discovered that:

- the fluid velocity, temperature and concentration increases when the Grashof number (Gr), modified Grashof number (G_c), the Soret number (S_r), angle of inclination (α), Dufour number (D_f), magnetic parameter (M) Increase and thus thickening the thermal boundary layer across the channel.
- the fluid velocity, temperature and concentration decreases with an increase in magnetic parameter (M), chemical reaction parameter (K_r), Schmidt number (Sc), Prandtl number (Pr) and the angle of inclination parameter (α) but has not much effect on Grashof number (Gr).
- increase in Prandtl number (Pr), radiation parameter (R), Grashof number (Gr), Increase the heat transfer at the wall and under the effect of the chemical reaction parameter (Q), Schmidt number (Sc), Grashof number (Gr), and heat source parameter (Q), enhance the rate of mass transfer.

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Heat and mass transfer analysis of magnetohydrodynamic (MHD) thermosolutal nanofluid flow over vertical and inclined porous media

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Abstract

In this study, the influence of heat and mass transfer on unsteady and steady MHD thermosolutal nanofluids flow in porous media in the presence of thermophoresis and Brownian motion effects was discussed. The governing boundary layer Partial Differential Equations (PDEs) were transformed into nonlinear Ordinary Differential Equations (ODEs) by using similarity transformation and then solved numerically using fourth order Runge-Kutta method with shooting technique. It is found that increase in heat source (γ) and thermal conductivity (α) decreases the heat transfer rate but increases the skin friction and mass transfer rates. It is further discovered that the increase in thermal Grashof (G_r) and solutal Grashof (G_c) increases skin friction, heat and mass transfer rates.

Keywords: nanofluid; MHD; chemical reaction; radiation

1 Introduction

The combined study of heat and mass transfer of Magnetohydrodynamics (MHD) in the nanoparticles nanofluids is of great practical standing in engineering and sciences because of many branches of science and engineering using cooling application of nanofluids in manufacturing processes. There had been affordable amount of work carried out by scientist and researchers on radiative heat transfer in nanofluids due to their abundant applications. The study of MHD flow through porous media is encountered in a wide range of engineering and industrial applications such as extraction of crude oil, geothermal systems, thermal insulation, heat exchangers, packed bed analytic reactors, atmospheric and oceanic circulations e.t.c.

Heat and mass transfer analysis of Nanofluid flow based on Cu, Al₂O₃, and TiO₂ over a moving rotating plate and impact of various nanoparticle shapes was discussed [1]. Chemical reaction and thermal radiation effects on boundary layer flow of nanofluid over a wedge with viscous and Ohmic dissipation was investigated [2]. Cross-diffusion effects on heat and mass transfer of magnetohydrodynamic flow in porous media over exponentially stretching surface. They found out

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that solutal Grashof, thermal Grashof, magnetic parameter, radiation parameter, Dufour and Soret effects had significant effects on MHD fluid flow in porous media stretching surface was investigated [3]. Finite element Soret Dufour effects on an unsteady MHD heat and mass transfer flow past an accelerated inclined vertical plate was investigated [4]. Slip effects on heat and mass transfer in MHD Visco-Elastic fluid flow through a Porous Channel was analyzed [5]. Numerical approach of heat and mass transfer of MHD Casson fluid under radiation over an exponentially permeable stretching sheet with chemical reaction and Hall Effect was analyzed numerically [6]. Analytical and numerical solution of viscous fluid flow with the effects of thermal radiation and chemical reaction past a vertical porous surface was analyzed [7]. Computational frame work on MHD dissipative flow over a stretched region with cross diffusion simultaneous solutions was discussed [8].

Influence of novel turbulator on efficiency of solar collector system was presented [9]. Numerical simulation of a time-dependent electroviscous and hybrid nanofluid with Darcy forchheimer effect between squeezing/compressing plates was investigated [10]. MHD and nonlinear thermal radiation effects on hybrid nanofluid past a wedge with heat source and entropy generation was presented [11]. Study of a nonuniform heat source over a Riga plate using n th order chemical reaction on Oldroyd-B nanofluid flow for two-dimensional motion was studied [12]. Influence of thermal radiation and viscous dissipation on MHD flow of UCM fluid over a porous stretching sheet with higher order chemical reaction was studied [13]. Maxwell nanofluid flow over an infinite vertical plate with ramped and isothermal wall temperature and concentration was discussed [14].

MHD flow of nano fluid past a vertical permeable semi-infinte moving plate with constant heat source was analyzed [15]. Computational treatment of Magnetohydrodynamics Maxwell nanofluid flow across a stretching sheet considering higher order chemical reaction and thermal radiation was computed [16].). Estimation of unsteady hydromagnetic Williamson fluid flow in a radiative surface through numerical and artificial neural network modeling was analyzed [17]. Statistical modeling for bioconvective tangent hyperbolic nanofluid towards stretching surface with zero mass flux condition was analyzed [18]. Non-linear thermal radiation and heat source effects on unsteady electrical MHD motion of nanofluid past a stretching surface with binary chemical reaction was investigated [19]. MHD non-Newtonian fluids flow past a stretching sheet under the influence of non-linear radiation and viscous dissipation was examined [20].

Analysis of heat and mass transfer of magnetohydrodynamics nanofluid flow in conducting field through a vertical plate in porous medium was presented [21]. Radiation and chemical reaction effect on MHD thermosolutal nanofluid flow over a vertical plate in porous medium was suggested [22]. Impact of Lorentz Force and Viscous Dissipation on Unsteady Nanofluid Convection Flow over an Exponentially Moving Vertical Plate [23]. The exact solutions of time-dependent free convection MHD flow of some nanofluids close to a moving vertical plate, saturated in porous medium incorporating radiative heat flux and heat sink have been studied [24]. Analytical series solutions for the unsteady boundary layer flow of Williamson nanofluids induced by a permeable stretching sheet embedded in a porous medium in the presence of magnetic field, thermal radiation, and chemical reaction have been discussed [25].

Three dimensional nanofluid stirring with non-uniform heat source/sink through an elongated sheet was studied [26]. Simulation of time-dependent radiative heat motion over a stretching/shrinking sheet of hybrid nanofluid: Stability analysis for dual solutions was analyzed [27]. Impact of thermal conductivity on the thermophysical properties and rheological behavior of nanofluid and hybrid nanofluid was presented [28]. Impact of hybrid nanofluids on MHD flow and heat transfer near a vertical plate with ramped wall temperature have been studied [29]. Thermal

Radiation, Chemical Reaction and Viscous Dissipation Effects on Unsteady Magnetohydrodynamics (MHD) Flow of Viscoelastic Fluid Embedded in a Porous Medium was studied [30].

2 Formulation of the problem

Consider unsteady two-dimensional incompressible laminar boundary layer MHD flow of a viscous nanofluid over a permeable inclined stretching sheet. The nanofluid is supplied heat by the stretching sheet and concentration of chemical species at uniform rates. It is assumed that the influence of density variation with temperature and concentration occurs only on the body force term and hence the changes in both concentration and temperature induce the buoyancy force. A uniform magnetic Field is applied normal to the surface of the stretching sheet. Further, it is assumed that a homogeneous first order chemical reaction with thermal radiation is taking place in the flow. It is assumed that the velocity of the stretching sheet is $u_w(x,t)$ in the direction of the force F applied along the x -axis and that of the mass transfer is $v_w(t)$ Normal to the stretched sheet.

It is also assumed that the surface wall temperature and concentration of the sheet are $T_w(x,t)$ and $C_w(x,t)$, respectively, while the uniform temperature and concentration far from the sheet are, respectively, T_∞ and C_∞ . In addition, it is assumed that the effect described by Fourier's and Fick's law is of higher order of magnitude than the effect due to Dufour and Soret and thus the Dufour and Soret effects are neglected. The fluid thermal conductivity and molecular diffusivity are assumed to vary as a linear function of temperature. The model flow diagram is illustrated in Figure 1.

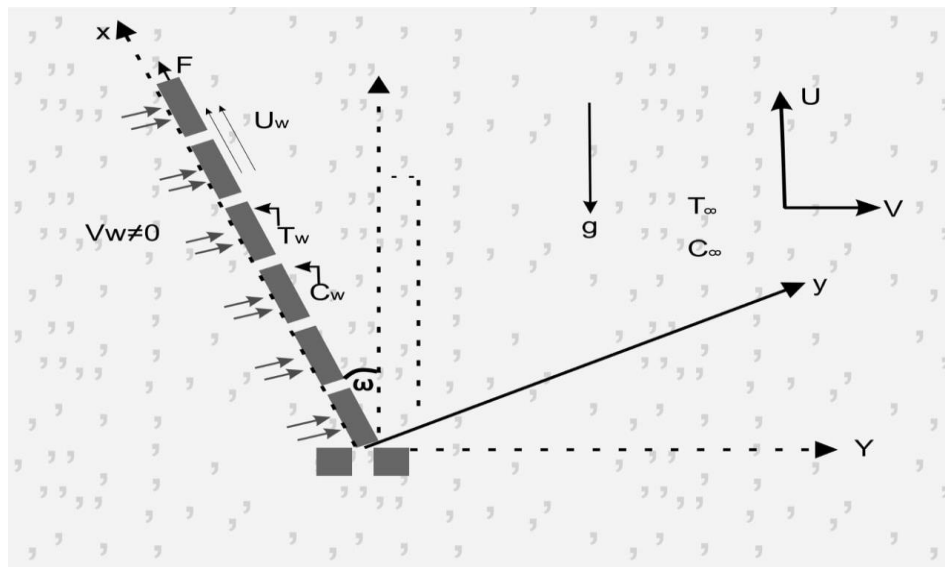


Figure 1: Physical Model and Coordinate system

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

Momentum equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} - \frac{\mu}{K_p} u + g\beta_c (C - C_\infty) \cos \omega + g\beta_T (T - T_\infty) \cos \omega, \quad (2)$$

Energy equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \left(\frac{D_T}{T_\infty} \right) \left(\frac{\partial T}{\partial y} \right)^2 \right] - \frac{1}{(\rho c_p)} \frac{\partial q_r}{\partial y} + \frac{Q}{\rho c_p} (T - T_\infty) \quad (3)$$

Concentration equation

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty} \right) \left(\frac{\partial^2 T}{\partial y^2} \right) - \lambda (C - C_\infty). \quad (4)$$

The relevant boundary conditions are:

$$u = u_w(x, t) = \frac{cx}{1 - \pi t}, v = v_w(x, t), T = T_w(x, t), C = C_\infty(x, t) \text{ at } y = 0 \quad (5)$$

$$u \rightarrow 0, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } y \rightarrow \infty.$$

The radiation heat flux q_r is modelled using Rosseland's approximation which gives

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y} \quad (6)$$

It is also assume that if the difference in the temperature within the flow is T^4 , then T^4 can be written as the linear combination of the temperature by expanding T^4 by Taylor's series about T_∞ gives

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + \dots \quad (7)$$

Neglecting the higher order beyond the first degree in $(T - T_\infty)$ then we have

$$T^4 \approx -3T_\infty^4 T + 4T_\infty^3 T, \quad (8)$$

where t is time, u is velocity in x direction, ρ is the Mass density of the fluid, C_p is Specific heat capacity at constant pressure, K_p is Permeability, λ is the Rate of chemical reaction, β_T is Coefficient of volumetric thermal expansion, β_c is Coefficient of volumetric expansion due to chemical species, T is Temperature of the species, C is Concentration of the species, T_∞ is Ambient temperature, C_∞ is Ambient concentration, μ is Kinematic viscosity of the ambient fluid, σ is Electrical conductivity, B_0 is Strength of external magnetic field, g is Acceleration due to gravity, q_r is Radiation heat flux, ω is Angle of inclination measured from the vertical axis to the stretching sheet, D_T is Thermophoretic diffusion coefficient, D_B is Brownian diffusion coefficient, c is Initial stretching sheet, w is Sheet

ambient condition, ∞ is Boundary layer edge ambient condition, v_0 is Constant that describe the wall mass transfer parameter, σ^* is the Stefan-Boltzmann constant, k^* is Rosseland mean absorption coefficient, α is Thermal diffusivity of the fluid.

In other to transform the governing Partial Differential Equation (PDE), (2 - 4), will be transform into system of Ordinary Differential Equation (ODE) by introducing the dimensionless (similarity) variable given below

$$\eta = \sqrt{\frac{c}{\mu(1-\pi)}}y, \psi(x, y) = \left(\frac{\mu c}{(1-\pi)}\right)xf(\eta), \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \quad (9)$$

Therefore, $T = (T_w - T_\infty) + T_\infty$ and $C = (C_w - C_\infty) + C_\infty$.

Introducing the stream function $\psi(x, y)$ which define the velocity component in the form

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \quad (10)$$

Using (10) in (1),

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} \right) = 0.$$

Continuity equation satisfies Cauchy Riemann equation, using (10) into (2) - (4), gives

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial y} \right) + \left(\frac{\partial \psi}{\partial y} \right) \left(\frac{\partial^2 \psi}{\partial x \partial y} \right) - \frac{\partial \psi}{\partial x} \left(\frac{\partial^2 \psi}{\partial y^2} \right) &= \mu \frac{\partial^3 \psi}{\partial y^3} - \frac{\sigma B_0^2}{\rho} \frac{\partial \psi}{\partial y} - \frac{\mu}{K_p} \frac{\partial \psi}{\partial y} \\ &+ g\beta_c(C - C_\infty)\cos\omega + g\beta_T(T - T_\infty)\cos\omega \end{aligned} \quad (11)$$

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} + \frac{Q}{\rho C_p} (T - T_\infty) \quad (12)$$

$$\frac{\partial C}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \left(\frac{\partial^2 T}{\partial y^2} \right) - \lambda(C - C_\infty) \quad (13)$$

In view of similarity variable (9) on equations (11) – (13) the following equation (14) to (16) were obtained

$$f''' - A \frac{\eta}{2} f'' - (M + \phi) f' - (f')^2 + ff'' + G_c \phi + G_r \theta = 0 \quad (14)$$

$$\left(1 + \frac{4}{3\alpha} R\right) \theta'' + \left[N_b \phi' - P_r \left(A \frac{\eta}{2} - f \right) \right] \theta' + N_t (\theta')^2 + P_r \gamma \theta = 0, \quad (15)$$

$$\phi'' - L_e \left(A \frac{\eta}{2} - f \right) \phi' + \frac{N_t}{N_b} \theta'' - L_e \delta \phi = 0 \quad (16)$$

The relevant boundary conditions are

$$f(0) = f_w, f'(0) = 0.3, \theta(0) = 0.3, \phi(0) = 0.3, f'(5) = 0, \theta(5) = 0, \phi(5) = 0 \quad (17)$$

where prime denotes differentiation with respect to η . The dimensionless parameter describing they physical properties that is affected by temperature, is defined as follows

$A = \frac{\pi}{c}$ is the unsteadiness parameter, $\varphi = \frac{\mu(1-\pi)}{cK_p}$ is the porous medium parameter,

$M = \frac{\sigma B_0^2(1-\pi)}{c\rho}$ is the magnetic parameter, $R = \frac{4T_\infty^3 \sigma^*}{3k^* \rho c_p}$ is the thermal radiation parameter,

$\delta = \frac{\lambda(1-\pi)}{c}$ is the chemical reaction parameter, $G_r = \frac{g\beta_T x(T_w - T_\infty) \cos \alpha}{u_w^2}$ is the thermal Grashof

number, $G_c = \frac{g\beta_C x(C_w - C_\infty) \cos \alpha}{u_w^2}$ is the concentration or solutal Grashof number,

$N_b = \frac{\tau D_B (C_w - C_\infty)}{\alpha}$ is the Brownian motion parameter, $N_t = \frac{\tau D_T (T_w - T_\infty)}{\alpha T_\infty}$ is the thermophoresis

parameter, $P_r = \frac{\mu}{\alpha}$ is the Prandtl number, $L_e = \frac{\mu}{D_B}$ is the Lewis number and $\gamma = \frac{Q(1-\pi)}{c\rho c_p}$ is the heat

source parameter.

3 Method of solution

The governing equations of heat and mass transfer effects on unsteady MHDs thermosolutal nanofluid flow are essentially PDEs transformed into nonlinear ODEs together boundary conditions are numerically solved using a computer embedded software code. The codes being of Runge-Kotta scheme with shooting techniques. This method has been proven to be adequate and seems to give accurate results and has be widely used [33], [34]. It seems to be the most flexible of the common methods. The scheme is also applicable to various types of boundary layer flow problems including the free and mixed connection flows.

In the numerical method employed in solving the model equations that is boundary value problem (BVP). Shooting techniques was also applied, this shooting techniques reformulates the BVP to initial value problem (IVP) by adding sufficient number of conditions at one end and adjust their

conditions until the given condition are satisfied at the other end while Runge-Kutta method solve the initial value problem.

Equation (14) to (16) were integrated as IVPs the values for $f'''(0), \theta'(0)$ and $\phi'(0)$ which were required to explain the skin friction, Nusselt and Sherwood numbers, but no such value exist at the boundary. The suitable guess values for $f''(0), \theta'(0)$ and $\phi'(0)$ were chosen and then intergration was carried out. The researchers compared the calculated values for $f''(0), \theta'(0)$ and $\phi'(0)$ at $\eta = 5$ with the given boundary conditions $f''(5) = 0, \theta'(5) = 0$ and $\phi'(5) = 0$. Then the adjusted and the estimated values for $f''(0), \theta'(0)$ and $\phi'(0)$, to give a better approximation for the solution. The researchers transform a series of computations to obtain values for $f''(0), \theta'(0)$ and $\phi'(0)$ and then applied a fourth order Runge-Kutta with shooting techniques with step size $h = 0.01$. The above procedure was repeated until the results came up the desired degree of accuracy (10^{-6})

The computation to obtain velocity, temperature and concentration profiles were presented to observe the effect on heat and mass transfer of unsteady MHDs thermophoresis on porous media. The proportional effects on the skin friction coefficient, Nusselt and Sherwood numbers were presented and determined for independent parameters. To explain the effect on heat and mass transfer of unsteady MHDs thermophoresis in porous media.

Quantities of physical and practical interest in this model are the skin or local friction factor (C_f) which represent the wall shear stress physically, the Nusselt or local Nusselt number (Nu_x) which defines the heat transfer rate physically and Sherwood or local Sherwood number (Sh_x) which defines the mass transfer rates which are represented as follows

$$C_f = \frac{2\tau_m}{\rho u_w^2}, Nu_x = \frac{xq_w}{K(T_w - T_\infty)}, Sh_x = \frac{xh_m}{D_B(C_w - C_\infty)} \quad (18)$$

$$\tau_m = \mu \left[\frac{\partial u}{\partial y} \right]_{y=0}, q_w = -K \left[\frac{\partial T}{\partial y} \right]_{y=0}, h_m = -D_B \left[\frac{\partial C}{\partial y} \right]_{y=0}$$

In which τ_m, q_w and h_m are the skin friction or shear stress, heat flux and mass flux from the sheet respectively. The skin friction, heat flux and mass flux are as follow

$$f''(0) = \frac{1}{2} \sqrt{Re_x} C_f, -\theta'(0) = \frac{Nu_x}{\sqrt{Re_x}}, -\phi'(0) = \frac{Sh_x}{\sqrt{Re_x}} \text{ Where } f''(0) \text{ is the skin friction, } -\theta'(0) \text{ is}$$

the Nusselt number and $-\phi'(0)$ is Sherwood number, this also implies that $Re_x = \frac{u_w x}{\mu} = \frac{cx^2}{\mu(1-\pi)}$ is the local Reynolds number.

4 Results and discussion

Table 1 represents the results of variation in Skin friction ($f''(0)$), Nusselt ($-\theta'(0)$) and Sherwood numbers ($-\phi'(0)$) at the surface with magnetic field parameter M , thermal Grashof number Gr , solutal or concentration Grashof number Gc , porosity at the wall f_w , Lewis number Le , thermal diffusivity α , chemical reaction δ , porous medium parameter φ , heat source γ , unsteadiness parameter A , radiation parameter R , Prandtl number Pr , which are of physical and engineering interest. Also the friction factor, local Nusselt and Sherwood numbers are discussed and presented through graphs and tables. The computation was carried out using shooting technique with Runge-Kutta iteration algorithm. These were used for couple nonlinear differential equations along with boundary conditions. In the numerical solution, a check was to confirm that smoothness conditions at the end of boundary layer were satisfied. For numerical computations and results we consider the non-dimensional parameter, with their values as follows $M = 1$, $Gr = Gc$, $f_w, Le, \alpha = 1$, $\delta = \varphi = \gamma = A = R = 0.5$, and $Pr = 0.4$, These values are considered to be constant throughout the entire research apart from the varied parameters shown in the figures.

Table 1: Numerical Approximation of the variation of parameters for $f''(0)$, $\theta'(0)$ and $\phi'(0)$

Parameter	Value	$f''(0)$	$-\theta'(0)$	$-\phi'(0)$
M	1	-0.048178	0.002663	0.223047
	4	-0.421512	-0.002680	0.212712
	5	-0.506547	-0.003437	0.210965
	7	-0.650434	-0.004452	0.208458
Gr	1	-0.044105	-0.025930	0.243175
	2.5	0.260343	-0.014428	0.250429
	3.5	0.450256	-0.008271	0.254167
	4.5	0.632298	-0.002930	0.257425
Gc	1	-0.044105	-0.025930	0.243175
	2.5	0.146030	-0.023193	0.247264
	3.5	0.269669	-0.021504	0.249778
	4.5	0.391145	-0.019900	0.252147
δ	0.5	-0.050851	-0.025757	0.279489
	1.5	-0.070576	-0.025074	0.413780
	2.5	-0.081585	-0.024661	0.513373
	3.5	-0.089143	-0.024388	0.596295
Pr	0.4	-0.044105	-0.025930	0.243175
	0.6	-0.034778	-0.089996	0.287858
	0.8	-0.017374	-0.204832	0.366537
	1	0.024284	-0.480675	0.555269

A	0.5	-0.044105	-0.025921	0.243175
	1.5	0.003179	-0.065406	0.186526
	2.5	0.028169	-0.088219	0.142952
	3.5	0.034934	-0.089555	0.116856
N_t	1	-0.044105	-0.025930	0.243175
	1.5	-0.053829	-0.029337	0.275758
	2	-0.064592	-0.032868	0.310066
	2.5	-0.076451	-0.036566	0.346120
N_b	1	1.361810	-0.019137	0.290769
	1.5	1.379045	-0.020744	0.272715
	2	1.389212	-0.022384	0.263783
	2.5	1.396505	-0.023912	0.258457
γ	0.5	-0.044105	-0.025930	0.243175
	0.8	-0.032522	-0.104795	0.297968
	1.1	-0.005723	-0.274607	0.412213
	1.3	0.081609	-0.821827	0.776237
ϕ	0.5	-0.044105	-0.025930	0.243175
	0.8	-0.096387	-0.027640	0.241806
	1.1	-0.143615	-0.029648	0.240580
	1.4	-0.186770	-0.030267	0.239477

Figures 2-4 present the influences of magnetic field parameter (M) on velocity, temperature and concentration profile respectively of the flow. It is evident from the figure 2 that an increase in the magnetic field parameter causes a decrease in the fluid velocity and 3 and 4 improves the distribution of temperature and concentration profile. It is due to the fact that an increase in magnetic field develops the force opposite to the direction of flow. But this force reduces the thermal and concentration boundary thickness, in this case, we see enhancement in the temperature and concentration profile.

Figures 5 - 7 shows the effect of thermal Grashof number (G_r) it is evident that increase in thermal Grashof parameter increases both nanofluid velocity, temperature and concentration profile. It was discovered that nanofluid velocity increased with increase in thermal Grashof parameter. This rely on the fact that thermal Grashof parameter is the ratio of thermal buoyancy force to the hydrodynamic force and this take place on the boundary layer due to changes in temperature and thus an increase in the thermal buoyancy effect on the nanofluid makes the fluid to cool the hot sheet, in this case thermal Grashof parameter slows down temperature and concentration.

Figures 8 - 10 display the effects of solutal Grashof parameter on fluid velocity, temperature and concentration profile respectively. The influence of solutal Grashof parameter on nanofluid velocity as shown in figure 8 shows that increase in solutal Grashof parameter increases the velocity profiles.

The values of temperature and concentration profile is increased due to variation in solutal Grashof parameter. In this case, temperature and concentration at the boundary layer gets thinner as solutal Grashof parameter increased.

Figures 11 - 13 illustrates the effect of chemical reaction parameter (δ) on velocity, temperature and concentration profile of the flow. It is evident that increase in δ decline nanofluid velocity flow thereby leads to increase in temperature and concentration profile. Figure 13 show that as δ increase, the concentration profile increases. Figure 14 – 16 present the influence of Prandtl number parameter (P_r) on velocity profile shown in figure 14 it can be noted that as the variation in P_r increases, the nanofluid velocity flow increases. It is important to note that from figure 15 P_r increases, it causes decrease in temperature of nanofluid, this is as a result that P_r is inversely proportional to the thermal diffusion and directly proportional to the momentum diffusivity and thereby as P_r increases, it lead to the thermal diffusion and concentration of the fluid increases.

Figures 17 – 19 depicts the typical effect of unsteadiness parameter (A) on velocity, temperature and concentration profiles of the fluid flow, respectively. It is observed that increase in A decreases the temperature and concentration profiles of the flow. It was discovered that there is increase in the nanofluid velocity flow. It is also evident that increase in unsteadiness parameter increases the temperature close to the stretching sheet thereby the wall temperature is higher than the temperature of the surrounding, and as a result of high wall temperature nanoparticles move to cooler area, resulting in the decrease of the concentration profiles.

Figures 20 – 22 present the effects of thermophoresis parameter (N_t) on velocity, temperature and concentration profile of the nanofluid flow. Increase in N_t decreases the temperature profile and causes increase in concentration profile. Figure 23 – 25 illustrate the influence of Brownian motion parameter (N_b) on velocity, temperature and concentration profile of the flow. It is also discovered the increase in N_b causes increase in the nanofluid velocity and also leads to decrease in temperature and concentration profile. It is also observed that enhancement in the N_b result in the increment of velocity, temperature and concentration boundary layer thickness. Figure 26 – 28 exhibit the effects of heat source parameter (γ) on the velocity, temperature and concentration profile of the flow respectively. It is clear that an increase in the γ increase the nanofluid velocity and concentration profile of the flow, but a decrease in temperature profile is evident. The negative values of γ act as heat absorption.

Figures 29 – 31 illustrate the effects porous medium parameter (φ) on nanofluid velocity, temperature and concentration profile. It is clear that as φ increases, the non-dimensional velocity decline this is due to the fact that as the tightness of the tightness of φ increases, the resistance against the flow increases in this case the fluid velocity decreases and in this way the heat transferred from the plate. It is evident that the distribution of porous medium parameter enhance the temperature and concentration boundary layer thickness.

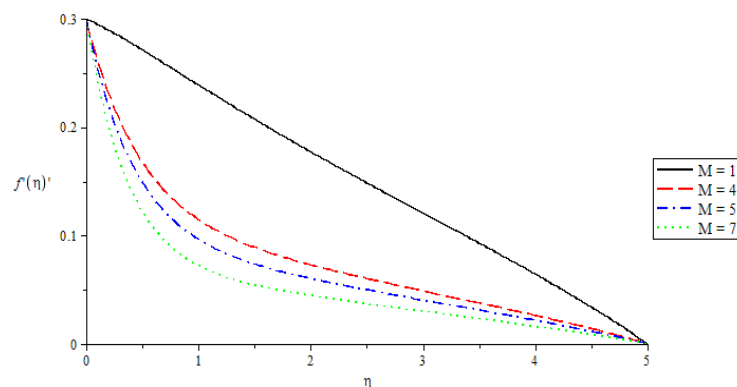


Figure 2: Variation of Hartmann number (M) on velocity profile

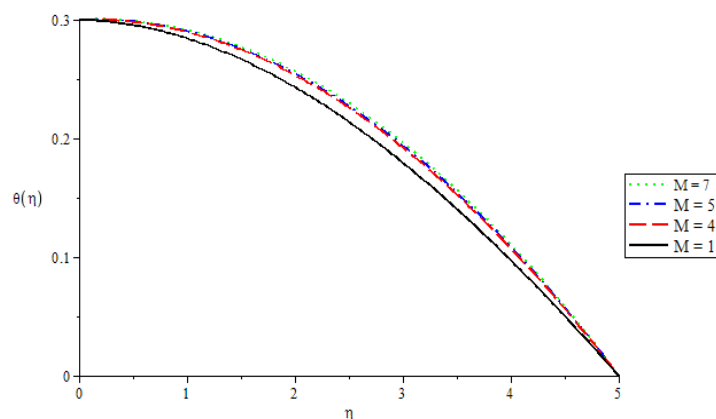


Figure 3: Variation of Hartmann number (M) on temperature profile

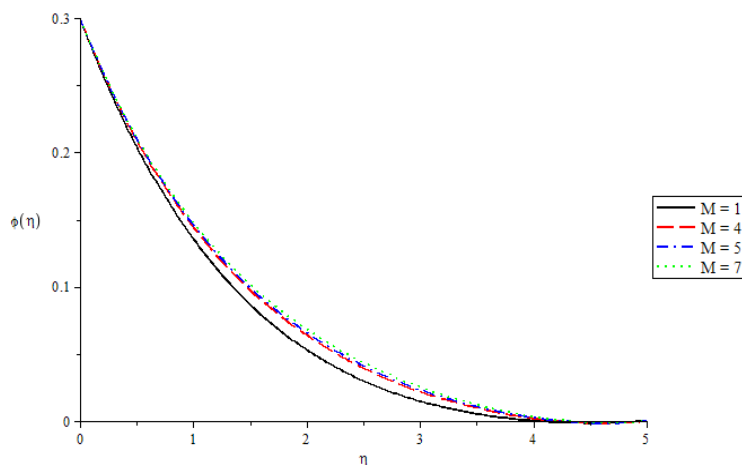


Figure 4: Variation of Hartmann (M) on concentration profile

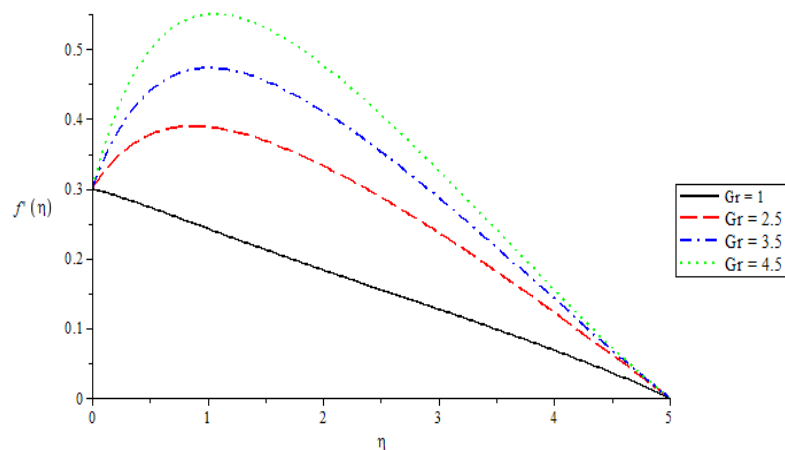


Figure 5: Variation of thermal Grashof number (G_r) on velocity profile

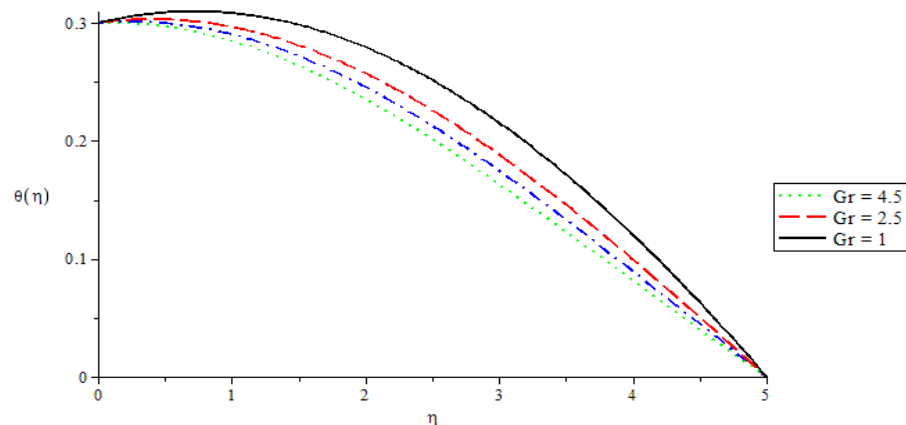


Figure 6: Variation thermal Grashof number (G_r) on temperature profile

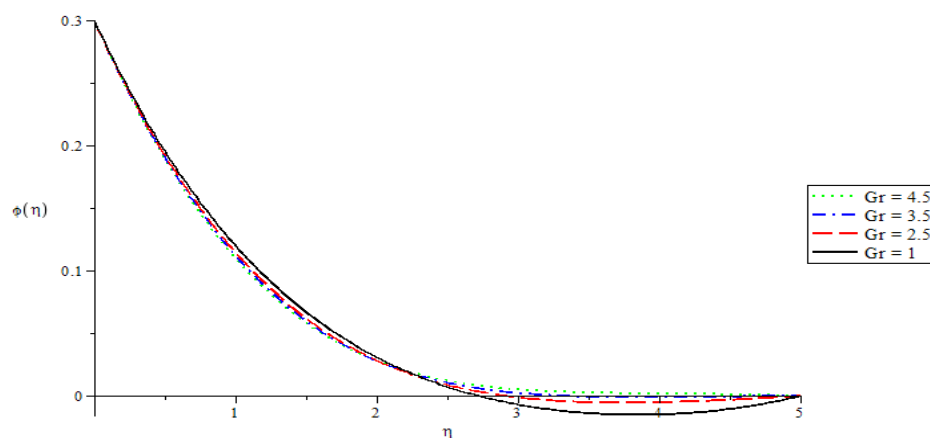


Figure 7: Variation of thermal Grashof number (G_r) on concentration profile

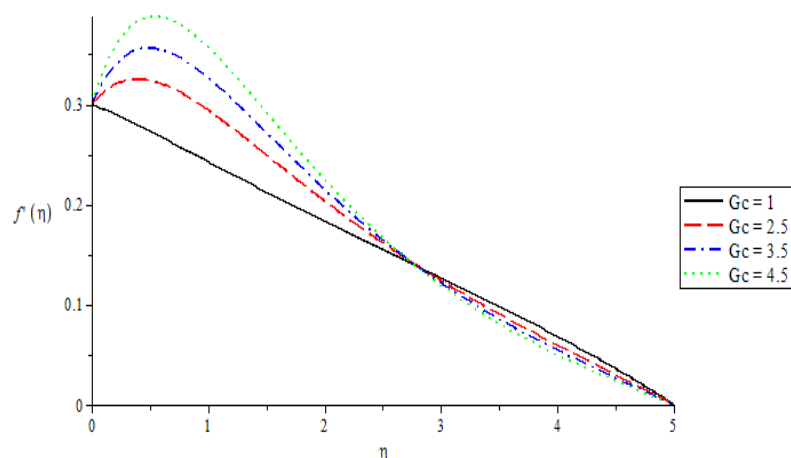


Figure 8: Variation of mass Grashof number (G_c) on velocity profile

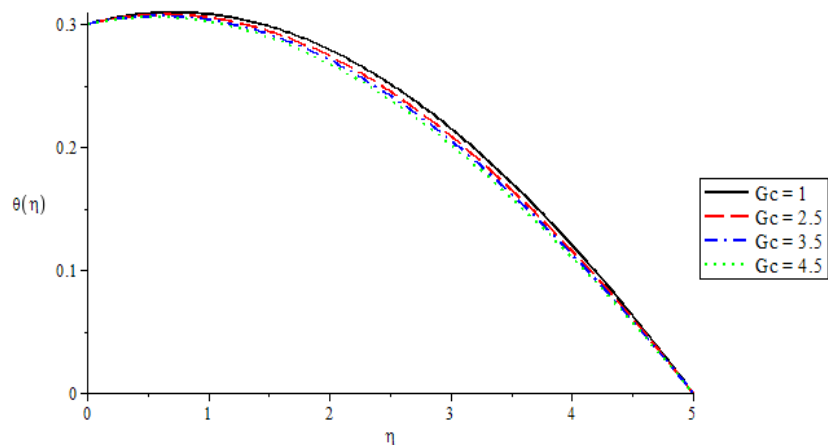


Figure 9: variation of mass Grashof number (G_c) on temperature profile

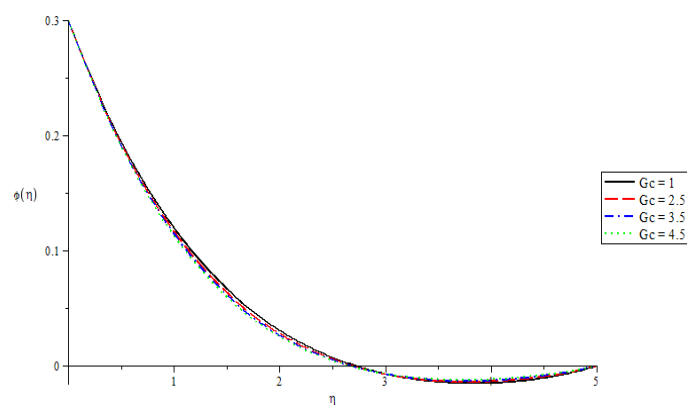


Figure 10: Variation of mass Grashof number (G_c) on concentration profile

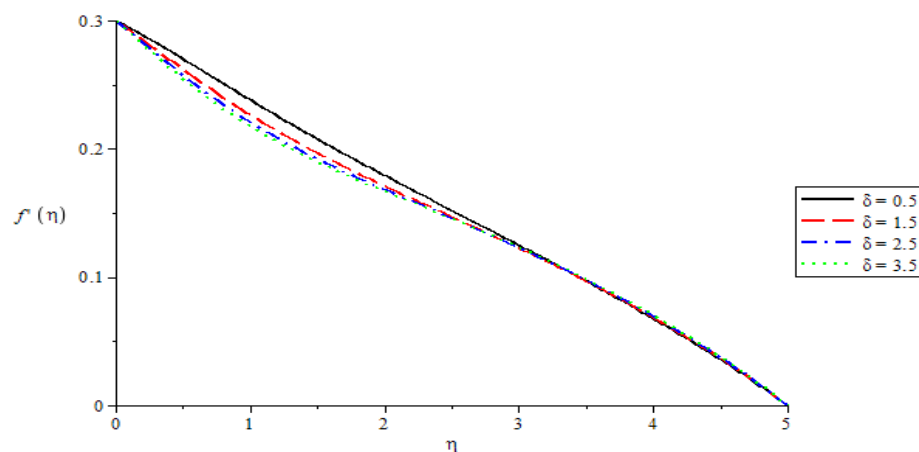


Figure 11: Variation of chemical reaction (δ) on velocity profile

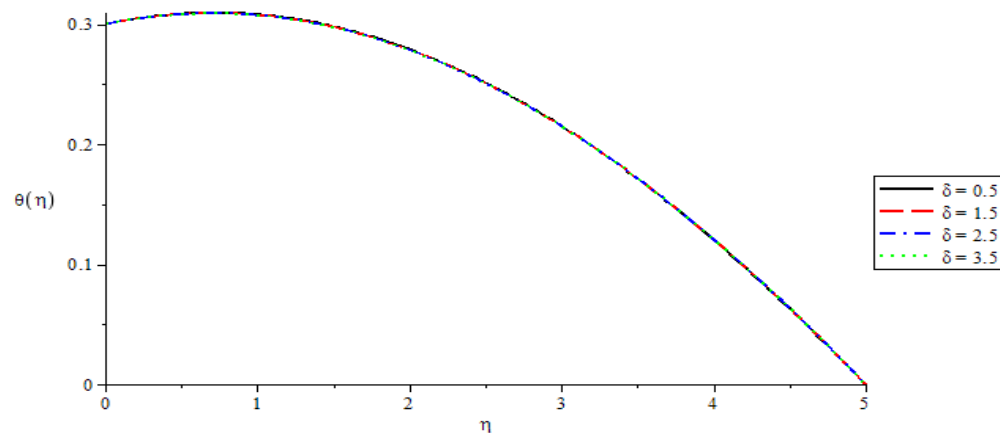


Figure 12: Variation of chemical reaction (δ) on temperature profile

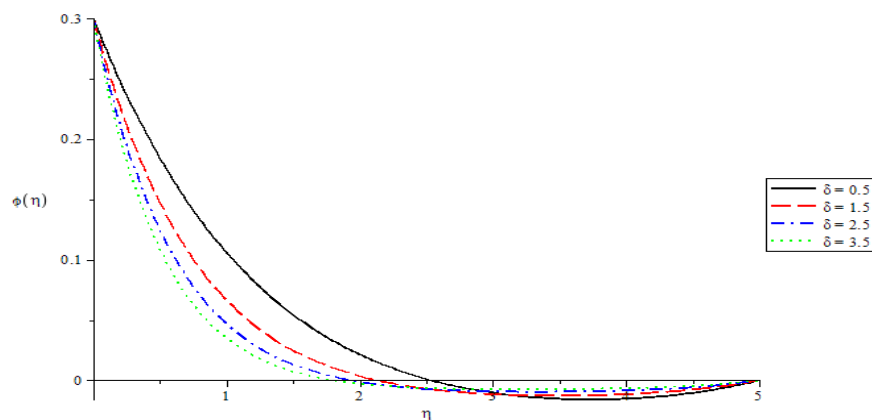


Figure 13: Variation of chemical reaction (δ) on concentration profile

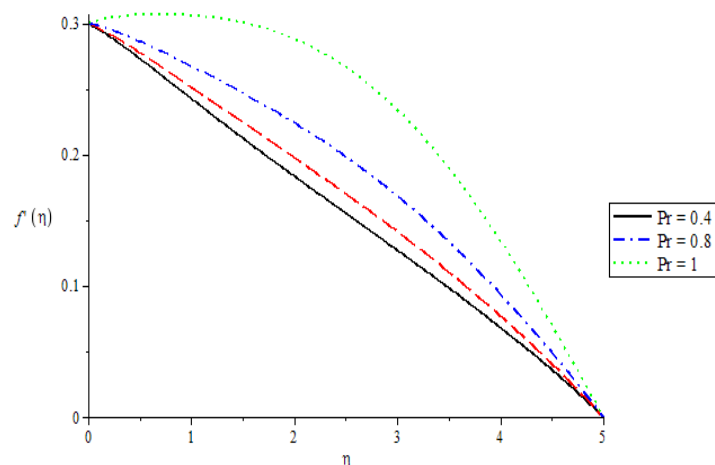


Figure 14: Variation of Prandtl number (P_r) on velocity profile

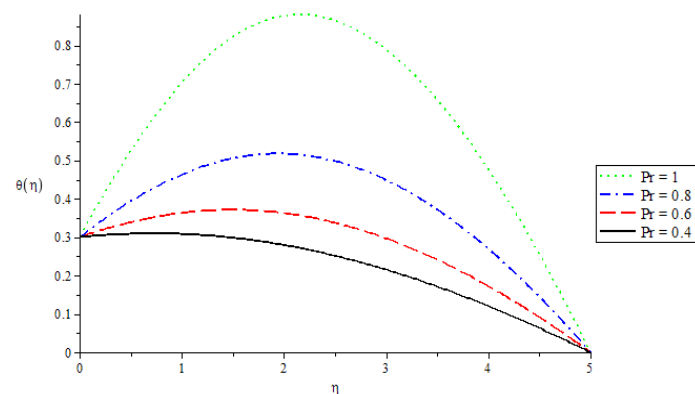


Figure 15: Variation of Prandtl number (P_r) on temperature profile

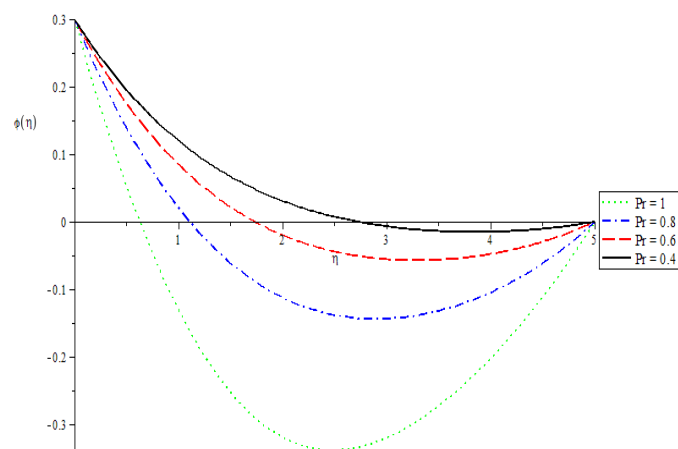


Figure 16: variation of Prandtl number (P_r) on concentration profile

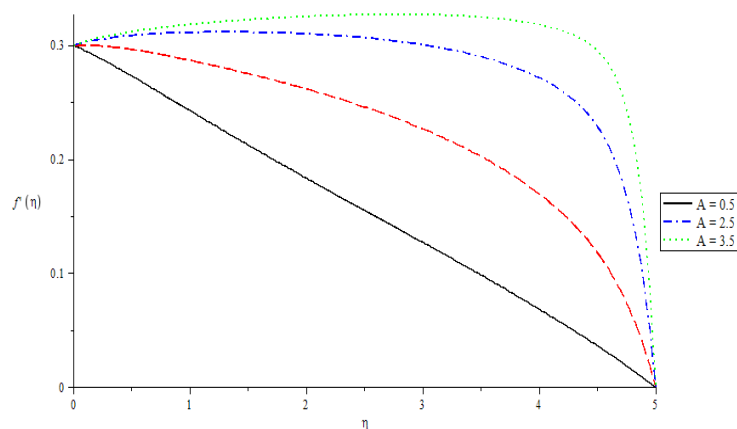


Figure 17: Variation of unsteadiness parameter (A) on velocity profile

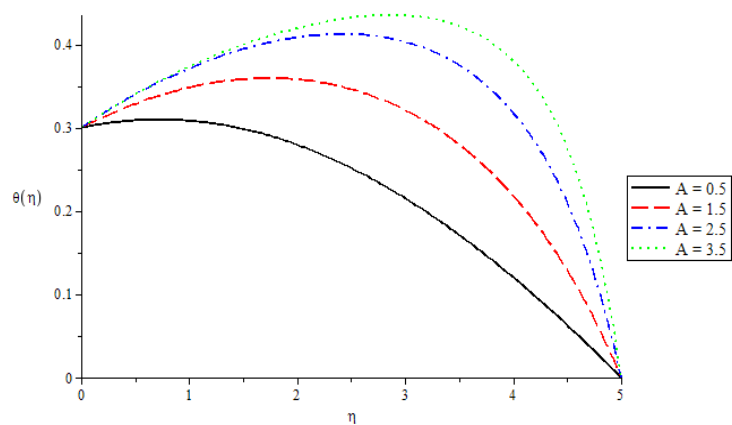


Figure 18: Variation of unsteadiness parameter (A) on temperature profile

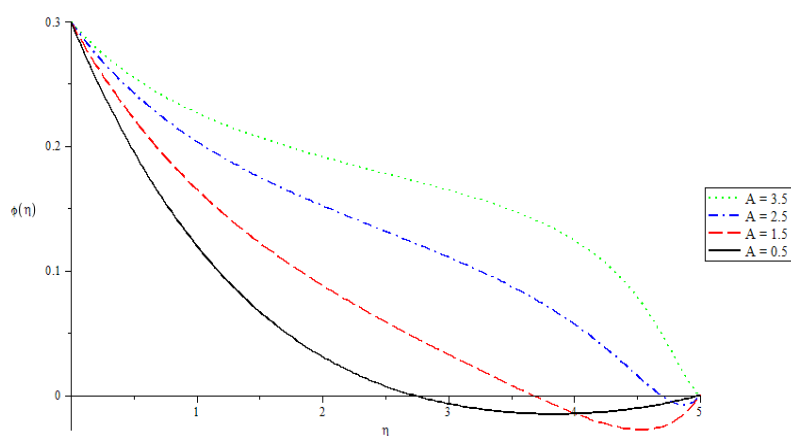


Figure 19: Variation of unsteadiness parameter (A) on concentration profile

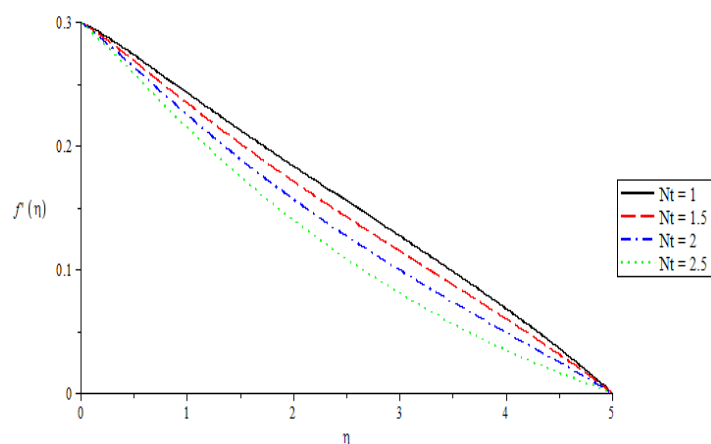


Figure 20: Variation of thermophoresis (N_t) on velocity profile

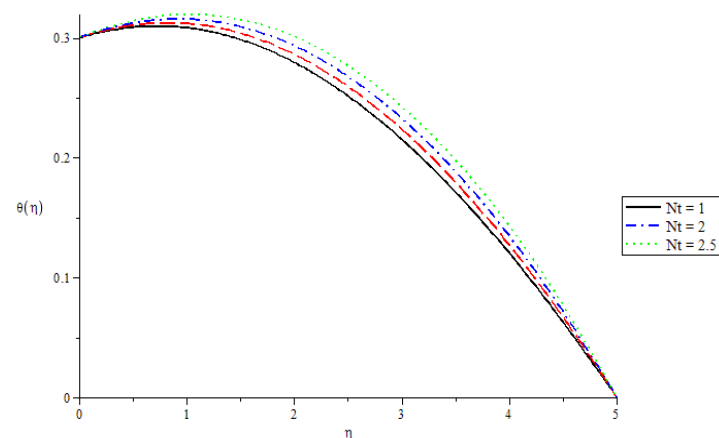


Figure 21: Variation of thermophoresis (N_t) on temperature profile

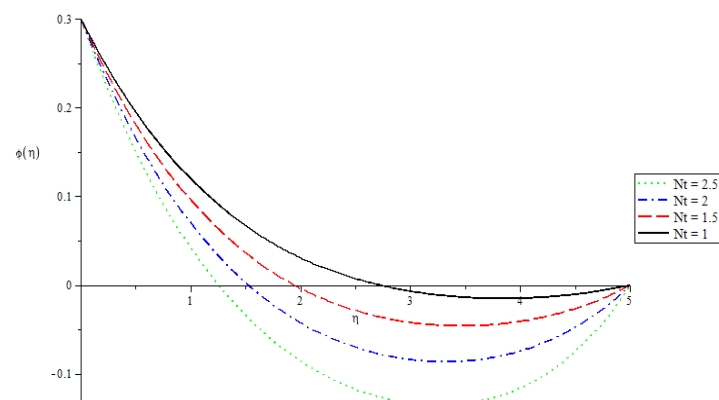


Figure 22: Variation of thermophoresis (N_t) on concentration profile

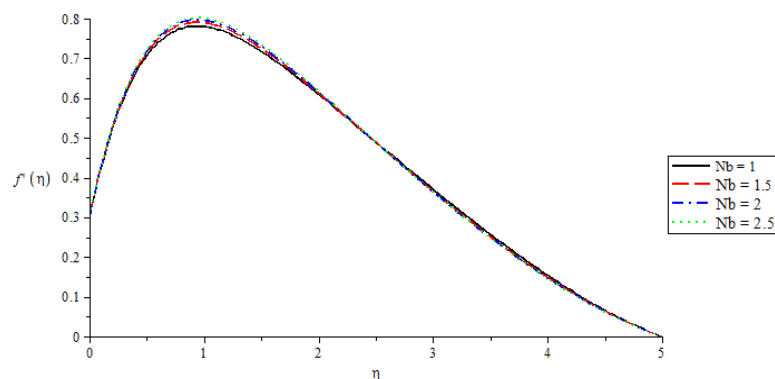


Figure 23: Variation of Brownian motion (N_b) on velocity profile

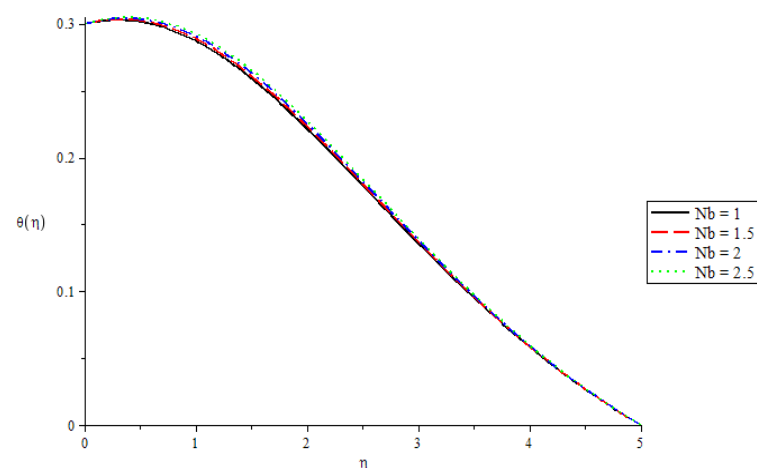


Figure 24: Variation of Brownian motion (N_b) on temperature profile

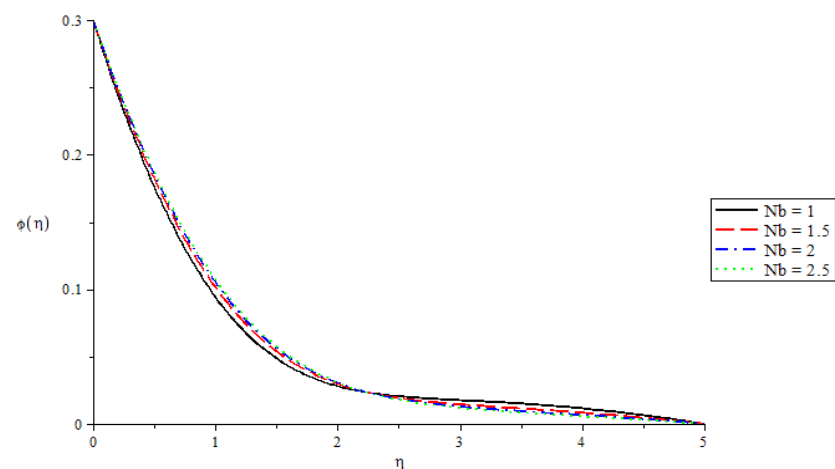


Figure 25: Variation of Brownian motion (N_b) on concentration profile

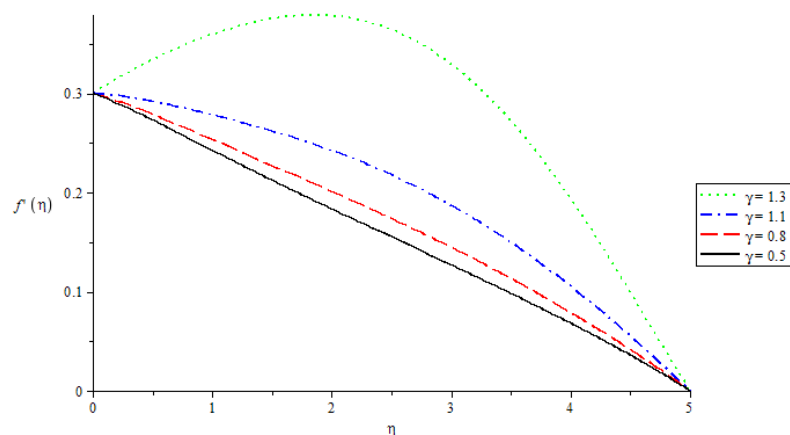


Figure 26: Variation of heat source (γ) on velocity profile

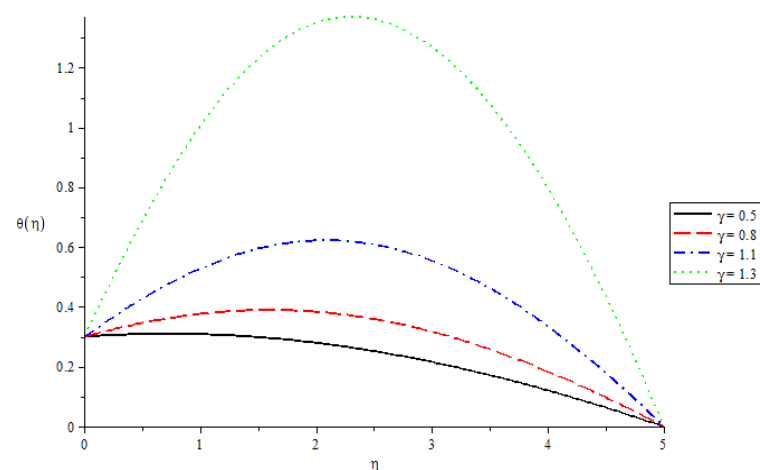


Figure 27: Variation of heat source (γ) on temperature profile

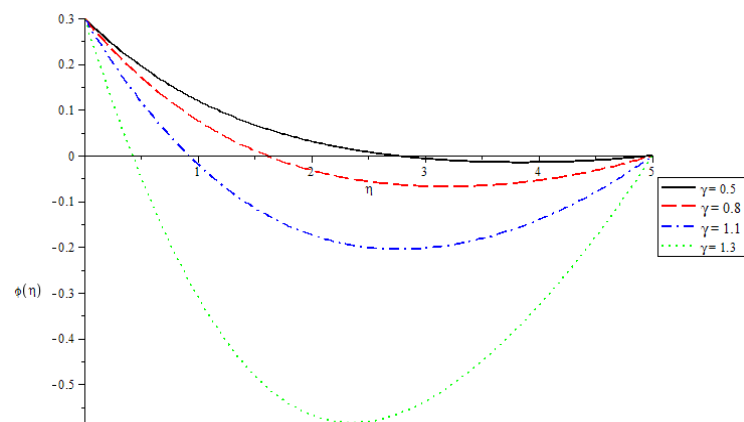


Figure 28: Variation of heat source (γ) on concentration profile

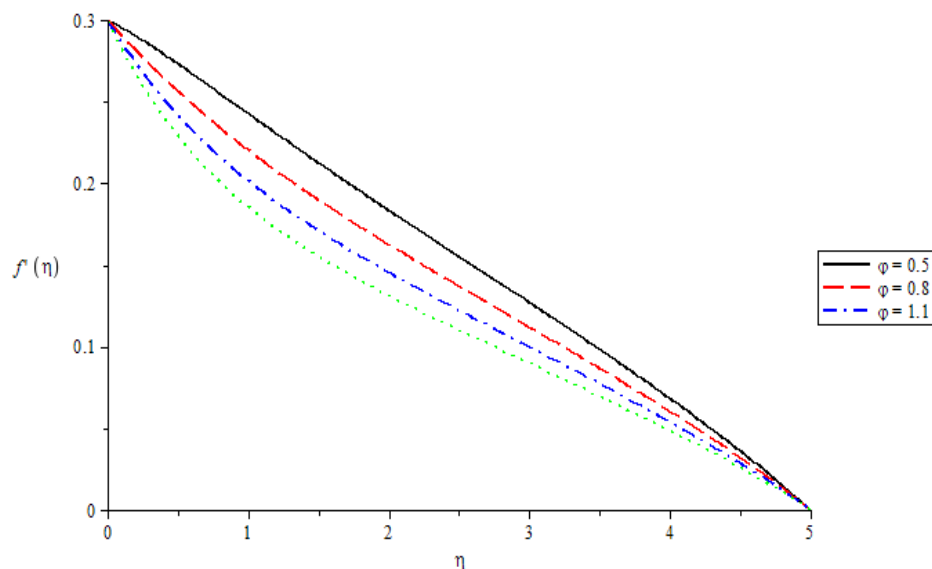


Figure 29: Variation of porous medium parameter (ϕ) on velocity profile

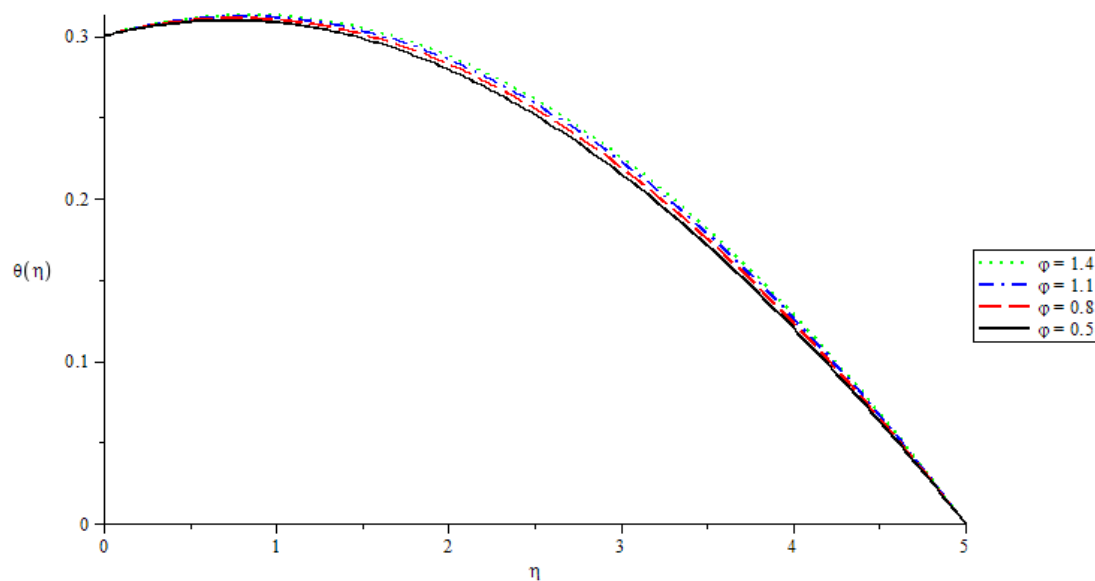


Figure 30: Variation of porous medium parameter (ϕ) on temperature

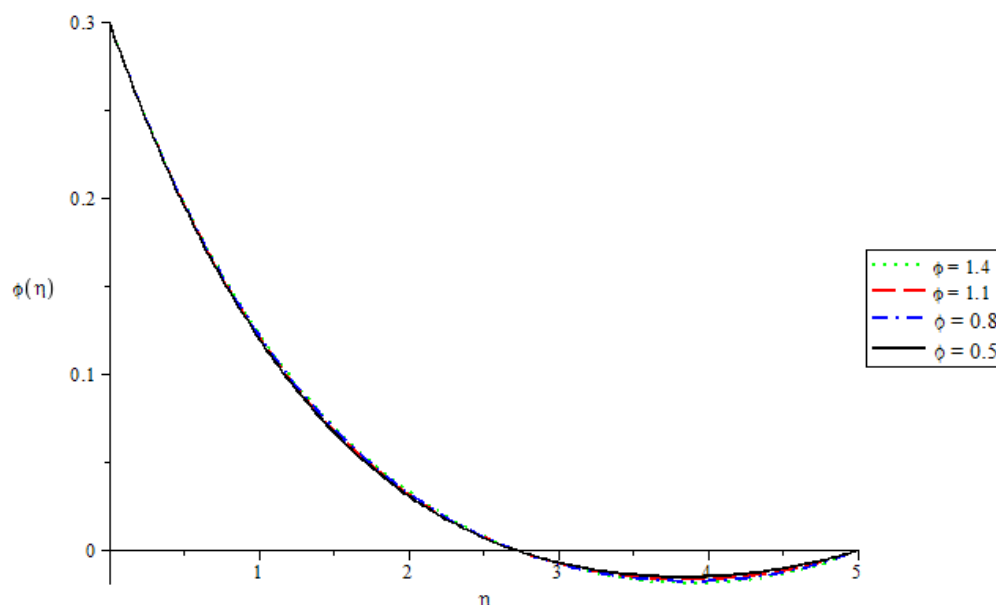


Figure 31: Variation of porous medium parameter (ϕ) on concentration profile

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Optimal vaccination and treatment strategies for control of rubella with vertical transmission

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Abstract

Rubella is a viral contagious disease associated with puffy cheeks, tender and swollen jaw. It spreads through direct contact with saliva or respiratory droplets from the mouth, nose or throat of infected persons. This work presents a system of nonlinear ordinary differential equations which models the transmission dynamics of the disease, with vertical transmission from infected mothers to their offspring. The model also incorporates vaccination against the disease, and treatment of infected individuals as control measures for the disease. It is shown that the disease-free equilibrium of the model is locally asymptotically stable when the control reproduction number, R_c is less than one. It is also shown that the model possesses a unique endemic equilibrium which exists when $R_c > 1$. This confirms the global stability of the disease-free equilibrium, and the absence of backward bifurcation in the model. Optimal control analysis is performed on the model to obtain the proportion of susceptible humans that will be vaccinated and infected humans that need to be isolated and treated for optimal control of the disease. Plots are presented to show the dynamics of the disease in the presence of the controls.

Keywords: rubella; mathematical model; vaccination; vertical transmission; isolation; treatment; control reproduction number; optimal control.

1 Introduction

Rubella is a viral infection that is very contagious and caused by the RuV virus [4]. Rubella is a neglected disease with no surveillance system in place and no national incidence figure in Nigeria [5]. [19] says that rubella is an acute infection affecting mainly children and young adults. It is also known as three-day measles or German measles and comes with a distinctive red rash that usually starts from the face and then spreads to other parts of the body [2][14]. Rubella is different from measles though both illnesses share some signs and symptoms like the red rash which appears in at most 80% of infected persons and sometimes wrongly diagnosed as measles or scarlet fever [11]. Rubella is not as severe as measles and is caused by a different virus than measles [2]. The diagnosis is usually confirmed with lab tests such as virus culture or blood test. The symptoms include mild fever, headache, runny nose, red itchy eyes, enlarged tender lymph nodes at the base of the skull, the back of the neck and behind the ears, a fine pink rash that begins on the face and quickly spreads to the trunk and then

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arms and legs before disappearing in the same order, aching joints especially in young women, general discomfort, etc. [5][14]. The symptoms appear generally between two to three weeks after exposure to the virus and last about one to five days [4][14]. About 25 – 50% of rubella patients are asymptomatic [2][4]. Rubella is transmitted from person to person through contact with infected mucus from the throat and neck, coughs and sneezes from infected persons, bloodstream from pregnant woman to her unborn child. Infected persons with rubella become infectious from one week before rashes appear till about one week after the rashes disappear [19]. That's why an infected person can spread the illness even before realizing he or she has it.

Rubella can be prevented by administering the Measles-Mumps-Rubella (MMR) vaccine to children between 12 – 15 months of age for first dose and 4 – 6 years of age for the second dose [4][14][19]. The vaccine grants permanent immunity to the infection. MMR vaccine is very safe and effective. One dose of the MMR vaccine is about 97% effective at preventing rubella [2]. It is reported that 168 of 194 countries in the world had introduced the rubella vaccines as of December, 2018 [19]. Also, if someone gets the infection, he or she should be isolated to avoid spreading the infection to others. Health care providers are encouraged to educate the public especially parents on the safety protocol over the disease. Children should stay off nursery or school while adults should stay off work for 5 days after the rash appears, avoiding close contact with anyone who is pregnant if infected, washing your hands often with soap and warm water and proper disposal of tissues after coughing or sneezing can all help prevent the spread [14]. Recovering from the disease also grants immunity to the disease just as vaccination [4][14].

There are cases of complications arising from rubella infections though very rare. Some women who have had rubella infections have been reported to experience arthritis in the fingers, wrists and knees which does not last more than one month [2][19]. In rare cases, inflammation of the brain or ear infection has been associated to rubella disease as well as miscarriage, fetal death, stillbirth in pregnant women especially within the first trimester [19]. It is reported that about 90% of infants born to mothers who had rubella during the first 12 weeks of pregnancy develop congenital rubella syndrome (CRS), [3][8][17][19]. Severe complications are seen on the unborn children of pregnant women where it is seen to cause one or more of the following syndromes; growth delays, deafness, cataracts, congenital heart defects, organ defects, problems with mental development and learning [8][17]. It is the leading vaccine-preventable cause of birth defects [19]. There is no treatment for rubella infection however, simple self-care measures are given to infected persons to minimize the effects of the symptoms. Such measures include given bed rest and drugs to relieve pains, aches and fever as well as drinking lots of fluids [14]. There are about 26,000 cases of rubella worldwide each year mostly common in Asia, Africa and the Middle East [4].

Mathematical modelling of infectious diseases has helped to study the mechanisms through which diseases spread, to make predictions on the future course of an outbreak and to evaluate methods of controlling them. [1] designed a mathematical model of measles and rubella with vaccination. They analyzed the individual infections as well as the coinfection of both diseases. [16] studied the dynamics of re-infection of rubella transmission model with vaccination. Their model allowed recovered humans to be re-introduced into the exposed population. [18] discussed modelling the dynamics of rubella disease with vertical transmission where the recovered humans are reintroduced into the susceptible population. They performed sensitivity analysis on the model which showed that increase in more vertical infection and contact rate with infected individuals increases the number of infected humans. [9] worked on the numerical solution for Rubella disease model with vaccination using adaptive Runge-kutta method. The work compared the Runge-kutta order 4 method and the Runge-kutta Fehlberg method in predicting the rate of population condition for Rubella

moves to the compartment, S_v . The exposed class, E_A becomes infectious at the incubation rate, δ_1 . The proportion, v_1 of the infectious class are isolated and receives treatment, while the proportion, v_2 , stays without isolation and treatment. In the infectious class, there is a proportion, γ of I_A that are women, who becomes pregnant at the rate, μ , and produces $\gamma\mu I_A$ infected fetuses due to automatic vertical transmission of the virus. After delivery, the infected fetuses become infectious at the rate, δ_2 , and becomes isolated for treatment at the rate, θ . The rates of recovery for persons in the treatment is ω_2 , while the non-treatment class recovers naturally at the rate, ω_1 . Both infected and uninfected compartments are affected by natural death, which occurs at the rate, τ_1 , while only the infected fetuses are affected by death rate, τ_2 , due to the infection.

Using the flow diagram, fig. (1) and the dynamics of the disease as described above, we arrive at the following system of nonlinear ordinary differential equations for transmission of rubella in a given population:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \alpha_1\beta_1 I_{NT}S - (\kappa + \tau_1)S, \\ \frac{dS_v}{dt} &= \kappa S - \tau_1 S_v, \\ \frac{dE_A}{dt} &= \alpha_1\beta_1 I_{NT}S - (\delta_1 + \tau_1)E_A, \\ \frac{dE_J}{dt} &= \gamma\mu I_A - (\delta_2 + \tau_1)E_J, \\ \frac{dI_A}{dt} &= \delta_1 E_A - (\gamma\mu + v_1 + v_2 + \tau_1)I_A, \\ \frac{dI_J}{dt} &= \delta_2 E_J - (\theta + \tau_1 + \tau_2)I_J, \\ \frac{dI_T}{dt} &= v_1 I_A + \theta I_J - (\omega_2 + \tau_1)I_T, \\ \frac{dI_{NT}}{dt} &= v_2 I_A - (\omega_1 + \tau_1)I_{NT}, \\ \frac{dR}{dt} &= \omega_1 I_{NT} + \omega_2 I_T - \tau_1 R\end{aligned}\tag{1}$$

where, $S(0) = S^0$, $S_v(0) = S_v^0$, $E_A(0) = E_A^0$, $E_J = E_J^0$, $I_J(0) = I_J^0$, $I_A(0) = I_A^0$, $I_T(0) = I_T^0$, $I_{NT}(0) = I_{NT}^0$, $R(0) = R^0$.

$$N = S + S_v + E_A + E_J + I_J + I_A + I_T + I_{NT} + R.\tag{2}$$

We assume that the solution to the system exists in the region Ω described by

$$\{S, S_v, E_A, E_J, I_A, I_J, I_T, I_{NT}, R\} \in \mathbb{R}^9 \mid S + S_v + E_A + E_J + I_J + I_A + I_T + I_{NT} + R \leq \frac{\Lambda}{\tau_1}.$$

Table 1: Description of Parameters

Parameters	Description
Λ	Level of recruitment
τ_1	Natural death rate
τ_2	Death due to the disease
α_1	Contact rate with Infectious humans not undergoing treatment
β_1	Rate of transmission from contacts with Infectious humans not undergoing treatment
δ_1	Rate of development of infectiousness from E_A to I_A
δ_2	Rate of development of infectiousness from E_J to I_J
κ	Proportion of Susceptible humans that are vaccinated
γ	Proportion of Infectious humans that are females
μ	Proportion of infectious that are pregnant
θ	Proportion of infectious newborns that are isolated and treated
v_1	Proportion of Infectious humans that are isolated and undergoing treatment
v_2	Proportion of Infectious humans that are not isolated and undergoing treatment
ω_1	Level of recovery of humans not isolated and treated
ω_2	Level of recovery of humans isolated and treated

3 Positivity and boundedness of solutions

Lemma 1: Let the initial data set for the model be $\{S^0 > 0, S_v^0 > 0, E_A^0 > 0, E_J^0 > 0, I_A^0 > 0, I_J^0 > 0, I_T^0 > 0, I_{NT}^0 > 0, R^0 > 0\}$. Then, the solution set $\{S(t), S_v(t), E_A(t), E_J(t), I_J(t), I_A(t), I_T(t), I_{NT}(t), R(t)\}$ of the model with the given initial data will remain positive and bounded for all time $t > 0$.

Proof: It can be shown from eqn. (1) that

$$\begin{aligned} \frac{dS}{dt} &\geq -\alpha_1\beta_1 I_{NT}S - (\kappa + \tau_1)S \\ \frac{dS_v}{dt} &\geq -\tau_1 S_v \\ &\vdots \\ \frac{dR}{dt} &\geq -\tau_1 R. \end{aligned} \tag{3}$$

Hence, solving the differential inequality with given initial solutions gives

$$\begin{aligned} S &\geq S^0 e^{-(\alpha_1\beta_1 I_{NT} + \kappa + \tau_1)t} > 0 \\ S_v &\geq S_v^0 e^{-\tau_1 t} \\ &\vdots \\ R &\geq R^0 e^{-\tau_1 t} \end{aligned} \tag{4}$$

Hence, given initial solutions that are individually positive, the solution to the model system remains positive $\forall t > 0$.

Further, we have that the total human population, N satisfies the equation

$$\frac{dN}{dt} = \Lambda - \tau_1 N. \quad (5)$$

Hence, as $t \rightarrow \infty$, $0 < N(t) \leq \frac{\Lambda}{\tau_1}$ showing that the human population is always bounded.

Therefore, the region Ω is positively invariant and is an attractor of all positive solution of the system.

4 Local stability of the disease-free equilibrium

The disease-free equilibrium (DFE), $E^0 = \left(\frac{\Lambda}{\kappa + \tau_1}, \frac{\kappa \Lambda}{\tau_1(\kappa + \tau_1)}, 0, 0, 0, 0, 0, 0 \right)$ of (1) is locally asymptotically stable if the all the eigenvalues of the Jacobian matrix of the model system at E^0 are negative or have negative real parts. Let $J(E^0)$ denote the Jacobian matrix evaluated at E^0 , then

$$J(E^0) = \begin{bmatrix} -(\kappa + \tau_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\alpha_1 \beta_1 \Lambda}{\kappa + \tau_1} & 0 \\ \kappa & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\delta_1 + \tau_1) & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1 \beta_1 \Lambda}{\kappa + \tau_1} & 0 \\ 0 & 0 & 0 & -(\delta_2 + \tau_1) & \gamma \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 & -(\gamma \mu + v_1 + v_2 + \tau_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & 0 & -(\theta + \tau_1 + \tau_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_1 & \theta & -(\omega_2 + \tau_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2 & 0 & 0 & -(\omega_1 + \tau_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_2 & \omega_1 & -\tau_1 & 0 \end{bmatrix}$$

Some of the eigenvalues of $J(E^0)$ are $-\tau_1, -\tau_1, -(\kappa + \tau_1), -(\omega_2 + \tau_1), -(\theta + \tau_1 + \tau_2), -(\delta_2 + \tau_1)$. The remaining eigenvalues, λ of $J(E^0)$ satisfies the cubic equation

$$Q_3 \lambda^3 + Q_2 \lambda^2 + Q_1 \lambda + Q_0 = 0 \quad (6)$$

where

$$\begin{aligned} Q_3 &= 1 \\ Q_2 &= (\delta + \tau_1) + (\gamma \mu + v_1 + v_2 + \tau_1) + (\omega_1 + \tau_1) \\ Q_1 &= (\delta + \tau_1)(\omega_1 + \tau_1) + (\delta + \tau_1)(\gamma \mu + v_1 + v_2 + \tau_1) + (\omega_1 + \tau_1)(\gamma \mu + v_1 + v_2 + \tau_1) \\ Q_0 &= (\delta + \tau_1)(\omega_1 + \tau_1)(\gamma \mu + v_1 + v_2 + \tau_1)[1 - R_c] \\ \text{where } R_c &= \frac{\alpha_1 \beta_1 \Lambda v_2 \delta_1}{(\kappa + \tau_1)(\delta_1 + \tau_1)(\omega_1 + \tau_1)(\gamma \mu + v_1 + v_2 + \tau_1)}. \end{aligned} \quad (7)$$

Since $Q_3 > 0$, $Q_2 > 0$, $Q_1 > 0$, Routh-Hurwitz criterion assures that all the eigenvalues are negative if $R_c < 1$. Hence, the disease-free equilibrium is locally asymptotically stable if $R_c < 1$. The quantity, $R_c = \frac{\alpha_1 \beta_1 \Lambda v_2 \delta_1}{(\kappa + \tau_1)(\delta_1 + \tau_1)(\omega_1 + \tau_1)(\gamma \mu + v_1 + v_2 + \tau_1)}$ is the control reproduction number of the disease. It gives the average number of secondary infectious cases that is caused by a single infection in total susceptible

population in the presence of the control measures, [13]. If $R_c < 1$, then the control measure is effective, otherwise it is ineffective.

5 Existence of endemic equilibrium point

Lemma. A unique endemic equilibrium for the model system exists if and only if $R_c > 1$.

The endemic equilibrium point (EEP) are steady-state solutions to the model system (1) when the disease persists in the population (i.e. all state variables are positive). Let $EEP = (S^*, S_v^*, E_A^*, E_J^*, I_J^*, I_A^*, I_T^*, I_{NT}^*, R^*)$, with

$$S^* = \frac{\Lambda}{\alpha_1 \beta_1 I_{NT}^* + \kappa + \tau_1}, \quad S_v^* = \frac{\kappa S^*}{\tau_1}, \quad E_A^* = \frac{\alpha_1 \beta_1 I_{NT}^* S^*}{\delta_1 + \tau_1}, \quad E_J^* = \frac{\gamma \mu I_A^*}{\delta_2 + \tau_1}, \quad I_A^* = \frac{\delta_1 E_A^*}{\gamma \mu + v_1 + v_2 + \tau_1},$$

$$I_J^* = \frac{\delta_2 E_J^*}{\theta + \tau_1 + \tau_2}, \quad I_T^* = \frac{v_1 I_A^* + \theta I_J^*}{\omega_2 + \tau_1}, \quad I_{NT}^* = \frac{v_2 I_A^*}{\omega_1 + \tau_1}, \quad R^* = \frac{\omega_1 I_{NT}^* + \omega_2 I_T^*}{\tau_1}, \text{ then the}$$

components of EEP satisfies the system

$$\begin{aligned} \Lambda - \alpha_1 \beta_1 I_{NT}^* S^* - (\kappa + \tau_1) S &= 0 \\ \kappa S^* - \tau_1 S_v^* &= 0 \\ \beta_1 I_{NT}^* S^* - (\delta_1 + \tau_1) E_A^* &= 0 \\ \gamma \mu I_A^* - (\delta_2 + \tau_1) E_J^* &= 0 \\ \delta_1 E_A^* - (\gamma \mu + v_1 + v_2 + \tau_1) I_A^* &= 0 \\ \delta_2 E_J^* - (\theta + \tau_1 + \tau_2) I_J^* &= 0 \\ v_1 I_A^* + \theta I_J^* - (\omega_2 + \tau_1) I_T^* &= 0 \\ I_A^* - (\omega_1 + \tau_1) I_{NT}^* &= 0 \\ I_{NT}^* + \omega_2 I_T^* - \tau_1 R^* &= 0. \end{aligned}$$

Substituting E_A^* into the fifth equation and simplifying, gives

$$\alpha_1 \beta_1 v_2 (\delta_1 + \tau_1) (\gamma \mu + v_1 + v_2 + \tau_1) I_A^{*2} + [(\kappa + \tau_1) (\delta_1 + \tau_1) (\omega_1 + \tau_1) (\gamma \mu + v_1 + v_2 + \tau_1) - \alpha_1 \beta_1 \Lambda v_2 \delta_1] I_A^* = 0.$$

This gives a quadratic equation of the form

$$\alpha_1 \beta_1 v_2 I_A^{*2} + (\kappa + \tau_1) (\omega_1 + \tau_1) (1 - R_c) I_A^* = 0,$$

whose solutions are $I_A^* = 0$ and $I_A^* = \frac{(\kappa + \tau_1) (\omega_1 + \tau_1) (R_c - 1)}{\alpha_1 \beta_1 v_2}$. The trivial solution, $I_A^* = 0$ implies the disease-free equilibrium which is locally asymptotically stable if $R_c < 1$. The solution, $I_A^* = \frac{(\kappa + \tau_1) (\omega_1 + \tau_1) (R_c - 1)}{\alpha_1 \beta_1 v_2}$ is positive if $R_c > 1$. This shows that the model has a unique endemic equilibrium

which exists when $R_c > 1$. The existence of a unique endemic equilibrium when $R_c > 1$, precludes the existence of backward bifurcation in the model. Therefore, the disease-free equilibrium is globally

asymptotically stable when $R_c < 1$. Epidemiologically, this shows that the disease can be eradicated by applying the two control measures irrespective of the initial infected populations.

6 Optimal control of rubella

In this case, we assume the rate of vaccination, κ and treatments, v_1 and θ are not constant but functions of time; $\kappa = \kappa(t)$, $v_1 = v_1(t)$ and $\theta = \theta(t)$ respectively. Therefore, we consider an optimal control problem with objective function given by

$$J(x; \kappa, v_1, \theta) = \int_0^T \left(A_1 S(t) + A_2 E_A(t) + A_3 E_J(t) + A_4 I_A(t) + A_5 I_J(t) + \frac{A_6}{2} \kappa^2(t) + \frac{A_7}{2} v_1^2(t) + \frac{A_8}{2} \theta^2(t) \right) dt \quad (8)$$

where the constants A_i ; $i = 1, 2, 3, 4, 5, 6, 7, 8$ are positive weights that help to balance each term in the integrand. The quadratic terms, $\frac{A_6}{2} \kappa^2(t)$, $\frac{A_7}{2} v_1^2(t)$ and $\frac{A_8}{2} \theta^2(t)$ represent the costs involved in vaccination, isolating and treating those that are symptomatically infectious at any time t . In the above optimal control problem, we assume that the control parameters, $\kappa^*(t)$, $v_1^*(t)$ and $\theta^*(t)$ are Lebesgue measurable with $a_1 < \kappa^*(t) \leq b_1$, $a_2 < v_1^*(t) \leq b_2$ and $a_3 < \theta^*(t) \leq b_3$, $\forall t \in [0, t_f]$. Therefore, we seek an optimal controls, $\kappa^*(t)$, $v_1^*(t)$ and $\theta^*(t)$ such that

$$J(x^*; \kappa^*(t), v_1^*(t), \theta^*(t)) = \min\{J(x; \kappa(t), v_1(t), \theta(t)) : (\kappa(t), v_1(t), \theta(t)) \in \Gamma\}, \quad (9)$$

where the control set Ω is given by

$\Gamma = \{(\kappa(t), v_1(t), \theta(t)) : \text{each control parameter is piecewise continuous on } [0, T], a_1 < \kappa^*(t) \leq b_1, a_2 < v_1^*(t) \leq b_2 \text{ and } a_3 < \theta^*(t) \leq b_3\}$, where $a_1, b_1, a_2, b_2, a_3, b_3$ are constants on $[0, 1]$.

As shown in [7], the existence of the optimal controls, $\kappa^*(t)$, $v_1^*(t)$ and $\theta^*(t)$ and the associated optimal solutions, $(S^*, S_v^*, E_A^*, E_J^*, I_A^*, I_T^*, I_{NT}^*, R^*)$ is guaranteed since the state solutions are bounded as proved earlier. The state system satisfies the Lipschitz property with respect to the state variables, since the solution exists and is unique.

For the convexity of the integrand, $L(t, S, E_A, E_J, I_A, I_J, \kappa, v_1, \theta) = A_1 S(t) + A_2 E_A(t) + A_3 E_J(t) + A_4 I_A(t) + A_5 I_J(t) + \frac{A_6}{2} \kappa^2(t) + \frac{A_7}{2} v_1^2(t) + \frac{A_8}{2} \theta^2(t)$, with respect to the control parameters, κ, v_1, θ in the set, Γ , we need to show that

$$L(t, S, E_A, E_J, I_A, I_J, \lambda\kappa + (1 - \lambda)\kappa', \lambda v_1 + (1 - \lambda)v_1', \lambda\theta + (1 - \lambda)\theta') \leq \lambda L(t, S, E_A, E_J, I_A, I_J, \kappa, v_1, \theta) + (1 - \lambda)L(t, S, E_A, E_J, I_A, I_J, \kappa', v_1', \theta'), \text{ for } \lambda \in (0, 1).$$

Note that

$$\begin{aligned} L(t, S, E_A, E_J, I_A, I_J, \lambda\kappa + (1 - \lambda)\kappa', \lambda v_1 + (1 - \lambda)v_1', \lambda\theta + (1 - \lambda)\theta') &= A_1 S(t) + A_2 E_A(t) + \\ &+ A_3 E_J(t) + A_4 I_A(t) + A_5 I_J(t) + \frac{A_6}{2} (\lambda\kappa + (1 - \lambda)\kappa')^2 + \frac{A_7}{2} (\lambda v_1 + (1 - \lambda)v_1')^2 + \\ &+ \frac{A_8}{2} (\lambda\theta + (1 - \lambda)\theta')^2 \end{aligned} \quad (10)$$

and

$$\lambda L(t, S, E_A, E_J, I_A, I_J, \kappa, v_1, \theta) + (1 - \lambda)L(t, S, E_A, E_J, I_A, I_J, \kappa', v_1', \theta') = A_1 S(t) + A_2 E_A(t) + A_3 E_J(t) + A_4 I_A(t) + A_5 I_J(t) + \frac{A_6}{2} \lambda \kappa^2 + \frac{A_7}{2} \lambda v_1^2 + \frac{A_8}{2} \lambda \theta^2 + \frac{A_6}{2} (1 - \lambda) \kappa'^2 + \frac{A_7}{2} (1 - \lambda) v_1'^2 + \frac{A_8}{2} (1 - \lambda) \theta'^2. \quad (11)$$

Therefore, we need to show from (10) and (11) that $(\lambda \kappa + (1 - \lambda) \kappa')^2 \leq \lambda \kappa^2 + (1 - \lambda) \kappa'^2$, $(\lambda v_1 + (1 - \lambda) v_1')^2 \leq \lambda v_1^2 + (1 - \lambda) v_1'^2$ and $(\lambda \theta + (1 - \lambda) \theta')^2 \leq \lambda \theta^2 + (1 - \lambda) \theta'^2$.

For $(\lambda \kappa + (1 - \lambda) \kappa')^2$, we have

$$\begin{aligned} (\lambda \kappa + (1 - \lambda) \kappa')^2 &= \lambda^2 \kappa^2 + 2\lambda(1 - \lambda) \kappa \kappa' + (1 - \lambda)^2 \kappa'^2 \\ &= \lambda(\lambda - 1 + 1) \kappa^2 + 2\lambda(1 - \lambda) \kappa \kappa' + (1 - \lambda)(1 - \lambda) \kappa'^2 \\ &= \lambda \kappa^2 + (1 - \lambda) \kappa'^2 - \lambda(1 - \lambda) [\kappa^2 - 2\kappa \kappa' + \kappa'^2] \\ &= \lambda \kappa^2 + (1 - \lambda) \kappa'^2 - \lambda(1 - \lambda) [\kappa - \kappa']^2 \\ &\leq \lambda \kappa^2 + (1 - \lambda) \kappa'^2 \end{aligned} \quad (12)$$

Using similar procedure for $(\lambda v_1 + (1 - \lambda) v_1')^2$ and $(\lambda \theta + (1 - \lambda) \theta')^2$, we obtain that

$$\begin{aligned} (\lambda v_1 + (1 - \lambda) v_1')^2 &\leq \lambda v_1^2 + (1 - \lambda) v_1'^2 \\ (\lambda \theta + (1 - \lambda) \theta')^2 &\leq \lambda \theta^2 + (1 - \lambda) \theta'^2. \end{aligned}$$

This proves the convexity of the integrand on the set, Γ .

By Pontryagin maximum principle [15], the Hamiltonian has the form

$$\begin{aligned} H(S, E_A, E_J, I_A, I_J, \kappa, v_1, \theta, \pi) &= A_1 S(t) + A_2 E_A(t) + A_3 E_J(t) + A_4 I_A(t) + A_5 I_J(t) + \frac{A_6}{2} \kappa^2(t) + \\ &\frac{A_7}{2} v_1^2(t) + \frac{A_8}{2} \theta^2(t) + \pi_1(t) (\Lambda - \rho(t) S(t) - (\kappa + \tau_1) S(t)) + \pi_2(t) (\kappa S - \tau_1 S_v) \\ &+ \pi_3(t) (\rho(t) S(t) - (\delta_1 + \tau_1) E_A(t)) + \pi_4(t) (\gamma \mu I_A - (\delta_2 + \tau_1) E_J) + \pi_5(t) (\delta_1 E_A - (\gamma \mu + v_1 + \\ &v_2 + \tau_1) I_A) + \pi_6(t) (\delta_2 E_J - (\theta + \tau_1 + \tau_2) I_J) + \pi_7(t) (v_1 I_A + \theta I_J - (\omega_2 + \tau_1) I_T) + \\ &\pi_8(t) (v_2 I_A - (\omega_1 + \tau_1) I_{NT}) + \pi_9(t) (\omega_1 I_{NT} + \omega_2 I_T - \tau_1 R) \end{aligned} \quad (13)$$

where $\pi_i(t), i = 1, 2, \dots, 9$ are the adjoint variables that satisfy the system of ordinary differential equations

$$\begin{aligned} \frac{d\pi_1}{dt} &= \rho(t) \alpha_1 \beta_1 I_{NT} (\pi_1 - \pi_3) + (\kappa + \tau) \pi_1 - \pi_2 \kappa - A_1 \\ \frac{d\pi_2}{dt} &= \pi_2 \tau_1 \\ \frac{d\pi_3}{dt} &= (\delta_1 + \tau_1) \pi_3 - \delta_1 \pi_5 - A_2 \\ \frac{d\pi_4}{dt} &= (\delta_2 + \tau_1) \pi_4 - \delta_2 \pi_6 - A_3 \\ \frac{d\pi_5}{dt} &= (\gamma \mu + v_1 + v_2 + \tau_1) \pi_5 - \gamma \mu \pi_4 - v_1 \pi_7 - v_2 \pi_8 - A_4 \end{aligned} \quad (14)$$

$$\begin{aligned}\frac{d\pi_6}{dt} &= (\theta + \tau_1 + \tau_2)\pi_6 - \theta\pi_7 - A_5 \\ \frac{d\pi_7}{dt} &= (\omega_2 + \tau_1)\pi_7 - \omega_2\pi_9 \\ \frac{d\pi_8}{dt} &= (\omega_1 + \tau_1)\pi_8 - \omega_1\pi_9 \\ \frac{d\pi_9}{dt} &= \tau_1\pi_9,\end{aligned}$$

with the transversality condition $\pi_i(T) = 0, i = 1, 2, \dots, 9$. Using the optimality condition in the Pontryagin maximum principle on the Hamiltonian, $H(S, E_A, E_J, I_A, I_J, \kappa, v_1, \theta, \pi)$, we obtain the optimal controls, $\kappa^*(t) = \frac{(\pi_1(t) - \pi_2(t))S(t)}{A_6}$, $v_1^*(t) = \frac{(\pi_5(t) - \pi_7(t))I_A(t)}{A_7}$, $\theta^*(t) = \frac{(\pi_6(t) - \pi_7(t))I_J(t)}{A_8}$, with their characterizations

$$\begin{aligned}\kappa^*(t) &= \max\left(a_1, \min\left(b_1, \frac{(\pi_1(t) - \pi_2(t))S(t)}{A_6}\right)\right) \\ v_1^*(t) &= \max\left(a_2, \min\left(b_2, \frac{(\pi_5(t) - \pi_7(t))I_A(t)}{A_7}\right)\right) \\ \theta^*(t) &= \max\left(a_3, \min\left(b_3, \frac{(\pi_6(t) - \pi_7(t))I_J(t)}{A_8}\right)\right).\end{aligned}\tag{15}$$

Therefore, at any time, t , (15) gives the number of people that need to be vaccinated, the number of infected persons that need to be isolated and treated, and the number of infectious newborns that need to be isolated and treated for optimal control of rubella in a given population.

7 Numerical experiments

Here, we obtain the numerical solution to the optimal control problem using forward-backward sweep technique. This method involves solving the optimal state equation using forward Runge-Kutta order 4 method with the initial guess for the optimal control parameter. This is followed by solving the adjoint system of equation with backward Runge-Kutta order 4 method, starting with the transversality conditions $\pi_i(T) = 0$. The optimal control is updated with the current state and adjoint solutions at the end of each iteration. The iteration is repeated until convergence criteria is reached. The initial solutions used are $S^0 = 250$, $S_v^0 = 20$, $E_A^0 = 100$, $E_J^0 = 100$, $I_J^0 = 150$, $I_A^0 = 100$, $I_T^0 = 100$, $I_{NT}^0 = 20$, $R^0 = 100$. We have also used $A_1 = 10, A_2 = 10, A_3 = 10, A_4 = 10, A_5 = 10, A_6 = 20, A_7 = 50, A_8 = 10, a = 0, b = 1$.

Table 2: Description of Parameters and Values

Parameters	Description	Source
Λ	250	Assumed
τ_1	0.0000432	[6]
τ_2	0.00001	Assumed
α	0.0175	Assumed
β	0.0125	Assumed
δ_1	0.143	[19]
δ_2	0.143	[19]
κ	0.005	Assumed
γ	0.05	Assumed
μ	0.01	Assumed
θ	0.5	Assumed
v_1	0.31	Assumed
v_2	0.062	Assumed
ω_1	0.07	Assumed
ω_2	0.0526	[2][4][7]

The results of the numerical experiments are shown in figures 2 – 10 below. In figure 2, we see the plot for the susceptible humans against time indicating that the population decreases with time under optimal control than when there was no control. As more people are treated and vaccinated against rubella, the susceptible human population reduces more than when there was no control against the disease. The vaccinated human population shown in fig. 3 also increases under optimal control than when there was no optimal control. That shows that vaccination will protect more people from the disease.

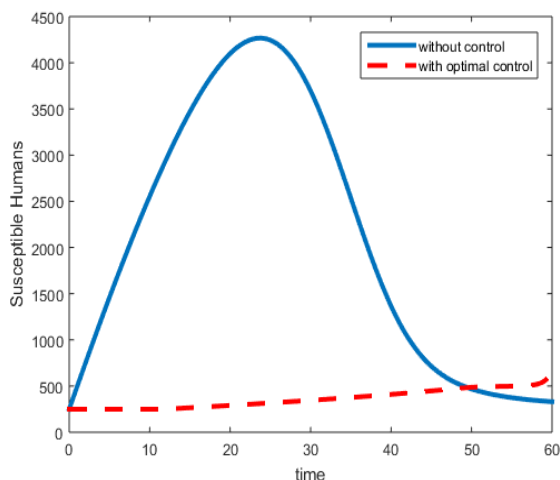


Figure 2: Susceptible Humans

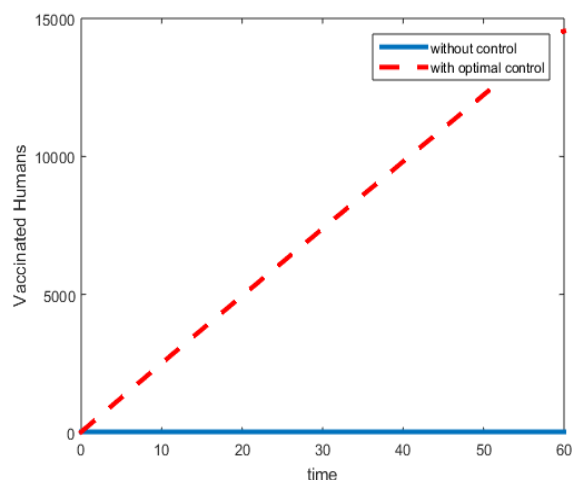


Figure 3: Vaccinated Humans

The exposed adult humans are shown in fig. 4 and decreased sharply under optimal control as compared to when there was no control. As more people are treated and vaccinated, it reduces the rate of infection and with time, the disease will be controlled. Consequently, the infectious adult humans also decrease sharply under optimal control than when there was no control as shown in fig. 5. The decrease is attributed to the reduction in the number of persons who are exposed to the disease due to vaccination and treatment.

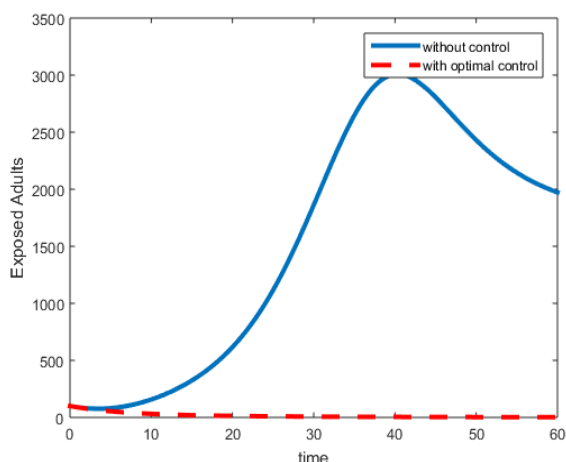


Figure 4: Exposed Adult Human

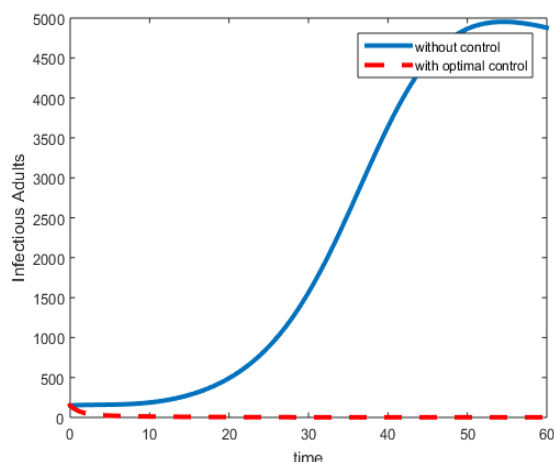


Figure 5: Infectious Adult Humans

The exposed fetus population shown in fig. 6 also decreased under control due to the reduction in the infectious human population than when there was no control. It shows that proper treatment and vaccination which will reduce the rate of exposure to the disease will also reduce and eliminate the rate at which fetus population can be infected. The infectious Newborns are also seen to decrease under control as shown in fig. 7. The decrease is due to the reduction of exposed fetus population as a result of vaccination of humans and treatment of infectious adults which causes a reduction in vertical transmission.

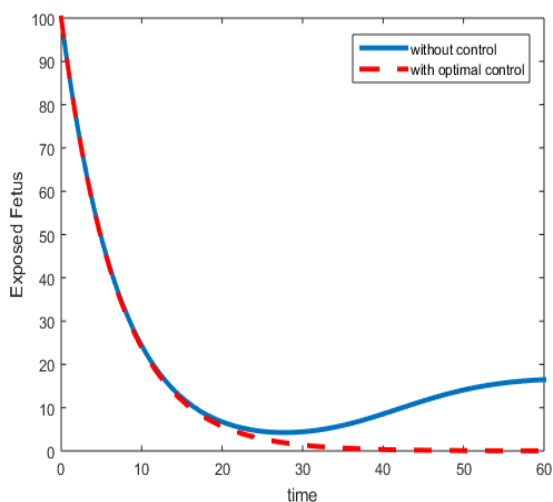


Figure 6: Exposed Fetus

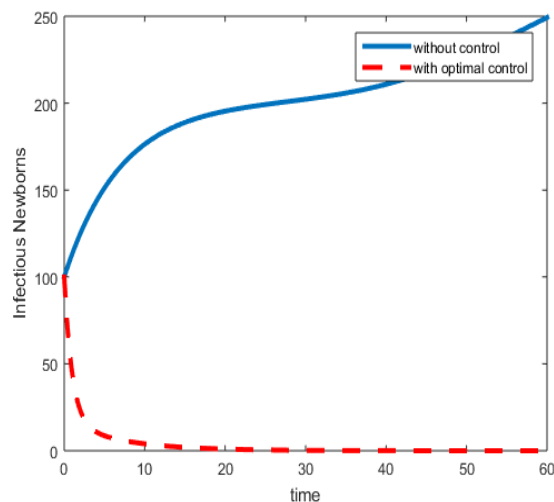


Figure 7: Infectious Newborns

The isolated and treated humans shown in fig. 8 experiences a sharp increase as more people are isolated and treated for rubella. The isolation and treatment reduce the rate of infection as it reduces the contact rate of infectious population with the susceptible humans as well as possibility of vertical transmission. The untreated humans shown in fig. 9 shows that optimal control of the disease causes the population of untreated humans to reduce drastically. As more people are vaccinated, isolated and treated, the untreated population reduces as expected.

The decrease in the rate of infection due to increase in vaccination, isolation and treatment of infectious humans causes the recovered human population to be reduced drastically as shown in Fig. 10. In general, we have shown the effectiveness of the optimal control analysis which reduces the rate of infection and shows how more people can be protected from the disease.

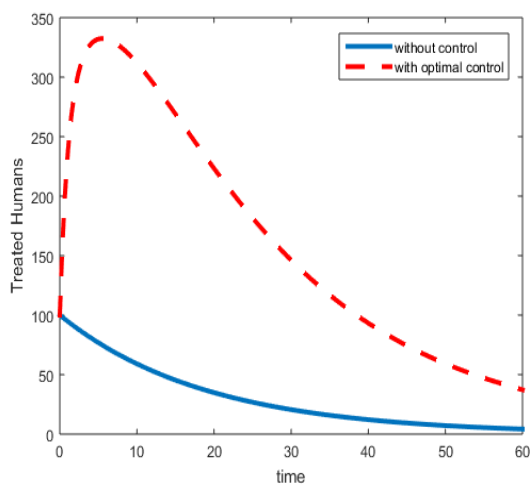


Figure 8: Isolated and Treated Humans

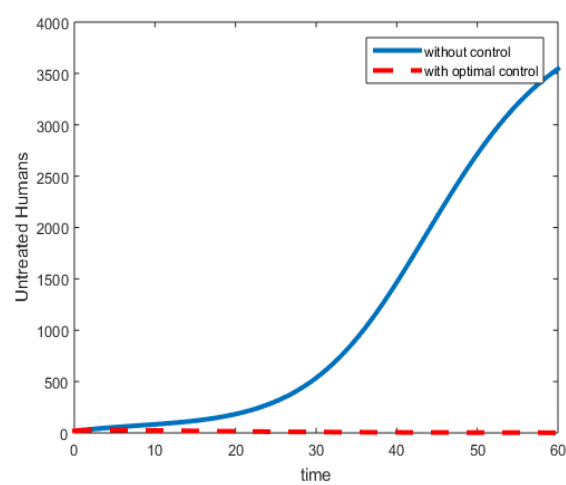


Figure 9: Untreated Humans

Fig. 11 gives the proportion of infectious humans that needs to be isolated and treated at any time t , in order to optimally control the disease in a given human population as well as the number of susceptible humans that need to be vaccinated. At the initial time, $t = 0$, no persons have been vaccinated nor isolated and treated since this is the point of commencement of the disease. The proportion κ , v_1 and δ increases afterwards as t increases, but begins to decrease after reaching its peak.

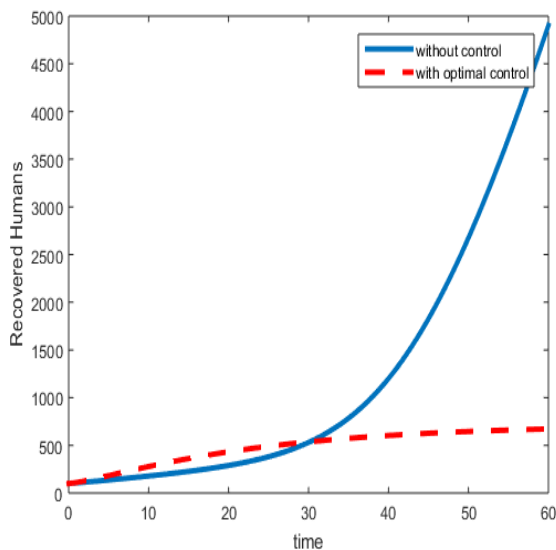


Figure 10: Recovered Humans

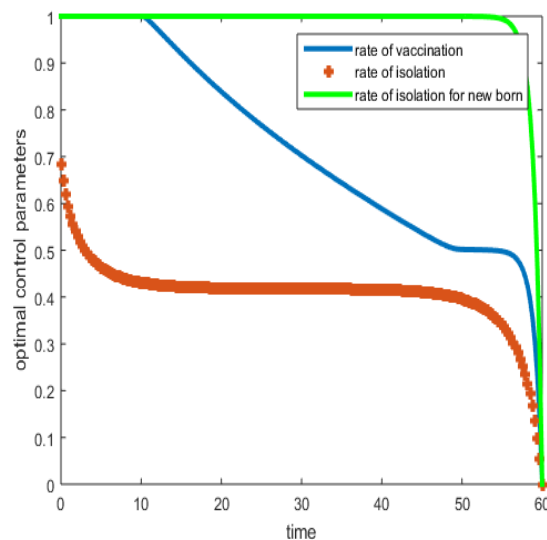


Figure 11: Graph of Optimal Control

The increase shows the time when the number of susceptible humans and infected persons is increasing, and hence, more proportion of susceptible humans need to be vaccinated and more proportion of infectious humans needs to be isolated and treated, for optimal control of the disease. The point where the graph begins to decrease shows that the optimal control effectively reduced the number of susceptible humans to the disease as well as the number of infectious humans viz-a-viz those that should be isolated. With these reductions in the populations of the susceptible, infected and infectious individuals, rubella will not spread in the population, thus the disease will be effectively controlled.

8 Conclusion

We have presented a mathematical model for transmission dynamics of rubella disease and its optimal control. It was shown that the disease-free equilibrium is both locally and globally asymptotically stable when the control reproduction number, $R_c < 1$. We also showed that a unique endemic equilibrium point of the model exists when $R_c > 1$. This confirms the global stability of the disease-free equilibrium, and the absence of backward bifurcation in the model. The application of optimal control methods in isolating infectious humans drastically reduced the spread of the disease.

We conclude that isolating infectious individuals effectively and treating them will help to control the spread of the disease and possibly eliminate it.

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Lassa fever disease: a sensitivity analysis of its treatment control model

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Abstract

A treatment model with seven mutually exclusive compartments related to Lassa fever is presented and the dynamics is completely determined by the effective reproduction number which was obtained using the next-generation approach. A sensitivity analysis of the treatment deterministic model is performed. This is done in order to determine the relative importance of the model parameters to the disease transmission. Stability and sensitivity based on such methods as Lyapunov functional, Comparison theorem etc. Numerical simulation analysis (using MATLAB and MAPLE) were carried out to support the theoretical analysis. The result of the sensitivity analysis shows that the most sensitive parameter is the treatment rate, followed by human recovery rate, then person-to-person contact. This suggests that control strategies should target effective drugs for treatment and education to reduce person to person contact.

Keywords: Lassa fever; mathematical modeling; sensitivity; treatment control

1 Introduction

Lassa fever (LF) is a serious and potentially deadly viral disease caused by Lassa virus (LASV) that is prevalent in West Africa including Benin, Sierra Leone, Guinea, Liberia, Mali, and Nigeria because LASV reservoir and vector (*Mastomys natalensis*) is abundantly seen in these areas [3]. LASV has 2-21 day or more than that incubation period, thus it increases the risk of its transmission from endemic to non-endemic regions [17]. One of the major concerns with LASV is its high case fatality rate, additionally, the virus can easily spread from human-human contact, which makes it even more dangerous. Another major concern with LASV is the lack of effective vaccines and therapeutics for treating the disease. This is further compounded by the fact that the virus can also spread through aerosol, making it even more difficult to contain [2].

Due to the severe nature of LASV and its potential to spread easily, it is classified as a Biosafety Level 4 agent. This means that it is considered to be one of the most dangerous pathogens known to mankind. Despite this, the overall impact of LASV on the increased threat of viral diseases in West Africa has not been fully understood [1]. However, it is known that LASV is a significant cause of yearly morbidity and mortality in many of Africa's poverty-stricken communities such as during 2022 in Nigeria around 189 deaths reported out of 1067 confirmed cases of LASV. Annually, around 1–3 million infections occur of LF with around 5000 deaths [7].

This Lassa fever epidemic, with its increased frequency indicates that our current knowledge of its dynamics and public health guidelines to control the disease is not adequate. [5] carried out a study

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by the use of mathematical model which is formulated based on the natural history of Lassa fever. They also carried out another study on the mathematics of Lassa fever treatment model [5] which is more comprehensive than previous mathematical model studies. However, they did not consider the sensitivity of the model, and this prompted the present study. Hence, we shall consider the Lassa fever epidemic under such sub-headings as transmission, signs and symptoms, method of diagnosis, etc. We shall also consider vaccine and treatments, natural host, prevention and control, and such other information that may be very necessary for proper understanding of our mathematical model and the subsequent results and analysis. Hence, the present study will present a mathematical treatment model on the spread of the Lassa fever disease with current features and the sensitivity analysis on the transmission dynamics.

In this study a compartmental mathematical treatment model is presented to investigate the sensitivity analysis of Lassa fever transmission dynamic model with treatment control. Sensitivity analysis in mathematical biology tells us how important each parameter is to disease transmission and is used to assess how sensitive a model is to variations in the value of the parameters of the model and to changes in the structure of the model [12]. It refers to technique used to determine how independent variable values will impact a particular dependent variable under a given set of assumptions. It is used to determine parameters that have high impact on the basic reproduction number so that it is directly targeted by intervention strategies. It is necessary to determine how sensitive the threshold quantity basic reproductive number with respect to its parameters. This analysis reveals how crucial each of the parameter is to the disease transmission.

Sensitivity analysis has been used for various parameterization tasks of models of biological systems, such as finding essential parameters for research prioritization [9], identifying insignificant parameters or model reduction or parameters clustering (Peter et al in [12]). Classically, sensitivity of the model is determined by the partial derivatives of the outcome with respect to its parameters. Sensitivity analysis methods based on such quantities are called local as the derivative is taken at a fixed point in the state space of model parameters – at the parameter face value. Moreover, these methods belong to the class of one-factor-at-a-time (OAT) methods, because the net effect of a parameter on the property of the outcome is taken while assuming that all other factors are fixed [12].

In recent times, sensitivity analysis of Lassa fever models has received attention of researchers in mathematical epidemiology and has become a subject of intense study [5, 8, 10, 11, 12, 13]. A good number of mathematical models has been developed to assess the sensitivity of each factor driving the disease in order to determine the major factors to be targeted by appropriate interventions in order to eradicate Lassa fever. The present work is aimed at conducting sensitivity analysis on a new Lassa fever treatment model due to [6], for effective management of Lassa fever spread. We reproduce their Lassa fever treatment control model here for ease of reference:

$$\left\{ \begin{aligned} \frac{dS_h}{dt} &= \Lambda_h + \phi(1-\nu)I_h + \psi R_h - \beta_1 \sigma I_r S_h - \beta_2 \varepsilon I_h S_h - \eta(1-e^{-rt})S_h - \mu_h S_h \\ \frac{dE_h}{dt} &= \beta_1 \sigma I_r S_h + \beta_2 \varepsilon I_h S_h + \eta(1-e^{-rt})S_h - (\kappa + \mu_h)E_h \\ \frac{dI_h}{dt} &= \kappa E_h - (\phi + \theta + \delta + \mu_h)I_h \\ \frac{dT_{h1}}{dt} &= \alpha \theta I_h - \mu_h T_{h1} \\ \frac{dT_{h2}}{dt} &= (1-\alpha)\theta I_h - (\delta + \mu_h)T_{h2} \\ \frac{dR_h}{dt} &= \phi \nu I_h + \varpi T_{h1} - (\psi + \mu_h)R_h \\ \frac{dS_r}{dt} &= \Lambda_r - \beta_3 \vartheta I_r S_r + \varphi R_r - \mu_r S_r \\ \frac{dI_r}{dt} &= \beta_3 \vartheta I_r S_r - (\omega + \mu_r)I_r \\ \frac{dR_r}{dt} &= \omega I_r - (\varphi + \mu_r)R_r \end{aligned} \right. \quad (1)$$

The definitions of above model variables and parameters are listed in Tables 1 and 2.

Table 1: Description of the state variables of the model

Variable	Description
S_h	Number of Susceptible humans
E_h	Number of Exposed humans
I_h	Number of Infectious humans
T_{h1}	Early stage treatment group
T_{h2}	Late stage treatment group
R_h	Number of Recovered humans
S_r	Number of Susceptible rodents
I_r	Number of Exposed rodents
R_r	Number of Infectious rodents

Table 2: Description of the parameters of the Lassa fever model

Parameters	Description	Values	Source
Λ_h	Recruitment level of humans	20	Assumed
Λ_r	Recruitment level of rodents	200	Assumed
δ	Per capita Lassa-induced death rate	0.2	[17]
ψ	Recovered human loss of immunity	0.9	[16]
ϕ	Spontaneous individual recovery	0.001	[17]
β_1	Transmission rate per contact by an infectious rodent	0.00002	[18]
β_2	Transmission rate per contact by an infectious through sexual activity	0.002	[18]
β_3	Transmission rate per contact from an infectious human through other means	0.08	[18]

η	Relative infectiousness of individuals with aerosol	0.03	[15]
μ_h	Natural mortality rate for humans	0.02041	[4]
μ_r	Natural mortality rate for rodents	0.06	[17]
κ	Progression rate of human from exposed to infected	0.05	Estimated
α	Progression rate of rodents from exposed to infected	0.42	Estimated
σ	Contact rate of rodent per human per unit time	0.005	[6]
$\mathcal{G}$	Relative human-to-rodent transmissibility of infect humans	0.6	Estimated
ε	Relative human-to-human transmissibility of infect humans	0.5	Estimated
r	Number of doses of airborne viral particles	Negligible	[6]
ω	Recovery rate of rodents	0.00001	[18]
φ	Recovered rodent loss of immunity	0.001	Estimated
ϖ	Rate of early stage treated group who recovers	0.2	Estimated
α	Fraction of early stage treatment group	(0,1)	Assumed
θ	Treatment rate of humans	0.8	Estimated

Source: [6]

2 Qualitative properties of the model

Since the model (1) monitors human population, all its associated parameters and state variables are assumed to be non-negative for all $t \geq 0$. Before analysing the model, it is instructive to show that the state variables of the model remain non-negative for all non-negative initial conditions. Thus, we claim the following result.

Theorem 1: Let the initial data $S_h(0), E_h(0), I_h(0), T_{h1}(0), T_{h2}(0), R_h(0), S_r(0), I_r(0), R_r(0)$ be non-negative. Then, the solutions $(S_h, E_h, I_h, T_{h1}, T_{h2}, R_h, S_r, I_r, R_r)$ of model (1) are positive and bounded for all $t > 0$, whenever they exists.

Proof:

The first equation (1) of the system can be written as:

$$\frac{dS_h}{dt} + [\lambda(S_h) + \mu_h] S_h = \Lambda_h + \phi(1-\nu) I_h + \psi R_h$$

$$\frac{d}{dt} [S_h(t) \eta(t)] = (\Lambda_h + \phi(1-\nu) I_h + \psi R_h) \eta(t),$$

where $\eta(t) = \exp\left(\int_0^t [\lambda(S_h) + \mu_h] t dt\right) > 0$ is the integrating factor. Hence, integrating this last relation from 0 to t_1 , we have

$$[S_h(t_1) \eta(t_1)] - S_h(0) \eta(0) = \int_0^{t_1} (\Lambda_h + \phi(1-\nu) I_h + \psi R_h) \exp[(\lambda(S_h) + \mu_h) p] dp,$$

$$\eta(0) = 1,$$

$$[S_h(t_1) \eta(t_1)] - S_h(0) = \int_0^{t_1} (\Lambda_h + \phi(1-\nu) I_h + \psi R_h) \exp[(\lambda(S_h) + \mu_h) p] dp$$

$$S_h(t_1) \eta(t_1) = S_h(0) + \int_0^{t_1} (\Lambda_h + \phi(1-\nu) I_h + \psi R_h) \exp[(\lambda(S_h) + \mu_h) p] dp$$

so that the division of both side by $\eta(t)$ yields

$$S_h(t_1) = \left[S_h(0) + \int_0^{t_1} (\Lambda_h + \phi(1-\nu)I_h + \psi R_h) \exp\left[(\lambda(S_h) + \mu_h)p\right] dp \right] \times \eta^{-1}(t_1) > 0.$$

The same arguments can be used to prove that $E_h(t), I_h(t), R_h(t), S_r(t), I_r(t), R_r(t) \geq 0$ for all $t > 0$.

Furthermore, let $N = S_h + E_h + I_h + T_{h1} + T_{h2} + R_h$. Then,

$$\begin{aligned} \dot{N}(t) &= \dot{S}_h + \dot{E}_h + \dot{I}_h + \dot{T}_{h1} + \dot{T}_{h2} + \dot{R}_h \\ &= \Lambda_h + \phi(1-\nu)I_h + \psi R_h - \lambda_h S_h - \mu_h S_h + \lambda_h S_h - (\kappa + \mu_h)E_h + \kappa E_h - (\phi + \theta + \delta + \mu_h)I_h \\ &\quad + \alpha\theta I_h - \mu_h T_{h1} + (1-\alpha)\theta I_h - (\delta + \mu_h)T_{h2} + \phi\nu I_h + \varpi T_{h1} - (\psi + \mu_h)R_h \\ &\leq \Lambda_h - \mu_h N_h - \delta I_h, \quad \delta = 0 \end{aligned}$$

This implies that as $t \rightarrow \infty, \sup N_h \leq \frac{\Lambda_h}{\mu_h}$. Also from (1), we have that as: $t \rightarrow \infty, \sup S_r(t) \leq \frac{\Lambda_r}{\mu_r}$.

This completes the proof.

Combining Theorem 1 with the trivial existence and uniqueness of a local solution for the model (1), we have established the following theorem which ensures the mathematical and biological well-posedness of system (1).

Theorem 2: The dynamics of model (1) is a dynamical system in the biological feasible compact set

$$\Gamma := \left\{ (S_h, E_h, I_h, T_{h1}, T_{h2}, R_h, S_r, I_r, R_r) \in \mathbb{R}_+^9 : 0 \leq S_h \leq \frac{\Lambda_h}{\mu_h}, N_h \leq \frac{\Lambda_h}{\mu_h}, N_r \leq \frac{\Lambda_r}{\mu_r} \right\} \quad (2)$$

2.1 Reproduction number and stability analysis

The model (1) has a disease-free equilibrium (DFE), obtained by setting the right hand sides of the equations in the model to zero, given by

$$\xi_0 = (S_h^0, E_h^0, I_h^0, T_{h1}^0, T_{h2}^0, R_h^0, S_r^0, I_r^0, R_r^0) = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, 0, 0, \frac{\Lambda_r}{\mu_r}, 0, 0 \right). \quad (3)$$

The stability of ξ_0 can be established using the next generation operator method on the system (1).

Using the notations in [16], the matrices F and V , for the new infection terms and the remaining transfer terms respectively, are given by

$$F = \begin{pmatrix} 0 & \beta_2 \varepsilon S_h^0 & 0 & 0 & \beta_1 \sigma S_h^0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 \vartheta S_r^0 \end{pmatrix}; \quad V = \begin{pmatrix} (\mu_h + \kappa) & 0 & 0 & 0 & 0 \\ -\kappa & (\phi + \mu_h + \delta + \theta) & 0 & 0 & 0 \\ 0 & \alpha\theta & \mu_h & 0 & 0 \\ 0 & -\alpha(1-\theta) & 0 & (\delta + \mu_h) & 0 \\ 0 & 0 & 0 & 0 & (\omega + \mu_r) \end{pmatrix}$$

In the calculation of matrices F and V , we took the infection variables to be E_h, I_h, T_{h1}, T_{h2} and I_r as explained in [5]. Thus

$$R_r = R_h + R_r = \frac{\beta_2 \kappa \varepsilon \Lambda_h}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)} + \frac{\beta_3 \vartheta \Lambda_r}{\mu_r (\omega + \mu_r)},$$

where R_T is obtained from $\rho(FV^{-1})$ with ρ being the spectral radius of the matrix FV^{-1} . Computing the partial derivatives of R_T with respect to the parameters under investigation (θ, ϖ) further reveals the effect of these parameters on Lassa fever control in the community. This gives

$$\frac{\partial R_T}{\partial \theta} = -\frac{\beta_2 \varepsilon \kappa \Lambda_h}{\mu_h (\mu_h + \kappa) (\phi + \mu_h + \delta + \theta)^2} < 0. \quad (4)$$

Clearly, it follows from (4) that the partial derivative is less than zero, unconditionally. Hence, treatment rate of humans will have a positive impact in reducing the burden of LF in the population, irrespective of the values of the other parameters in the expressions on the right-hand sides of (4). Assuming treatment control can reduce the basic reproduction number reduces R_0 by a factor q , then effective reproduction number [6]:

$$R_T = (1-q) R_0. \quad (5)$$

This implies that no outbreak if $R_T \leq 1$, ie, if $q \geq 1 - \frac{1}{R_0}$, we obtain that

$$\begin{aligned} \frac{R_T}{R_0} &= \frac{\phi + \mu_h + \delta}{\phi + \mu_h + \delta + \theta} = 0.88 \Rightarrow R_T = (1-q) R_0 \\ \therefore 1-q &= 0.88 \Rightarrow q = 0.12 \end{aligned}$$

Thus, introduction of treatment control significantly reduces the transmission of LF required for successful eradication of the disease (requirement of at least 12% effectiveness). This means that the treatment reproduction number R_T is 12% smaller than the basic reproduction number R_0 , indicating that the use of treatment as a control strategy is 12% effective on reducing LF transmission. This implies that early detection is of paramount importance in the control of Lassa fever in a population. However, this is an indication that the early stage group treatment program alone at this rate is not enough to eliminate the disease. So, multiple control strategy requires urgent assessment.

Lemma 1: The DFE of the model (1), ξ_0 , is locally asymptotically stable if $R_T < 1$ and unstable if $R_T > 1$. The threshold quantity R_T is the effective reproduction number under treatment control for the Lassa fever model. Biologically speaking, Lemma (1) implies that Lassa fever can be eliminated from the community (when $R_T < 1$) if the initial sizes of the subpopulation of the model are in the basin of attraction of ξ_0 in the presence of treatment.

2.2 Global Asymptotic Stability: Special Case $\sigma = \eta = 0$.

Consider the model (1) with $\sigma = \eta = 0$. We claim the following:

Theorem 3: The DFE of the model (1) with $\sigma = \eta = 0$ is globally-asymptotically stable (GAS) whenever $R_T < 1$.

Proof:

Consider the model (1) with $\sigma = \eta = 0$. Further, consider the following linear Lyapunov function

$$F = \kappa E_h + (\kappa + \mu_h) I_h$$

with Lyapunov derivative (where a dot represents differentiation with respect to t):

$$\begin{aligned}\dot{F} &= \kappa \dot{E}_h(t) + (\kappa + \mu_h) \dot{I}_h(t) \\ &= \kappa (\lambda_h S_h - (\kappa + \mu_h) E_h) + (\kappa + \mu_h) (\kappa E_h - (\phi + \theta + \delta + \mu_h) I_h) \\ &= (\kappa + \mu_h) (\phi + \theta + \delta + \mu) \left[\frac{\beta_2 \varepsilon \kappa S_h}{(\kappa + \mu_h) (\phi + \theta + \delta + \mu)} - 1 \right] I_h \\ \dot{F} &\leq (\kappa + \mu_h) (\phi + \theta + \delta + \mu_h) [R_T - 1] I_h\end{aligned}$$

Hence, $\dot{F} \leq 0$ if $R_T \leq 1$ with $\dot{F} = 0$ if and only $I_h = 0$. Therefore F is a Lyapunov function in Γ and it follows Salle's Invariance Principle [6], that every solution to the equations in (1) (with $\sigma = \eta = 0$) with initial conditions in Γ converges to ξ_0 as $t \rightarrow \infty$. i.e.,

2.3 Effect of the effective treatment rate

In order to investigate the impact of the effective treatment rate on the spread of LF, we carry out some numerical simulations to show the contribution of effective treatment rate during the whole infection. From Figure 2, we can observe that infected individuals reach a lower point level as θ increases. This Figure 1 illustrates the great influence of the effective treatment rate as shown in the threshold analysis.

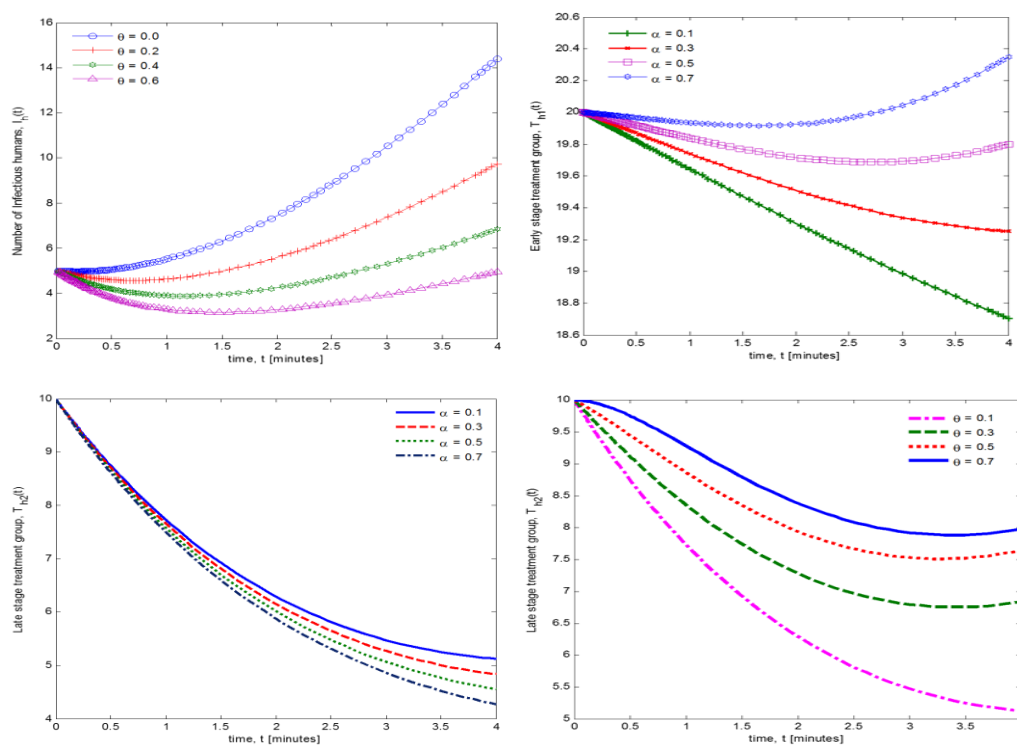


Figure 1. Impact of effective treatment rate on the spread of LF.

3 Sensitivity analysis

Sensitivity indices allow us to measure the relative change in a variable when a parameter changes. If the model is simple, it may be possible to differentiate the outcome with respect to each parameter in turn. The derivatives are the rate of changes of predictions with respect to the parameters. This work adopts the normalized forward sensitivity index to conduct the sensitivity index of a variable with respect to a parameter is the ratio of the change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivative

[5]. For instance, the normalized forward sensitivity index on R_T which depends differentially on a parameter φ , is defined by

$$\Pi_{\varphi}^{R_T} = \frac{\partial R_T}{\partial \varphi} \times \frac{\varphi}{R_T} \quad (6)$$

The sensitivity index of parameters with respect to R_T is given as:

$$\begin{aligned} \Pi_{\Lambda_h}^{R_T} &= \frac{\beta_2 \varepsilon \kappa}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)} \times \frac{\Lambda_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = 1 > 0 \\ \Pi_{\beta_2}^{R_T} &= \frac{\varepsilon \kappa \Lambda_h}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)} \times \frac{\beta_2 (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = 1 > 0 \\ \Pi_{\varepsilon}^{R_T} &= \frac{(1 - u_1) \kappa \Lambda_h}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)} \times \frac{\varepsilon (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = 1 > 0 \\ \Pi_{\kappa}^{R_T} &= \frac{\beta_2 \varepsilon \Lambda_h \mu_h}{(\kappa + \mu_h)^2 [(\phi + \mu_h + \delta + \theta) \mu_h]} \times \frac{\kappa (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = 0.667 > 0 \\ \Pi_{\theta}^{R_T} &= -\frac{\beta_2 \varepsilon \kappa \Lambda_h}{\mu_h (\mu_h + \kappa) (\phi + \mu_h + \delta + \theta)^2} \times \frac{\theta (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = -1.784 < 0 \\ \Pi_{\delta}^{R_T} &= \frac{-\beta_2 \varepsilon \kappa \Lambda_h}{\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)^2} \times \frac{\delta (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = -0.743 < 0 \\ \Pi_{\mu_h}^{R_T} &= \frac{-\beta_2 \varepsilon \kappa \Lambda_h [3\mu_h + 2\mu_h (2\kappa(\phi + \delta + \theta) + \kappa(\phi + \delta + \theta) + 2\kappa(\phi + \delta + \theta))] }{[\mu_h (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)]^2} \\ &\times \frac{\mu_h^2 (\kappa + \mu_h) (\phi + \mu_h + \delta + \theta)}{\beta_2 \varepsilon \kappa \Lambda_h} = -1.603 < 0 \\ \Pi_{\phi}^{R_T} &= -\frac{\beta_2 \varepsilon \kappa \Lambda_h}{\mu_h (\mu_h + \kappa) (\phi + \mu_h + \delta + \theta)^2} \times \frac{\phi (\phi + \mu_h + \delta + \theta) \mu_h}{\beta_2 \varepsilon \kappa \Lambda_h} = -1.482 < 0 \end{aligned}$$

Table 3: Sensitivity indices of the effective reproduction number to model parameters

Parameter	Estimated values	Sensitivity index
μ_h	0.02041	-1.603
θ	0.8	-1.784
δ	0.2	-1.784
ϕ	0.001	-1.482
Λ_h	20	1
β_2	0.002	1
ε	0.5	1
κ	0.01	-0.667

Source: Researchers' Computation

It is observed in Table 3 that these parameters have either positive or negative effects on the effective reproduction number. Positive index indicates that the value of R_T will increase with increase in the parameter and the one with negative index will decrease the value of R_T with increase in the parameter values.

4 Discussion of results

This study is concerned with an epidemiological treatment model of Lassa fever (LF). The effect of treatment rate of humans on the model is investigated and the main results are given in terms of local and global stability. It has been shown that that treatment rate of humans will have a positive impact in reducing the burden of LF in the population. Our model predicts that treatment control can reduce the population level transmission by up to 12% alone without existing interventions. Therefore, treatment has significant effect on LF transmission, but it may not be able to eliminate the disease unless a multiple control strategy is adopted. Accordingly, we should reduce the reproduction number periodically so as to keep the number of infected below the capacity for treatment. Finally, some numerical simulations are carried out to support our theoretical results. Based on this, we carry out some numerical simulations to show the contribution of effective treatment rate during the whole infection. From the simulation results, we observed that infected individuals reach a lower point level as θ increases. This figure illustrates the great influence of the effective treatment rate as shown in the threshold analysis. It is hoped that this study would motivate public health epidemiologists to collect relevant data for further and better understanding of the effect of treatment on the dynamics of LF.

From Table 3, the most significant parameters affecting the effective reproduction number are per capital induce death rate, treatment rate and natural recovery rate. These parameters should be targeted by control intervention strategies. Increasing these parameters will decrease the effective reproduction number for the control model. Similarly, if the values are decreased, the effective reproduction number will be increased and as such the spread of Lassa fever increases. An increase in the positively indexed parameters like rate of exposure to aerosol, transmission rate per contact by an infective through sexual activity and relative human-to-human transmissibility rate of infected humans increase the value of the effective reproduction number and as such the disease invades the population. The results in Table 3 have far reaching implications to Lassa fever transmission and control. Firstly, it shows that treatment plays an indispensable role in curtailing the spread of Lassa fever outbreak. This collaborates the earlier works of [5, 10, 13] who concluded that the best strategies against the disease are maintaining good community hygiene to control the mastomys and taking care of the infective under specific isolation precaution. This result shows that with individuals' total adherence to effective use of personal protection against disease-induce death, little treatment efforts will then be required by the community in the control of the spread of the disease. Secondly, the lowest sensitivity index, which is the relative infectiousness of individuals with aerosol does not imply that this parameter is not relevant to Lassa fever control. It implies that Lassa fever prevention is achievable provided that relative infectiousness of individuals with aerosol is low enough to cause outbreak. This could be done by keeping the environment very ventilated. Ventilation represents a primary infectious disease control strategy through dilution of room air around a source and removal of infectious agents [5].

5 Conclusion

In this paper, we have successfully carried out the sensitivity analysis of treatment model of Lassa fever transmission in human and rodent population. Mathematically, we have shown that it is possible to control Lassa fever disease in Nigeria by treatment measure. We therefore conclude that early treatment measures have some influence in reducing the burden of Lassa fever. Also, it was concluded that the most sensitive parameter is the treatment rate, followed by human recovery rate, Lassa fever induced-death then person-to-person contact. The following recommendations are made:

- We recommend possible public health interventions based on the result of this study. This result obtained in this work can be a useful guide for the local national control programme on Lassa fever, decide on the framework for planning and designing the cost-effective strategies for the best intervention packages in eliminating Lassa fever in Nigeria.

- We recommend that relevant health agencies of government (such as NCDC) should bring about rapid control of Lassa fever through the results of this study as well as increasing capacity for early detection. More so, we recommend a research collaboration between mathematical epidemiologists and other medical researchers such as epidemiologists, virologists, microbiologists, etc.

The result of the sensitivity analysis shows that the most sensitive parameter is the treatment rate, followed by human recovery rate, then person to person contact. This suggests that control strategies should target effective drugs for treatment and education to reduce person to person contact.

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Time series analysis of crime deterrence on drug abuse in Jalingo Local Government Area, Taraba State, Nigeria

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Abstract

Time Series on crime deterrence on drug abuse was carried out in order to determine the trend and tendency of crime and the effect of drug abuse, unemployment, and number of arrest on crime. After careful analysis, it was discovered that in the year 2017, the number of crime reported is 291 which was less than other figures used in the analysis and more forecast for subsequent years 2018, 2019 and 2020 with reported forecast rising to a high level of 414, 484 and 454 respectively. 77.3% on the variation in the crime was explained by the number of arrest, unemployment rate and drug abuse cases. The coefficients indicate that drug abuse cause increase in crime while previous arrest cause decrease in crime rate. This could be as a result of using previous information.

Keywords: autocorrelation; forecast; arrest; drug-abuse; unemployment.

1 Introduction

One of the continuous problems that have bedeviled the existence of mankind is crime. Since early days of man's existence, crime had been a disturbing threat to personality, property and lawful authority [1]. The relationship between crime and evolution of mankind is considered a historical one as Cain (first son of Adam and Eve) committed the first violent crime when he murdered his brother Abel because of jealousy [2]. Today, in the modern complex world, the situation is most highly disturbing. It started in the primitive days as a simple and less organized issue, and ended today as very complex and organized. Due to the complex nature of the subject of crime, its causes and consequences, various academic disciplines such as criminology, sociology, geography, psychology and demography study it from their own perspective [3].

The aim of this research is to determine the trends in crime rate in Taraba state and forecast the future possible crime rate in the state. To determine the trend of crime rate in Taraba State With a view to see the impact of drug abuse, unemployment rate, and number of arrests on crime rate. This research was carried out in Taraba state, and it is a secondary data on crimes provided by the Taraba State Police Divisional Head Quarters, Jalingo, from 2000 to 2016.

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2 Research methodology

To realise the objectives of this research, simple time domain techniques (ARIMA model) O.I. The work in [10] will be used to forecast and to determine the trend of crime rates. The Multiple Linear Regression Analysis will also be performed in order to determine the impact of drug abuse, unemployment rate, and the number of arrests. SPSS version 20 and E-views shall be used to carry out all the analysis. The ARIMA encompass three components, AR, MA, and integrated series. AR stands for the autoregressive model i.e. regressing the dependent variable with linear combination of its past values or lagged values [4], MA stands for moving average model i.e. regressing the dependent errors with linear combination of its past error or lagged error or innovation and I stands the differencing order, that is number of difference applied on the stochastic process before attaining stationary [9]. Hence, the mathematical form of the model with the parameters is:

$$Crime = Constant + \beta_1 * Arrest + \beta_2 * Unemp + \beta_3 * Drug + \beta_4 * Prev.Arrest.$$

There are steps we will take to achieve our aims and these are listed below: Model Estimation, Model Identification, Model Diagnostic and Forecasting Accuracy.

2.1 Model estimation

We estimate the parameters of ARIMA model using Statistical Package for social sciences (SPSS).

2.2 Model identification

The major thing to do here is to test for stationarity of the series using the approach of Phillips-Perron test statistic [6].

2.3 Model diagnosis

If we fit the model, the model diagnosis will aid us to know the forecasting, reliability, accuracy ability which will be judge under the coefficient of determination or through the use of the smallest mean square error or other smallest measurement tools like MAE (Mean Absolute Error), MAPE (Mean Absolute Percentage Error), RMSE (Root Mean Square Error), MSE (Mean Square Error). Box and Jenkin [5] model building strategy entails a diagnosis of the adequacy of the model. More importantly, it is necessary to ascertain in what way a model is adequate or inadequate. And this entails being sure that;

- The model converged upon minimum sum of square error
- The sum of square error (residuals) should be quite small so that R^2 of the model could be quite large [8].

3 Data analysis and interpretation

3.1 Time series analysis

Table 3.1 Minimization phase section

Itn	Error Sum					
No.	of Squares	Lambda	AR(1)	AR(2)	MA(1)	MA(2)
0	264452	0.1	0.1	0.1	0.1	0.1
1	169655	0.1	-0.2273186	-5.678551E-02	0.4330371	0.2286701
2	1.329453E+15	0.04	5.681635E-02	0.4535415	0.9430795	0.7431056
3	1.881917E+07	0.4	-0.2168886	0.1127902	0.6257304	0.520533
4	161098.2	4	-0.2458111	-5.257529E-02	0.4841563	0.2917979
5	160097.9	1.6	-0.2662098	-4.102172E-02	0.5505252	0.3357024
6	157076.8	0.64	-0.2816789	-5.006321E-02	0.5395117	0.3182929
7	156547.3	0.256	-0.2935411	-6.232658E-02	0.5448436	0.2975857
8	155869.4	0.1024	-0.2564089	-7.762863E-02	0.5881894	0.2610659
9	153828.9	0.04096	-0.1462758	-0.1128353	0.6924632	0.173146
10	146632.2	0.016384	0.1247474	-0.1988176	0.9505295	-4.271355E-02
11	348883.9	0.0065536	0.630744	-0.377059	1.469549	-0.4616326
12	140005.8	0.065536	0.2199221	-0.3662845	1.024166	-8.783563E-02
13	128638.5	0.0262144	0.4119723	-0.4661938	1.229158	-0.2448201
14	7.40718E+14	1.048576E-02	0.4711801	-0.258344	1.823251	-0.1722613
15	6.844108E+12	0.1048576	0.4453836	-0.3274333	1.718337	-0.1312337
16	4.772269E+08	1.048576	0.4053402	-0.4492193	1.457659	-0.1190874
17	118132.7	10.48576	0.4084557	-0.4678892	1.273622	-0.2139967
18	116883.2	4.194304	0.4086467	-0.4698515	1.280031	-0.210033
19	115139.2	1.677722	0.409653	-0.4749367	1.289584	-0.205186
20	112053.4	0.6710886	0.4117909	-0.4835263	1.310181	-0.1966075

The above table displays the algorithms progress toward a solution. The Error Sum of Squares presents the sum of the squared residuals. This is the value that is being minimized by the algorithm. Lambda is the value of Marquart's lambda parameter. AR(1), AR(2), MA(1) and MA(2) are the values of the autoregressive and moving average parameters.

Table 3.2 Model description section

Series	Crime rate-MEAN	
Model	Regular(2,1,2)	Seasonal(No seasonal parameters)
Mean	409.8	
Observations	15	
Iterations	20	
Pseudo R-Squared	19.897974	
Residual Sum of Squares	112053.4	
Mean Square Error	11205.34	
Root Mean Square	105.8553	

Table 3.2 displays summary information about the solution. The series shows the name of the variable being analyzed, which is crime rate. The Model contain the phrase *Regular* (p,d,q) gives the highest order of the regular ARIMA parameters. The *Seasonal* (P,D,Q) gives the highest order of the seasonal ARIMA parameters, if they were used (SARIMA is not used in this work), where;

2 is the highest order auto regression parameter in the model.

1 is the number of times the series was differenced.

2 is the highest order moving average parameter in the model.

The mean is the average of the series, which is 409.8 crimes per year. Observations is the number of observations (rows) in the series. Iterations is the number of iterations before the algorithm converged or was aborted. Pseudo R-Squared is the value that generates a statistic that acts like the R-Squared value in multiple regression, a value near zero indicates a poorly fitting model, while a value near one indicates a well-fitting model, this value (19.90%) indicate that about 20% of the crime rate is determined by time. Residual Sum of Squares presents the sum of the squared residuals this is the value that is being minimized by the algorithm. Mean Square Error is the average squared residual (MSE), which is a measure of how closely the forecasts track the actual data. The statistic is popular because it shows up in analysis of variance tables. However, because of the squaring, it tends to exaggerate the influence of outliers (points that do not follow the regular pattern). Root Mean Square is the square root of MSE. This statistic is popular because it is in the same units as the time series.

Table 3.3 Model estimation section

Parameter	Parameter	Standard		Prob
Name	Estimate	Error	T-Value	Level
AR(1)	0.4117909	0.1477031	2.7880	0.005304
AR(2)	-0.4835263	0.3768807	-1.2830	0.199503
MA(1)	1.310181	0.4818036	2.7193	0.006542
MA(2)	-0.1966075	0.4494427	-0.4374	0.661787

It can be seen from Table 3.3 that the estimates of the ARIMA model parameters can be given as follows:

- AR(1) The 1st-order autoregressive parameter is 0.4117909.
 AR(2) The 2nd-order autoregressive parameter is -0.483562.
 MA(1) The 1st-order moving average parameter is 1.310181.
 MA(2) The 2nd-order moving average parameter is -0.1966075.

The ARIMA model is therefore given by the equation:

$$(1 - L)y_t = 0.411y_{t-1} - 0.484y_{t-2} + 1.310\epsilon_{t-1} - 0.196\epsilon_{t-2}.$$

Parameter estimate is the estimated parameter value. The Standard Error is the estimate of the standard error of the parameter value. The T-value is the t-test value testing whether the parameter is statistically significant (different from zero). The Probability Level is the probability level for the corresponding t-test. The p-value would have to be less than 0.05 in order for the parameter to be considered statistically different from zero. The above analysis shows that the p-values of the AR(1) and MA(1) (0.005 and 0.006 respectively) are less than 0.05 indicating that the 1st-order autoregressive parameter and 1st-order moving average parameter are significant, but the p-values of the AR(2) and MA(2) (0.199 and 0.661 respectively) are greater than 0.05 indicating that the 2nd-order autoregressive parameter and 2nd-order moving average parameter is not significant.

Table 3.4 Asymptotic correlation matrix of parameters

	AR(1)	AR(2)	MA(1)	MA(2)
AR(1)	1.000000	0.989966	0.587114	-0.399626
AR(2)	0.989966	1.000000	0.518744	-0.430849
MA(1)	0.587114	0.518744	1.000000	-0.685189
MA(2)	-0.399626	-0.430849	-0.685189	1.000000

Table 3.4 gives the asymptotic estimates of the correlation between the parameter estimates. If some of the correlations are greater than 0.9999, the appropriate parameters are removed, in this analysis, the highest correlation is 0.9899 indicating that none of the parameters shall be removed.

2	0.151336	5	-0.292838	8	-0.037631	11	0.014799
3	-0.354306	6	0.235722	9	-0.108505	12	0.105290

Significant if |Correlation| > 0.516398

The autocorrelations are all non-significant, implying that the residuals are white noise and therefore the model is fit for use.

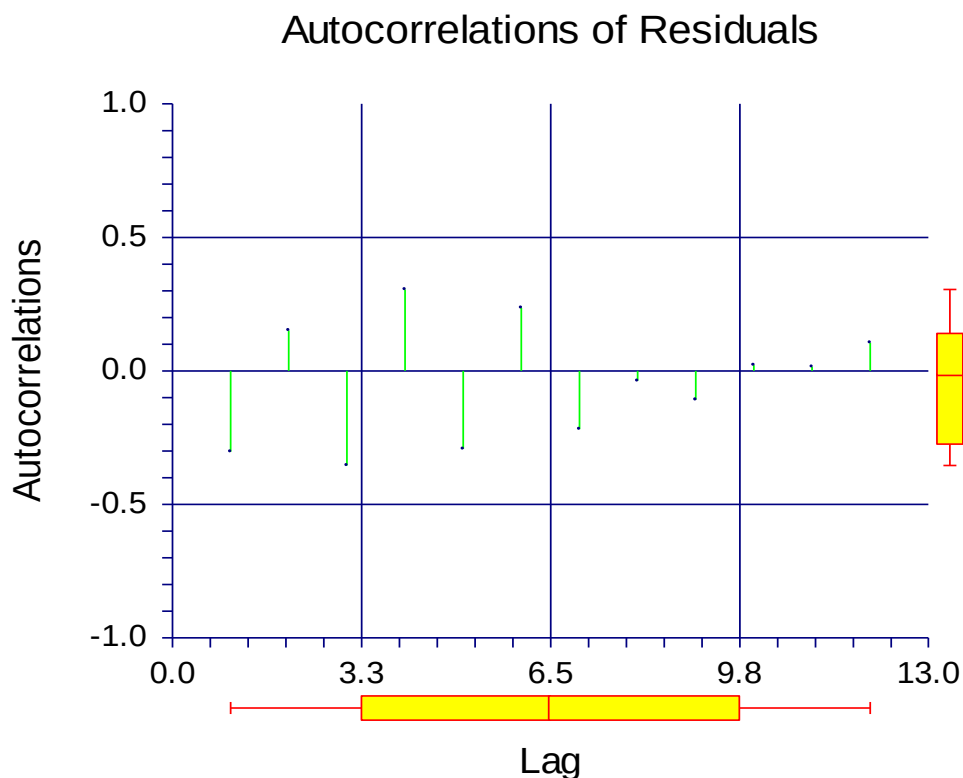


Figure 3.1 Autocorrelation plot

Fig. 3.1 plot is the key diagnostic to determine if the model is adequate. Since no autocorrelations or a pattern of autocorrelations is found in the residuals, the model is assumed to be as good as possible and we proceed to use the forecasts.

Table 3.5 Forecast section of crime rate

Row	Date	Actual	Residual	Forecast	Lower 95% Limit	Upper 95% Limit
1	2002	444.0	27.8	416.2	207.7	624.8
2	2003	483.0	53.8	429.2	220.6	637.7
3	2004	405.0	4.8	400.2	191.7	608.7
4	2005	389.0	30.7	358.3	149.8	566.8
5	2006	333.0	-47.9	380.9	172.3	589.4
6	2007	357.0	-29.4	386.4	177.9	594.9
7	2008	368.0	-55.1	423.1	214.5	631.6
8	2009	462.0	34.7	427.3	218.8	635.9
9	2010	307.0	-132.1	439.1	230.6	647.7
10	2011	453.0	75.4	377.6	169.1	586.2
11	2012	262.0	-201.3	463.3	254.8	671.9
12	2013	361.0	-30.4	391.4	182.8	599.9
13	2014	532.0	37.7	494.3	285.8	702.9
14	2015	659.0	159.8	499.2	290.7	707.8

15	2016	332.0	-94.7	426.7	218.1	635.2
	Forecast					
16	2017			291.4	82.9	499.9
17	2018			414.2	175.9	652.4
18	2019			484.4	232.7	736.1
19	2020			453.9	202.2	705.6

Table 3.5 presents the forecasts and the 100(1-alpha)% prediction limits of the forecast, the analysis indicates that in the year 2017 the number of reported crime is expected to be 291 (the number can at least become 83 crimes and at most 500 crimes) while in 2018, 2019 and 2020 the reported crime is forecasted to rise to a high level of 414, 484 and 454 respectively all within prediction limits..

Table 3.6 Autocorrelations of residuals of crime rate-mean

Lag	Correlation	Lag	Correlation	Lag	Correlation	Lag	Correlation
1	-0.302900	4	0.304680	7	-0.218449	10	0.021190
2	0.151336	5	-0.292838	8	-0.037631	11	0.014799
3	-0.354306	6	0.235722	9	-0.108505	12	0.105290

Significant if $|\text{Correlation}| > 0.516398$

The autocorrelations are all non-significant, implying that the residuals are white noise and therefore the model is fit for use.

Regression Analysis

Multiple linear regression analysis is used to investigate the effect of drug abuse, unemployment and arrests on the crime rate

Table 3.7 Model summary

Model	R	R Square	Adjusted Square	Std. Error of the Estimate
1	.918 ^a	.843	.773	49.25537

a. Predictors: (Constant), Number of previous arrest, Number of arrests, unemployment rate, drug abuse cases recorded

The analysis in Table 3.7 shows that the multiple correlations between the independent variables (number of arrests, unemployment rate, drug abuse cases, and previous number of arrest) and the crime rate is 0.918, implying a strong linear relationship. The adjusted coefficient of determination R-Square is 0.773 indicating that 77.3% of the variation in the crime rates is explained by the number of arrests, unemployment rate, drug abuse cases, and previous number of arrest.

Table 3.8: ANOVA^a

Model	Sum of Squares	df	Mean Square	F	Sig.
1 Regression	116800.390	4	29200.098	12.036	.001 ^b
Residual	21834.824	9	2426.092		
Total	138635.214	13			

a. Dependent Variable: Crime Rate

b. Predictors: (Constant), Number of previous arrest, Number of arrests, unemployment rate, drug abuse cases recorded

Table 3.8 shows that the p-value of the ANOVA of this regression model (0.001) is less than 0.05 we therefore conclude that the model is significant and therefore fit for use

Table 3.9 Regression coefficient

Model	Unstandardized Coefficients		Standardized Coefficients	t	Sig.
	B	Std. Error	Beta		
1 (Constant)	-148.500	694.715		-.214	.836
Number of arrests	-.429	.224	-.448	-1.911	.088
unemployment rate	7.535	3.899	.298	1.932	.085
drug abuse cases recorded	2.808	.936	.651	2.999	.015
Number of previous arrest	-.518	.167	-.456	-3.096	.013

a. Dependent Variable: Crime Rate

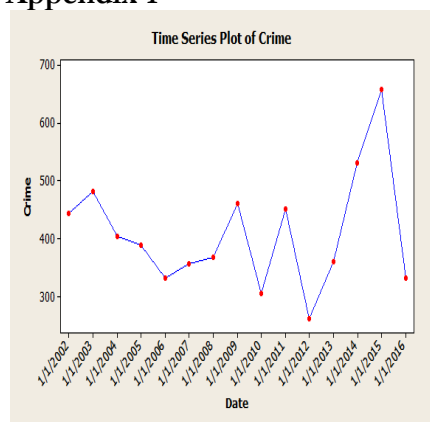
$$\text{Crime} = -148.5 - 0.429 * \text{Arrest} + 7.535 * \text{Unemp} + 2.808 * \text{Drug} - 0.518 * \text{Prev\_Arrest}$$

$y = -148.5 - 0.429 * x_1 + 7.535 * x_2 + 2.808 * x_3 - 0.518 * x_4$, where y is the total number of crime rate, x_1 is the Number of arrests, x_2 is the unemployment rate, x_3 is the drug abuse cases, and x_4 is the number of arrests in the previous year.

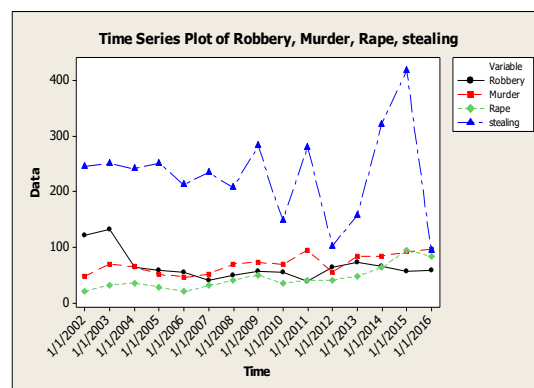
4 Discussion and conclusion

The p-value of the coefficients of x_3 and x_4 in the model (0.015, and 0.013 respectively) are less than 0.05. We therefore conclude that drug abuse, and arrests in the previous year have significant effect on crime. The signs of the coefficients indicate that drug abuse caused an increase in crime rate while previous arrests caused a decrease in crime rate.

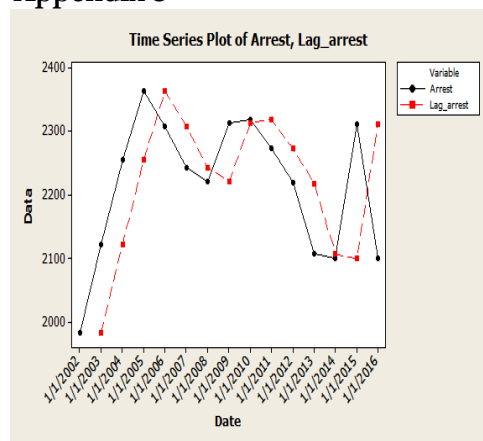
Appendix 1



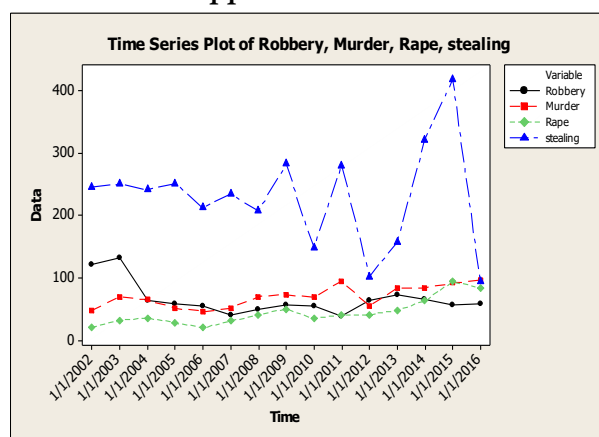
Appendix 2



Appendix 3

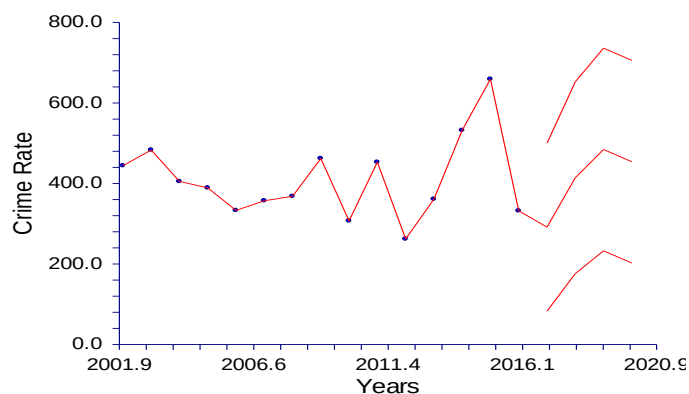


Appendix 4



Appendix 5

Crime Rate and Forecasts



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A stochastic model of asset pricing function with additive effects series for capital market prices

I.U. Amadi*, L.E. Ebakpa†, and P.A. Azor‡

Abstract

In this paper, a combination of two stochastic systems namely: Stochastic Differential Equation (SDE) and Stochastic Delay Differential Equation (SDDE) respectively were considered. A closed form analytical solutions were presented in details which determined asset values through additive effects series. The validity of the analytical solution were obtained and verified using initial stock prices; which were explicitly analysed using Kolmogorov-Smirnov and results showed that the two asset values do not come from a common distribution. Finally, the impressions of other key stock quantities were discussed and graphical results presented using MATLAB.

Keywords: asset value; Kolmogorov –Smirnov (KS); additive effects; SDE and SDDE.

1 Introduction

The investment of assets in businesses creates a room where the returns or profits would be used for the management of the daily cost of the running businesses in financial investments. So, an investor might potentially require more capital investment so as to bring about the improvement of the investment proceedings. Accordingly, financial resources are invested in essence to obtain more asset investments for the purpose of expansion and growth of the business. This create a room for improved unit productivities, creation of innovativeness for new products or bring in values to the investments and to look for modern technology that improves on productivity and minimizes cost and replace outdated assets. If capital assets are skipped, stock exchange will definitely be confronted with the challenges of getting off the ground. There has been progressive growth of stock market activities and performances in the financial market as investors earn incomes to have a living.

At the same time, the stock market is typically avowed for its high returns. Returns on investment for some capital market are approaches for aptly evaluating returns on an investment. Investors adopt these measures to evaluate or assess the significance of expenditure [1]. In the capital market, the fluctuations of stock prices are caused by changes of their values over time. As a result, it is absolutely essential to understand properly the type of physical quantities to model so that realistic results of the stock variables could be obtained; [2] looked at the stochastic model of price changes on the floor of stock market. Thus far, the equilibrium price and the market growth

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rate of assets were found out. [3] studied the stochastic analysis of stock market expected returns and growth-rates. The precise conditions for obtaining the drifts, volatilities, Growth-Rates of four different assets were looked at. [4] studied on the stochastic model of the changes or variations of stock market prices. Conditions for finding out the equilibrium price, adequate circumstances for dynamic stability and convergence to equilibrium of the growth rate of the value function of shares. Furthermore, [5] studied the unstable nature of stock market forces using proposed differential equation model. [6] looked at stochastic analysis of the behaviour of stock prices. Results show that the proposed model is efficient for the prediction of stock prices. [7] worked on the solution of differential equations and stochastic differential equations of time-varying investment returns; where precise conditions were obtained which governs stock return rates through multiplicative effect and multiplicative inverse trends series. [8] considered the stochastic model of some selected stocks in the Nigerian Stock Exchange (NSE), in this study, the drift and volatility coefficients for the stochastic differential equations were determined. [9] worked on the problem of stock price fluctuations using stochastic differential equations, principal component analysis and KS goodness of fit test were studied. Details of the analytical solutions were given; the computational and graphical results were presented and discussed respectively. [10] considered the stochastic analysis of Markov Chain in Finite state. The transition Matrix replicated the use of 3-State transition probability matrix which enables them to present precise condition of obtaining expected mean rate of returns of each stock.

This paper combined two stochastic systems such as: Stochastic Differential Equation (SDE) and Stochastic Delay Differential Equation (SDDE) respectively which gave precise conditions of determining asset values through additive effects series. The validity of the analytical solution was verified using initial stock prices. Hence, this paper compliments the works of [11] and [12] respectively by incorporating time dependent delay parameter to realistically assess asset pricing function with additive effects series for capital market price changes which have not been seen elsewhere.

The arrangement of this paper is set as follows: Section 2.1 presents the mathematical framework, Results and discussion are seen in Section 3.1 and paper is concluded in Section 4.1.

2 Mathematical framework

At this stage we present few intricate definitions touching this dynamic area of study, hence we have the following:

Definition 1: Probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω represents a set of sample space, $\mathcal{F}$ represents a collection of subsets of Ω , while $\mathbb{P}$ is the probability measure defined on each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ -algebra or σ -field such as $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the arbitrary unions and finite intersections. Hence it is called probability measure when the following condition holds:

$$(i) \quad P(A) \geq 0 \text{ for all } A \subset \Omega \quad (1)$$

$$(ii) \quad P(\Omega) = 1 \quad (2)$$

$$(iii) \quad A, B \subset \Omega, A \cap B = \emptyset \text{ then } P(A \cup B) = P(A) + P(B) \quad (3)$$

Definition 2: Normal Distribution: A normal distribution function is a peculiar distribution in probability theory and is usually used for modeling asset returns. A normal distribution is used in the Black-Scholes Partial differential equation to value European options. A normal distribution depends on two parameters.

Mean, $\mu \in \mathbb{R}$, is the expectation of a random variable normal distribution.

Variance, $\sigma^2 > 0$, deals with the magnitude of the spread from the mean.

In Black-Scholes formula, normal distributions are used. The cumulative distribution, usually denoted as $\Phi(X)$, is the probability that X will be equal to or less than X , expressed as $F_x(x) = P(X \leq x)$. A standard normal cumulative distribution function is defined as.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt. \quad (4)$$

A normal distribution is a symmetric distribution, which means that it touches around a vertical axis of symmetry. Obviously, there is a connection between any given points with same distance to the vertical axis. This relationship is defined in equation (5):

$$\phi(x) = 1 - \phi(-x). \quad (5)$$

Definition 3: A σ -algebra is a set F of subsets of Ω with the following axioms:

$$(i) \quad \emptyset, \Omega \in F. \quad (6)$$

$$(ii) \quad \text{If } A \in F, \text{ then } A^c \in F. \quad (7)$$

$$(iii) \quad \text{If } A_1, A_2, \dots \in F, \text{ then } \bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in F. \quad (8)$$

Clearly $A^c := \Omega - A$ is the complement of A .

Definition 4: If F is a σ -algebra in Ω , then Ω is called a measurable space and the members of F are called the measurable sets in Ω .

Definition 5: Let (Ω, M) be a measurable space. A map $\mu: M \rightarrow \mathbb{R} = [0, \infty) \cup \{\infty\}$ is called a measure provided that

$$(i) \quad \mu(\emptyset) = 0 \quad (9)$$

$$(ii) \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \quad (10)$$

Definition 6: Stochastic process: It can also be seen as a statistical event that evolves time in accordance to probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete.

Definition 7: A stochastic process whose finite dimensional probability distributions are all Gaussian (Normal distribution), where X_k are independent and identically distributed random variables.

Therefore μ is not known completely which is subject to environmental effects.

Therefore the stock dynamics governing this process can be written as :

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t), \quad (11)$$

where σ is the volatility, dZ is the Brownian motion which is random term, the stochastic term added to (12) gives (3.15) which makes it stochastic differential equation.

Definition 8: A Stochastic Differential Equation is a differential equation with stochastic term.

Therefore assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and $W(t) = (W_1(t), W_2(t), \dots, W_m(t))^T, t \geq 0$ an m -dimensional Brownian motion on the given probability space. We have SDE in coefficient functions of f and g as follows

$$dX(t) = f(t, X(t))dt + g(t, X(t))dZ(t), \quad 0 \leq t \leq T, \quad (12)$$

$$X(0) = x_0, \quad (13)$$

where $T > 0$, x_0 is an n -dimensional random variable and coefficient functions are in the form $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$. SDE can also be written in the form of integral as follows:

$$X(t) = x_0 + \int_0^t f(S, X(S))dS + \int_0^t g(S, X(S))dZ(S) \quad (14)$$

where dX, dZ are known as stochastic differentials. The $\mathbb{R}^n$ is a valued stochastic process $X(t)$ satisfying (11).

Theorem 1.1: let $T > 0$, be a given final time and assume that the coefficient functions $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ are continuous. Moreover, $\exists$ finite constant numbers λ and β such that $\forall t \in [0, T]$ and for all $x, y \in \mathbb{R}^n$, the drift and diffusion term satisfy

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq \lambda \|x - y\|, \quad (15)$$

$$\|f(t, x)\| + \|g(t, x)\| \leq \beta(1 + \|x\|). \quad (16)$$

Suppose also that x_0 is any $\mathbb{R}^n$ -valued random variable such that $E(\|x_0\|^2) < \infty$. then the above SDE has a unique solution X in the interval $[0, T]$. Moreover, it satisfies $E\left(\sup_{0 \leq t \leq T} \|X(t)\|^2\right) < \infty$. the proof of the theorem 1.1 is seen in [17].

Theorem 3.2:(Ito's lemma). Let $f(S, t)$ be a twice continuous differential function on $[0, \infty) \times A$ and let S_t denotes an Ito's process

$$dS_t = a_t dt + b_t dz(t), t \geq 0,$$

Applying Taylor series expansion of F gives:

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (dS_t)^2 + \text{higher order terms (h.o.t)},$$

So, ignoring h.o.t and substituting for dS_t we obtain

$$dF_t = \frac{\partial F}{\partial S_t} (a_t dt + b dz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (a_t dt + b dz(t))^2 \quad (17)$$

$$= \frac{\partial F}{\partial S_t} (a_t dt + b dz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b^2 dt, \quad (18)$$

$$= \left(\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b^2 \right) dt + \frac{\partial F}{\partial S_t} b_t dz(t) \quad (19)$$

More so, given the variable $S(t)$ denotes stock price, then following GBM implies (3.15) and hence, the function $F(S, t)$, Ito's lemma gives:

$$dF = \left(\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dz(t) \quad (20)$$

2.1 Mathematical formulation

On the verge of assessing asset value and other financial market quantities there must be a delay τ in such assessments in one way or the other. Therefore, incorporating delay term in a model gives more realistic and better model. Therefore delay term is added in the stochastic differential equation which gives :

$$dX_i(t) = (t - \tau) X_i(t) dt + \sigma X_1(t) dZ(t) \quad (21)$$

where μ is an expected rate of returns on stock, σ is the volatility of the stock, dt is the relative change in the price during the period of time and Z is a Wiener process. Following the method of [7] and [11] on rate of returns gives as:

$$R_t := (\lambda_1 + \lambda_2)^2, \dots, \quad (22)$$

where $t = 1, 2, \dots$

Using (22) in (21) gives the following system of equations which is SDDE and SDE respectively:

$$dX_2(t) = (t - \tau) (\theta_1 + \theta_2)^2 X_2(t) dt + \sigma X_2(t) dZ^{(2)}(t) \quad (23)$$

$$dX_4(t) = (\theta_1 + \theta_2)^2 X_4(t) dt + \sigma X_4(t) dZ^{(4)}(t), \quad (24)$$

where $X_2(t)$, and $X_4(t)$ are underlying stocks with the following initial conditions:

$$\left. \begin{aligned} X_2(0) &= X_0 \\ X_4(0) &= X_0 \end{aligned} \right\} \quad (25)$$

where, $X_2(t)$, $X_4(t)$, are asset prices, The expression dZ , which contains the randomness that is certainly a characteristic of asset prices is called a Wiener process or Brownian motion. λ_1 and λ_2

represents rate of returns of first and second investments respectively, $\lambda_1 \lambda_2$ is multiplicative effects, $\lambda_1 + \lambda_2$ is additive effects.

2.2 Method of solution

The model (23) and (24) is made up of a system of stochastic delay differential equations whose solutions are not trivial. We implement the methods of Ito's lemma in solving for $X_2(t)$, and $X_4(t)$

To grab this problem we note that we can forecast the future worth of the asset with sureness.

From (23) Let $f(X_2, t) = \ln X_2$ so differentiating partially gives

$$\frac{\partial f}{\partial X_2} = \frac{1}{X_2}, \quad \frac{\partial^2 f}{\partial X_2^2} = -\frac{1}{X_2^2}, \quad \frac{\partial f}{\partial t} = 0 \quad (26)$$

According to Ito's gives

$$df(X_2, t) = \sigma X_2 \frac{\partial f}{\partial X_2} dZ(t) + \left((\theta_1 + \theta_2)^2 X_2(t) \frac{\partial f}{\partial X_2} + \frac{1}{2} \sigma^2 X_2^2 \frac{\partial^2 f}{\partial X_2^2} + \frac{\partial f}{\partial t} \right) dt \quad (27)$$

Substituting (26) and (27) into (23) gives

$$\begin{aligned} &= \sigma X_2 \frac{1}{X_2} dZ(t) + \left((t - \tau)(\theta_1 + \theta_2)^2 X_2(t) \frac{1}{X_2} + \frac{1}{2} \sigma^2 X_2^2 \left(-\frac{1}{X_2^2} \right) + 0 \right) dt \\ &= \sigma dZ(t) + \left((t - \tau)(\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) dt. \end{aligned}$$

Integrating both sides, taking upper and lower limits gives

$$\begin{aligned} \int_0^t d \ln X_2 &= \int_0^t df(X_2, u) = (t - \tau) \int \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dZ(t) \\ \ln X_2 - \ln X_0 &= (t - \tau) \left((\theta_1 + \theta_2)^2 u - \frac{1}{2} \sigma^2 u \right) \Big|_0^t + (\sigma Z u) \Big|_0^t \\ \ln \left(\frac{X_2}{X_0} \right) &= (t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t) \end{aligned} \quad (28)$$

Taking the ln of the both sides

$$X_2(t) = X_0 \exp \left((t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t) \right) \quad (29)$$

From (24), let $f(X_4, t) = \ln X_4$ so differentiating partially gives

$$\frac{\partial f}{\partial X_4} = \frac{1}{X_4}, \quad \frac{\partial^2 f}{\partial X_4^2} = -\frac{1}{X_4^2}, \quad \frac{\partial f}{\partial t} = 0 \quad (30)$$

According to Ito's gives

$$df(X_4, t) = \sigma X_4 \frac{\partial f}{\partial X_4} dZ(t) + \left((\theta_1 + \theta_2)^2 X_4(t) \frac{\partial f}{\partial X_4} + \frac{1}{2} \sigma^2 X_4^2 \frac{\partial^2 f}{\partial X_4^2} + \frac{\partial f}{\partial t} \right) dt \quad (31)$$

Substituting (30) and (31) into (24) gives

$$\begin{aligned} &= \sigma X_4 \frac{1}{X_4} dZ(t) + \left((\theta_1 + \theta_2)^2 X_4(t) \frac{1}{X_4} + \frac{1}{2} \sigma^2 X_4^2 \left(-\frac{1}{X_4^2} \right) + 0 \right) dt \\ &= \sigma dZ(t) + \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) dt. \end{aligned}$$

Integrating both sides, taking upper and lower limits gives

$$\begin{aligned} \int_0^t d \ln X_4 &= \int_0^t df(X_4, u) = \int \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dZ(t) \\ \ln X_4 - \ln X_0 &= \left((\theta_1 + \theta_2)^2 u - \frac{1}{2} \sigma^2 u \right) \Big|_0^t + (\sigma Z u) \Big|_0^t \\ \ln \left(\frac{X_4}{X_0} \right) &= \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t) \end{aligned} \quad (32)$$

Taking the ln of the both sides

$$X_4(t) = X_0 \exp \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t). \quad (33)$$

Table 1: The impact of time delay in the assessment of Asset values with additive effects through the

solution below: $X_2(t) = X_0 \exp \left\{ (t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma dz(t) \right\}$ $t = 2, \quad dz = 1$

τ	S_0	σ	$(\theta_1 + \theta_2)$	$X_2(t)$	$(\theta_1 + \theta_2) X_2(t)$	
0.25	5.00	0.2	1.0000	194.3067	2.0000	7056346.005
	6.00	0.2	1.0000	233.1681	2.0000	8467615.206
	5.70	0.2	1.0000	221.5097	2.0000	8044234.446
0.35	5.20	0.2	1.0000	165.4483	2.0000	3297445.466
	4.60	0.2	1.0000	146.3581	2.0000	2916790.99
	4.00	0.2	1.0000	127.2679	2.0000	2536496.513
0.45	3.70	0.2	1.0000	96.3833	2.0000	1054242.249
	3.90	0.2	1.0000	101.5932	2.0000	1111228.317
	3.50	0.2	1.0000	91.1734	2.0000	997256.1817
0.55	3.30	0.2	1.0000	70.5942	2.0000	423770.8709
	3.75	0.2	1.0000	79.9783	2.0000	480102.9504
	4.00	0.2	1.0000	85.3102	2.0000	512109.8138

Table 1 shows the various levels of asset value increase as a result of delay parameter. This remark is realistic because as you delay in selling- off your landed properties, Estates, companies etc over time; will eventually yield more monies compared to when it is sold for today. That is why investors buy these assets and allow it for some time to appreciate in value before they are sold.

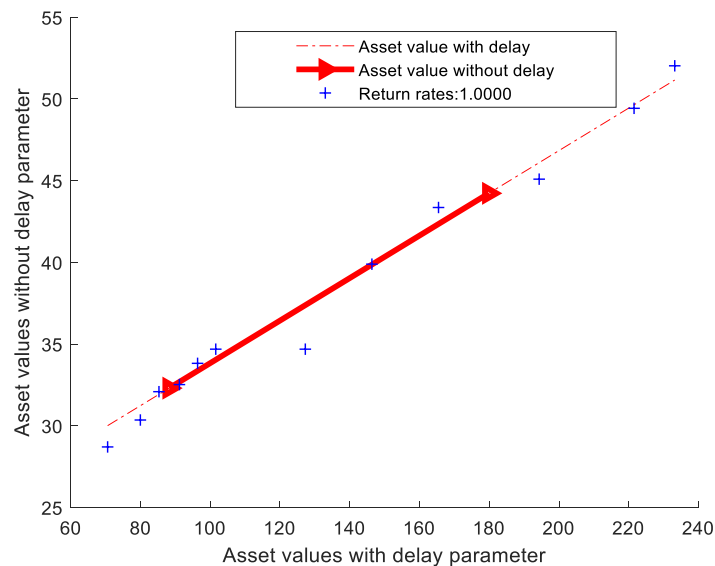


Figure 1: Graphical representation of two asset values (with and without delay) parameter which follows additive effects with:1.0000.

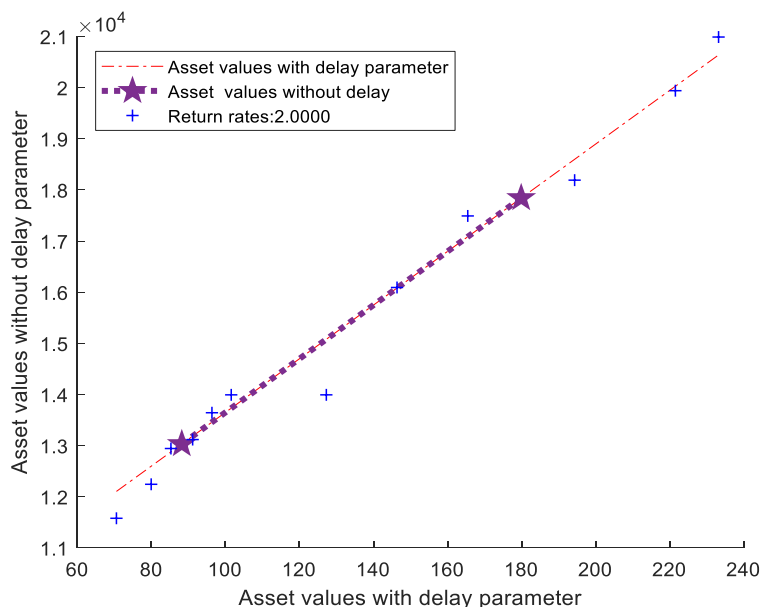


Figure 2: Graphical representation of two asset values (with and without delay) parameter which follows additive effects with:1.0000.

Figures 1 and 2 are quantile-quantile (q-q) of two asset values under considerations; it shows that two asset values do not come from a population with a common distribution. These results agree with our findings in Tables 1 and respectively using Kolmogorov –Smirnov (KS).

Table 2: The assessment of Asset values with additive effects through the solution below:

$$X_4(t) = X_0 \exp \left\{ \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma dz(t) \right\}, dz = 1 \text{ and } t = 2$$

S_0	σ	$(\theta_1 + \theta_2)$	$X_4(t)$	$(\theta_1 + \theta_2)$	$X_4(t)$
5.00	0.2	1.0000	43.3557	2.0000	17490.9330
6.00	0.2	1.0000	52.0268	2.0000	20989.1196
5.70	0.2	1.0000	49.4255	2.0000	19939.6636
5.20	0.2	1.0000	45.0899	2.0000	18190.5703
4.60	0.2	1.0000	39.8872	2.0000	16091.6584
4.00	0.2	1.0000	34.6846	2.0000	13992.7464
3.70	0.2	1.0000	32.0832	2.0000	12943.2904
3.90	0.2	1.0000	33.8174	2.0000	13642.9278
3.50	0.2	1.0000	30.3490	2.0000	12243.6531
3.31	0.2	1.0000	28.7015	2.0000	11578.9977
3.75	0.2	1.0000	32.5168	2.0000	13118.1998
4.00	0.2	1.0000	34.6846	2.0000	13992.7464

Table 2 displays asset values earned without delay. There is no much profit here because goods and services are sold as they arrived without minding the future outcome; hence asset value is a function of time and money. This remark informs investors or traders on how effectively they can grow more wealth in their investments.

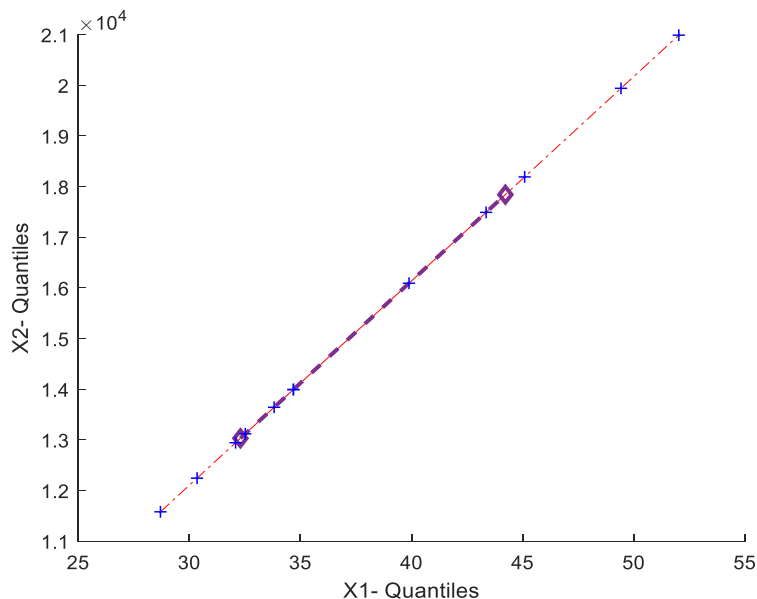


Figure 3: Graphical representation of two asset values (with and without delay) parameter which follows additive effects .

H_0 : The asset value functions $X_2(t)$ and $X_4(t)$ comes from a common distribution

H_1 : They are not from a common distribution

Table 3: Goodness of fit test for assessment of two asset values with and without delay when its return rates is:1.0000.

$(\theta_1\theta_2)$	$X_4(t)$	$X_2(t)$	Mean	STD	KSStat	P-value	Decision
1.0000	43.3557	194.3067	118.8312	106.7385	1.0000	0.2890	Accept
1.0000	52.0268	233.1681	142.5975	128.0862	1.0000	0.2890	Accept
1.0000	49.4255	221.5097	135.4676	121.6819	1.0000	0.2890	Accept
1.0000	45.0899	165.4483	105.2691	85.1062	1.0000	0.2890	Accept
1.0000	39.8872	146.3581	93.1227	75.2863	1.0000	0.2890	Accept
1.0000	34.6846	127.2679	80.9762	65.4663	1.0000	0.2890	Accept
1.0000	32.0832	96.3833	64.2332	45.4670	1.0000	0.2890	Accept
1.0000	33.8174	101.5932	67.7053	47.9247	1.0000	0.2890	Accept
1.0000	30.3490	91.1734	60.7612	43.0093	1.0000	0.2890	Accept
1.0000	28.7015	70.5942	49.6478	29.6226	1.0000	0.2890	Accept
1.0000	32.5168	79.9783	56.2476	33.5603	1.0000	0.2890	Accept
1.0000	34.6846	85.3102	59.9974	35.7977	1.0000	0.2890	Accept

Following Tables 3 and 4 the alternative hypothesis of KS was accepted because it is the hypothesis we are will to accept if we reject the null hypothesis hence the p-values are not significant. The level of significance $\alpha = 0.01$ shows that the two asset values are not from a common distribution which

has financial meaning as it affects asset returns; with these traders or investors are properly guided in terms of decision making. Clearly, we can boldly say that there is significant dissimilarity among the two asset values $(X_2(t), X_4(t))$ throughout the trading days. In another dimension, the mean rates which can be seen in columns 5 of Tables 3 and 4 respectively defines the average levels of asset values when additive rate of return is 1.0000 and 2.0000. The standard deviation measures the degree of each level of changes in respect to asset values in financial markets.

Table 4: Goodness of fit test for assessment of two asset values with and without delay when it return rates is: 2.0000 .

(θ_1, θ_2)	$X_4(t)$	$X_2(t)$	Mean	STD	KSStat	P-value	Decision
2.0000	17490.9330	7056346.005	3.3369e+06	4.9772e+06	1.0000	0.2890	Accept
2.0000	20989.1196	8467615.206	4.2443e+06	5.9727e+06	1.0000	0.2890	Accept
2.0000	19939.6636	8044234.446	4.0321e+06	5.6740e+06	1.0000	0.2890	Accept
2.0000	18190.5703	3297445.466	1.6578e+06	2.3188e+06	1.0000	0.2890	Accept
2.0000	16091.6584	2916790.99	1.1664e+06	2.0511e+06	1.0000	0.2890	Accept
2.0000	13992.7464	2536496.513	1.2752e+06	1.7837e+06	1.0000	0.2890	Accept
2.0000	12943.2904	1054242.249	5.3359e+05	7.3631e+05	1.0000	0.2890	Accept
2.0000	13642.9278	1111228.317	5.6244e+05	7.7611e+05	1.0000	0.2890	Accept
2.0000	12243.6531	997256.1817	5.0475e+05	6.9651e+05	1.0000	0.2890	Accept
2.0000	11578.9977	423770.8709	2.1767e+05	2.9146e+05	1.0000	0.2890	Accept
2.0000	13118.1998	480102.9504	2.4661e+05	3.3021e+05	1.0000	0.2890	Accept
2.0000	13992.7464	512109.8138	2.6305e+05	3.5222e+05	1.0000	0.2890	Accept

3 Conclusion

Generally, an assessment of asset values and its return rates are very vital in every capital investment. The non-existence of assessment of asset values may have a hard time crumbling which does not benefit any trader or investor in a time varying investments. Therefore , this paper studied two stochastic systems which results showed as follows: (i) : increase in delay parameter dominantly increases the value of asset (ii) the solution of SDDE has more impact over solution of SDE in terms of asset value estimates, (iii) higher initial stock price has substantial effects on the value of asset over time (iv) a higher asset value also have a higher level of asset changes according to standard deviation rates(v) the two asset values under-consideration do not come from a common distribution . More so, comparing the existence and uniqueness of the models will be an interesting study to explore.

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Yoruba language text-based hate detection model for social media

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Abstract

There is an emerging problem of hate speech sharing and spreading associated with Nigerian social media. This work was targeted to develop classification model for text based hate speech in Yoruba language. Support Vector Machine (SVM) was used on crowd sourced datasets comprising 1,420 Yoruba extracted from social media platforms. Annotated 71% Yoruba text datasets were used to train the SVM classifier. The data sets were preprocessed using the Python libraries. The technologies used for the front end are HTML version 5.0, CSS version 3.0, Java script, jQuery, and Bootstrap. The back end involved the use of Laravel version 9.0, a PHP framework, and MySQL. 1008 and 412 datasets were used to train and test the Yoruba SVM classifier, respectively. The confusion matrix indicated that the trained SVM Classifier for Yoruba was able to identify Yoruba hate speech. Accuracy, Precision, Recall, F1-Score and AUC are 75.49%, 75.21%, 0.0954, 0.8410, and 0.60, respectively for the Yoruba SVM Classifier. The model performance compared with other existing models showed that for Yoruba database; Naïve Bayes gave 69.09%, accuracy, and 72.64% precision. Logistic Regression gave 66.51% accuracy, and 68.89% precision. KNN gave 75.52% accuracy, and 86.91% precision. Random Forest gave 54.88% accuracy, and 50.65% precision.

Keywords: classification model; confusion metrics; hate speech; support vector machine; yoruba language

1 Introduction

The misuse of social media is on the rise, with hate speeches and abusive language spreading on social media and it is assuming a dangerous and alarming dimension in Nigeria. Hate speech is now ubiquitous as a result of social media, which has encouraged the spreading and sharing of hate speech and is gradually becoming part of our society.

There are many definitions of hate speech from Facebook, Twitter, (Davidson *et al.*, 2017; Fatuna and Nunes, 2018 and Bhardwaj *et al.*, 2020). Common to these and other definitions of hate speech is that it is directed at an individual or group and that it is based on some aspect of the individual or group identity.

To solve this problem, we worked on artificial intelligence that can identify hate speech. The model is made up of supervised learning algorithms trained on capturing different aspects of Yoruba

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hate speech. Notably it has a deep learning layer that converts a social media post's language into numerical value and accesses this sequence. In this way, the social media post content can be understood by SVM Classifier which helps it decide whether the post is hateful or not. Some of the past approaches with promising results for detecting hate speech in textual content in indigenous languages (Gaikwad *et al.*, 2021; Matamoros-Fernandez and Farkas, 2021 and Saini, 2020) but were not on text based Yoruba hate speech detection using support vector machine (SVM).

2 The classifier

Support Vector Machine (SVM) is a supervised learning algorithm primarily used for classification problems in machine learning. It creates decision boundaries or hyperplane which segregates dimensional space into classes. The hyperplane classifies the comments into either hate or not hate being a binary classification. Hyperplane is a dot in a one dimensional space (Figure 2.0), a land in two dimensional space (Figure 3.0) and a plane in three dimensional space (Figure 4.0). SVM helps to choose the best separating hyperplane with maximum margin Support Vector determines the hyperplane to choose.

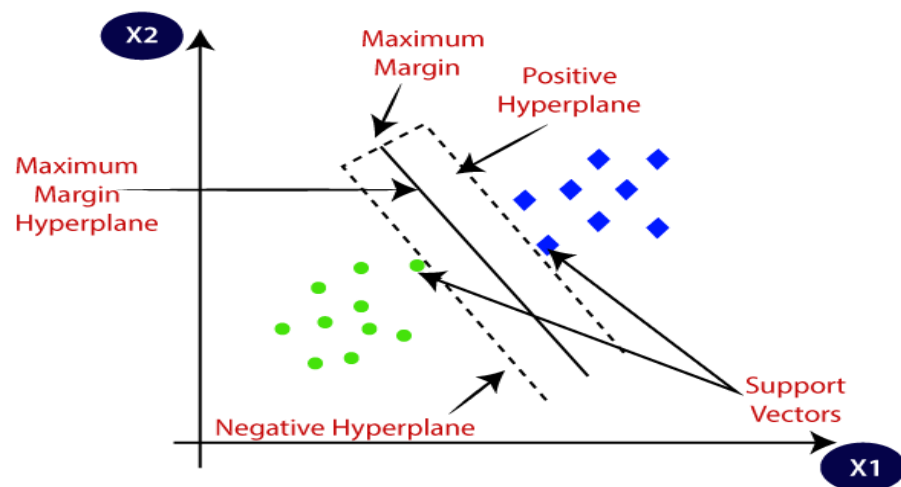


Figure 1: Maximum Margin in SVM (Saini, 2020)

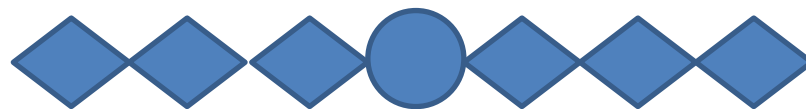


Figure 2: Hyperplane as a dot in one dimensional space

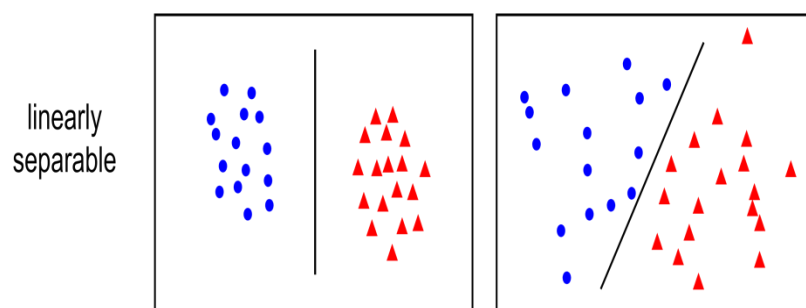


Figure 3: Hyperplane as a line in two dimensional space

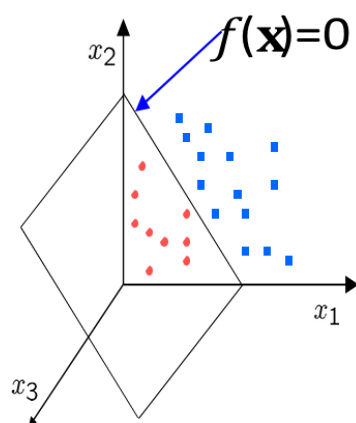


Figure 4: Hyperplane is a plane in three dimensional space

Support Vectors are the data points that lie closest to the hyperplane. They have decision on the linear boundary.

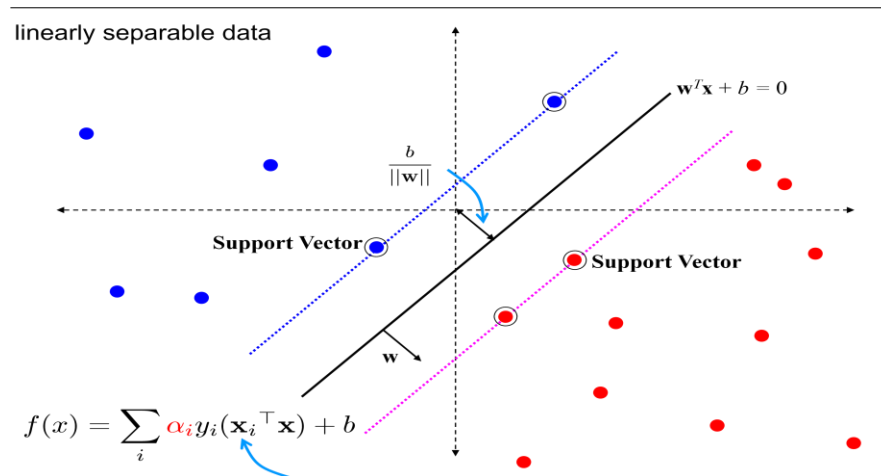


Figure 5: Support Vectors (SVM JavatPoint)

SVM Predicts Outputs

- During inference time, SVM predicts the output Y with input X
 - The inputs are set of features x for each sample
 - W is the weight vector for each feature
- Only support vector determines the weight
- Only w is needed to classify new data

Through linear combination of input and weight, output Y is predicted

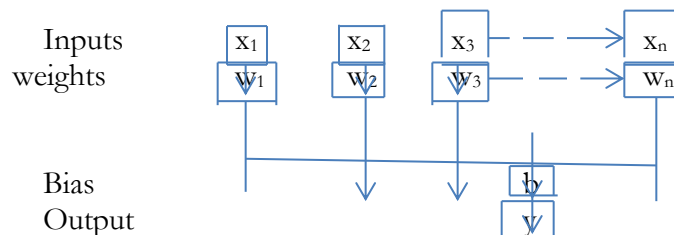


Figure 6: SVM Architecture

Linear Classifier

Given training data (x_i, y_i) for $i = 1, 2, \dots, N$, with $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$,
linear classifier $f(x)$

such that

$$f(x_i) \geq 0 \quad y_i = +1, \text{ and}$$

$$f(x_i) < 0 \quad y_i = -1$$

- The linear classifier has the form

$$f(x) = w^T x + b$$

- W is known as the weight vector while the b is the bias

- Only w is needed for classifying the unlabeled comments or test data
- The training data is used to learn w and then discarded
 - For a new or unlabeled comments, vectorized unlabeled comments would be applied to the classifier equation such that if
 - $w^T x_i + b \geq 1$ it is classified as positive (hate)
 - $w^T x_i + b \leq -1$ it is classified as negative (not hate)

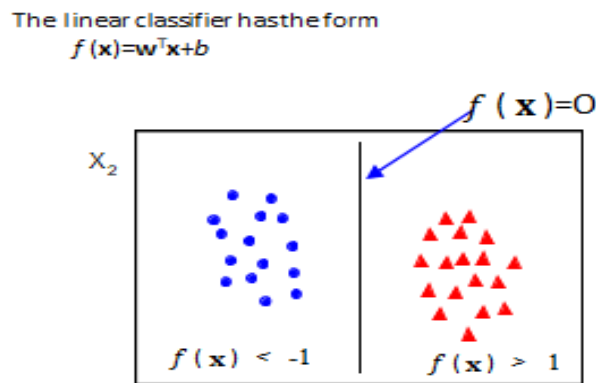


Figure 7.0: SVM Classifier in Binary Classification

$w^T x_i + b \geq 1$ it is classified as positive (hate)
 $w^T x_i + b \leq -1$ it is classified as negative (not hate)

3 Methodology

3.1 Data sources and collection

Yoruba text comments posted by social media users in response to high conversation on Facebook and twitter social media were the data set source. Yoruba text comments were collected using web scraping in python with twint library and imported into CSV file from Facebook and twitter. The publicly available crowd sourced access was created to be able to copy the content needed by creating secured connection with the source site (my page on face book) where contents were fetched into the document object module (DOM). Value dash pair was used to sort the content of the DOM and data of interest were selected and stored in CSV file. The data set were divided into training and test of data sets using resembling statistical method which is available in scikit-learn.

3.2 Data annotation

The Yoruba text data set (71%) were uploaded and distributed via crowd flower into a dedicated web site created from acquired domain for three annotators to manually tag the text comments into either hate or not hate (binary classification). The manually tagged text comments became the training data set used to train the support vector machine (SVM) and optimize the model

parameters. The other part that was not tagged was used as test data set 29% for Yoruba language. Every document which contained the text data set was classified into either hate or not-hate.

3.3 Data cleaning and preprocessing

Preprocessing of input text can play a key role in text classification by reducing dimensionality and removing unnecessary content (Husain and Uzuner, 2022). Different Preprocessing techniques using Natural language processing (NLP), which is a python library that provides modules for processing text comments in other to filter noisy and non-informative features from the Yoruba text comments. The following were carried out: Removals, replacement, translation, tokenization, lemmatization, stemming, vectorization, data scaling and kernel selection.

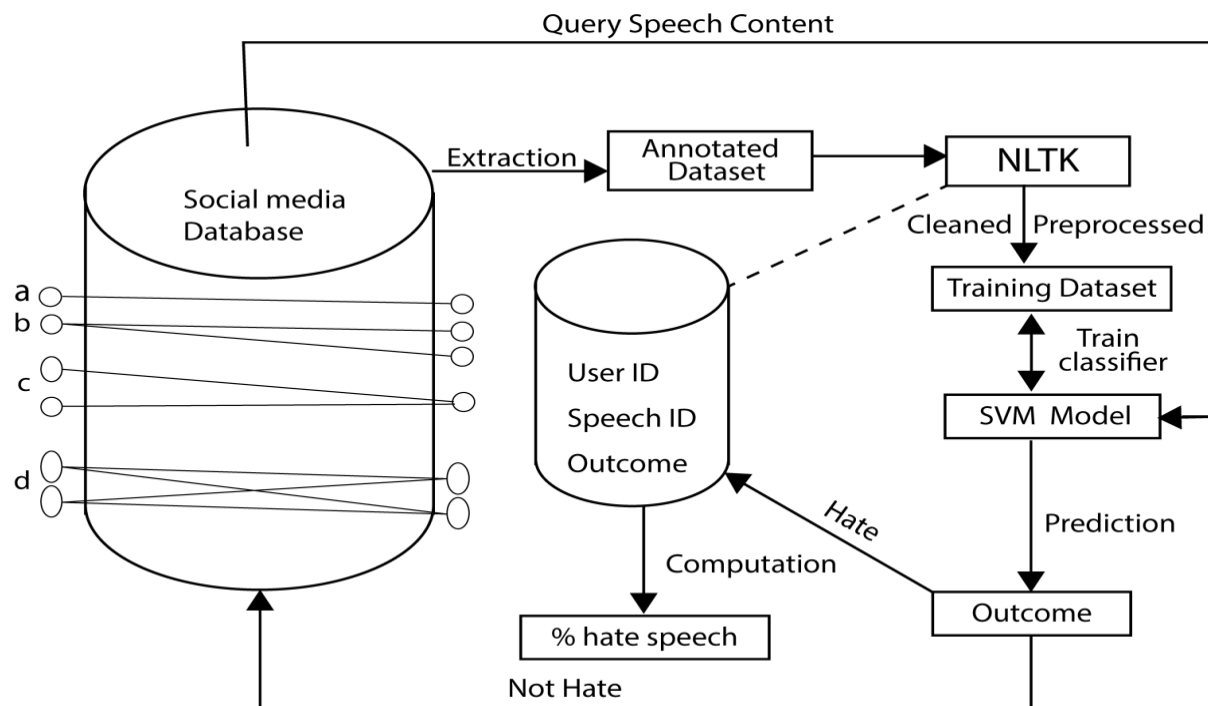


Figure 8.: SVM Classifier for detection of Nigerian indigenous language hate speech in Nigeria Social Media

4 Results and discussion

The data set were divided into training (71%) and test (29%) data sets. 732 and 277 data sets were used to train hate and not hate, respectively as shown in Table 1.0 respectively. The data set that was not tagged by annotators (29%) made up of 280 hate dataset and 132 not hate dataset were used as test the SVM Classifier for Yoruba language.

A total of 355 data sets were predicted hate out of which 267 data sets were true hate (true positive) and 88 data sets are not hate but predicted hate (false positive) as shown in Table 2. 13 hate data sets were predicted not hate (false negative) while 44 not hate data sets were predicted not hate (true negative) (Table 2)

Table 1: Yoruba language text Data set Distribution

Dataset	Hate	Not hate	Row Total
Training	732	277	1009
Test	280	132	412
Column Total	1012	409	1420

Table 2: Yoruba Language Data Set Predicted.

Test Data Labels	SVM Predicted Hate	SVM Predicted Not Hate	Row Total
Hate	267(tp)	13(fn)	280
Not Hate	88 (fp)	44(tn)	132
Colum Total	355	57	412

Yoruba Language Data Set Confusion Matrix

$$\begin{bmatrix} 267 & 13 \\ 88 & 44 \end{bmatrix}$$

$$\% \text{ Trained Yoruba dataset} = \frac{1008}{1420} \times 100 = 71\%$$

$$\% \text{ Test Yoruba dataset} = \frac{412}{1420} \times 100 = 29\%.$$

Keys:

p = positive (hate)

n = negative (not hate)

tp = true positive (positive and predicted positive)

tn = true negative (negative and predicted negative)

fp = false positive (negative but predicted positive, false alarm, type II error or overestimation)

fn = false negative (positive but predicted negative, missed, type I error or underestimation).

Based on the values in the SVM binary classification confusion matrix the following measurements were carried out. Precision, which calculates total accurate prediction as the percentage of the total instances i.e. % of positives predictions that are correct.

$$\text{Yoruba SVM model precision} = \frac{tp}{tp+fr} = \frac{267}{269+88} = 0.7521,$$

$$\% \text{ of predictions that are correct} = \text{accuracy} = \frac{tp}{tp+tn+fn+fp}$$

$$\text{Yoruba SVM model accuracy} = \frac{267+44}{267+13+14+88} = \frac{311}{412} = 0.7549.$$

Accuracy and precision are not enough to determine if the SVM model is correct. Consequently, F1 score, which is the harmonic mean area of precision, was calculated. In order to calculate the harmonic mean of precision, recall, which is another common metric for classification problem was measured. Recall is also known as sensitivity, it is % of positive cases that were predicted as positive.

$$\text{Recall} = \frac{tp}{tp+fn}.$$

$$\text{SVM model recall for Yoruba language} = \frac{267}{267+13} = \frac{267}{280} = 0.9536.$$

$$F_1\text{-score} = 2 \cdot \frac{\text{recall} \cdot \text{precision}}{\text{recall} + \text{precision}}$$

$$\text{SVM model } F_1\text{-score for Yoruba} = 2 \cdot \frac{0.7172}{1.7057} = 0.8410.$$

The F_1 -score gives a better and unbiased representation of the performance measurement than accuracy measurement even for unbalanced data.

$$\text{Misclassification rate of SVM Classifier for Yoruba Language} = \frac{fp+fn}{tp+tn+fp+fn} = 0.23$$

$$\text{True positive rate (TPR) of SVM Classifier for Yoruba Language} = \frac{tp}{p} = 1 - \text{FNR} = 0.75$$

$$\text{True negative rate (TNR) of SVM Classifier for Yoruba Language} = \frac{tn}{n} = 1 - \text{FPR} = 0.77$$

$$\text{False positive rate (FPR) of SVM Classifier for Yoruba Language} = \frac{fp}{fp+tn} = 1 - \text{TNR} = 0.67$$

$$\text{False negative rate (FNR) of SVM Classifier for Yoruba Language} = \frac{fn}{fn+tp} = 1 - \text{TPR} = 0.05$$

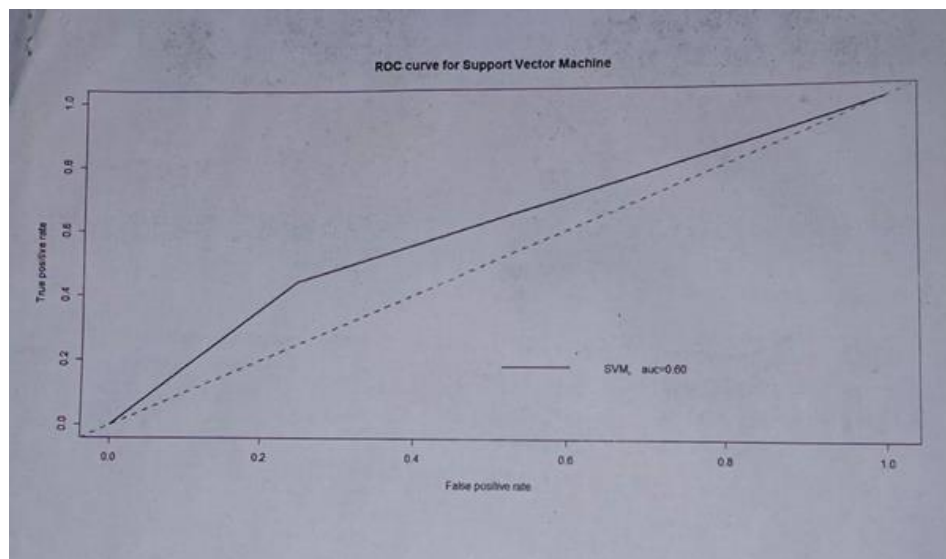


Figure 9: ROC Curve for SVM Classifier for Yoruba

Area under curve (AUC) represents the degree of separability. It tells how much the model is capable of distinguishing between hate and not hate. AUC uses the true positive rates and false positive rates of the models and makes it better in assessment than accuracy. The value of AUC of 0.60 was observed for Yoruba SVM Model (Figure 9.0).

The value of AUC of Yoruba means that the Yoruba model is discriminatory and as such was able to detect Yoruba hate speech. The machine learning classifiers for Yoruba language performed well as shown in Figure 10.0 and other statistical indices.

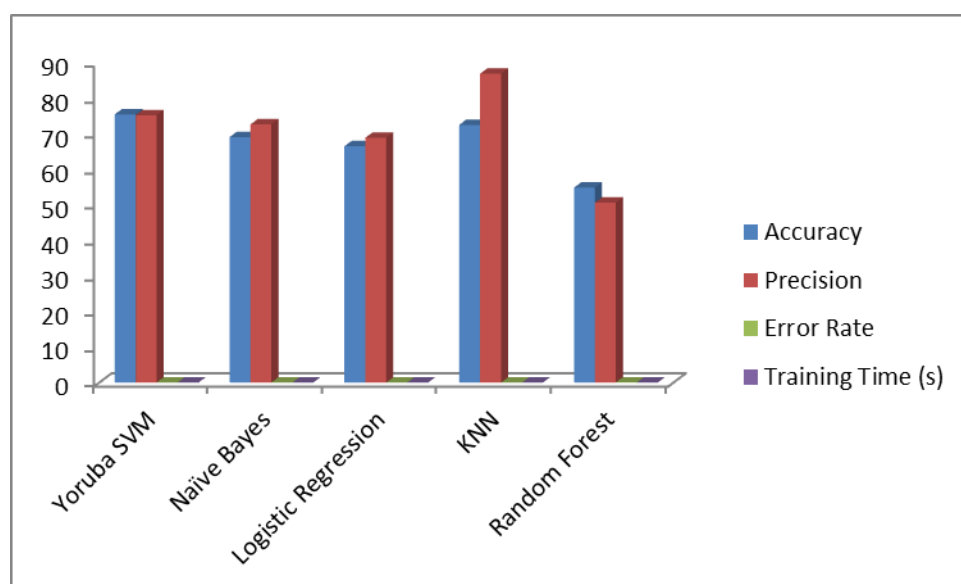


Figure 10: Yoruba SVM Classifier against others

5 Conclusion

Support Vector Machine (SVM) classifier developed was able to detect hate speeches in Yoruba text data set. The SVM classifier displayed better performance compared with other classifiers. The deployment of the model as backend could help alleviate or reduce the incidence of hate speech on Nigeria social media in Yoruba languages. Error analysis can be carried out to examine what could have been responsible for wrong prediction of some data sets by the SVM classifier.

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Solution of stochastic delay differential equation on asset value changes for capital market price

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Abstract

An asset values in trading business creates opportunity where returns or profits would be used at every periods. As a result, a stochastic model of asset value function is considered only in periodic events. Close form analytical solutions were obtained which gave precise measures and conditions of generating asset values through additive effects series. The effects of time delay for all periods were critically examined. Normality tests were conducted to show the impact analysis of initial stock prices and asset values that followed some probability distributions. To this end, the Tables, graphs and other stock quantities were discussed for the purpose of investment plans as it affects asset value functions.

Keywords: asset value; normality test; additive effects; SDE and SDDE

1 Introduction

Mathematical models are produced from equations which stock price models cannot be omitted as a result of multiple application in our day-to-day business transactions. These applications consist of banking and finance, accountancy, quantitative finance, etc. In several areas in science and engineering, the design, correct evaluation, and analysis of system exposed to practical or lifelike condition should consider the prospect of “white noise” random forces that must influence the system or inaccurate assessments or analysis in the system. Uncertainty is integral to the mathematical formulation of several circumstances or situations like stock market fluctuations, noise in population systems, irregular fluctuations or communication networks in observed signals. The economic crunch in our nations are resultant effect of stock price fluctuations, as a result of randomness pertaining to stock market or exchange, investors government, policy makers, etc, are unable to give a precise prediction of the future.

Stock price fluctuations create fear and as a result, people resort to criminality so as to meet individual needs, and snowballs to buying out of anxiety in fact, financial analysis who invest in financial market are typically clueless of the behaviour of stock market; hence they go through this stock trading problem, and also faces the challenge of not knowing the type of stocks to be bought and sold for profit maximization. Hence relevant information on regular basis are required or needed by both financial analysis and potential investors for the prediction of stock price behaviour.

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The unstable characteristic and other significant influences like liquidity on stock returns, because the abrupt changes in share prices is erratic and happens regularly. So, in order to help investors and owners of corporations take decisions on the level of their investment in stock market [1], researchers are curious and fascinated in studying the behaviour of the unstable market variables.

Nonetheless, the price evolution of ab risky assets are generally modelled as a path or track of a risky assets that are generally of a diffusion process defined on several basic probability space, with the Geometric Browner motion, the main tool used as the established reference model, [2] Many researcher have modelled stock market prices with several ways obtained results. For example,[3] studied the unstable feature of stock market forces, making use of suggested differential equation model. In the research of [4], stability analysis of stochastic model of price change at the floor of a stock market was considered and precise conditions were obtained which determined the equilibrium price and growth rate of asset shares. Stochastic analysis of the behaviour of stock prices was studied by [1], and results showed that the proposed model was efficient for predicting stock prices. Similarly, [5] considered the stochastic model of some selected stocks in the Nigerian Stock Exchange (NSE), in this research, the drift and volatility coefficients for the stochastic differential equations were obtained and the Euler-Maruyama method for system of SDEs was utilized to invigorate the stock prices [6], developed the geometric Brownian motion and looked at the exactness of the model with comprehensive analysis of assumed data.

The work in [7] studied the stochastic modelling of stock prices applying a method of Brownian motion model to explain the stock price time series. Result from the study revealed that as long as a model based upon the white noise is fitted to the market values, the two interpretations will give different estimates of the parameters, but identical values pertaining to the predicted stock prices. Nevertheless [2] studied stochastic model of the fluctuation of stock market price. Conditions for finding out the equilibrium price, adequate conditions for robust stability and convergence to equilibrium of the growth rate of the value function of shares. However, [8] looked at a stochastic model of price changes at the floor of stock market. In the work of [9], the equilibrium price and the market growth rate of shares we determined. See [10-14] for other articles on stock prices.

Past researches for example, [5] examined a stochastic model of several selected stocks in the Nigerian Stock Exchange (NSE) where the Euler-Maruyam method for system of (SDE) was utilized to invigorate the stock prices and result revealed that stock1 yields the best returns on investment compared to stock2, stock3 and stock4. The merit of this research over [5] is that the present research models the effect of growth-rates on stock market price estimation or forecast regarding to volatility and the drift.

Aforementioned studies have consequently considered similar problems of SDE. For instance, [22],[20] and [23] have only focused on Growth rate of assets, stability of stock prices, asset value and its return rates In all, none has provided any asset values with time dependent delay, with variations of initial closing stock prices, to guarantee the consistency of such solutions and goodness of test to show some levels of probability distributions.

In this study, we considered SDE with time delay parameter in the model which gave three different solutions with aid of initial conditions The effects of time delay for all asset values were critically examined. Normality tests were conducted to show the impact analysis of initial stock prices and asset values that followed some probability distributions. Furthermore, the Tables, graphs and other stock quantities were discussed.

To the best of our understanding this is the first study that has one model equation with three initial conditions (initial closed stock prices) to produce three investment analytical solutions to realistically assess the dynamics of asset values as a result of delay parameter.

The organization of this paper is set as follows: Section 2.1 presents the preamble, Results and discussion are seen in Section 3.1 and paper is concluded in Section 4.1.

2 Preamble

For us to adequately understand stochastic processes, we present few intricate definitions on probability spaces rudiment of this dynamic area of study:

Definition 1: Probability space is a triple $(\Omega, \mathcal{F}, \tilde{A})$ where Ω represents a set of sample space, $\mathcal{F}$ represents a collection of subsets of Ω , while $\tilde{A}$ is the probability measure defined on each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ -algebra or σ -field such as $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the arbitrary unions and finite intersections. Hence it is called probability measure when the following condition holds.

- (i) $P(A) \geq 0$ for all $A \subset \Omega$
- (ii) $P(\Omega) = 1$
- (iii) $A, B \subset \Omega, A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

Definition 2: Normal Distribution: A normal distribution function is a peculiar distribution in probability theory and is usually used for modeling asset returns. A normal distribution is used in the Black-Scholes Partial differential equation to value European options. A normal distribution depends on two parameters, Westergren and Rade (2003).

Mean, $\mu \in \mathbb{R}$, is the expectation of a random variable normal distribution.

Variance, $\sigma^2 > 0$, deals with the magnitude of the spread from the mean.

The cumulative distribution, usually denoted as $\phi(X)$, is the probability that X will be equal to or less than X , expressed as $F_x(x) = P(X \leq x)$. A standard normal cumulative distribution function is defined as.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt.$$

A normal distribution is a symmetric distribution, which means that it touches around a vertical axis of symmetry. Obviously, there is a connection between any given points with same distance to the vertical axis.

Definition 3: A σ -algebra is a set $\mathcal{F}$ of subsets of Ω with the following axioms:

- (i) $\emptyset, \Omega \in \mathcal{F}$
- (ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
- (iii) If $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in \mathcal{F}$

Clearly $A^c := \Omega - A$ is the complement of A .

Definition 4: If $\mathcal{F}$ is a σ -algebra in Ω , then Ω is called a measurable space and the members of $\mathcal{F}$ are called the measurable sets in Ω [16].

Definition 5: Let $(\Omega, \mathcal{M})$ be a measurable space. A map $\mu: \mathcal{M} \rightarrow \mathbb{R} = [0, \infty) \cup \{\infty\}$ is called a measure provided that

- (i) $\mu(\emptyset) = 0$
- (ii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Definition 6: Stochastic process: A stochastic process $X(t)$ is a relations of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each t in the index set T , $X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t . In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

Definition 7: A stochastic process whose finite dimensional probability distributions are all Gaussian.(Normal distribution).

Definition 8: Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point $x = a$ to a point $x = b$. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k=1}^n X_k \quad (1)$$

where X_k are independent and identically distributed random variables.

Definition 9: (Differential Equation): is an equation which has functions and their derivatives. In reality the functions is associated to real quantities whereas the derivatives denotes rate of change. Example of differential equation is follows

$$\frac{dS(t)}{dt} = \mu S(t) \quad (2)$$

$$S(0) = S_0 \quad (3)$$

where $S(t)$ represent asset price, μ rate of return, $\frac{dS(t)}{dt}$ is the rate of change of asset price and S_0 is the initial stock price; (2) and (3) can be obtained using variable separable which gives:

$$S(t) = S_0 e^{\mu t}. \quad (4)$$

Therefore μ is not known completely which is subject to environmental effects. Therefore (2) can be written as

$$dS(t) = \mu S(t) dt + \sigma S(t) dZ(t) \quad (5)$$

where σ is the volatility, dZ is the Brownian motion or Wiener's process which is random term, the stochastic term added to (2) gives (5) which makes it stochastic differential equation.

Definition 10: A Stochastic Differential Equation is a differential equation with stochastic term. Therefore assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and $W(t) = (W_1(t), W_2(t), \dots, W_m(t))^T, t \geq 0$ an m -dimensional Brownian motion on the given probability space. We have SDE in coefficient functions of f and g as follows

$$dX(t) = f(t, X(t))dt + g(t, X(t))dZ(t), 0 \leq t \leq T, \quad (6)$$

$$X(0) = x_0, \quad (7)$$

where $T > 0$, x_0 is an n -dimensional random variable and coefficient functions are in the form $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$. SDE can also be written in the form of integral as follows:

$$X(t) = x_0 + \int_0^t f(S, X(S))dS + \int_0^t g(S, X(S))dZ(S) \quad (8)$$

where dX, dZ are terms known as stochastic differentials. The $\mathbb{R}^n$ is a valued stochastic process $X(t)$ satisfying (6).

Theorem 1.1: let $T > 0$, be a given final time and assume that the coefficient functions $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ are continuous. Moreover, $\exists$ finite constant numbers λ and β such that $\forall t \in [0, T]$ and for all $x, y \in \mathbb{R}^n$, the drift and diffusion term satisfy

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq \lambda \|x - y\|, \quad (9)$$

$$\|f(t, x)\| + \|g(t, x)\| \leq \beta(1 + \|x\|). \quad (10)$$

Suppose also that x_0 is any $\mathbb{R}^n$ -valued random variable such that $E(\|x_0\|^2) < \infty$. then the above SDE has a unique solution X in the interval $[0, T]$. Moreover, it satisfies $E\left(\sup_{0 \leq t \leq T} \|X(t)\|^2\right) < \infty$. the proof of the theorem 1.1 is seen in [17].

Theorem 3.2:(Ito's lemma). Let $f(S, t)$ be a twice continuous differential function on $[0, \infty) \times \mathbb{A}$ and let S_t denotes an Ito's process

$$dS_t = a_t dt + b_t dz(t), t \geq 0.$$

Applying Taylor series expansion of F gives:

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (dS_t)^2 + \text{higher order terms (h.o, t)}.$$

So, ignoring h.o.t and substituting for dS_t we obtain

$$dF_t = \frac{\partial F}{\partial S_t} (a_t dt + b dz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (a_t dt + b dz(t))^2 \quad (11)$$

$$= \frac{\partial F}{\partial S_t} (a_t dt + b dz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b^2 dt, \quad (12)$$

$$= \left(\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} b^2 \right) dt + \frac{\partial F}{\partial S_t} b_t dz(t). \quad (13)$$

More so, given the variable $S(t)$ denotes stock price, then following GBM implies (5) and hence, the function $F(S, t)$, Ito's lemma gives:

$$dF = \left(\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dz(t) \quad (14)$$

3 Mathematical formulation of the problem

The stochastic analysis on the value valuation of asset value with time delay and additive effects as stock rate of returns is considered. The positive influence of time delay on the value of asset is equally considered. The volatility dynamics of stock prices was taken to be constant throughout the trading days. The initial stock price which is assumed to follow different trend series was categorized. The entire origin of stock dynamics is found in a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite time investment horizon $T > 0$. Therefore, we have the following stochastic delay differential equation below;

$$dX_i(t) = (t - \tau) X_i(t) dt + \sigma X_i(t) dZ(t) \quad (15)$$

where μ is an expected rate of returns on stock, σ is the volatility of the stock, dt is the relative change in the price during the period of time and Z is a Wiener process. Following the method of [18-19] on rate of returns gives as:

$$R_t := (\lambda_1 + \lambda_2)^2, \dots, \quad (16)$$

where $t = 1, 2, \dots$

Using (15) and (16) gives the following system of delay stochastic differential equations :

$$dX_1(t) = (t - \tau)(\theta_1 + \theta_2)^2 X_1(t) dt + \sigma X_1(t) dZ^{(1)}(t), \quad (17)$$

where $X_1(t), X_2(t), S_3(t)$ and $S_4(t)$ are underlying stocks with the following initial conditions:

$$X_1(0) = X_0, t > 0 \quad (18)$$

$$X_2(0) = e^t, t > 0 \quad (19)$$

$$X_3(0) = e^{-t}, t > 0 \quad (20)$$

where, $X_1(t), X_2(t)$ and $X_3(t)$ are asset prices, The expression dZ , which contains the randomness that is certainly a characteristic of asset prices is called a Wiener process or Brownian motion. λ_1 and λ_2 represents rate of returns of first and second investments respectively, $\lambda_1 + \lambda_2$ is additive effects

3.1 Method of solution

The model (17)-(20) is made up of a system of stochastic delay differential equations whose solutions are not trivial. We implement the methods of Ito's lemma in solving for $X_1(t), X_2(t)$ and $X_3(t)$ To grab this problem we note that we can forecast the future worth of the asset with sureness.

From (17) Let $f(X_1, t) = \ln X_1$ so differentiating partially gives

$$\frac{\partial f}{\partial X_1} = \frac{1}{X_1}, \quad \frac{\partial^2 f}{\partial X_1^2} = -\frac{1}{X_1^2}, \quad \frac{\partial f}{\partial t} = 0. \quad (21)$$

According to Ito's gives

$$df(X_1, t) = \sigma X_1 \frac{\partial f}{\partial X_1} dZ(t) + \left((\theta_1 + \theta_2)^2 X_1(t) \frac{\partial f}{\partial X_1} + \frac{1}{2} \sigma^2 X_1^2 \frac{\partial^2 f}{\partial X_1^2} + \frac{\partial f}{\partial t} \right) dt. \quad (22)$$

Substituting (17) and (21) into (22) gives

$$\begin{aligned} &= \sigma X_1 \frac{1}{X_1} dZ(t) + \left((t - \tau)(\theta_1 + \theta_2)^2 X_1(t) \frac{1}{X_1} + \frac{1}{2} \sigma^2 X_1^2 \left(-\frac{1}{X_1^2} \right) + 0 \right) dt \\ &= \sigma dZ(t) + \left((t - \tau)(\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) dt. \end{aligned}$$

Integrating both sides, taking upper and lower limits gives

$$\begin{aligned} \int_0^t d \ln X_1 &= \int_0^t df(X_u, u) = (t - \tau) \int \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dZ(t) \\ \ln X_1 - \ln X_0 &= (t - \tau) \left((\theta_1 + \theta_2)^2 u - \frac{1}{2} \sigma^2 u \right) \Big|_0^t + (\sigma Z u) \Big|_0^t \\ \ln \left(\frac{X_1}{X_0} \right) &= (t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma Z(t) \end{aligned} \quad (23)$$

Taking the ln of the both sides

$$X_1(t) = X_0 \exp\left((t-\tau)\left((\theta_1 + \theta_2)^2 - \frac{1}{2}\sigma^2\right)\right) + \sigma Z(t). \quad (24)$$

Applying other initial stock in (19)- (20) yields

$$X_2^*(t) = e^t e^{(t-\tau)\left((\theta_1 + \theta_2)^2 - \frac{1}{2}\sigma^2\right)} + \sigma Z(t) \quad (25)$$

$$X_3^{***}(t) = \frac{1}{e^t} e^{(t-\tau)\left((\theta_1 + \theta_2)^2 - \frac{1}{2}\sigma^2\right)} + \sigma Z(t) \quad (26)$$

3.2 Results and discussion

This Section presents results whose solutions are in (24)-(26), Hence we have the following:

Table 1: The impact of time delay in the assessment of additive Asset values through the solution

below: $X_1(t) = X_0 \exp\left\{(t-\tau)\left((\theta_1 + \theta_2)^2 - \frac{1}{2}\sigma^2\right)\right\} + \sigma dz(t)$ $t = 2, dz = 1$

τ	S_0	σ	$(\theta_1 + \theta_2)$	$X_1(t)$	$(\theta_1 + \theta_2)$	$X_1(t)$
0.1	1.0000	0.2	1.0000	50.6025	2.0000	4519898.067
0.2	1.0000	0.2	1.0000	41.5958	2.0000	2039061.07
0.3	1.0000	0.2	1.0000	34.1923	2.0000	919881.3745
0.4	2.0000	0.2	1.0000	56.2130	2.0000	829971.9471
0.5	2.0000	0.2	1.0000	46.2077	2.0000	374425.1444
0.6	2.0000	0.2	1.0000	37.9833	2.0000	168914.3702
0.7	3.0000	0.2	1.0000	46.8341	2.0000	114303.4792
0.8	3.0000	0.2	1.0000	38.4982	2.0000	51565.7148
0.9	3.0000	0.2	1.0000	31.6460	2.0000	23262.8347
1.0	4.0000	0.2	1.0000	34.6846	2.0000	13992.7464
1.1	4.0000	0.2	1.0000	28.5111	2.0000	6312.5460
1.2	4.0000	0.2	1.0000	23.4365	2.0000	2847.7781

Table 2: The impact of time delay in the assessment of additive Asset values through the solution

below: $X_2^*(t) = e^t \exp \left\{ (t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma dz(t) \right\}$ $t = 2, dz = 1$

τ	S_0	σ	$(\theta_1 + \theta_2)$	$X_1(t)$	$(\theta_1 + \theta_2)$	$X_1(t)$
0.1	7.38	0.2	1.0000	373.9066	2.0000	33397978.81
0.2	7.38	0.2	1.0000	307.3558	2.0000	15066826.15
0.3	7.38	0.2	1.0000	252.6502	2.0000	6797095.465
0.4	7.38	0.2	1.0000	207.6816	2.0000	3066372.857
0.5	7.38	0.2	1.0000	170.7168	2.0000	1383332.417
0.6	7.38	0.2	1.0000	140.3313	2.0000	624062.5866
0.7	7.38	0.2	1.0000	115.3540	2.0000	281533.2794
0.8	7.38	0.2	1.0000	94.8224	2.0000	127008.0744
0.9	7.38	0.2	1.0000	77.9452	2.0000	57297.1373
1.0	7.38	0.2	1.0000	64.0719	2.0000	25848.4506
1.1	7.38	0.2	1.0000	52.6679	2.0000	11661.0084
1.2	7.38	0.2	1.0000	43.2937	2.0000	5260.6293

Undoubtedly in Tables 1 and 2 show a little increase in time delay significantly reduces asset values through additive effects. This is physically consistent because as time passes some certain asset values continue to depreciates over time; example of such assets are cars, machines etc. This remark is an eye opener to an investor on how to manage such depreciating assets.

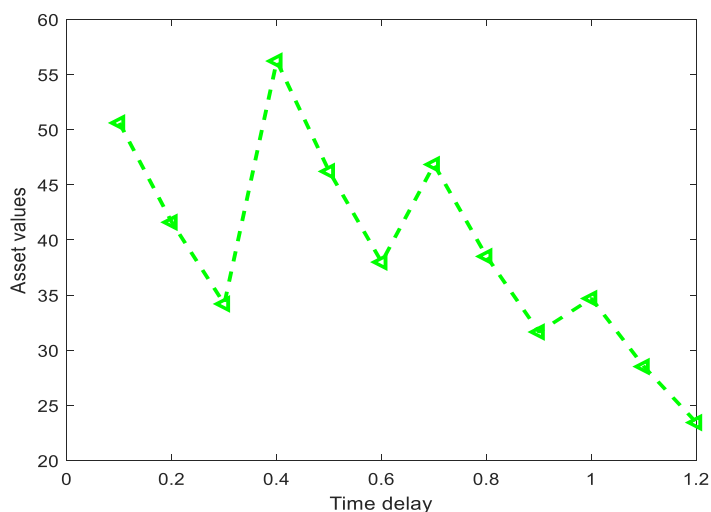


Figure 1: Profile of asset values with variations of time delay when initial price (X_0)

It can be seen in Figure 1, the graphical trend and impact of time delay to accurately assess asset value in time varying investments which is capital intensive. It is obviously informative to an investor whose aim and passion is to maximize profit hence it specifies time interval on the trend lines throughout the trading days.

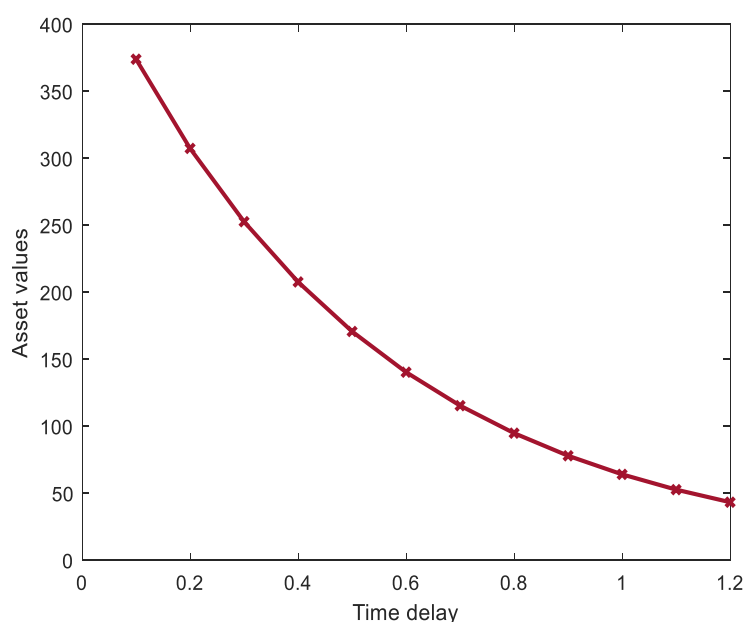


Figure 2: Profile of asset values with variations of time delay when the initial stock is exponential

Figures 2 and 3 describe a growth pattern that indicates dominantly, incremental improvements as time passes. Having an exponential growth in the investment means the trading business is highly indexed with millions of naira over time. This type of situation can interest investor to be more courageous in their economic investment, hence it grows exponentially.

Table 4: The impact of time delay in the assessment of additive Asset values through the

$$X_1(t) = \frac{1}{e^t} \exp \left\{ (t - \tau) \left((\theta_1 + \theta_2)^2 - \frac{1}{2} \sigma^2 \right) t + \sigma dz(t) \right\}$$

solution below

$$\text{where } dz = 1, t = 2 \text{ and } S_0 = \frac{1}{e^t}$$

τ	S_0	σ	$(\theta_1 + \theta_2)$	$X_1(t)$	$(\theta_1 + \theta_2)$	$X_1(t)$
0.	0.135	0.	1.0000	6.8465	2.0000	611542.2085
0.	0.135	0.	1.0000	5.6279	2.0000	275884.9628
0.	0.135	0.	1.0000	4.6262	2.0000	124459.95
0.	0.135	0.	1.0000	3.8028	2.0000	56147.6022
0.	0.135	0.	1.0000	3.1260	2.0000	25329.8610
0.	0.135	0.	1.0000	2.5695	2.0000	11427.0572
0.	0.135	0.	1.0000	2.1122	2.0000	5155.0869
0.	0.135	0.	1.0000	1.7363	2.0000	2325.6137
0.	0.135	0.	1.0000	1.4272	2.0000	1049.1539
1.	0.135	0.	1.0000	1.1732	2.0000	473.3047
1.	0.135	0.	1.0000	0.9644	2.0000	213.5219
1.	0.135	0.	1.0000	0.7927	2.0000	96.3261

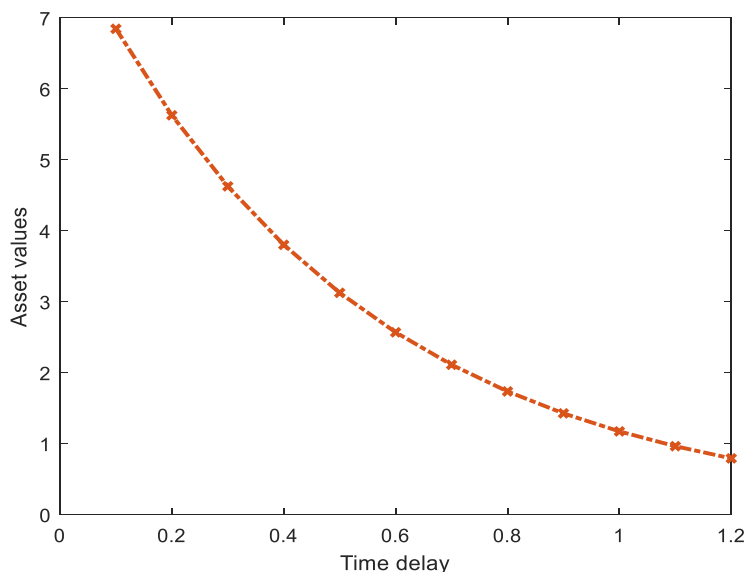


Figure 3 : Profile of Asset values with variations of Time Delay when the initial stock is exponential inverse

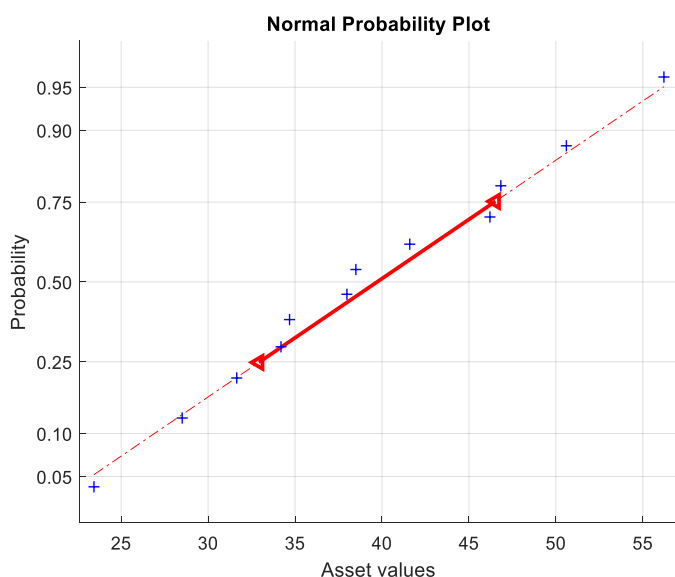


Figure 4 : Normal probability plot on Asset values when the initial stock price is (X_0)

The graphical description of Figure 4 shows that the two asset values through additive effects of Table 1 come from the identical distribution. They are statistically significant and correlated. The plot portrays a significant sure event which is highly beneficial to investor over in a time varying investments and with this scenario; profit margin is unavoidable and decisions can be properly taken to enrich decent management of the business.

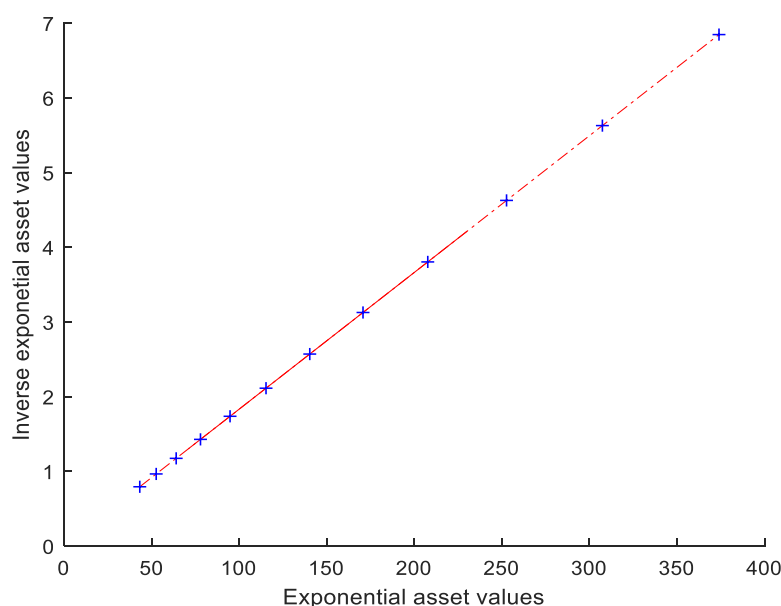


Figure 5 : Quantile-Quantile (QQ) plot assessment on two different Asset values when the initial stock prices is both exponential and exponential inverse

It can be seen in Figure 5 that the QQ plot indicates that the two exponential and inverse exponential asset values with additive effects of Tables 2 and 3 comes from a common distributions. They are statistically significant, correlated and have lots of financial benefits hence it waves around the normal distribution.

4 Conclusion

In this study, the stochastic analysis of variations of initial stock prices was successfully analysed through additive effects with the influence of time delay parameter. The asset values were obtained which all follows exponential trend of business over time. The normality probability and QQ plots all were statistically significant to address the issues in financial market for an assessment of asset pricing for economic investments.

Though, the present paper considered SDDE with additive effects to realistically assess asset values, future study should incorporate multiplicative effects as well and to be solved as stochastic systems.

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A family of non-linear rational integrators for the solution in ordinary differential equations

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Abstract

In this research work, we derived a Family of Non-Linear rational integrators for the solution of ordinary differential equations. We establish the various integrator formula with an interpolants of order $m = 2$ and 3 respectively with the use of MAPLE-18 and MATLAB softwares. Our method was compared with an explicit fourth-stage fourth-order Runge-Kutta methods and the Extended Block Integrator for First-Order Stiff and Oscillatory Differential Equations. Our results show good improvement over these recent existing methods compared with, and also ascertain the convergence and consistency of our scheme. Our result shows that our integrator is stable analytically and computationally.

Keywords and Phrases: rational integrator; stiff; singular and oscillatory problems; stability function and region of absolute stability (RAS)

1 Introduction

This research work is concerned with the determination of solution of initial value problems in Ordinary Differential Equations (ODEs). Many mathematical models have been formulated in terms of the rate of change of one or more physical variables and these naturally lead to differential equations, [5], [7]. The basis for many numerical integrators used in solving the differential equation in finding the solution of initial value problems in the form:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b, \quad (1)$$

where $f(x, y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region $D \subset [a, b]$ that are either Stiff or Singular.

Lipschitz constant according to [2], from (1) above, $f(x, y)$ is said to satisfy a Lipschitz condition in y , over the region D , if there exist a constant L such that

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$$\|f(x, y_1) - f(x, y_2)\| \leq L \|y_1 - y_2\|, \quad (2)$$

where L is called the Lipschitz constant and $f(x, y)$ is said to be Lipschitzian. By virtue of the relation.

Definition 1: Stiffness [6] The system (1) is said to be stiff over the interval $[a, b]$ if for every $x \in [a, b]$, the eigenvalues $\{\lambda_s(x), s = 1, \dots, m\}$ of the Jacobian matrix

$$J = \frac{\partial f}{\partial y} \quad (3)$$

satisfy the following conditions:

$$\operatorname{Re} \lambda_s(x) < 0, s = 1, \dots, m$$

where $\operatorname{Re} \lambda_s(x)$ represents the real component of the complex number $\lambda_s(x)$

$$\text{Stiffness Ratio} = \frac{\lambda_r(x)}{\lambda_s(x)} \gg 1, r, s = 1, \dots, m.$$

For the linear initial value problem

$$y' = Ay + g, \quad y(x_0) = y_0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} = A$$

is $m \times m$ matrix and $g \in \mathbb{R}^m$.

Definition 2: [7]: A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $\operatorname{Re} h\lambda < 0$.

However, A-stable is a severe requirement to ask of a numerical method for stiffness to be satisfied.

Definition 3: [11]: A numerical method is said to be stiffly-stable if (i) the region of absolute stability contains $\mathfrak{R}_1$ and $\mathfrak{R}_1$ and (ii) it is accurate for all $h \in \mathfrak{R}_2$ when applied to the test equation $y' = \lambda y, \lambda$ a complex constant with $\operatorname{Re} h\lambda < 0$, where

$$\mathfrak{R}_1 = \{[h\lambda] \operatorname{Re} h\lambda < -a\},$$

$$\mathfrak{R}_2 = \{[h\lambda] - a \leq \operatorname{Re} h\lambda \leq b, -c \leq \operatorname{Im} h\lambda \leq c\},$$

and a, b and c are positive constants.

2 Derivation of the method

The rational interpolant $U: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^m q_i x_{n+1}^i} \quad (4)$$

where p_i and q_i are called the integrator parameters, and r is an inhomogeneous constant.

Case 1: At $m = 2$, we have

$$y_{n+1} = r + \frac{\sum_{i=0}^1 p_i x_{n+1}^i}{1 + \sum_{i=1}^2 q_i x_{n+1}^i} \quad (5)$$

where p_0, p_1 and q_1, q_2 are called the integrator parameters and r is an inhomogeneous constant. By the use of binomial expansion for rational functions and employing the Taylor's series expansion on y_{n+1} , as

$$y_{n+1} = \sum_{k=0}^{\infty} \frac{h^k y_n^{(k)}}{k!}. \quad (6)$$

Comparing coefficients in terms of h , and putting in matrix form, we have

$$\begin{bmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = - \begin{bmatrix} \frac{y_n^{(4)}}{4!(n+1)^4} \\ \frac{y_n^{(3)}}{3!(n+1)^3} \end{bmatrix} \quad (7)$$

Applying the Cramers rule to determine q_1, q_2 we have

$$\text{where } q_i = \frac{x_i}{|A|}, \text{ where } i = 1, 2 \quad (8)$$

We employ Maple 18 software compiler and obtained the determinant of A , x_1 and x_2 respectively

$$|A| = \begin{vmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{vmatrix} \quad (9)$$

$$|A| = (n+1)2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}$$

$$|x_1| = \begin{vmatrix} -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ -\frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(1)}}{(n+1)} \end{vmatrix} \quad (10)$$

$$|x_1| = 2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}$$

$$|x_2| = \begin{bmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(4)}}{4!(n+1)^4} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & -\frac{y_n^{(3)}}{3!(n+1)^3} \end{bmatrix} \quad (11)$$

$$|x_2| = 4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}.$$

At the integration point $x = x_{n+1}$,

$$U(x_{n+1}) = \sum_{k=0}^{\infty} c_k x_{n+1}^k. \quad (12)$$

With these definitions for c_r 's, the determinant $|A|$ can be written as follows:

$$|A| = \frac{h^4}{x_{n+1}^4} [(n+1)[2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}], \quad (13)$$

$$|x_1| = \frac{h^5}{2x_{n+1}^5} [2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}], \quad (14)$$

$$|x_2| = \frac{h^6}{-12x_{n+1}^6} [4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}]. \quad (15)$$

Since q_i 's are defined by (8), we have

$$q_1 x_{n+1}^1 = \frac{h}{2} \frac{[2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}]}{[(n+1)[2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}]}, \quad (16)$$

$$q_2 x_{n+1}^2 = \frac{h^2}{-12} \frac{[4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}]}{[(n+1)[2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}]} \quad (17)$$

Now, recall from (4),

$$\begin{aligned} y_n &= r + p_0 \\ r &= y_n - p_0 \end{aligned} \quad (18)$$

$$\begin{aligned} p_1 &= \frac{hy_n^{(1)}}{1!x_{n+1}} + p_0 q_1 \\ p_1 x_{n+1} &= hy_n^{(1)} + p_0 q_1 x_{n+1} \\ p_0 q_2 &= \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} + \frac{hy_n^{(1)}}{1!x_{n+1}} q_1 \end{aligned} \quad (19)$$

$$p_0 = \frac{-1}{q_2} \left[\frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_1 x_{n+1} \right] \quad (20)$$

$$r = y_n - \frac{1}{q_2} \left[\frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_1 x_{n+1} \right]. \quad (21)$$

Putting equation (16 – 21) into 5, our integrator formula for $m = 2$ becomes

$$y_{n+1} = \frac{\sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + B y_n}{1 + A + B}. \quad (22)$$

Case 2: At $m = 3$, from (4) we have

$$y_{n+1} = r + \frac{\sum_{i=0}^2 p_i x_{n+1}^i}{1 + \sum_{i=1}^3 q_i x_{n+1}^i} \quad (23)$$

where p_0, p_1, p_2 and q_1, q_2, q_3 are called the integrator parameters and r is an inhomogeneous constant. By the use of binomial expansion for rational functions and employing the Taylor's series expansion on y_{n+1} , as $y_{n+1} = \sum_{k=0}^{\infty} \frac{h^k y_n^{(k)}}{k!}$ (24)

Comparing coefficients in terms of h , and putting in matrix form, we have

$$\begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = - \begin{bmatrix} \frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(4)}}{4!(n+1)^4} \end{bmatrix} \quad (25)$$

Applying the Cramers rule to determine q_1, q_2, q_3 we have

$$\text{where } q_i = \frac{x_i}{|A|}, \text{ where } i = 1, 2, 3 \quad (26)$$

we employ Maple 18 software compiler and obtained

The determinant of A

$$|A| = \begin{vmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{vmatrix} \quad (27)$$

$$|A| = (n+1)18y_n^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y_n^{(4)^2}y_n^{(1)} + 40y_n^{(3)^3}$$

$$|x_1| = \begin{vmatrix} -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{vmatrix} \quad (28)$$

$$|x_1| = 6y_n^{(2)^2} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} - 15y_n^{(4)^2}y_n^{(2)} + 6y_n^{(1)}y_n^{(5)}y_n^{(4)} - 4y_n^{(1)}y_n^{(6)}y_n^{(3)} + 120y_n^{(3)^2}y_n^{(4)}$$

$$|x_2| = \begin{vmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(1)}}{(n+1)} \end{vmatrix} \quad (29)$$

$$|x_2| = 15y_n^{(2)}y_n^{(5)}y_n^{(4)} - 10y_n^{(2)}y_n^{(6)}y_n^{(3)} - 16y_n^{(5)^2}y_n^{(1)} + 5y_n^{(1)}y_n^{(6)}y_n^{(4)} + 20y_n^{(3)^2}y_n^{(5)} - 25y_n^{(4)^2}y_n^{(3)}$$

$$|x_3| = \begin{vmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & -\frac{y_n^{(4)}}{4!(n+1)^4} \end{vmatrix} \quad (30)$$

$$|x_3| = 36y_n^{(5)^2}y_n^{(2)} - 30y_n^{(2)}y_n^{(4)}y_n^{(6)} - 120y_n^{(3)}y_n^{(5)}y_n^{(4)} + 75y_n^{(4)^3} + 40y_n^{(3)^2}y_n^{(6)}$$

At the integration point $x = x_{n+1}$.

$$U(x_{n+1}) = \sum_{i=0}^{\infty} c_k x_{n+1}^k.$$

Since q_i 's are defined by (26), we have

$$q_1 x_{n+1}^1 = \frac{h^1}{-2} \frac{[6y_n^{(2)^2} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} - 15y_n^{(4)^2}y_n^{(2)} + 6y_n^{(1)}y_n^{(5)}y_n^{(4)} - 4y_n^{(1)}y_n^{(6)}y_n^{(3)} + 120y_n^{(3)^2}y_n^{(4)}]}{[(n+1)18y_n^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y_n^{(4)^2}y_n^{(1)} + 40y_n^{(3)^2}]} \quad (31)$$

$$q_2 x_{n+1}^2 = \frac{h^2}{-10} \frac{[15y_n^{(2)}y_n^{(5)}y_n^{(4)} - 10y_n^{(2)}y_n^{(6)}y_n^{(3)} - 16y_n^{(5)^2}y_n^{(1)} + 5y_n^{(1)}y_n^{(6)}y_n^{(4)} + 20y_n^{(3)^2}y_n^{(5)} - 25y_n^{(4)^2}y_n^{(3)}]}{[(n+1)18y_n^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y_n^{(4)^2}y_n^{(1)} + 40y_n^{(3)^2}]} \quad (32)$$

$$q_3 x_{n+1}^3 = \frac{h^3}{-120} \frac{[36y_n^{(5)^2}y_n^{(2)} - 30y_n^{(2)}y_n^{(4)}y_n^{(6)} - 120y_n^{(3)}y_n^{(5)}y_n^{(4)} + 75y_n^{(4)^3} + 40y_n^{(3)^2}y_n^{(6)}]}{[(n+1)18y_n^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y_n^{(4)^2}y_n^{(1)} + 40y_n^{(3)^2}]} \quad (33)$$

Therefore, our general integrator formula for $m = 3$ becomes

Now, recall from (4)

$$\begin{aligned} y_n &= r + p_0 \\ r &= y_n - p_0 \end{aligned} \quad (34)$$

$$\begin{aligned} p_1 &= \frac{hy_n^{(1)}}{1!x_{n+1}} + p_0q_1 \\ p_1x_{n+1} &= hy_n^{(1)} + p_0q_1x_{n+1} \end{aligned} \quad (35)$$

$$\begin{aligned} p_2 &= \frac{h^2y_n^{(2)}}{2!x_{n+1}^2} + \frac{hy_n^{(1)}}{1!x_{n+1}} + p_0q_2 \\ p_2x_{n+1}^2 &= \frac{h^2y_n^{(2)}}{2!} + hy_n^{(1)}q_1x_{n+1} + p_0q_2x_{n+1}^2 \end{aligned} \quad (36)$$

$$p_0q_3 = \frac{h^3y_n^{(3)}}{3!x_{n+1}^3} + \frac{h^2y_n^{(2)}}{2!x_{n+1}^2}q_1 + \frac{h^1y_n^{(1)}}{1!x_{n+1}}q_2 \quad (37)$$

$$p_0 = \frac{-1}{q_3} \left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x_{n+1}^2 \right] \quad (38)$$

$$r = y_n - \frac{1}{q_3} \left[\frac{h^3y_n^{(3)}}{3!} + \frac{h^2y_n^{(2)}q_1x_{n+1}}{2!} + hy_n^{(1)}q_2x_{n+1}^2 \right] \quad (39)$$

Now from (4), the rational integrator is expanded to give

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + r + r q_1 x_{n+1} + r q_2 x_{n+1}^2 + r q_3 x_{n+1}^3}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3}. \quad (40)$$

Then,

$$p_0 + r = r + p_0 = y_n \quad (41)$$

$$\begin{aligned} p_1 x_{n+1} + r q_1 x_{n+1} &= h y_n^{(1)} - \frac{q_1 x_{n+1}}{q_3 x_{n+1}^3} \left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \right] \\ &\quad + q_1 x_{n+1} y_n + \frac{q_1 x_{n+1}}{q_3 x_{n+1}^3} \left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \right] \end{aligned}$$

$$p_1 x_{n+1} + r q_1 x_{n+1} = h y_n^{(1)} + q_1 x_{n+1} y_n \quad (42)$$

$$\begin{aligned} p_2 x_{n+1}^2 + r q_2 x_{n+1}^2 &= \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} - \frac{q_2 x_{n+1}^2}{q_3 x_{n+1}^3} \left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \right] \\ &\quad + q_2 x_{n+1}^2 y_n + \frac{q_2 x_{n+1}^2}{q_3 x_{n+1}^3} \left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \right] \end{aligned}$$

$$p_2 x_{n+1}^2 + r q_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} + q_2 x_{n+1}^2 y_n \quad (43)$$

$$r q_3 x_{n+1}^3 = q_3 x_{n+1}^3 y_n + \frac{q_3 x_{n+1}^3}{q_3 x_{n+1}^3} \left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \right] \quad (44)$$

Substituting equations (31) – (43) into (5), we obtain our general integrator for $m = 3$

$$y_{n+1} = \frac{\sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + C y_n}{1 + A + B + C}. \quad (45)$$

3 Convergence, consistency and stability

For any numerical methods to be useful, such method must satisfy some basic properties. According to [8], one- step methods like a family of non linear rational integrator for the solution of ordinary differential equations are normally described symbolically by

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h) \quad (46)$$

where $\Phi(x_n, y_n; h)$ is called the increment function, x_n the mesh point and h the mesh size. [9] state that a rational integrator is said to be consistent if the increment function is consistent with the initial value problem (IVP) that is if $\Phi(x, y, 0) = f(x, y)$.

Theorem 1: A non linear rational integrator of order $m = 2$, for the solution of ordinary differential equations.

$$y_{n+1} = \frac{\sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + B y_n}{1 + A + B} \quad (47)$$

is convergent and consistent.

Proof:

Subtracting y_n from both sides of equ. (47), we have

$$\begin{aligned} y_{n+1} - y_n &= \frac{\sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + B y_n}{1 + A + B} - y_n \\ y_{n+1} - y_n &= \frac{\sum_{i=1}^2 \frac{h^i y_n^{(i)}}{i!} + A h^i y_n^{(i)}}{1 + A + B} \\ \frac{y_{n+1} - y_n}{h} &= \frac{y_n^{(1)} + \sum_{i=2}^3 \frac{h^{i-1} y_n^{(i)}}{i!} + A h y_n^{(1)}}{1 + A + B + C} \end{aligned}$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right): \text{Limit}_{x \rightarrow 0} A = 0,$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)},$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)} = f(x_n, y_n) \text{ as required.}$$

Theorem 2: A non linear rational integrator of $m = 3$, for the solution of ordinary differential equations

$$y_{n+1} = \frac{\sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + C y_n}{1 + A + B + C} \quad (48)$$

is convergent and consistent.

Proof:

Subtracting y_n from both sides of equ. (48), we have

$$y_{n+1} - y_n = \frac{\sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + C y_n}{1 + A + B + C} - y_n$$

$$y_{n+1} - y_n = \frac{\sum_{i=1}^3 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=1}^2 \frac{h^i y_n^{(i)}}{i!} + B h^i y_n^{(i)}}{1 + A + B + C}$$

$$\frac{y_{n+1} - y_n}{h} = \frac{y_n^{(1)} + \sum_{i=2}^3 \frac{h^{i-1} y_n^{(i)}}{i!} + A \sum_{i=1}^2 \frac{h^{i-1} y_n^{(i)}}{i!} + B h y_n^{(i)}}{1 + A + B + C}$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right): \text{Limit}_{x \rightarrow 0} A = \text{Limit}_{x \rightarrow 0} B = 0$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)}$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)} = f(x_n, y_n) \text{ as required.}$$

The Stability Analysis of the Jordan Curve at $m = 2$:

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

We take various degrees of θ from 0° to 360° and get a corresponding values of R.

Table 1: Stability Analysis of the Jordan Curve at $m = 2$

θ°	R	Approximation for Curve Sketch	θ°	R	Approximation for Curve Sketch
0	5.61396669241456	5.61	195	-5.32137805328698	-5.32
15	5.32137805328698	5.32	210	-5.01260658797262	-5.01
30	5.01260658797262	5.01	225	-4.83406362171214	-4.83
45	4.83406362171214	4.83	240	-4.52694187340751	-4.53
60	4.52694187340751	4.53	255	-4.21018688238941	-4.21
75	4.21018688238941	4.21	270	0.00000000000000	0

90	0.0000000000000000	0	285	4.21018688238941	4.21
105	-4.21018688238941	-4.21	300	4.52694187340751	4.53
120	-4.52694187340751	-4.53	315	4.83406362171214	4.83
135	-4.83406362171214	-4.83	330	5.01260658797262	5.01
150	-5.01260658797262	-5.01	345	5.32137805328698	5.32
165	-5.32137805328698	-5.32	360	5.61396669241456	5.61
180	-5.61396669241456	-5.61			

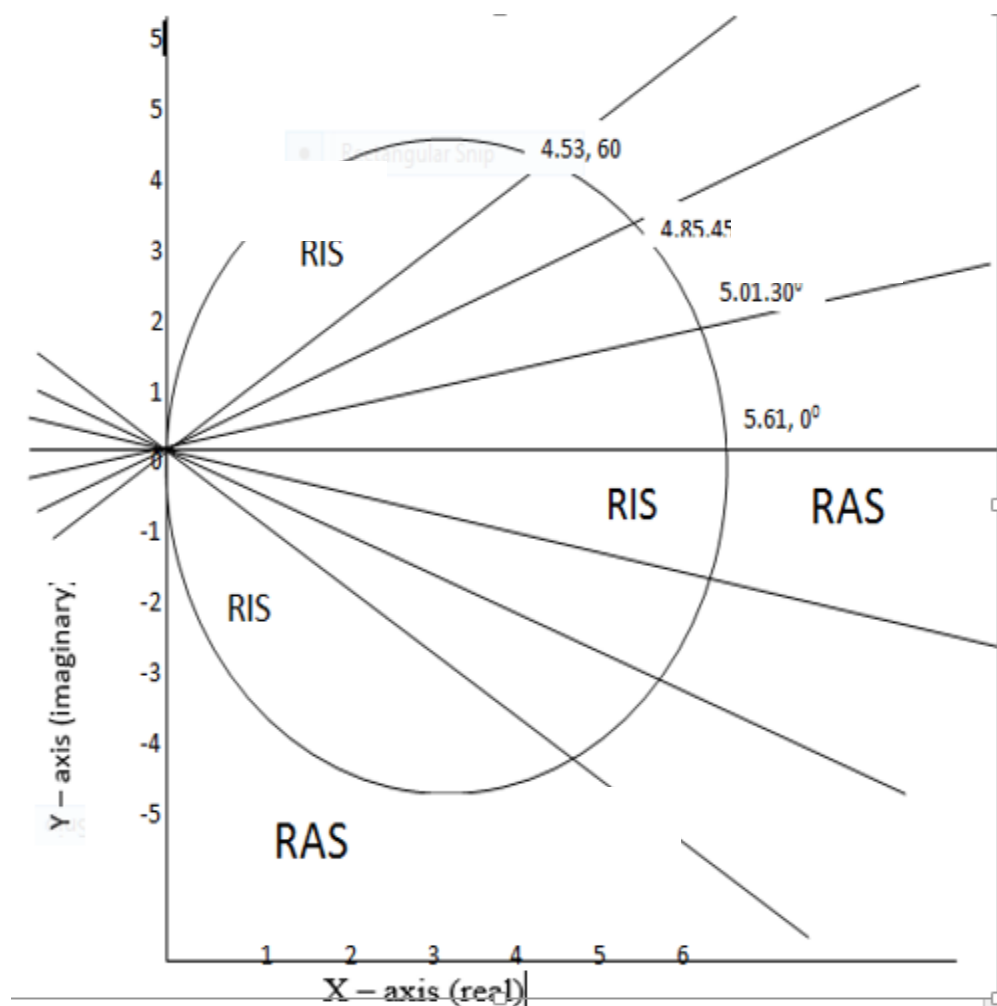


Figure 1: RAS and RIS for the case $m = 2$.

Hence the integrator is A-stable and so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

The Stability Analysis of the Jordan Curve $m = 3$:

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

We take various degrees of θ from 0^0 to 360^0 and get a corresponding values of R.

Table 2. Stability analysis of the Jordan Curve at $m = 3$

θ^0	R	Approximation for Curve Skehch	θ^0	R	Approximati onfor Curve Skehch
0	7.74596669241483	7.74	195	-7.67137805328621	-7.7
15	7.67137805328621	7.7	210	-7.46260444797262	-7.5
30	7.46260444797262	7.5	225	-7.16406362171214	-7.2
45	7.16406362171214	7.2	240	-6.84694187340751	-6.9
60	6.84694187340751	6.8	255	-6.60018688238941	-6.6
75	6.60018688238941	6.6	270	0.00000000000000	0
90	0.00000000000000	0	285	6.60018688238941	6.6
105	-6.60018688238941	-6.6	300	6.84694187340751	6.9
120	-6.84694187340751	-6.9	315	7.16406362171214	7.2
135	-7.16406362171214	-7.2	330	7.46260444797262	7.5
150	-7.46260444797262	-7.5	345	7.67137805328621	7.7
165	-7.67137805328621	-7.6	360	7.74596669241483	7.74
180	-7.74596669241483	-7.74			

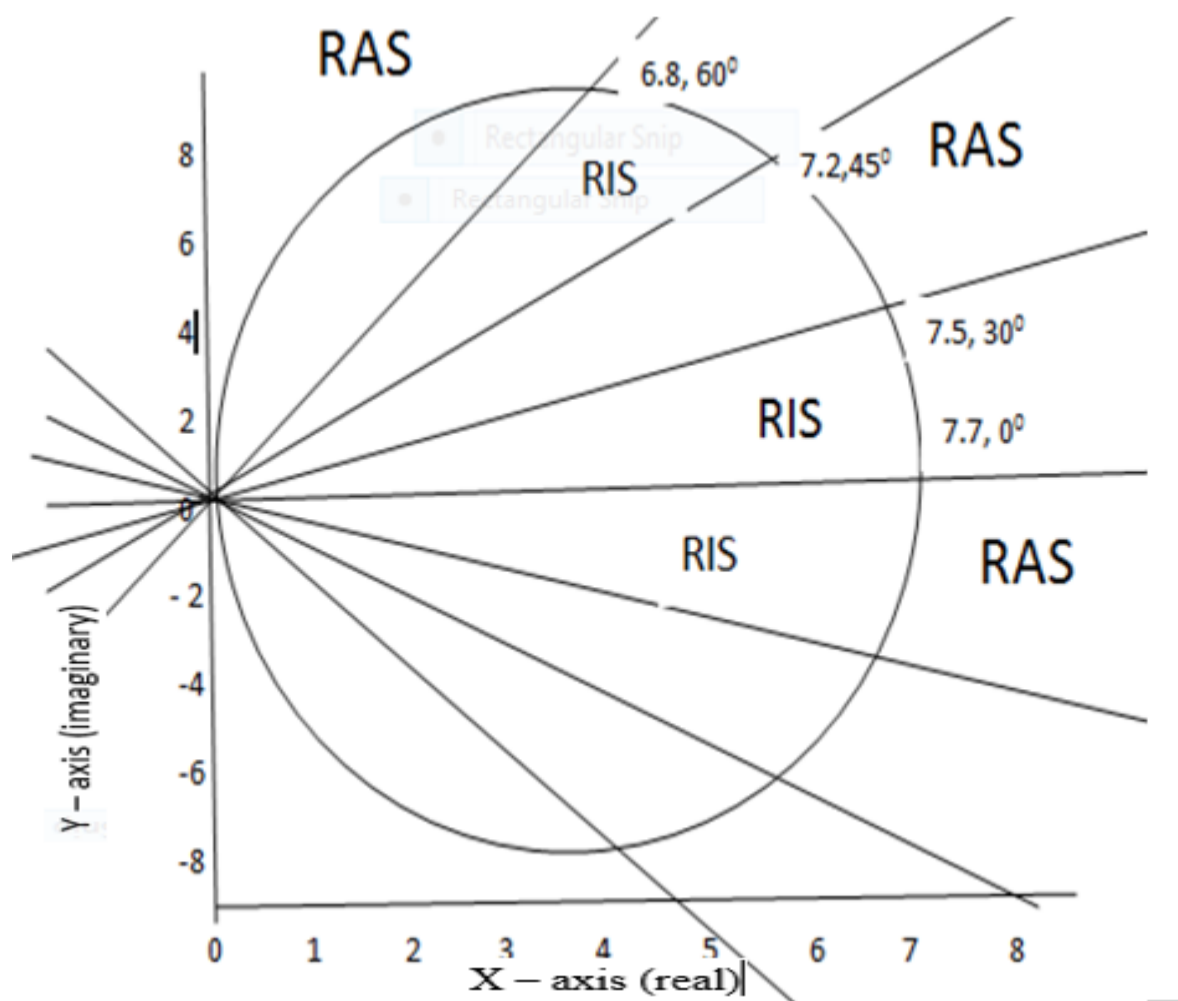


Figure 2: RAS and RIS for the case $m = 3$.

Hence the integrator is A-stable and so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

4 Numerical experiments

Problem 1: [3]

Consider $y' = y - y^2$; $y(0) = 0.5$; $h = 0.1$

The exact solution is $\frac{1}{(1+e^{-x})}$. Implicit 4th Order RKF

Table 3a: Numerical integration

XN	TSOL	Agbeboh & Esekhaigbe (2016)	New Rational Integrator of order m - 1	
			m=2	m=3
1.00E-01	5.249E-01	0.5249D+00	5.249E-01	5.249E-01
2.00E-01	5.498E-01	0.5498D+00	5.498E-01	5.498E-01
3.00E-01	5.744E-01	0.5744D+00	5.744E-01	5.744E-01
4.00E-01	5.986E-01	0.5986D+00	5.986E-01	5.986E-01
5.00E-01	6.224E-01	0.6224D+00	6.224E-01	6.224E-01
6.00E-01	6.456E-01	0.6456D+00	6.456E-01	6.456E-01
7.00E-01	6.681E-01	0.6681D+00	6.681E-01	6.681E-01
8.00E-01	6.899E-01	0.6899D+00	6.899E-01	6.899E-01
9.00E-01	7.109E-01	0.7109D+00	7.109E-01	7.109E-01
1.00E+00	7.310E-01	0.7310D+00	7.310E-01	7.310E-01

Table 3b: Error in numerical integration

XN	Agbeboh & Esekhaigbe (2016)	New Rational Integrator of order m - 1	
		m = 2	m = 3
1.00e-01	0.130D-08	-2.730e-07	-2.106e-11
2.00e-01	0.264D-08	-5.273e-07	-1.757e-10
3.00e-01	0.405D-08	-6.886e-07	-3.780e-10
4.00e-01	0.559D-08	-6.854e-07	-3.622e-10
5.00e-01	0.728D-08	-4.887e-07	-1.008e-09
6.00e-01	0.915D-08	-2.672e-07	-2.112e-07
7.00e-01	0.112D-07	-1.193e-06	-2.115e-07
8.00e-01	0.134D-07	-1.235e-05	-2.098e-07
9.00e-01	0.158D-07	-5.584e-04	-2.089e-07
1.00e+01	0.184D-07	-3.713e-03	-2.100e-07

Table 3 above shows the performance of our new numerical integrators. Our new integrator compete favorably well with the Implicit 4th Order RKF of [3], with a very high rate of convergence at m = 3.

Problem 2: [12]

Consider $y' = -\sin x - 200(y - \cos x)$; $y(0) = 0$; $h = 0.001$.

The exact solution is $y(x) = \cos x - e^{-200x}$.

Table 4a: Numerical integration

XN	TSOL	Sunday, et al (2013)	New Rational Integrator of order m - 1	
			m = 2	m = 3
1.00e-03	1.81268e-01	0.1812e-01	1.8126e-01	1.8126e-01
2.00e-03	3.29677e-01	0.3296e-01	3.2967e-01	3.2967e-01
3.00e-03	4.51183e-01	0.4511e-01	4.5118e-01	4.5118e-01
4.00e-03	5.50663e-01	0.5506e-01	5.5066e-01	5.5066e-01
5.00e-03	6.32108e-01	0.6321e-01	6.3210e-01	6.3210e-01
6.00e-03	6.98787e-01	0.6987e-01	6.9878e-01	6.9878e-01
7.00e-03	7.53378e-01	0.7533e-01	7.5337e-01	7.5337e-01
8.00e-03	7.98071e-01	0.7980e-01	7.9807e-01	7.9807e-01
9.00e-03	8.34660e-01	0.8346e-01	8.3466e-01	8.3466e-01
10.0e+03	8.64614e-01	0.8641e-01	8.6461e-01	8.6461e-01

Table 4b: Error in numerical integration

XN	Sunday, et al (2013)	New Rational Integrator of order m - 1	
		m = 2	m = 3
1.00e-03	1.8270e-07	6.4086e-05	4.1388e-09
2.00e-03	1.4085e-07	1.0493e-04	5.5915e-09
3.00e-03	5.5609e-07	1.2886e-04	5.5869e-09
4.00e-03	3.9270e-07	1.4066e-04	4.8676e-09
5.00e-03	2.2589e-07	1.4395e-04	3.8687e-09
6.00e-03	1.8561e-07	1.4142e-04	2.8328e-09
7.00e-03	1.5193e-07	1.3508e-04	1.8852e-09
8.00e-03	1.2601e-07	1.2639e-04	1.0808e-09
9.00e-03	1.1594e-07	1.1641e-04	4.3446e-10
1.00e+03	1.6619e-07	1.0589e-04	6.1043e-11

Table 4 above shows the performance of our new numerical integrator. The trend shows that for a given mesh point, the errors in numerical integrator decreases as m increases. Our new integrator competes fairly well with the Extended Block Integrator method of [12].

5 Conclusion

The use of numerical integrators in solving initial value problems have been demonstrated using a family non- linear rational integrator and have proven more effective. The numerical tables shows that our new integrator converges quickly to the analytic solution at each corresponding mesh point than its counterparts. The integrator compares favourably well with the recent work like [3] and [12].

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Mathematical modeling of the effect of kidnapping in Nigeria

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Abstract

This paper proposes a mathematical model for kidnapping using infectious disease dynamics to imitate its menace in the society with the goal of controlling its spread. The human population of interest is sub-divided into five compartments of susceptible $K_s(t)$, Kidnapper, $K_I(t)$, kidnap victim, $K_v(t)$, kidnapper enters detention facility, $K_D(t)$, Kidnapper's rehabilitation at detention yield repentance from crime $K_R(t)$. Further, the basic reproduction number R_0 is calculated. Thus $R_0 < 1$ and $R_0 > 1$ respectively represent a decrease and increase in kidnapping activities in the population of discussion. Particularly, $R_0 < 1$ indicates a kidnap free state (KFE) that is locally and globally asymptotically stable. An $R_0 > 1$ indicates a kidnap endemic equilibrium (KEE) that is locally and globally asymptotically stable. From the model analysis, a reduction in kidnapping activities could be achieved through effective counter-kidnapping activities by the law enforcement agencies with minimal incidence of death during rescue operations of victims and enforcement of maximum penalties for the perpetrators of the crime as seen in the numerical simulations represented in figures 1-4.

Keywords: kidnapper; mathematical modeling; anti-kidnapping operations; dynamics.

1 Introduction

What started as an agitation in the Niger Delta to draw the attention of the Nigerian government, the international community and oil companies operating within the region of the degradations of the environment by oil and oil related activities with the seizing of foreign and local workers for ransom payment has now spread to every part of the country, the federal capital territory (FCT) not spared the evil with its attendant devastating consequences on the victim and the society at large. Kidnapping, terrorism and banditry have become a serious menace in our society Nigeria, as in many African and third world countries rapidly, spreading across the nooks and crannies of the country and would require a coordinated action by all right thinking individuals, groups, government

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at local, sub-national and national level, foreign missions, development agencies, non-governmental organisation to stem its tides in the face of the grueling and harrowing experiences encountered by its victims which include but not limited to ransom payment, maiming , psychological torture, displacements of families, dire economic consequences and death in severe cases .

Responsible for the growing spate of the 'disease' which is now endemic in our society are; unemployment, underemployment, the wanton display of opulence by political office holders and their cronies in a society where the poverty rate is on the upswing, failure of government to formulate and implement appropriate policies that will attract investment leading to creation of jobs ,official corruptions by the law enforcement agencies, ineptitude by the security agencies and so on [5].

In his SBM intelligence's report titled 'The Economics of Kidnap Industry in Nigeria' covering data analysis from June 2011 -March 2020, [8] asserts that at least a total sum of USD 18.34million (N10.27 billion)was paid to kidnappers as ransom. He noted further in his 2022 report that more than 500 incidences of kidnapping occurred with about 3420 individuals abducted across Nigeria in nearly all the states, including the federal capital territory (FCT) between July 2021 to June 2022 involving 564 individuals killed in violence associated with kidnapping and a huge amount of 6.531billion naira demanded and 653.7million paid as ransom in the ensuing period as seen in [9].

Similarly, [4], opines that one of the mysteries of today's Nigeria is the flourishing of kidnap-for-ransom enterprise which has assumed an industrial scale and the status of the subsector of the economy alarmingly putting the country based on the Armed Conflict Location and Event Data Project, ACLEDP miles ahead of Mexico and Columbia in incidences of mass kidnapping and abduction in it 2021 report. He further highlights the whole kidnapping value chain to include but not limited to negotiators, banks that receive the ransom payment, compromised security agents , the police that charges fees from kidnap victim's relatives for tracking the kidnappers, a state intelligence network that is either abdicating its roles or bowing to the superior intelligence of the criminals, and a host of others.

That kidnapping is rising in the Nigerian society is known to many authors ([1]; [5]; [10]; [11];[14]). The key challenge is that no nation can develop sustainably in the face of burgeoning crime and criminality and hence a united effort towards limiting the effect of organised crime like trafficking, kidnapping, terrorism, armed banditry in the society is welcomed and will spare the nation the negative consequences of the phenomenon of kidnapping as submitted in [10].

According to [2], mathematical model uses abstraction to represent real life phenomenon acting as a pivotal tool in interpreting dynamical systems with parameters, variables and equations descriptions for the creation of relationships between components to be studied and understood . [6]employs the deterministic mathematical model to understand unemployment dynamics in Nigeria dividing the population into four compartments with dataset from the National Bureau of Statistics (NBS) which revealed the need more individuals to be involved with entrepreneurship to create employment for the teaming unemployed youths across the country.

Many researchers and authors have used mathematical modeling to represent real-life phenomena like ,unemployment, ecological systems, corruption, trafficking and such like ([12];[1];[3]), but not much is done on modeling kidnapping particularly depicting it as an infectious disease transmission model in the local environment due to paucity of reliable data save for the works as seen in [11] and [1] .

Whereas[11], proposed a mathematical model of kidnapping using the transmission process with a system of equations mirroring the modified SIR model with control parameters like the

rescue rate as the fulcrum of controlling the spread of kidnapping based on successful rescue of victims by law enforcement agents without incorporating other measures to curb the activities of the kidnappers within the domain of discourse, [1] modeled terrorism as a transmission and spread model with controlling measures including detention and rehabilitation. Both work parallels ideologically. Though ransom kidnapping and terrorism have some semblance in terms of unlawful forceful abduction of victims against their consent and such likes, terrorism ideology is rooted in religious and political beliefs.

By extending the work seen in [11] to incorporate detention and rehabilitation parameters a mathematical model directed by a system of five non-linear differential equations is proposed. This will provide insight on prosecution and rehabilitation effects on the dynamical system. The paper is structured as follow: section 2 deals with the mathematical formulations, while section 3 is concerned with the analysis of the equilibria solution and section 4 addresses the simulation and discussion of the results and the paper concludes in section 5 with insight on direction for research.

2 Mathematical formulations and assumptions

Due to interaction between the kidnapper and its victims, it assumed that a susceptible individual in the human society will either embrace the idea of kidnapping as a business or become a victim of kidnapping and thus the population of discussion is characterized by susceptible $K_s(t)$, Infected individual (kidnapper), $K_I(t)$, Kidnap victim, $K_V(t)$, quarantined (enter detention facility) $K_D(t)$ and recovered $K_R(t)$. Susceptible individual transition to kidnapper compartment at the rate $\gamma K_I \frac{K_s}{N}$ and $\frac{K_s}{N}$ is the probability that a contact is with susceptible individuals and γ is the rate

constant. Supposed the rate of kidnapping is constant at α such that the transition rate $K_V \frac{K_s}{N}$ into

kidnap victims $K_V(t)$ is proportional to the contact between susceptible individuals and the kidnapper. As a result of anti-kidnapping operations by law enforcement agencies some individuals are quarantined ϕ (enter detention facility, $K_D(t)$). Let μ be the natural death in each compartment and η be the death due to torture at the detention facility or during anti-kidnapping operations by law enforcement agents or rescue operations. It is decreased by the repentance/reformation at the detention facility by ψ . As rescued victims becomes susceptible again, it is assumed that recruitment into the susceptible class is by birth, through rescued kidnap victims and repentant/reformed individual. We denote λN , ωK_V and ψK_D to represent respectively the per capita birth rate, the rate where victims are rescued and the individual repentant/reformed rate constants.

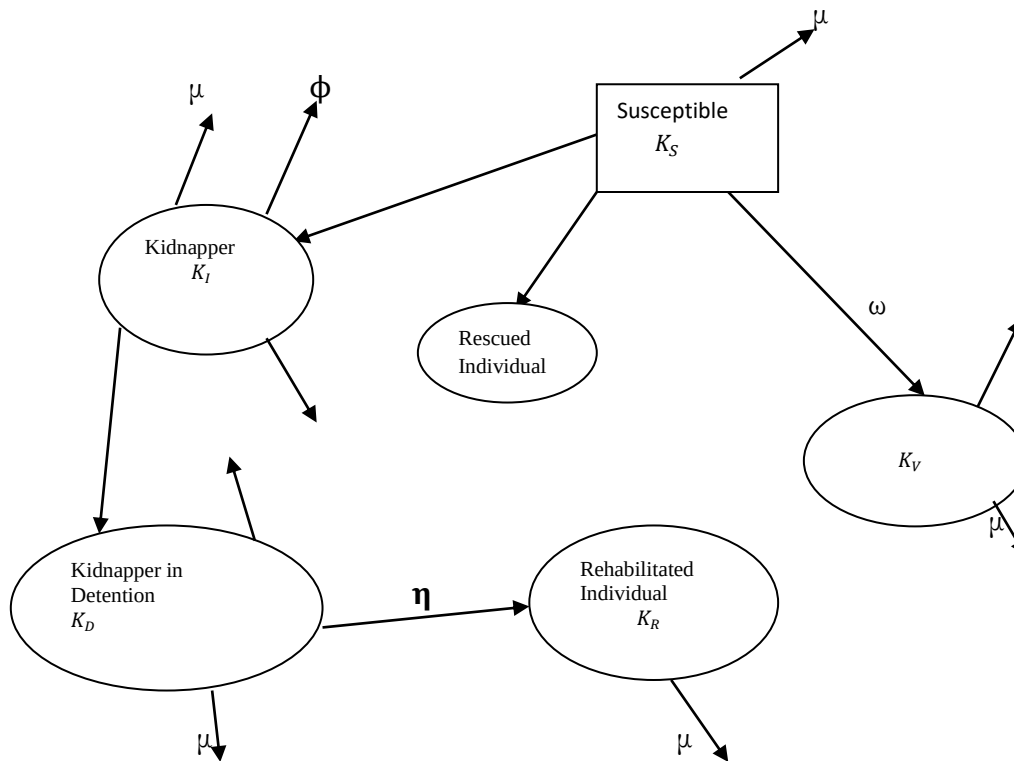


Figure 1. Kidnapping model flow diagram

Armed with the above assumptions and based on the idea as seen in [1] and [11], the mathematical structure becomes;

$$\begin{aligned}
 (i) \quad \frac{dk_s}{dt} &= \lambda N + \omega K_V + \psi K_D - \mu K_S - \gamma K_I \frac{K_S}{N} - \alpha K_V \frac{K_S}{N} \\
 (ii) \quad \frac{dK_I}{dt} &= \gamma K_I \frac{K_S}{N} - \mu K_I - \beta_1 K_I K_V - \phi K_I \\
 (iii) \quad \frac{dK_V}{dt} &= \alpha K_V \frac{K_S}{N} - (\mu + \omega) K_V - \beta_2 K_I K_V \\
 (iv) \quad \frac{dK_D}{dt} &= \phi K_I - (\eta + \mu + \psi) K_D \\
 (v) \quad \frac{dK_R}{dt} &= \eta K_D - \mu K_R
 \end{aligned} \tag{2.1}$$

2.1 Parameter values

Parameter values are listed in Table 2 with the various sources acknowledged. While some of the parameter values may display some global semblances, regional variations could be adapted to suit specific regions.

Table 1: Description of model's parameters and state variables

State Variables	Descriptions	Parameter	Descriptions
K_S	Susceptible individual (victim/crime propagator)	λ	Per capita birth rate
K_I	Kidnapper	μ	Natural death
K_V	Kidnap victim	ϕ	Detention rate
K_D	Kidnapper in detention facility	β_1	Death due to anti kidnapping operations by security agents
K_R	Reformed kidnapper (Recovered)	β_2	Kidnapping induced death
		α	Kidnapping rate
		η	Detention(torture) induced death
		ω	Rescue rate of kidnap victims
		ψ	Reformed or repentant rate at detention facility
		γ	Kidnapper's recruitment rate

Table 2: Model parameter and their dimensions

Parameter	Descriptions	Value	Unit	Source
λ	Per capita birth rate	0.000104	day^{-1}	[10]
μ	Natural death	0.0000356	day^{-1}	[10]
ϕ	Detention rate	0.009	day^{-1}	Assumed
β_1	Death due to anti kidnapping operations by security agents	0.000000000305	$humanday^{-1}$	[11]
β_2	Kidnapping induced death	0.000000000118	day^{-1}	[11]
α	Kidnapping rate	1.52	day^{-1}	[11]
η	Detention(torture) induced death	0.000000000100	$humanday^{-1}$	Assumed
ψ	Reformed or repentant rate at detention facility	0.000000000110	$humanday^{-1}$	Assumed

γ	Kidnapper's recruitment rate	0.305	day^{-1}	[11]
ω	Victim's rescue rate	0.098	day^{-1}	(as cited in [11])

3 The basic properties of the kidnapping model

3.1 Positivity of solutions

For the model in system (2.1) to make epidemiological (criminological) and mathematical meaning then all its state variables must be shown to be positive for all time. Thus the solution to the model system (2.1) with non-negative initial data remain positive for all $t \geq 0$.

Theorem 1. Supposed that $K_S(0) \geq 0, K_I(0) \geq 0, K_V(0) \geq 0, K_D(0) \geq 0, K_R(0) \geq 0$, then the solutions $K_S(t), K_I(t), K_V(t), K_D(t)$ and $K_R(t)$ of the system in (2.1) are all positive for all $t \geq 0$

Proof:

It follows from equation 2.1(i) that

$$\frac{dk_S}{dt} = \lambda N + \omega K_V + \psi K_D - \mu K_S - (\gamma + \alpha) K_I \frac{K_S}{N}$$

$$\frac{dk_S}{dt} \geq -\mu K_S - \pi K_S, \quad \pi = (\gamma + \alpha) \frac{K_I}{N}$$

$$\frac{dk_S}{dt} + (\mu + \pi) K_S \geq 0, \text{ by letting } f(t) = \pi + \mu \Rightarrow$$

$$\frac{dk_S}{dt} + f(t) K_S(t) \geq 0. \text{ Using integrating factor method and multiplying both sides of the inequality}$$

$$\text{by } \exp\left(\int_0^t f(s) ds\right)$$

$$\exp\left(\int_0^t f(s) ds\right) \frac{dk_S}{dt} + f(t) \exp\left(\int_0^t f(s) ds\right) K_S(t) \geq 0, \text{ we have}$$

$$\frac{d}{dt} (K_S(t) \exp(\int_0^t f(s) ds)) \geq 0, \text{ integrating the inequality from 0 to } t \text{ yield,}$$

$$\frac{d}{dt} \int_0^t (K_S(s) \exp(\int_0^t f(\mu + \pi) ds)) ds \geq 0. \text{ This implies that}$$

$$K_S(t) \geq K_S(0) \exp(\int_0^t f(\mu + \pi) ds). \text{ Thus } K_S(t) \geq 0, \forall t \geq 0. \text{ Similarly, it can be shown that}$$

$$K_I(t) \geq 0, K_V(t) \geq 0, K_D(t) \geq 0, \text{ and } K_R(t) \geq 0, \text{ this is done using the remaining equations in (2.1). Thus, equation 2.1(ii)}$$

$$\frac{dK_I}{dt} = \gamma K_I \frac{K_S}{N} - \beta_1 K_I K_V - (\phi + \mu) K_I$$

$$\frac{dK_I}{dt} = \gamma \rho K_I - \tau K_I - (\phi + \mu) K_I, \rho = \frac{K_S}{N}, \tau = \beta K_V$$

$$\frac{dK_I}{dt} - (\gamma \rho - \tau - (\phi + \mu)) K_I \geq 0 \text{ by letting } f(s) = \gamma \rho - \tau - (\phi + \mu) \Rightarrow$$

$$K_I(t) = K_I(0) \exp\left(\int_0^t f(\gamma \rho - (\tau - \phi - \mu)) ds\right) \text{ for all } t > 0. \text{ Repeating for equation 2.1(iii)}$$

$$\frac{dK_V}{dt} = \alpha K_V \frac{K_S}{N} - (\mu + \omega) K_V - \beta_2 K_I K_V$$

$$\frac{dK_V}{dt} = \alpha \kappa K_V - (\mu + \omega) K_V - \eta K_V, \kappa = \frac{K_S}{N}, \theta = \beta_2 K_I$$

$$\frac{dK_V}{dt} - (\alpha \kappa - (\mu + \omega + \theta)) K_V \geq 0, f(s) = (\alpha \kappa - (\mu + \omega + \theta))$$

$$K_V(t) = K_V(0) \exp\left(\int_0^t f(\alpha \kappa - (\mu + \omega + \theta)) ds\right). \text{ In the same vein, equation 2.1(iv) restated below}$$

$$\frac{dK_D}{dt} = \phi K_I - (\mu + \psi) K_D$$

becomes

$$K_D(t) = K_D(0) \exp\left(\int_0^t f((\mu + \psi)) ds\right). \text{ Finally, equations 2.1(v) yield}$$

$$K_R(t) = K_R(0) \exp\left(\int_0^t f(\mu) ds\right). \text{ This completes the proof.}$$

3.2 Invariant region

The population of interest represented by N at time t can be determined by

$N(t) = K_S(t) + K_I(t) + K_V(t) + K_D(t) + K_R(t)$, or by taking the derivatives of the state variables with respect to time we have

$$\frac{dN}{dt} = \frac{dK_S}{dt} + \frac{dK_I}{dt} + \frac{dK_V}{dt} + \frac{dK_D}{dt} + \frac{dK_R}{dt} \quad (3.1)$$

Upon substitution of equations (2.1) we have

$$\frac{dN}{dt} = \lambda N + \omega K_V + \psi K_D - \mu K_S - \gamma K_I \frac{K_S}{N} - \alpha K_V \frac{K_S}{N} + \gamma K_I \frac{K_S}{N} - \mu K_I - \beta_1 K_I K_V - \phi K_I$$

$$+ \alpha K_V \frac{K_S}{N} - (\mu + \omega) K_V - \beta_2 K_I K_V + \phi K_I - (\mu + \eta) K_D + \eta K_R - \mu K_R$$

$$\frac{dN}{dt} = (\lambda - \mu) N - (\beta_1 + \beta_2) K_I K_V \quad (3.2)$$

$$\frac{dN}{dt} = (\lambda - \mu) N. \quad (3.3)$$

(Assume no interactions between the kidnapper and the victim and imposing $t = 0, N = N_0$ as the initial human population). We now consider the analysis of system (2.1) in a feasible region.

Lemma 3.1: The model system (2.1) has solutions in the feasible region, Ω following the idea in [1] and [12].

Theorem 3.2: The solutions for system (2.1) is feasible for $t \leq 0$ if the solutions enter into the invariant region, Ω .

Proof: Let $\{K_S, K_I, K_V, K_D, K_R \in \mathbb{R}_+^5\}$ be any solutions to system (2.1) with non-negative initial conditions

$$\frac{dN}{dt} = (\lambda - \mu) N \quad \text{separating the variables and integrating,}$$

$$\int \frac{dN}{N} = \int (\lambda - \mu) dt \quad (3.4)$$

$$\ln N = (\lambda - \mu)t + C \quad (3.5)$$

$$N(t) = N(0) \exp(\lambda - \mu)t \quad (3.6)$$

Case I: $\lambda \neq \mu$ we have an open and varying population.

Case II: $\lambda = \mu$ we have a constant population at all time t .

Hence as $t \rightarrow \infty$ in equation (3.6) the human population N approaches $N(0)$ thus all feasible solutions set of the human population of system (2.1) enters the feasible region

$$\Omega = \{K_S, K_I, K_V, K_D, K_R \in \mathbb{R}_+^5 : K_S, K_I, K_V, K_D, K_R \geq 0; N \leq N(0)\} \quad (3.7)$$

This implies that the flow generated by system (2.1) is a positively invariant set and consequently the system (2.1) is criminologically and mathematically well-posed.

3.3 Existence of kidnap free equilibrium (KFE) and basic reproduction number

Kidnap free equilibrium (KFE) is a steady state where there are no kidnappers corresponding to $(K_S^* = 1, K_I^* = 0, K_V^* = 0, K_D^* = 0, K_R^* = 0)$. The Routh-Hourwitz criteria are used to establish asymptotic stability for non-linear differential equations. As demonstrated by [6], as seen in [1], the Routh-Hourwitz criteria provides not only the necessary but also the sufficient conditions for all

roots of characteristic polynomial containing a negative real part therefore means asymptotic stability. The local stability may be found from the Jacobian matrix at equilibrium points;

$$E_0 = (1, 0, 0, 0, 0). \quad (3.8)$$

Let $f_j(x)$ be the rate whereby new kidnappers transition into compartment i and $v_j(x)$ the net removal rate of kidnappers in compartment j .

If $\mathcal{F} = (f_1, \dots, f_{2n})$ and $\mathcal{V} = (v_1, \dots, v_{2n})$. Partitioning the derivatives we have

$$\Delta \mathcal{F}(E_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \Delta \mathcal{V}(E_0) = \begin{pmatrix} V & 0 \\ J_1 & J_2 \end{pmatrix} \text{ where}$$

$$F = \begin{bmatrix} \frac{\partial F_1}{\partial K_I} & \frac{\partial F_1}{\partial K_V} & \frac{\partial F_1}{\partial K_D} \\ \frac{\partial F_2}{\partial K_I} & \frac{\partial F_2}{\partial K_V} & \frac{\partial F_2}{\partial K_D} \\ \frac{\partial F_3}{\partial K_I} & \frac{\partial F_3}{\partial K_V} & \frac{\partial F_3}{\partial K_D} \end{bmatrix} x_0 \text{ and } V = \begin{bmatrix} \frac{\partial F_1}{\partial K_I} & \frac{\partial F_1}{\partial K_V} & \frac{\partial F_1}{\partial K_D} \\ \frac{\partial F_2}{\partial K_I} & \frac{\partial F_2}{\partial K_V} & \frac{\partial F_2}{\partial K_D} \\ \frac{\partial F_3}{\partial K_I} & \frac{\partial F_3}{\partial K_V} & \frac{\partial F_3}{\partial K_D} \end{bmatrix} x_0$$

The next generation matrix method is employed to derive the basic reproduction (Kidnapping production) number for the spread of kidnapping following the idea as cited in [11]). Clearly system (2.1) has a kidnapping free equilibrium (KFE) at $E_0 = (1, 0, 0, 0, 0)$.

Let $K = (K_S, K_I, K_V, K_D, K_R)^T$, so that system (2.1) can be written as
 $K' = F(K) - V(K)$
 where

$$F = \begin{bmatrix} \frac{\gamma}{N} & 0 & 0 \\ 0 & \frac{\alpha}{N} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} (\mu + \phi) & 0 & 0 \\ 0 & (\mu + \omega) & 0 \\ \phi & 0 & (\mu + \psi + \eta) \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{\mu + \phi} & 0 & 0 \\ 0 & \frac{1}{\mu + \omega} & 0 \\ \frac{1}{\phi} & 0 & \frac{1}{(\mu + \psi + \eta)} \end{bmatrix}.$$

The basic reproduction number is given by $R_0 = \rho FV^{-1}$ and ρ is the spectral radius as seen in [13]

$$\text{Thus } R_0 = \rho FV^{-1} = \begin{bmatrix} \frac{\gamma}{N} & 0 & 0 \\ 0 & \frac{\alpha}{NB} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\mu + \phi)} & 0 & 0 \\ 0 & \frac{1}{(\mu + \omega)} & 0 \\ \phi & 0 & \frac{1}{(\mu + \psi + \eta)} \end{bmatrix}.$$

$$FV^{-1} = \begin{bmatrix} \frac{\gamma}{A} & 0 & 0 \\ 0 & \frac{\alpha}{B} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \mu + \phi, N = S, B = \mu + \omega \quad \text{implies that } R_0 = \frac{\gamma\alpha}{AB} = \frac{\alpha\gamma}{(\mu+\phi)(\mu+\omega)}.$$

It should be noted however that the kidnapping free equilibrium (KFE), $E_0 = (1, 0, 0, 0, 0)$ is locally asymptotically stable if the quantity $R_0 < 1$ that is, kidnapping washes out in the human society or unstable if and only if $R_0 > 1$, that is, kidnapping persist in the human society. The threshold quantity R_0 is basic reproductive (kidnapping propagation) number of kidnappers. It is the transmission of a new kidnapper and victims produced by potential kidnapping. The question that arise bothers on the possibility of eradicating completely kidnapping? While this may seems possible as suggested by the assumptions of $R_0 < 1$, it is not feasible to achieve this since human vulnerabilities to crime and criminality put a spot on this assumption and hence our papers only propose some control measures which will minimize the spread of the complex nature of the 'disease'.

3.4 Existence of local stability of kidnap free equilibrium (KFE) point

Linearizing the system (2.1) about the equilibrium points results thus;

$$\begin{aligned} F_1(K_S, K_I, K_V, K_D, K_R) &= \lambda N + \omega K_V + \psi K_D - \mu K_S - \gamma K_I \frac{K_S}{N} - \alpha K_V \frac{K_S}{N} \\ F_2(K_S, K_I, K_V, K_D, K_R) &= \gamma K_I \frac{K_S}{N} - \mu K_I - \beta_1 K_I K_V - \phi K_I \\ F_3(K_S, K_I, K_V, K_D, K_R) &= \alpha K_V \frac{K_S}{N} - (\mu + \omega) K_V - \beta_2 K_I K_V \\ F_4(K_S, K_I, K_V, K_D, K_R) &= \phi K_I - (\mu + \eta + \psi) K_D \\ F_5(K_S, K_I, K_V, K_D, K_R) &= \eta K_D - \mu K_R \end{aligned} \tag{3.9}$$

By letting $X = \{K_S, K_I, K_V, K_D, K_R\}$. Since system (2.1) have continuous partial derivatives, evaluating the partial derivatives at equilibrium points, the Jacobian matrix becomes

$$J_{(X)} = \begin{bmatrix} \frac{\partial F_1}{\partial K_S} & \frac{\partial F_1}{\partial K_I} & \frac{\partial F_1}{\partial K_V} & \frac{\partial F_1}{\partial K_D} & \frac{\partial F_1}{\partial K_R} \\ \frac{\partial F_2}{\partial K_S} & \frac{\partial F_2}{\partial K_I} & \frac{\partial F_2}{\partial K_V} & \frac{\partial F_2}{\partial K_D} & \frac{\partial F_2}{\partial K_R} \\ \frac{\partial F_3}{\partial K_S} & \frac{\partial F_3}{\partial K_I} & \frac{\partial F_3}{\partial K_V} & \frac{\partial F_3}{\partial K_D} & \frac{\partial F_3}{\partial K_R} \\ \frac{\partial F_4}{\partial K_S} & \frac{\partial F_4}{\partial K_I} & \frac{\partial F_4}{\partial K_V} & \frac{\partial F_4}{\partial K_D} & \frac{\partial F_4}{\partial K_R} \\ \frac{\partial F_5}{\partial K_S} & \frac{\partial F_5}{\partial K_I} & \frac{\partial F_5}{\partial K_V} & \frac{\partial F_5}{\partial K_D} & \frac{\partial F_5}{\partial K_R} \end{bmatrix}$$

$$J_{(0,0,0,0,0)} = \begin{bmatrix} -\mu & 0 & \omega & 0 & 0 \\ 0 & -\mu - \phi & 0 & 0 & 0 \\ \mu & 0 & -\mu - \omega & 0 & 0 \\ 0 & \phi & 0 & -\mu - \eta - \psi & 0 \\ 0 & 0 & 0 & \eta & -\mu \end{bmatrix} \quad (3.91)$$

The characteristic equation is $|J - \lambda I| = 0 \Rightarrow$

$$\begin{vmatrix} -\mu - \lambda & 0 & \omega & 0 & 0 \\ 0 & -\mu - \phi - \lambda & 0 & 0 & 0 \\ 0 & 0 & -\mu - \omega - \lambda & 0 & 0 \\ 0 & \phi & 0 & -\mu - \eta - \psi - \lambda & 0 \\ 0 & 0 & 0 & \eta & -\mu - \lambda \end{vmatrix} = 0 \quad (3.92)$$

On evaluation the Jacobian at equilibrium implies

$$(-\mu - \lambda)(-\mu - \phi - \lambda)(-\mu - \omega - \lambda)(-\mu - \eta - \psi - \lambda)(-\mu - \lambda) = 0 \quad (3.93)$$

$$-[\lambda^5 + (\omega + \psi + 4\mu + \phi + \eta)\lambda^4 + (10\mu^2 + 4\phi\mu + 4\omega\mu + 4\psi\mu + 4\eta\mu + \phi\omega + \eta\omega + \phi\psi + \phi\eta + \omega\psi + \omega + \mu)\lambda^3 + (8\mu^3 + 6\phi\mu^2 + 6\omega\mu^2 + 5\eta\mu^2 + 5\psi\mu^2 + 2\phi\mu^2 + 3\eta\omega\mu + 3\psi\phi\mu + 3\phi\omega\mu + 3\eta\phi + 2\phi\omega\mu + 3\phi\omega\mu + 2\eta\mu + 2\omega\mu + \phi\psi\omega + \phi\eta\omega + \psi\omega\mu)\lambda^2 + (4\mu^3 + 3\phi\mu^3 + 3\eta\mu^3 + 4\omega\mu^3 + 3\eta\mu^3 + 3\omega\phi\mu^2 + 3\eta\omega\mu^2 + \eta\omega\mu + 2\phi\omega\eta\mu + \eta\mu^2 + \phi\eta\mu^2)\lambda + \mu^5 + \omega\mu^4 + \psi\mu^4 + \eta\mu^4 + \phi\mu^4 + \phi\psi\mu^3 + \psi\omega\mu^3 + \eta\phi\mu^3 + \eta\omega\mu^3 + \phi\psi\omega\mu^2 + \eta\phi\omega\mu^2]. \quad (3.94)$$

Collecting the coefficient of the eigenvalues will yield

$$\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \quad (3.95)$$

where

$$a_0 = \mu^5 + \eta\mu^4 + \phi\mu^4 + \psi\mu^4 + \omega\mu^4 + \phi\psi\eta^3 + \psi\omega\mu^3 + \phi\eta\mu^3 + \eta\omega\mu^3 + \phi\psi\omega\mu^2 + \phi\omega\mu^2 + \phi\eta\omega\mu^2,$$

$$a_1 = 4\mu^3 + 3\phi\mu^3 + 3\eta\mu^3 + 4\omega\mu^3 + 3\eta\mu^3 + 3\omega\phi\mu^2 + 3\eta\omega\mu^2 + \eta\omega\mu + 2\phi\omega\eta\mu + \eta\mu^2 + \phi\eta\mu^2,$$

$$a_2 = 8\mu^3 + 6\phi\mu^2 + 6\omega\mu^2 + 5\eta\mu^2 + 5\psi\mu^2 + 2\phi\mu^2 + 3\eta\omega\mu + 3\psi\phi\mu + 3\phi\omega\mu + 3\eta\phi + 2\phi\omega\mu + 3\phi\omega\mu + 2\eta\mu + 2\omega\mu + \phi\psi\omega + \phi\eta\omega + \psi\omega\mu,$$

$$a_3 = 10\mu^2 + 4\phi\mu + 4\omega\mu + 4\psi\mu + 4\eta\mu + \phi\omega + \eta\omega + \phi\psi + \phi\eta + \omega\psi + \omega + \mu,$$

$$a_4 = \omega + \psi + 4\mu + \phi + \eta.$$

Applying the Routh-Hourwitz criteria, it follows that the eigenvalues have negative real part and as a result the kidnap free equilibrium is locally asymptotically (LAS) stable since there is no kidnapper and if it satisfies equation (3.96)

$$a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, \frac{a_3 a_4^2 - a_2 a_4}{a_1 a_4 - a_0} > 0, \frac{a_2 a_3 a_4 - a_2^2}{a_3 a_4 - a_2} > 0, \frac{a_3 a_4 - a_2}{a_4} > 0. \quad (3.96)$$

3.5 Existence of endemic equilibrium (EEE)

The endemic equilibrium point is a positive steady state where the crime of kidnapping persists in the population. It means that system (2.1) has endemic equilibrium E_e . Hence the model equation (2.1) is locally asymptotically stable since kidnappers still exist. It follows that the crime propagation number (basic reproduction number) $R_0 > 1$ implies a persistent kidnapping scenario in the human population according to [13]. While $R_0 < 1$ signifies a reduction in kidnapping crime in the human population.

4 Numerical simulations

The numerical simulation was carried out using MATLAB's ode45 with parameter values as depicted in Table 2 with appropriate initial conditions and defined as follow:

$$\begin{aligned} N &= 1, \lambda = 0.000104, w = 0.098 \\ \beta_1 &= 0.000000000235, \beta_2 = 0.000000000235, \phi = 0.009 \\ \eta &= 0.0000000001, K_S(0) = 0.8, K_I(0) = 0.1, K_V(0) = 0.05, K_D(0) = 0.03, K_R(0) = 0.02 \end{aligned}$$

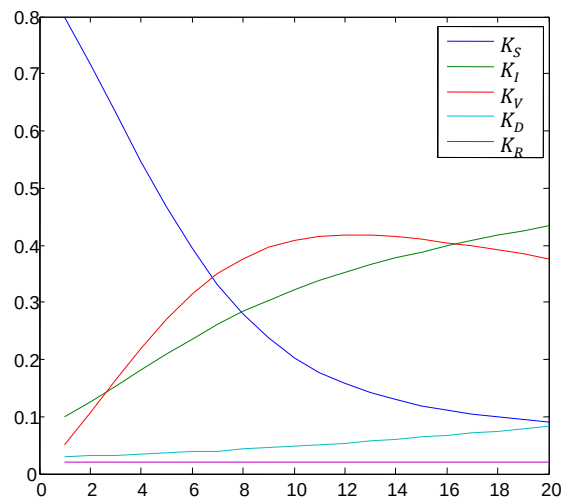


Figure 1: Assessing the impact of 50% kidnapping rate (α) in a typical Nigeria human society

Figure 1 shows that a 50% increase in the kidnapping rate (α) after a kidnapper (K_I) is introduced into a susceptible human population (K_S) led to a gradual rise in the number of Kidnappers and kidnap victims which peaked between 60days and 120day with a sharp decrease in the susceptible population. As could be seen from the graph that a steady rise in the recruitment rate will result in

the rise in kidnapping activities with more persons becoming victims. If unchecked, such a human society would be difficult to habit; thus, social and business activities will be adversely affected.

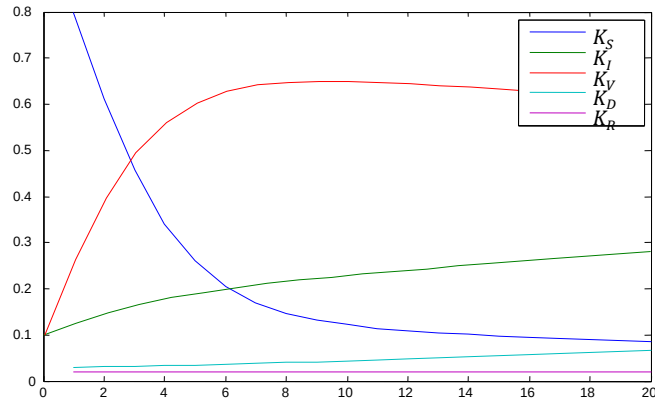


Figure2: Assessing the impact of 150% kidnapping rate (α) in a typical Nigeria human society.

Figure 2 presents a scenario where a 150% increase in the recruitment rate evolves into an endemic kidnapping situation, as could be seen in the sharp rise in victims of the 'disease' peaking between 60-100 days and remaining stable, leading to the decline in the susceptible population. Such a situation will require pragmatic steps such as law enforcement agencies ramping up rescue operations, proactive anti-kidnapping raiding of identified kidnappers' hideouts, forfeiture of known assets of perpetrators of the crime, and such measures, detention, and even capital punishment.

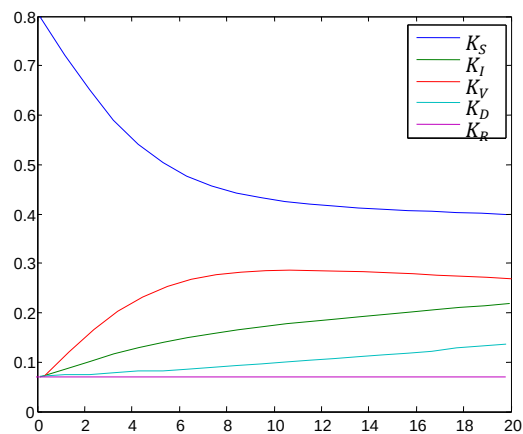


Figure3. Impact of 50% death due to anti-kidnapping operations by security agents (β_1) in a typical Nigeria human society

Figure 3 showcases the impact of a 50% increase in the death rate due to anti-kidnapping operation activities by the law enforcement agents with a rise in the gradual rise in the number of death of kidnappers(beta), detention of kidnappers and rescue of victims, a decline in the number of victims of the 'disease' is observed with the recovery of the susceptible subpopulation. This is akin to a situation where the crime of kidnapping gradually becomes unattractive due to sustained anti-kidnapping operations from law enforcement agents and the result is the witnessing of the gradual decline in victims accompanied with a corresponding increase in the susceptible class.

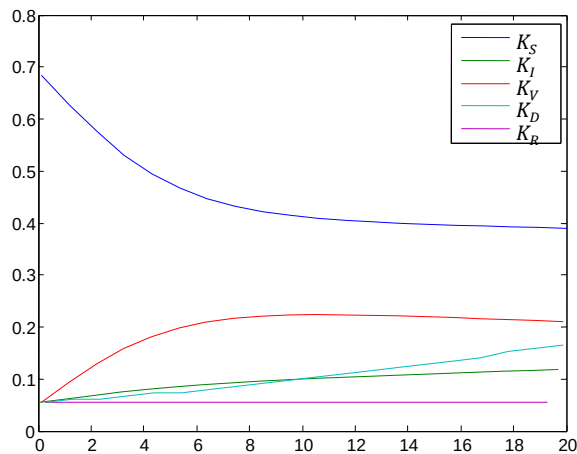


Figure 4. Assessing the impact of 150% death due to anti-kidnapping operations by security agents (beta) in a typical Nigeria human society.

Figure 4 depicts the impact of a one hundred and fifty (150%) percent death due to anti-kidnapping operations by law enforcement agents which resulted from a combination of rescue of victims and the rise of the number of kidnappers in detention ultimately leading to a decrease in the number of victims of the crime thus the susceptible human population instead of experiencing a stable decline as initially seen adjusted reacting to the effectiveness of rescue operations and rise in detention of kidnappers.

5 Discussion of results

Our model attempt to illustrate a real life phenomenon of kidnapping that has bedecked a typical Nigerian society. We used the infectious disease model to mirror a typical kidnapping situation whereby the susceptible human population due to their interactions with the infectious individuals (kidnaper) becomes a victim of kidnapping or embrace the business of kidnapping. From equation (3.95) a kidnap free human population is achieved where equation (3.96) is satisfied in line with the Routh-Hourwitz criteria. This will require a combination of policies such as a quick response by law-enforcement agents to rescue victims with minimal incidence of death and payment of ransom, prosecution of perpetrators of the crime. Weeding out security operatives who aid and abet the crime.

We also carried out numerical simulation using MATLAB's ODE45 a variable order Runge-Kutta method with respective absolute and relative tolerance levels of 10^{-9} and 10^{-8} . A (50%) increase in the kidnapping recruitment rate (α) led to a steady rise in the victims of the crime in human population witnessing in the process a reduction in the susceptible subpopulation (Figure 1). Similarly, a 150% increase in the recruitment rate (α) also witnessed an evolution of the crime of kidnapping into an endemic situation as could be seen in Figure 2. Thus a high proportion of kidnap victims will be experienced in such a human society. In this instance more than half of the human population became victims of the 'disease'. The implication of this scenario is that apart from the negative impact on global security and decline in productivity, the harrowing experiences of the victims of this social malaise can be best imagined; ranging from torture, maiming, rapping of victims, forceful recruitment into the crime against the individual's wishes and a host of others.

Similar simulations were done to assess the impact of beta (β_1) using a 50% and 150% increase to evaluate the impact of death due to anti-kidnapping operations on the human population and the results indicated that an increase in number of kidnappers' death and detention as well as prompt rescue of victims by the law enforcement agents will ultimately decrease the victims rate and the susceptible human subpopulation will instead of steady decline as initially seen in the simulation will adjust and remained stable (see figures 3 & 4). This scenario is ideal for the survival of such human society and sustainable where the right environment is created for copious employment opportunities to thrive through a mix of policies that weeds out law enforcement agents that aid and abet such crimes, prompt prosecution of offenders, the sincerity of state actors like political office holders seen to be above board in the discharge of their responsibilities to the citizens, ban of sales of hard drugs in our society and a host of others.

6 Conclusion and further research

In this model, we attempt to use mathematical model in proffering solutions to the menace of kidnapping for ransom payment in a typical Nigeria society. The vital information obtained from the analysis is the possibility of eradicating kidnapping crime by discouraging the recruitment of kidnappers and enhancing prompt rescue of kidnap victims with minimal casualty on the part of the victim as well as decimating the kidnappers through sustained anti-kidnapping operations by law enforcement agents leading to persecution and detention as illustrated in figures (1-4). Most of the kidnapping cases are carried out under the influence of hard drugs by armed bandit, terrorist and armed robbers. Thus, illicit drug dealers should be treated as enemies of the society. Though it is a difficult task to dissuade a hard drug addict from committing crime, great care should be exercised during rescue operations since a kidnap free state that is globally asymptotically stable is achieved where there is additional conditions (Theorem 3.2) apart from the basic crime reproduction number $R_0 < 1$. This by implication means that families of kidnap victims will prefer to pay ransom for the releases of their relatives than rely on the rescue strategies of law enforcement agents especially where there are casualties of kidnap victims during rescue operations and hence the likelihood of the perpetuation of the kidnapping crime will persist. Also, a combination of policies thrust is required by the various governments at all levels to create an enabling environment for employment to thrive. Further research could be done on the bifurcation of the crime of kidnapping which was not considered in this paper.

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On The Applications of Type II Half Logistic Extended Exponential distribution to Modeling Failure data

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Abstract

This article introduces a three-parameter extension of the Exponentiated exponential distribution called Type II half logistic Extended Exponential distribution. The proposed model is quite flexible and adaptable to model any kind of life-time data. Its probability density function may sometimes be unimodal and its corresponding hazard rate may be of monotone or non-monotone shape. Standard statistical properties such as its ordinary and incomplete moments, quantile function, moment generating function, reliability function, order statistics, and Renyi entropy are obtained. The maximum likelihood method is used to obtain the estimates of the model parameters. Two practical examples of failure data sets are presented to demonstrate the flexibility of the developed model.

Keywords and phrases: maximum likelihood estimation; incomplete moments; hazard rate; type ii half logistic extended exponential distribution

1 Introduction

In a more recent time, several distributions have been developed and studied which serves life-time models in applied sciences. The most popular among the life-time model is the so-called generalized beta-g family of distributions developed and studied by [15], the others includes: Kumaraswamy generalized family was developed by [8], generalized transmuted-G family was introduced by [18], exponentiated generalized family studied by [9], the type I half-logistic family introduced by [11], Transformed-Transformer (T-X) developed by [2], Weibull-G family was introduced by [6], Exponentiated half-logistic family by [10], the Kumaraswamy Weibull family developed by [13], odd Lomax-G developed by [12], modified Kies-G family by [4], The Half-Logistic Generalized Weibull Distribution by [20] and generalized linear failure rate-G family was studied and developed by [3].

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The main motivation of the study is to further enhance the scope of applications of the Exponentiated Exponential distribution to cope with different shapes of the failure rate.

1.1 Extended exponential distribution

If the random variable X has the distribution function that follows the Extended Exponential (EE) distribution, then the corresponding density function is given as

$$g(x) = \lambda \theta e^{-\lambda x} (1 - e^{-\lambda x})^{\theta-1}, \quad x > 0; \lambda, \theta > 0, \quad (1)$$

and its cumulative density function and hazard function is respectively given by

$$G(x) = (1 - e^{-\lambda x})^{\theta-1}, \quad x > 0; \lambda, \theta > 0, \quad (2)$$

and

$$h(x) = \frac{\lambda \theta e^{-\lambda x} (1 - e^{-\lambda x})^{\theta-1}}{1 - (1 - e^{-\lambda x})^{\theta-1}}, \quad (3)$$

where λ , a positive scale parameter and θ is a positive shape parameter. The shape of the hazard function of EE can take different forms, it has an increasing or decreasing hazard function $\theta > 1$ or $\theta < 1$ and for $\theta = 1$, it exhibits a constant hazard.

2 Type II Half logistic Extended Exponential distribution

The Type II half logistic-G (TIIHL-G) family of distributions was developed by [13] with the CDF given by

$$\begin{aligned} F(x; \theta, \varphi) &= 1 - \int_0^{-\log G(x; \psi)} \frac{2\alpha e^{-at}}{(1 + e^{-at})^2} dt \\ &= \frac{2[F(x; \psi)]^\alpha}{1 + [F(x; \psi)]^\alpha}, \quad x > 0; \alpha > 0, \end{aligned} \quad (4)$$

where $G(x; \varphi)$ the CDF of baseline model with parameter is vector φ and $G(x; \alpha, \varphi)$ is CDF obtained by using the T-X generator developed by [1]. The PDF of the TIIHL-G family is given by

$$f(x; \theta, \varphi) = \frac{2\theta g(x; \varphi)[G(x; \psi)]^{\alpha-1}}{[1 + [G(x; \psi)]^\alpha]^2}, \quad x > 0, \theta > 0. \quad (5)$$

A random variable (r.v) X follows Type II half logistic Frechet (TIIHLEE) distribution if its PDF is obtained by inserting (1) and (2) in (5) as follows:

$$f(x) = \frac{2\alpha \lambda \theta e^{-\lambda x} (1 - e^{-\lambda x})^{\theta\alpha-1}}{[1 + (1 - e^{-\lambda x})^{\theta\alpha}]^2}, \quad (6)$$

and its corresponding CDF is given by

$$F(x) = \frac{2(1 - e^{-\lambda x})^{\theta\alpha}}{1 + (1 - e^{-\lambda x})^{\theta\alpha}}. \quad (7)$$

The graph of the CDF, PDF of TIIHLEE distribution is given in figure (1) and (2), respectively for various values of the parameters.

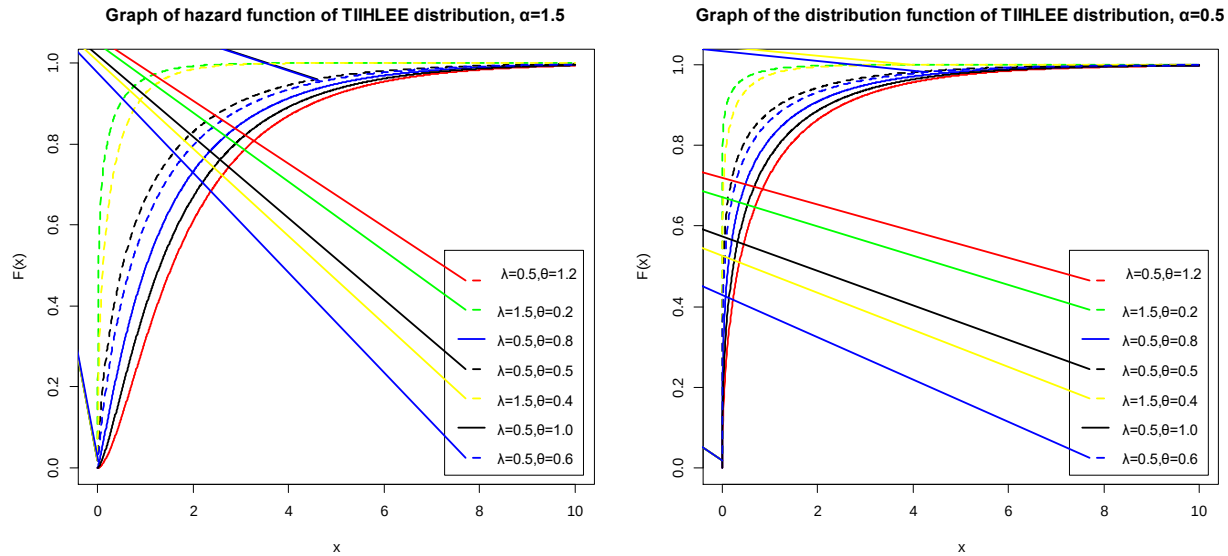


Figure 1: Graph of the PDF of TIHLEE distribution

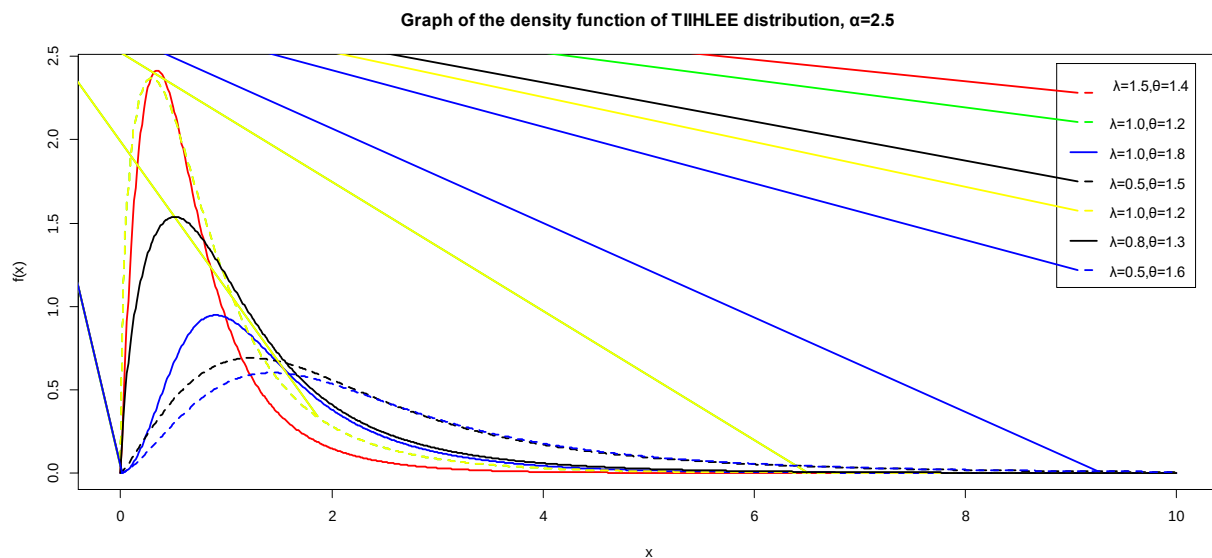


Figure 2: Graph of the PDF of TIHLEE distribution

Correspondingly, an expression for the survival ($S(x)$) and the hazard ($h(x)$) function of TIHLEE distribution is given respectively as

$$S(x) = \frac{1 - (1 - e^{-\lambda x})^{\theta\alpha}}{1 + (1 - e^{-\lambda x})^{\theta\alpha}}, \quad (8)$$

and

$$h(x) = \frac{2\alpha\lambda\theta e^{-\lambda x}(1 - e^{-\lambda x})^{\theta-1}(1 - e^{-\lambda x})^{\theta(\alpha-1)}}{[1 - (1 - e^{-\lambda x})^{\theta\alpha}][1 + (1 - e^{-\lambda x})^{\theta\alpha}]} \quad (9)$$

The graph of the $S(x)$, $h(x)$ of TIHLEE distribution is given in figure (3) and (4) for various values of the parameters, respectively.

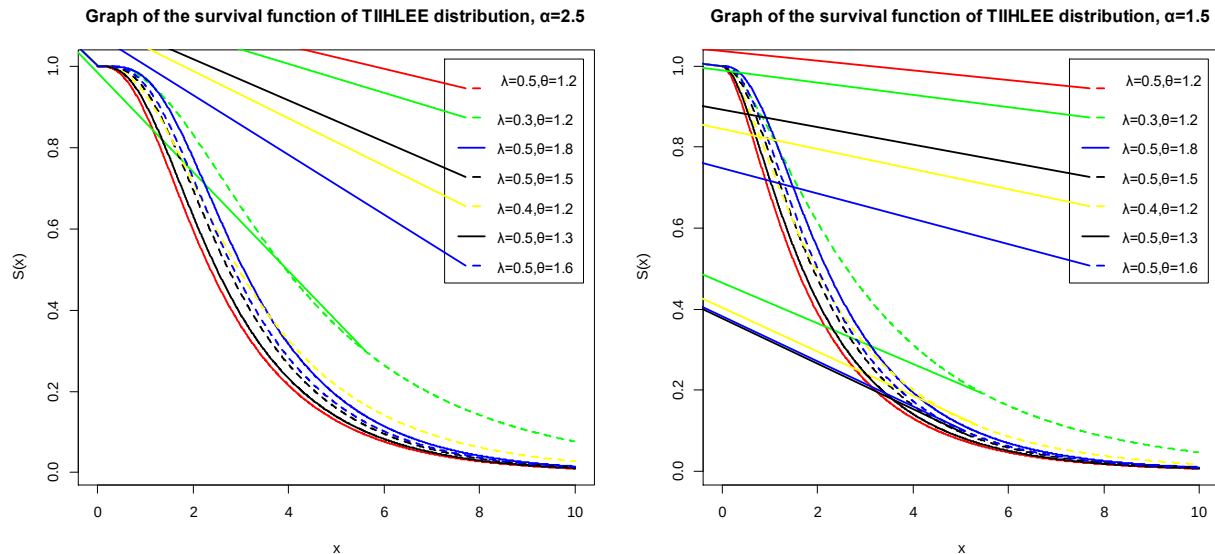


Figure 3: Graph of the $S(x)$ TIHLEE distribution

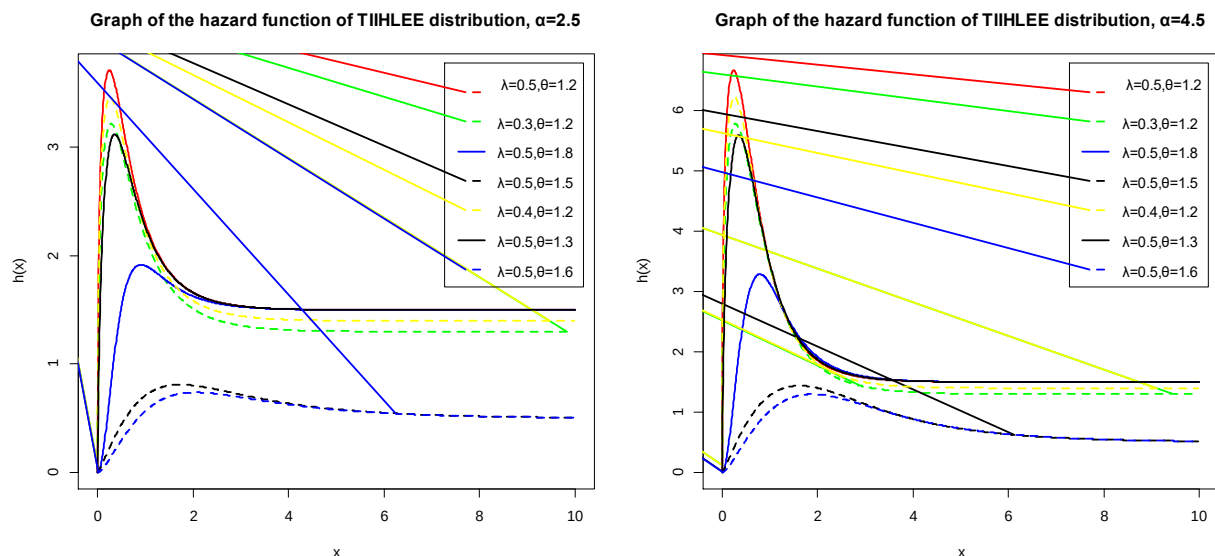


Figure 4: Graph of the $h(x)$ of TIHLEE distribution

3 Important representations

In this section, an important tool for the expansion of the PDF for *TIHLEE* is provided. From the generalized binomial series given by

$$(1 + v)^{-w} = \sum_{i=0}^{\infty} (-1)^i \binom{w + i - 1}{i} v^i. \quad (10)$$

For $|v| < 1$ and w is a positive real non-integer. Then, by applying the binomial theorem (10) in (6), the density function of *TIHLEE* distribution becomes

$$f(x) = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} e^{-\lambda x(j+1)} \quad (11)$$

Here, we derived some properties of *TIHLEE* distribution.

3.1 Quantile function

The quantile function of X is denoted by $Q(u)$, defined as

$$Q(u) = -1/\lambda \left[1 - \frac{1}{\alpha\theta} \left(\frac{u}{2-u} \right) \right]. \quad (12)$$

By taking the value of $u = 0.5, 0.75$, in (12) we obtain an expression for the median (m) and the upper quartile (UQ) of the *TIHLEE* distribution respectively, given as

$$m = -1/\lambda \left[1 - \frac{1}{5\alpha\theta} \right], \quad (13)$$

$$QU = -1/\lambda \left[1 - \frac{3}{5\alpha\theta} \right]. \quad (14)$$

3.2 Moments of *TIHLEE* distribution

The moment plays an important role in distribution study and real data application(s). Now, we obtain the s^{th} moment of the *TIHLEE* distribution under certain regularity condition; the s^{th} moment of *TIHLEE* distribution is obtained as

$$\mu'_s = \int_0^{\infty} x^s f(x) dx = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} J(x), \quad (15)$$

where

$$J(x) = \int_0^{\infty} x^s e^{-\lambda x(j+1)} dx. \quad (16)$$

Taking $w = \lambda x(j+1)$, $x = \frac{w}{\lambda(j+1)}$, $dw = \lambda(j+1)dx$ and substitute in (16), we have

$$J(x) = [\lambda(j+1)]^{-(s+1)} \int_0^\infty w^s e^{-w} dw. \quad (17)$$

Finally, we have an expression for the k th moment of *TIHLEE* distribution given as

$$\mu'_s = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)} \right]^{s+1} \Gamma(s+1). \quad (18)$$

An expression for the mean (μ'_s) the second (μ'_2) and the third (μ'_3) moments of *TIHLEE* distribution can be derived respectively as

$$\mu'_1 = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)} \right]^2, \quad (19)$$

$$\mu'_2 = 4\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)} \right]^3 \quad (20)$$

and

$$\mu'_3 = 12\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)} \right]^4, \quad (21)$$

the coefficient of skewness (q_s) and kurtosis (q_k) of *TIHLEF* distribution are given by

$$k_s = \frac{\mu_3}{(\mu'_2 - [\mu'_1]^2)^{3/2}}, \quad k_k = \frac{\mu_4}{(\mu'_2 - [\mu'_1]^2)^2}.$$

Table 1 gives the numerical values of mean (μ'_1), variance (σ^2), skewness (k_s) and kurtosis (k_k) for various values of the parameters of *TIHLEE* distribution.

Table 1: Table of first six moments, variance, skewness and kurtosis of *TIHLEE* distribution.

	$\theta = 0.2, \lambda = 1.2$				
Moments	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.8$	$\theta = 1.2$	$\theta = 1.6$
μ_1	0.0135	0.0563	0.1191	0.1887	0.2650
μ_2	0.0166	0.0674	0.1372	0.2095	0.2843
μ_3	0.0375	0.1508	0.3038	0.4590	0.6165
μ_4	0.1199	0.4808	0.9646	1.4515	1.9414
μ_5	0.4904	1.9638	3.9334	5.9087	7.8897
μ_6	2.4310	9.7289	19.4712	29.2269	38.9961
σ^2	0.0164	0.0642	0.1230	0.1739	0.2141
k_s	17.5090	8.5864	5.9833	4.8796	4.3180
k_k	437.382	108.6142	54.9103	37.8987	30.3941

3.3 Incomplete moments of *TIHLEE* distribution

The incomplete moment of *TIHLEE* distribution can be obtained using (11) as

$$\eta'_s = \int_0^t x^s f(x) dx = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} J(x) \quad (22)$$

where

$$J(x) = \int_0^t x^s e^{-\lambda x(j+1)} dx. \quad (23)$$

Taking $w = \lambda x(j+1)$, $x = \frac{w}{\lambda(j+1)}$, $dw = \lambda(j+1)dx$ and substitute in (23), we have

$$J(x) = [\lambda(j+1)]^{-(s+1)} \int_0^{\lambda x(j+1)} w^s e^{-w} dw. \quad (24)$$

Finally, we have an expression for the k th moment of *TIHLEE* distribution given as

$$\eta'_s = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)} \right]^{s+1} \Gamma(s+1, \lambda t(j+1)) \quad (25)$$

3.4 Renyi entropy

Renyi entropy was proposed by [17]. It can be obtained by

$$I_\eta(x) = \frac{1}{1-\eta} \log V \quad (26)$$

where

$$V = \int_{-\infty}^{\infty} f^\eta(x) dx, \quad \eta > 0, \eta \neq 1. \quad (27)$$

Plugging (6) in (27), then applying binomial theorem given in (10), we have

$$\begin{aligned} V &= 2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} \int_0^\infty e^{-\lambda x(\eta+j)} dx \\ &= 2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} [\theta(\eta+j)]^{-1}. \end{aligned} \quad (28)$$

Finally, we have an expression for the Renyi entropy of *TIHLEE* distribution as

$$I_\eta(x) = \frac{1}{1-\eta} \log \left(2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} [\theta(\eta+j)]^{-1} \right) \quad (29)$$

3.5 Order statistics of *TIHLEE* distribution

Order statistics is widely applied in statistical theory. Suppose $X_1, X_2, \dots, X_n$ be a random sample having CDF $F(x)$. Let $x_{1:n}, x_{2:n}, \dots, x_{n:n}$ be the ordered sample of size n , then the density of r^{th} order statistics is given as

$$f_{r:n}(x) = w \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} f(x) F(x)^{l+r-1} \quad (30)$$

where $w = \frac{n!}{(n-r)!r!}$. The PDF of the r^{th} order statistics of *TIHLEE* distribution is obtained by putting (6) and (7) in (30), changing s with $l+r-1$ followed by simple algebraic manipulation, we have

$$f_{r:n}(x) = 4\alpha\lambda\theta w \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} e^{-\lambda x} (1 - e^{-\lambda x})^{\theta\alpha(l+r)} \left[1 + (1 - e^{-\lambda x})^{\theta\alpha} \right]^{-(l+r+3)} \quad (31)$$

By taking the value of $r = 1$ and n in equation (31), we respectively obtain the first and the n^{th} order statistics of *TIHLEE* distribution.

3.6 Maximum likelihood estimation

Given a random sample of $x_1, x_2, \dots, x_n$ from the *TIHLEE* distribution, the likelihood function for $\psi = (\alpha, \lambda, \theta)$ is

$$l(\alpha, \lambda, \theta) = \log(2\alpha\lambda\theta) + \lambda \sum_{i=1}^n x_i + (\theta\alpha - 1) \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) - 2\theta \sum_{i=1}^n \log \left[1 + (1 - e^{-\lambda x_i})^{\theta\alpha} \right]. \quad (32)$$

Maximizing $l(\alpha, \lambda, \theta)$ with respect to α, λ , and θ , we obtain the following system of nonlinear equations:

$$\frac{n}{\alpha} + \theta \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) - 2 \sum_{i=1}^n \left(\frac{(1 - e^{-\lambda x_i})^{\theta\alpha} \log(1 - e^{-\lambda x_i})}{[1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]} \right) = 0 \quad (33)$$

$$\frac{n}{\lambda} - \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})} + 2 \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i}) [1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]} = 0 \quad (34)$$

$$\frac{n}{\theta} + \alpha \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) - 2\alpha \sum_{i=1}^n \frac{\log(1 - e^{-\lambda x_i})}{[1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]} = 0 \quad (35)$$

respectively. The solution of this system has to be calculated numerically to get the estimates $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$.

4 Application

In this section, we compare the results of fitting the TIHLEE (α, λ, θ) and EE (λ, θ) and E (λ) distributions to two real data sets. All the computations done using the R programming language (R Development Core Team, First, we consider the data set consisting of the length of intervals times at which vehicles pass a point on a road. The first data set is given as follows: 2.50, 2.60, 2.60, 2.70, 2.80, 2.80, 2.90, 3.00, 3.00, 3.10, 3.20, 3.40, 3.70, 3.90, 3.90, 3.90, 4.60, 4.70, 5.00, 5.60, 5.70, 6.00, 6.00, 6.10, 6.60, 6.90, 6.90, 7.30, 7.60, 7.90, 8.00, 8.30, 8.80, 8.80, 9.30, 9.40, 9.50, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31.0, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8. Total Time on Test plot is given in Figure 3a which indicates that the data exhibits non-monotone failure rate and Figure (3b) is the graph of empirical density function for data I, shows that the data is positively skewed. From Table 2.0, it could be observed that the data is over-dispersed leptokurtic.

The second data set introduced by [14]: 112, 68, 84, 109, 153, 143, 60, 70, 98, 164, 63, 63, 77, 91, 91, 66, 70, 77, 63, 66, 66, 94, 101, 105, 108, 112, 115, 126, 161, 178. These data represent the numbers of tumor-days of 30 rats fed with unsaturated diet. [5] and [7] used the Gompertz distribution for these data set in order to obtain exact confidence intervals and joint confidence regions for the parameters based on two different statistical analysis. Figure 4 indicates that the second data set exhibits increasing failure rate and moderately positive skewed. From Table 2.0, it could be observed that the data is over-dispersed mesokurtic.

The MLEs of the model parameters errors in parentheses) and the values of the AIC (Akaike Information CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion)

Table 2 : Exploratory data analysis for data set I

min	q_1	median	Mean	q_3	Max.	variance	kurtosis	Skewness	Range
2.5	5.9	12.1	21.6	29.9	119.8	574.6	6.6	1.9	117.3

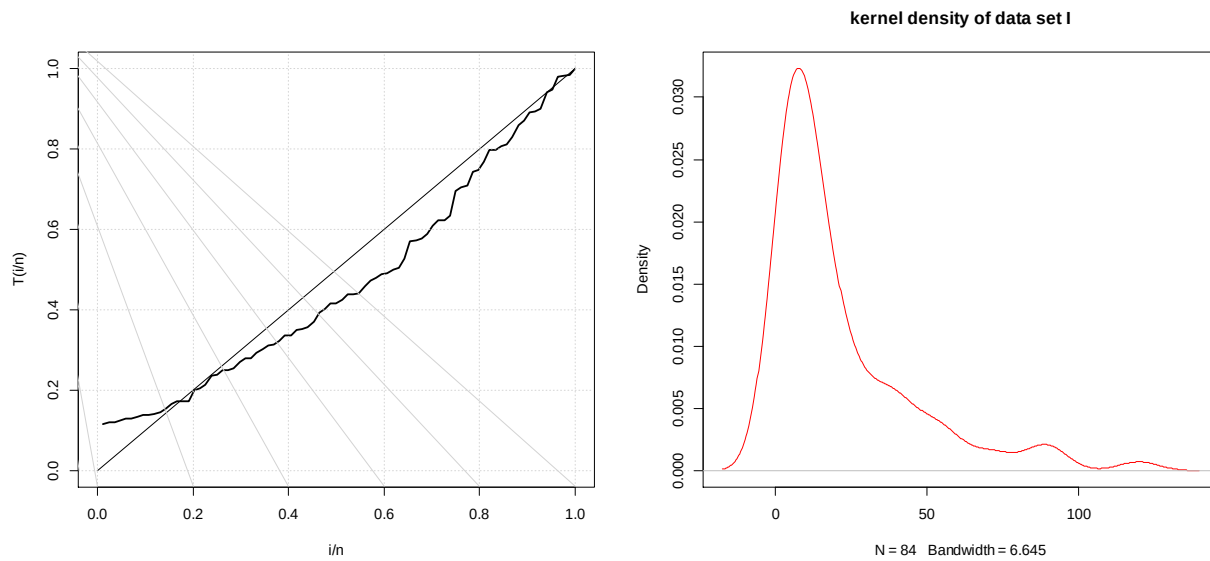


Figure 5. TTT plot and empirical density plots for data set 1

Table 3.0 MLEs, standard error (in braces) and Measures of goodness of fit for the parameters of the models for dataset I

Model	Parameters Estimates			Measures of goodness of fit			
	α	λ	θ	l	AIC	BIC	CAIC
<i>TIHLEE</i>	12.76 (1.33)	0.01 (0.00)	0.52 (0.05)	53.96	113.93	121.22	114.23
<i>EE</i>	— (—)	1.11 (3.35)	0.04 (0.13)	342.12	688.23	693.09	688.38
<i>E</i>	— (—)	0.0 (0.01)	— (—)	342.12	686.23	688.66	686.28

Table 4: Exploratory Data Analysis for data set II

Min	q_1	median	Mean	q_3	Max.	variance	kurtosis	Skewness	Range
60	68.5	92.5	98.48	112.0	178.0	1150.5	2.7	0.8	118.0

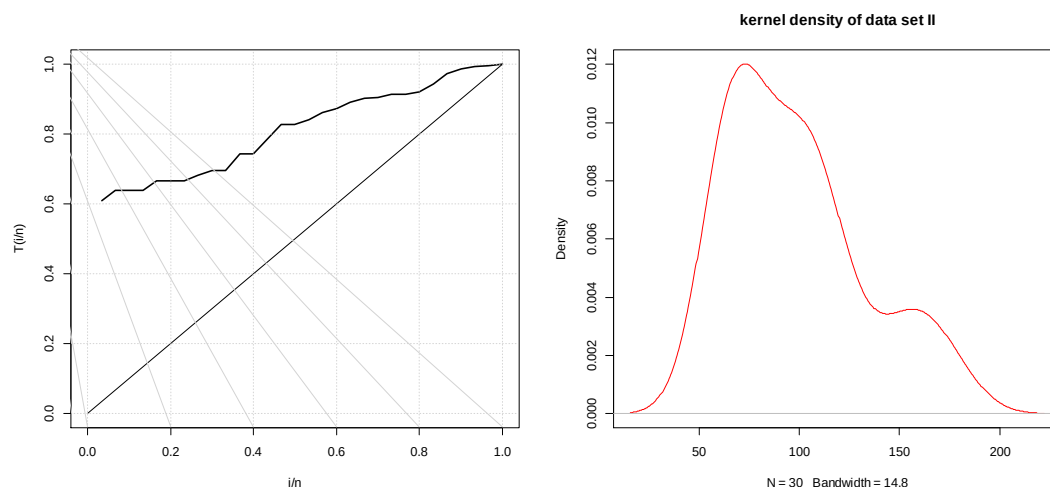


Figure 6. TTT plot and empirical density plots for data set II

Table 5.0 MLEs, standard error (in braces) and Measures of goodness of fit for the parameters of the models for dataset II

Model	Parameters Estimates			Measures of goodness of fit			
	α	λ	θ	l	AIC	BIC	CAIC
<i>TIHLEE</i>	8.34 (1.37)	0.01 (0.00)	0.94 (0.20)	84.89	175.78	179.98	176.70
<i>EE</i>	— (—)	0.02 (0.00)	5.44 (1.51)	149.65	303.30	306.10	303.74
<i>E</i>	— (—)	0.03 (0.01)	— (—)	155.45	312.90	314.40	313.03

5 Conclusion

In this paper, we studied and developed a new generalization of extended exponential distributions called Type II half-logistic Extended Exponential distribution, which is developed based using Type II half-logistic-G family. The Type II half-logistic Exponentiated Exponential distribution generalizes the Exponentiated Exponential and Exponential distribution. Some properties of the newly developed distribution such as moments, incomplete moments, reliability and hazard rate functions, order statistics, Renyi entropy, and moment generating functions are derived. The maximum likelihood estimators are used to obtain the estimates of the parameters of the distribution. An application of the *TIHLEE* distribution to two lifetime data sets shows that the new distribution provides a reasonable fit than other types of models when used to model life-time data.

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A machine learning model and application for heart disease prediction using prevalent risk factors in Nigeria

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Abstract

Over the past two decades, heart disease remains the principal source of death globally. According to the World Health Organization, about 17.9 million deaths occur every year internationally as a result of heart disease. The vast number of deaths is common amongst low and middle-income countries and Nigeria is at the top on the list of the most affected sub-Saharan African Nations. With the ravaging impact of heart disease in developing countries, there is need to have a reliable, accurate and efficient approach to make an early diagnosis of the disease to achieve prompt therapy or management. The purpose of this research is to develop a machine learning model and application for heart disease prediction using prevalent heart disease risk factors in Nigeria. Quantitative, qualitative, and experimental research methods were used in implementing this research. The experiment for the training and testing of the models were carried out using Decision Tree (DT), Random Forest (RF), Naïve Bayes (NB) and Voting Ensembles Classifier (VEC) with NB, DT, and RF as base learners. The DT, NB, RF and VEC machine learning algorithms gave an accuracy and error rate values of (95.83%, 0.042), (97.22%, 0.028), (97.92%, 0.021) and (96.53%, 0.035) respectively. This research developed a heart disease prediction model and application using RF and Django python web framework respectively. The heart disease dataset used in training the model was gotten from Federal Medical Centre Keffi, Nassarawa State, Nigeria. The developed system acts as a decision support tool for cardiologists.

Keywords and phrases: machine learning; supervised learning machine learning algorithm; heart disease prediction; prevalent risk factors

1 Introduction

According to World Health Organization (WHO), heart disease also referred to as cardiovascular disease is currently the number one killer disease in the world and also three-quarters of the world's deaths from cardiovascular disease occur in low and middle-income nations [17]. This is because it most of the time shows no known symptoms, making it difficult to detect on time, which is why it is called a silent killer. Nigeria is top amongst the most affected Sub-Saharan African countries and there are projections that the figures will keep rising due to urbanization, poor eating habits, and stressful lifestyles [18]. The currently existing traditional ways of diagnosing cardiovascular disease in Nigeria sometimes cause vague, inaccurate prognoses, and lack prompt prescription for the disease or its risk

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factors. This is due to human errors, lack of modern-day cardiovascular disease detection equipment, work pressure, and stress on the few available cardiologists.

Several risk factors are associated with cardiovascular disease and it has become imperative to provide an accurate and innovative approach to make an early diagnosis of the risk or presence to achieve prompt treatment or management of the disease. Machine Learning (ML) is a commonly used technique for processing enormous data in the healthcare sector. Researchers in the field of Artificial Intelligence (AI) apply several ML techniques to analyze huge and complex medical data, helping healthcare professionals to predict several diseases [2]. With the large amount of locally available primary cardiovascular disease datasets generated from heart disease risk factors daily in cardiovascular care clinics in teaching hospitals and medical centers across the country, an accurate, efficient, affordable, simple to use, and readily available ML decision support tool for the early detection of cardiovascular disease presence and its risk was developed. This decision support tool is used by cardiologists to reduce the escalating prevalence of morbidities, disabilities, and mortalities due to cardiovascular disease in Nigeria to a manageable level.

2 Literature review

In today's digital and Artificial Intelligence-driven world, several clinical decision support systems on heart disease prediction have been developed to simplify and ensure efficient diagnosis. Das et al. [2] developed a model using machine algorithms such as Support Vector Machine (SVM), Decision Tree (DT), Random Forest (RF) and k-Nearest Neighbors (kNN) for prediction of heart diseases. The highest accuracy level was achieved by the kNN classifier, with an accuracy score of 86.84%. Shah et al. [16] developed a ML model for heart disease prediction using 4 ML techniques, kNN ML method achieved the highest accuracy score. Arumugam et al. [1] in a study that developed a fine-tuned ML model for forecasting the likelihood of heart disease in diabetic individuals used Naïve Bayes (NB), SVM, and DT ML techniques. Olaniyi et al. [10] developed and implemented an intelligent heart disease prediction system that is modeled on feed forward multilayer perceptron Artificial Neural Network (ANN) and SVM. Their findings showed that SVM gives a better result with an accuracy score of 87.5% using the Cleveland Heart Disease University of California Institute Repository Dataset (CHDUCIRD). DT produced the highest accuracy score of 90.5%.

Desai et al. [3] developed a system for monitoring the health status of heart disease patients using SVM, kNN, ANN, Logistic Regression (LOR), and Gradient Boosting Trees (GBT) ML algorithms and cloud computing technology. The LOR ML model gave the best accuracy score of 85.96% and was used in developing the system using the CHDUCIRD dataset. Gupta et al. [9] developed a heart disease prediction models using LOR, SVM, NB, DT, kNN and RF supervised learning ML algorithms and the CHDUCIRD dataset. LOR produced the best model with an accuracy score of 92.30%. Qureshi and Warke [14] developed heart disease prediction models using RF, ANN, NB, and DT supervised learning ML algorithms. The RF model produced the best model with an accuracy score of 82.25% using the CHDUCIRD dataset.

Sarah et al. [15] carried out a comparative study on the performance of supervised learning ML algorithms for heart disease prediction using LOR, SVM, DT, kNN, NB, and RF supervised learning ML algorithms. LOR and SVM algorithms produced the best models with an accuracy score of 85.25% respectively. Dutta et al. [5] carried out a comparative study on the performance of kNN, DT, XG Boost (XGB), and RF ML algorithms in the prediction of heart disease. kNN produced the best accuracy score of 85.71% using the CHDUCIRD dataset.

Based on the reviewed literature above, none of the previous works used Voting Ensembles Classifier (with NB, DT and RF as the base classifiers) ML algorithm in predicting the presence of heart disease. Secondly, health data is demographic in nature [8], but from the extant literature, most of the existing projects carried out in the prediction of heart disease using ML algorithms and risk factors causing heart disease used CHDUCIRD, a secondary dataset from University of California Institute dataset database existing online that were gotten from other climes. Therefore, this research is based basically on locally available primary heart disease dataset and its risk factors gotten from Nigeria.

3 Methodology

Quantitative, qualitative, and experimental research methodologies were adopted in implementing this research. The experiment for the training and testing of the models were carried out using Decision tree (DT), Random Forest (RF), Naïve Bayes (NB) and Voting Ensembles Classifier (VEC) with NB, DT, and RF as base classifier algorithms as shown in the flowchart in figure 1. The performances of the algorithms were measured by the accuracy and error rate gotten from Confusion Matrix.

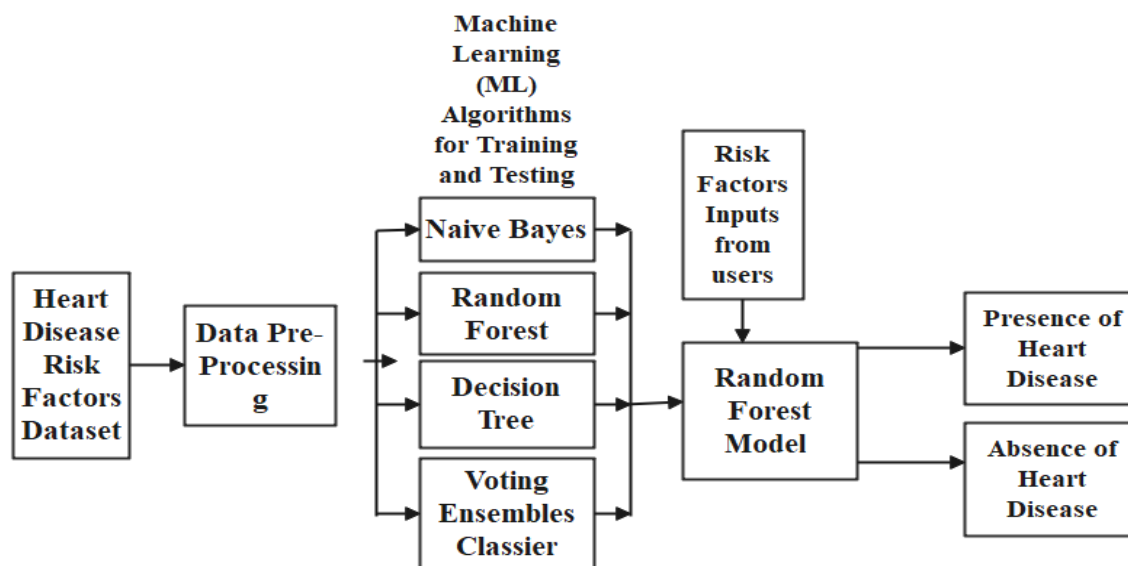


Figure 1: Flowchart showing the flow of activities in implementing the machine learning model for heart disease prediction

Decision Tree (DT): According to Shah et al. [16], DT is a frequently used data mining ML method that has the ability to solve classification and regression problems, but is more commonly used for resolving classification tasks. This algorithm categorizes a population into branch-like segments that construct a reversed tree with a root node, internal nodes, and leaf nodes. The ML method is non-parametric and can competently deal with large, complex, and complicated data without imposing a complex parametric structure. DT learning is a method for approximating discrete-valued target functions, in which the learned function is represented by a DT. Learned trees can also be shown as

sets of if-then rules to improve human legibility. The DT model makes analysis based on three basic nodes, namely:

- Root node: principal node, based on this node all-other nodes functions.
- Interior node: handles various features or attributes of the dataset.
- Leaf node: represent the outcome of each test, i.e., the class of the dependent variable in classification problems.

The DT algorithm divides the data into two or more analogous sets based on the most significant indicators. The entropy of each feature is computed, and then the dataset is divided, with predictors having the minimum entropy. The formula for calculating the entropy of an attribute is shown in Eq. (1).

$$Entropy(s) = \sum_{i=1}^C -p_i \log_2 p_i \quad (1)$$

where in Eq. (1), C is the total number of classes and the probability of samples belonging to a class at a given node can be denoted as p_i . A simple basic structure of a DT showing the three basic nodes is shown in figure 2.

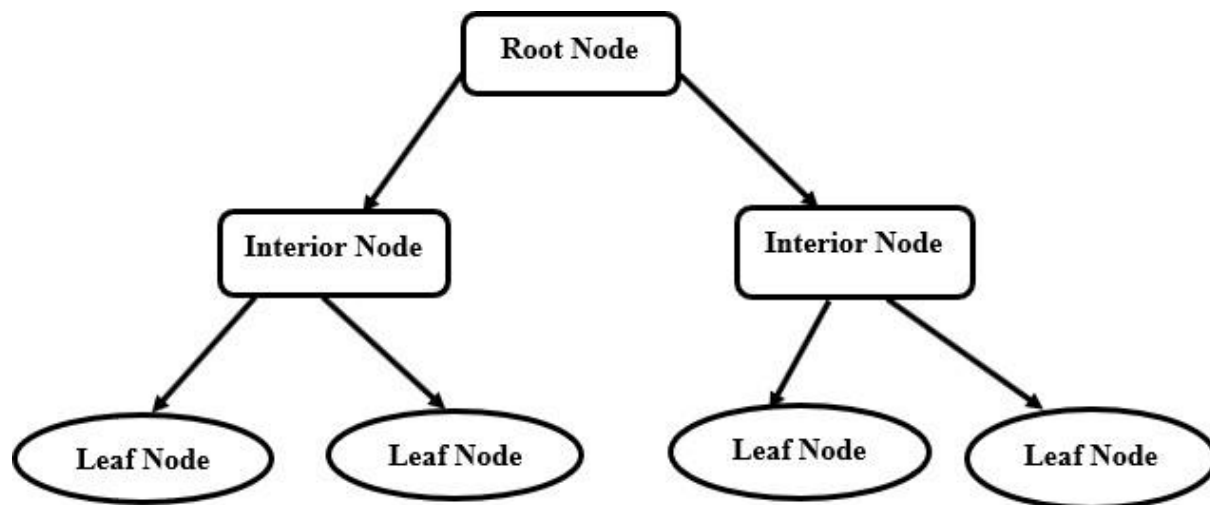


Figure 2: Structure of a Decision Tree showing the root node, interior nodes represented by a rectangle, and leaf nodes represented by an oval shape.

Random Forest (RF): RF is an ensemble ML algorithm. This algorithm works by combining many DT into a forest [11]. In the RF algorithm, each separate tree outputs a class expectation, and the class with the most votes becomes the model's forecast. The greater the number of trees in a RF classifier, the better the chance of increased accuracy. It can be used for both classification and regression tasks, although it excels more at classification problems and can overcome the shortcomings of DT and other ML methods, such as overfitting and missing values.

Naive Bayes (NB): According to Diwakar et al. [4], NB is a classification and regression supervised learning ML technique based on Bayes' theorem with the assumption of strong (Naive)

independence amongst predictors. In simple terms, an NB classifier believes that the existence of one attribute in a class is unrelated to the presence of any other attribute. For example, if a fruit is green, spherical, and roughly 4 inches in diameter, it is considered an orange. Even if these attributes depend on each other or upon the existence of the other attributes, a NB classification technique would consider all of these characteristics to independently contribute to the probability that this fruit is an orange. NB is a simple, easy-to-implement, and efficient algorithm that handles non-linear and complex data. Mathematically, the Bayes theorem is written as shown in Eq. 2.

$$P(A/B) = \frac{P(B/A) * P(A)}{P(B)} \quad (2)$$

where in Eq. (2), $P(A/B)$ is the posterior probability, $P(B)$ is the predictor prior probability, $P(A)$ is the class prior probability, $P(B/A)$ is the possible probability of predictor.

Voting Ensembles Classifier (VEC): Similarly, to the RF ML algorithm, the VEC classifier estimates several base models. It uses voting to combine the separate independent predictions or classifications from the various classifiers to arrive at the final results. However, the main difference lies in the base models. The VEC model does not mandate or necessitate the base models to be homogenous. Put differently, we can train different base learner ML models to arrive at the final result [6]. Both hard and soft voting mechanisms are implemented in the VEC algorithm. We forecast the final class label using the hard voting approach as the class label that has been forecasted the most frequently by the classification trained models. Soft voting, on the other hand, predicts class labels by averaging class probabilities. Mathematically, hard voting is represented by Eq. (3).

$$\hat{y} = \text{mode}\{A_1(x), A_2(x), A_3(x)\} \quad (3)$$

In Eq. (3), $\hat{y}$ is the target variable class label. A_1 , A_2 and A_3 are they base classifiers used in the VEC classifier respectively. x is the data points or records. Figure 3 shows the framework of the VEC algorithm using NB, RF, and DT as the base models or classifiers.

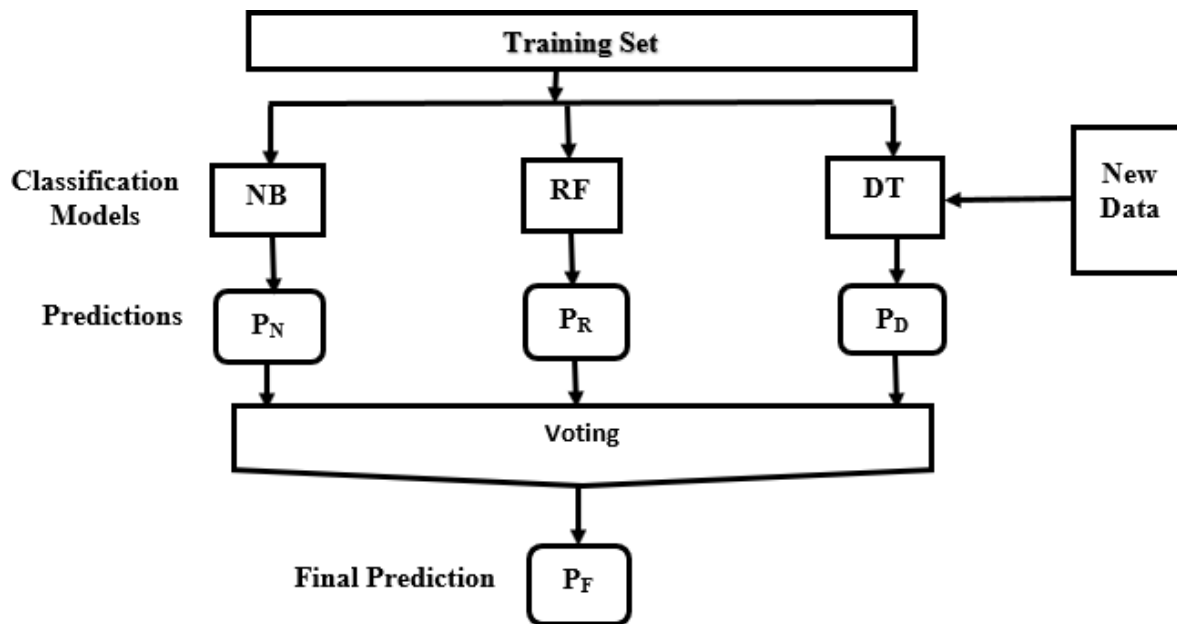


Figure 3: Figure showing the framework of the VEC algorithm using NB, RF and DT as base models.

In figure 3, P_N is the prediction value for NB classifier, P_R is the prediction value for RF classifier, P_D is the prediction value for DT classifier and P_F is the final prediction for the VEC model. Training Set is the dataset used for training the model, while New Data is the dataset used for testing the model.

4 Data preprocessing

The heart disease dataset used for this thesis was collected from heart disease clinic in Federal Medical Centre Keffi. The collected dataset is made up of 720 instances and 13 variables. 12 are independent variables while 1 is a dependent variable. The heart disease risk factors inputs data considered for this research are; sex, age, hypertension status, weight, height, physical activity level, smoking habit, alcohol consumption habit, diabetes status, drug abuse status, genetic history and dyslipidaemia status as shown in details in table 1. The format of the heart disease dataset gotten from Federal Medical Centre Keffi is shown in table 1 below.

Table 1: Attributes and details of dataset of heart disease gotten from Federal Medical Centre Keffi.

Sr.no.	Attribute	Details
1	Sex	1= male; 0=female
2	Age	Patients age in years
3	Diabetic	1-Yes; 0-No
4	Weight	Patients weight in kilogram(kg)
5	Height	Patients height in metres(m)
6	Hypertensive	1-Yes; 0-No
7	Dyslipidaemia	Level of good or bad lipids or cholesterol. 1-Yes; 0-No
8	Genetic history	2=Parent; 1=Other than parents; 0=None

9	Abuse drugs	1-Yes; 0-No
10	Smoke	3=Daily; 2=Occasionally; 1=Used to Not now; 0=Never
11	Alcohol	3=Significantly; 2=Occasionally; 1=Used to Not now; 0=Never
12	Physical activity level	0=Active; 1=Moderately Active; 2=Sedentary
13	Heart disease prediction	0=Absence of heart disease; 1=Presence of heart disease

4.1 Applying Pearson Correlation

In this section, the correlations between attributes of the FMC Keffi dataset and heart disease prediction are analyzed and applicability to the classification analysis is examined. Correlation coefficients are used in statistics to measure how strong a relationship is between two variables. The correlation between two variables is a measure of how well the variables are related. The most common measure of correlation in statistics is the Pearson Correlation which shows the linear relationship between two variables. Pearson correlation coefficient analysis produces a result between -1 and 1. A result of -1 means that there is a perfect negative correlation between the two values, while a result of 1 means that there is a perfect positive correlation between the two variables. Results between 0.5 and 1.0 indicate high correlation. For the purpose of this research, independent variables of the heart disease dataset (risk factors) with Pearson correlation coefficient values from, 0.3 (30%) and above with the dependent variable (Heart disease prediction) shown in the Seaborn Heatmap visualizing FMC Keffi data correlations (Figure 4) is used for training the model.

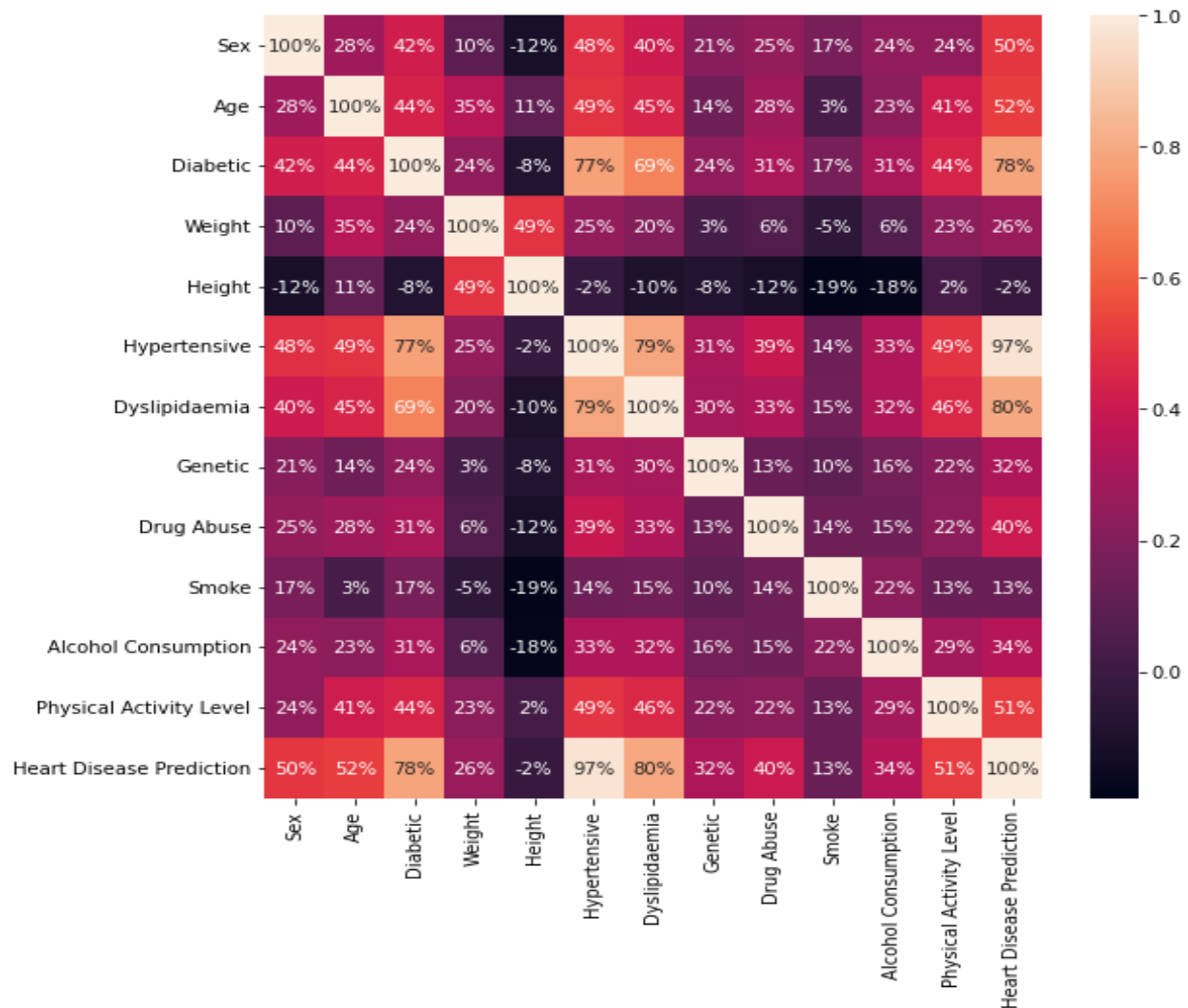


Figure 4: Seaborn Heatmap visualizing FMC Keffi data correlations

5 Model evaluation

For evaluating the model, the dataset is divided into 2 parts. 80 percent was used in training the model, while the remaining 20 percent was used in testing the model.

5.1 Evaluation metrics

The evaluation metrics used for this research such as accuracy and error rate were gotten from the confusion matrix table shown in table 2. In the field of machine learning and precisely the problem of classification, a confusion matrix, also known as an error matrix, is a specific table layout that allows visualization of the performance of machine learning algorithm. The template for any binary confusion matrix is made up of the followings; true positives, false negatives, false positives, and true negatives along with the positive and negative classifications as shown in table 2 [7][12-13].

Table 2: Binary Confusion Matrix.

	Predicted Condition	
	Positive (PP)	Negative (PN)
Positive (P)	True Positive (TP)	False Negative (FN)
Negative (N)	False Positive (FP)	True Negative (TN)

From table 2 above, condition Positive (P) is the number of real positive cases in the data, condition Negative (N) is the number of real negative cases in the data, True Positive (TP) a test result that correctly indicates the presence of a condition or characteristic, True Negative (TN) a test result that correctly indicates the absence of a condition or characteristic, False Positive (FP) a test result which wrongly indicates that a particular condition or attribute is present, False Negative (FN) a test result which wrongly indicates that a particular condition or attribute is absent.

The performance metrics used in this thesis are;

Accuracy: Accuracy (ACC) is calculated as the number of all correct predictions divided by the total number of the dataset. The best accuracy is 1.0, whereas the worst is 0.0. It can also be calculated by $1 - \text{ERR}$. The best accuracy in percentage is 100%. Mathematically it is shown in Eq. (4).

$$ACC = \frac{TP + TN}{TP + TN + FN + FP} \quad 4$$

Error Rate: Error rate (ERR) is calculated as the number of all incorrect predictions divided by the total number of the dataset. The best error rate is 0.0, whereas the worst is 1.0. Mathematically it is shown in Eq. 5.

$$ERR = \frac{FP + FN}{TP + TN + FN + FP} \quad 5$$

6 Results

This thesis implemented the ML based model using Python programming language. It trained they ML algorithms used in this thesis with heart disease risk factors and their corresponding output datasets gotten from Federal Medical Centre, Keffi. The dataset was divided into two parts, 80 percent for training and 20 percent for testing. The accuracy score gotten from the testing of the various ML algorithms used in this thesis is shown in the Bar graph in figure 5. RF ML algorithm gave the most accurate result with an accuracy score of 97.92 %. RF was used in developing the heart disease prediction application because of its high accuracy result as shown in table 3. While, the output, is the presence or absence of heart disease. The application development was carried out using Django, a python web development framework.

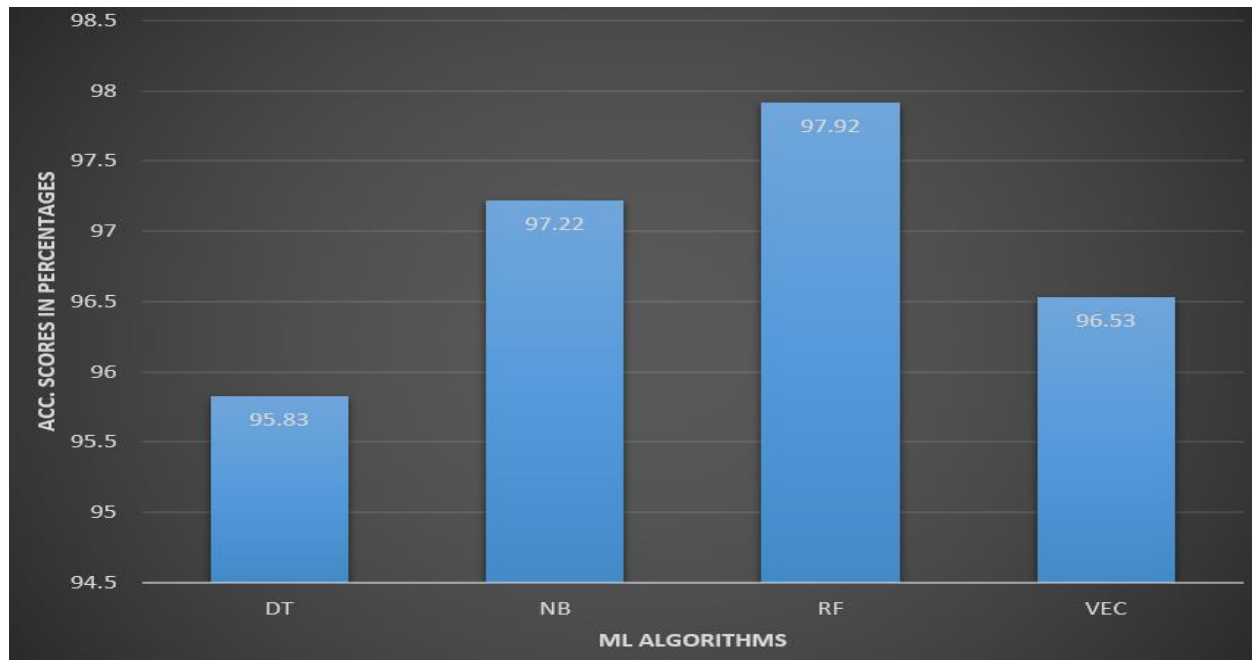


Figure 5: A Bar Chart showing the Comparison of ML Algorithms in terms of Accuracy Score in Percentage after Testing of the Models.

Table 3: Comparison of the performance of all the 4 machine learning algorithms used in this research.

Algorithms	Accuracy (%)	Error Rate	Training Time (s)
Decision Tree (DT)	95.83	0.042	0.02
Naïve Bayes (NB)	97.22	0.028	0.01
Random Forest (RF)	97.92	0.021	0.11
Voting Ensembles Classifier (VEC) with DT, RF and NB as base classifiers	96.53	0.035	0.02

7 Conclusion

This research used the RF trained machine learning model for implementing the heart disease prediction application. The heart disease prediction application was developed using the django python web framework. The heart disease dataset used in training the model was gotten from Federal Medical Centre Keffi, Nassarawa State, Nigeria. The developed system acts as a decision support tool for cardiologists.

Further works can be carried out in the area of heart disease dataset collection by involving more heart disease clinics in different geographical locations and regions or climate.

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The stability of an inhomogeneous rational integrator for the solution in ordinary differential equations

F. Ebhohimen^{*†} and G.A. Akhanolu[‡]

Abstract

This research work is concerned with the determination of solution to different classes of problems in Ordinary Differential Equations (ODEs). We derived an inhomogeneous rational integrators for the solution of ordinary differential equations. The convergence and consistency and scheme in the interpolant of order $m = 4$, was established. The stability function and the analysis of the method was carried out with the use of MAPLE-18 and MATLAB software. We compared our new integrator with some recent existing methods and discovered that ours compete favorably well with a very high rate of convergence..Our result shows that the new integrator is stable analytically and computationally.

Keywords and Phrases: rational integrator; stiff; singular and oscillatory problems; stability function and region of absolute stability (RAS)

1 Introduction

In physical sciences, mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equation are called differential equation, [13]. Differential equations do not only arise in the physical sciences but also in diverse fields as economics, medicine, engineering, psychology, operation research and even in areas such as biological simulations and anthropology, population models, electrical, network, nuclear reactors, mechanical oscillations, chemical kinetics, engineering work, process control [5].

According to [8], one of the main directions to construct methods with higher order and extensively stability region, is using higher derivatives of the solutions and some methods that have been introduced that have good properties, especially for stiff problems in [2],

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b, \quad (1)$$

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where $f(x, y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region $D \subset [a, b]$ that are either Stiff or Singular.

Lipschitz constant according to [2], from (1) above, $f(x, y)$ is said to satisfy a Lipschitz condition in y , over the region D , if there exist a constant L such that

$$||f(x, y_1) - f(x, y_2)|| \leq L||y_1 - y_2|| \quad (2)$$

where L is called the Lipschitz constant and $f(x, y)$ is said to be Lipschitzian by virtue of the relation.

Definition 1: Stiffness [6]

The system (1) is said to be stiff over the interval $[a, b]$ if for every $x \in [a, b]$, the eigenvalues $\{\lambda_s(x), s = 1, \dots, m\}$ of the Jacobian matrix

$$J = \frac{\partial f}{\partial y} \quad (3)$$

Satisfy the following conditions:

$$\text{Re } \lambda_s(x) < 0, s = 1, \dots, m,$$

where $\text{Re } \lambda_s(x)$ represents the real component of the complex number $\lambda_s(x)$

$$\text{Stiffness Ratio} = \frac{\lambda_r(x)}{\lambda_s(x)} \gg 1, r, s = 1, \dots, m.$$

For the linear initial value problem

$$y' = Ay + g, \quad y(x_0) = y_0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f'}{\partial y} = A$$

is $m \times m$ matrix and $g \in R^m$.

Definition 2: [7]: A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $\text{Re } h\lambda < 0$.

However, A-stable is a severe requirement to ask of a numerical method for stiffness to be satisfied.

Definition 3: [11]: A numerical method is said to be stiffly-stable if

the region of absolute stability contains $\mathfrak{R}_1$ and $\mathfrak{R}_1$ and

(ii) it is accurate for all $h \in \mathfrak{R}_2$ when applied to the test equation $y' = \lambda y$, λ a complex constant with $\text{Re } h\lambda < 0$, where

$$\mathfrak{R}_1 = \{[h\lambda] \text{Re } h\lambda < -a\},$$

$$\mathfrak{R}_2 = \{[h\lambda] - a \leq \text{Re } h\lambda \leq b, -c \leq \text{Im } h\lambda \leq c\},$$

and a, b and c are positive constants.

Definition 4: [1]: A one-step numerical method is said to be L-stable if it is A-stable and, in addition, when applied to the scalar test equation $y' = \lambda y$, λ is a complex constant with $\text{Re } h\lambda < 0$, it yields $y_{n+1} = \text{Re } (h\lambda)y_n$, where $|\text{Re } (h\lambda)| \rightarrow 0$ as $\text{Re } (h\lambda) \rightarrow \infty$.

2 Derivation of the method

The rational interpolant $U : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^m q_i x_{n+1}^i} \quad (4)$$

where p_i and q_i are called the integrator parameters, and r is an inhomogeneous constant.

At $m = 4$, from (4) we have

$$y_{n+1} = r + \frac{\sum_{i=0}^3 p_i x_{n+1}^i}{1 + \sum_{i=1}^4 q_i x_{n+1}^i} \quad (5)$$

where p_0, p_1, p_2, p_3 , and q_1, q_2, q_3, q_4 are called the integrator parameters and r is a constant. Also

$$\begin{aligned} y_{n+1} &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3][1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4]^{-1} \\ &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3]\{\sum_{i=0}^{\infty} (-1)^i [q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4]^i\}. \end{aligned}$$

By the use of binomial expansion for rational functions, we have,

$$\begin{aligned} y_{n+1} &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3][1 - (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4) + (q_1 x + q_2 x^2 \\ &\quad + q_3 x^3 + q_4 x^4)^2 - (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4)^3 + (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4)^4 - \dots] \\ y_n &= r + p_0 \\ r &= y_n - p_0 \end{aligned} \quad (6)$$

$$\begin{aligned} p_1 &= \frac{h y_n^{(1)}}{1! x_{n+1}} + p_0 q_1 \\ p_1 x_{n+1} &= h y_n^{(1)} + p_0 q_1 x_{n+1} \end{aligned} \quad (7)$$

$$\begin{aligned} p_2 &= \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} + \frac{h y_n^{(1)}}{1! x_{n+1}} + p_0 q_2 \\ p_2 x_{n+1}^2 &= \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} + p_0 q_2 x_{n+1}^2 \end{aligned} \quad (8)$$

$$p_3 = \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_1 + \frac{h y_n^{(1)}}{1! x_{n+1}} q_2 + p_0 q_3$$

$$p_3 x_{n+1}^3 = \frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 + p_0 q_3 x_{n+1}^3$$

$$- p_0 q_4 = \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4} + \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} q_1 + \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_2 + \frac{h^1 y_n^{(1)}}{1! x_{n+1}} q_3$$

$$p_0 = \frac{-1}{q_4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right]$$

$$r = y_n + \frac{-1}{q_4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right].$$

Now from equ. (5), the rational integrator is expanded to give

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + p_3 x_{n+1}^3 + r + r q_1 x_{n+1} + r q_2 x_{n+1}^2 + r q_3 x_{n+1}^3 + r q_4 x_{n+1}^4}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3 + q_4 x_{n+1}^4}.$$

Then

$$p_0 + r = r + p_0 = y_n$$

$$p_1 x_{n+1} + r q_1 x_{n+1} = h y_n^{(1)} - \frac{q_1 x_{n+1}}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right]$$

$$+ q_1 x_{n+1} y_n + \frac{q_1 x_{n+1}}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right]$$

$$p_1 x_{n+1} + r q_1 x_{n+1} = h y_n^{(1)} + q_1 x_{n+1} y_n$$

$$p_2 x_{n+1}^2 + r q_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} -$$

$$\frac{q_2 x_{n+1}^2}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right] + q_2 x_{n+1}^2 y_n$$

$$+ \frac{q_2 x_{n+1}^2}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right]$$

$$p_2 x_{n+1}^2 + r q_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} + q_2 x_{n+1}^2 y_n$$

$$p_3 x_{n+1}^3 + r q_3 x_{n+1}^3 = \frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 -$$

$$\begin{aligned} & \frac{q_3 x_{n+1}^3}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right] + q_3 x_{n+1}^3 y_n \\ & + \frac{q_3 x_{n+1}^3}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right] \\ p_3 x_{n+1}^3 + r q_3 x_{n+1}^3 &= \frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}^1}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 + q_3 x_{n+1}^3 y_n \end{aligned} \quad (16)$$

$$r q_4 x_{n+1}^4 = q_4 x_{n+1}^4 y_n + \frac{q_4 x_{n+1}^4}{q_4 x_{n+1}^4} \left[\frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 \right] \quad (17)$$

Substituting (13) – (17) into (12), we obtain our general integrator in (19),

$$\begin{aligned} & y_{n+1} \\ & y_n + h y_n^{(1)} + q_1 x_{n+1} y_n + \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} + q_2 x_{n+1}^2 y_n + \frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}^1}{2!} + h y_n^{(1)} q_2 x_{n+1}^2 \\ & + q_3 x_{n+1}^3 y_n + \frac{h^4 y_n^{(4)}}{4!} + \frac{h^3 y_n^{(3)} q_1 x_{n+1}^1}{3!} + \frac{h^2 y_n^{(2)} q_2 x_{n+1}^2}{2!} + h y_n^{(1)} q_3 x_{n+1}^3 + q_4 x_{n+1}^4 y_n \\ & = \frac{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3 + q_4 x_{n+1}^4}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3 + q_4 x_{n+1}^4} \end{aligned}$$

Now, we let

$$A = q_1 x_{n+1}, \quad B = q_2 x_{n+1}^2, \quad C = q_3 x_{n+1}^3 \quad \text{and} \quad D = q_4 x_{n+1}^4. \quad (18)$$

Therefore, our general integrator formula for $m = 4$ becomes

$$y_{n+1} = \frac{\sum_{i=0}^4 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + C \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + D y_n}{1 + A + B + C + D} \quad (19)$$

3 The convergence and consistency

For any numerical methods to be useful, such method must satisfy some basic properties. One- step methods like the stability of an inhomogeneous rational integrator for the solution of ordinary differential equations are normally described symbolically by

$$y_{n+1} = y_n + h \Phi(x_n, y_n; h), \quad (20)$$

where $\Phi(x_n, y_n; h)$ is called the increment function, x_n the mesh point and h the mesh size. [9] and also confirmed by [6] state that a rational integrator is said to be consistent if the increment function is consistent with the initial value problem (IVP) that is if $\Phi(x, y, 0) = f(x, y)$

Theorem: The stability of an inhomogeneous rational integrator for the solution of ordinary differential equations

$$y_{n+1} = \frac{\sum_{i=0}^4 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + C \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + D y_n}{1 + A + B + C + D} \quad (21)$$

where the functions A, B, C and D are defined by equation and stated in equation (18) is convergent and consistent.

Proof:

Subtracting y_n from both sides of equ. (21), we have

$$\begin{aligned} y_{n+1} - y_n &= \frac{\sum_{i=0}^4 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^3 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=0}^2 \frac{h^i y_n^{(i)}}{i!} + C \sum_{i=0}^1 \frac{h^i y_n^{(i)}}{i!} + D y_n}{1 + A + B + C + D} - y_n \\ y_{n+1} - y_n &= \frac{\sum_{i=1}^4 \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=1}^3 \frac{h^i y_n^{(i)}}{i!} + B \sum_{i=1}^2 \frac{h^i y_n^{(i)}}{i!} + C h y_n^{(1)}}{1 + A + B + C + D} \\ \frac{y_{n+1} - y_n}{h} &= \frac{y_n^{(1)} + \sum_{i=2}^4 \frac{h^{i-1} y_n^{(i)}}{i!} + A \sum_{i=1}^3 \frac{h^{i-1} y_n^{(i)}}{i!} + B \sum_{i=1}^2 \frac{h^{i-1} y_n^{(i)}}{i!} + C h y_n^{(1)}}{1 + A + B + C + D} \end{aligned}$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right): \text{Limit}_{x \rightarrow 0} A = \text{Limit}_{x \rightarrow 0} B = \text{Limit}_{x \rightarrow 0} C = \text{limit}_{x \rightarrow 0} D = 0$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)}$$

$$\text{Limit}_{h \rightarrow 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n^{(1)} = f(x_n, y_n) \text{ as required.}$$

4 Stability considerations

To effectively solve initial value problems in ordinary differential equations which is stiff, we need numerical integrators that possess special stability properties such as inhomogeneous function. To test for stability, we apply the integrator to the test equation.

$$y' = \lambda y \quad (22)$$

where by induction, will give $y_n^{(k)} = \lambda^k y_n$ and $\lambda h = \bar{h}$. We have

$$A = \frac{\bar{h}}{-2}, \quad B = \frac{3\bar{h}^2}{-28}, \quad C = \frac{-\bar{h}^3}{84}, \quad \text{and} \quad D = \frac{\bar{h}^4}{1680}. \quad (23)$$

Putting (23) into (21) we have

$$y_n + hy_n^{(1)} + \frac{h^2 y_n^{(2)}}{2} + \frac{h^3 y_n^{(3)}}{6} + \frac{h^4 y_n^{(4)}}{24} + Ay_n + Ah y_n^{(1)} + A \frac{h^2 y_n^{(2)}}{2} + A \frac{h^3 y_n^{(3)}}{6} +$$

$$+ By_n + Bh y_n^{(1)} + B \frac{h^2 y_n^{(2)}}{2} + Cy_n + Ch y_n^{(1)} + Dy_n$$

$$y_{n+1} = \frac{1 + A + B + C + D}{1 + A + B + C + D}$$

$$y_n + hy_n^{(1)} + \frac{h^2 y_n^{(2)}}{2} + \frac{h^3 y_n^{(3)}}{6} + \frac{h^4 y_n^{(4)}}{24} - \frac{\bar{h}}{2} y_n - \left(\frac{\bar{h}}{2}\right)(\bar{h} y_n) - \left(\frac{\bar{h}}{2}\right) \frac{h^2 y_n^{(2)}}{2} - \left(\frac{\bar{h}}{2}\right) \frac{h^3 y_n^{(3)}}{6}$$

$$+ \left(\frac{3\bar{h}^2}{28}\right) y_n + \left(\frac{3\bar{h}^2}{28}\right) h y_n + \left(\frac{3\bar{h}^2}{28}\right) \frac{h^2 y_n^{(2)}}{2} - \left(\frac{\bar{h}^3}{84}\right) y_n - \left(\frac{\bar{h}^3}{84}\right) h y_n + \left(\frac{\bar{h}^4}{1680}\right) y_n$$

$$y_{n+1} = \frac{1 + A + B + C + D}{1 + A + B + C + D}$$

$$y_n + hy_n^{(1)} + \frac{h^2 y_n^{(2)}}{2} + \frac{h^3 y_n^{(3)}}{6} + \frac{h^4 y_n^{(4)}}{24} - \frac{\bar{h}}{2} y_n - \left(\frac{\bar{h}}{2}\right)(\bar{h} y_n) - \left(\frac{\bar{h}^3}{4}\right) y_n - \left(\frac{\bar{h}^4}{12}\right) y_n$$

$$+ \left(\frac{3\bar{h}^2}{28}\right) y_n + \left(\frac{3\bar{h}^2}{28}\right) h y_n + \left(\frac{3\bar{h}^4}{56}\right) y_n - \left(\frac{\bar{h}^3}{84}\right) y_n - \left(\frac{\bar{h}^3}{84}\right) h y_n + \left(\frac{\bar{h}^4}{1680}\right) y_n$$

$$y_{n+1} = \frac{1 + A + B + C + D}{1 + A + B + C + D}$$

$$y_n + \left(\bar{h} - \frac{\bar{h}}{2}\right) y_n + \left(\frac{\bar{h}^2}{2} - \frac{\bar{h}^2}{2} + \frac{3\bar{h}^2}{28}\right) y_n + \left(\frac{\bar{h}^3}{6} - \frac{\bar{h}^3}{4} + \frac{3\bar{h}^3}{28} - \frac{\bar{h}^3}{84}\right) y_n$$

$$+ \left(\frac{\bar{h}^4}{24} - \frac{\bar{h}^4}{12} + \frac{3\bar{h}^4}{56} - \frac{\bar{h}^4}{84} + \frac{\bar{h}^4}{1680}\right)$$

$$y_{n+1} = \frac{1 + A + B + C + D}{1 + A + B + C + D}$$

$$y_{n+1} = \frac{y_n + \frac{\bar{h} y_n}{2} + \frac{3\bar{h}^2 y_n}{28} + \frac{\bar{h}^3 y_n}{84} + \frac{\bar{h}^4 y_n}{1680}}{1 + A + B + C + D}$$

$$\frac{y_{n+1}}{y_n} = \frac{y_n + \frac{\bar{h} y_n}{2} + \frac{3\bar{h}^2 y_n}{28} + \frac{\bar{h}^3 y_n}{84} + \frac{\bar{h}^4 y_n}{1680}}{1 - \frac{\bar{h}}{2} + \frac{3\bar{h}^2}{28} - \frac{\bar{h}^3}{84} + \frac{\bar{h}^4}{1680}}$$

Simplifying the numerator and the denominator, we get

$$\frac{y_{n+1}}{y_n} = \frac{1680 + 840\bar{h} + 180\bar{h}^2 + 20\bar{h}^3 + \bar{h}^4}{1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4}$$

$$\zeta(\bar{h}) = \frac{y_{n+1}}{y_n} = \frac{1680 + 840\bar{h} + 180\bar{h}^2 + 20\bar{h}^3 + \bar{h}^4}{1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4} \quad (24)$$

5 Analysis of the stability formular

Our new integrator is said to be Absolutely Stable if $|\zeta(\bar{h})| \leq 1$,

$$\text{If we let } \zeta(\bar{h}) = \frac{1680+840\bar{h}+180\bar{h}^2+20\bar{h}^3+\bar{h}^4}{1680-840\bar{h}+180\bar{h}^2-20\bar{h}^3+\bar{h}^4} = \frac{\phi(\bar{h})}{\Psi(\bar{h})},$$

it implies $|\zeta(\bar{h})| \leq 1 \Leftrightarrow \left\| \frac{\phi(\bar{h})}{\Psi(\bar{h})} \right\| \leq 1$.

To analyse the formula, we set $\bar{h} = u + iv$ where $i^2 = -1$. Hence $\|\phi(\bar{h})\| \leq \|\Psi(\bar{h})\| \therefore \|\phi(\bar{h})\| - \|\Psi(\bar{h})\| \leq 0$,

where $\phi(\bar{h}) = 1680 + 840\bar{h} + 180\bar{h}^2 + 20\bar{h}^3 + \bar{h}^4$ and

$$\Psi(\bar{h}) = 1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4.$$

Let $\phi(u, v) = A(u, v) + iB(u, v)$ and set $\bar{h} = u + iv$ where $i^2 = -1$ $\Psi(u, v) = C(u, v) + iD(u, v)$

$$\begin{aligned} \Rightarrow \phi(u, v) &= 1680 + 840(u + iv) + 180(u + iv)^2 + 20(u + iv)^3 + (u + iv)^4 \\ \phi(u, v) &= 1680 + 840(u + iv) + 180(u^2 + 2uvi + v^2i^2) + 20(u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3) \\ &+ (u^4 + 4u^3vi + 6u^2v^2i^2 + 4uv^3i^3 + v^4i^4). \end{aligned}$$

On expansion, we get

$$\begin{aligned} A(u, v) &= u^4 - 6u^2v^2 + v^4 + 20u^3 - 60uv^2 + 180u^2 - 180v^2 + 840u + 1680 \\ B(u, v) &= 4u^3v - 4uv^3 + 60u^2v - 20v^3 + 360uv + 840v \end{aligned}$$

Hence, the inequality becomes

Simplifying further, the inequality above holds if and only if

$$\begin{aligned} A^2 &= u^8 - 12u^6v^2 + 38u^4v^4 - 12u^2v^6 + v^8 + 40u^7 - 360u^5v^2 + 760u^3v^4 - 120uv^6 \\ &+ 760u^6 - 4920u^4v^2 + 6120u^2v^4 - 360v^6 + 8880u^5 - 38880u^3v^2 + 23280uv^4 + \\ &69360u^4 - 185760u^2v^2 + 35760v^4 + 539600u^3 - 504000uv^2 + 1310400u^2 - \\ &604800v^2 + 2822400u + 2822400 \\ B^2 &= 16u^6v^2 - 32u^4v^4 + 16u^2v^6 + 480u^5v^2 - 640u^3v^4 + 160uv^6 + 6480u^4v^2 \\ &- 5280u^2v^4 \\ + 400v^6 &+ 49920u^3v^2 - 21120uv^4 + 230400u^2v^2 - 33600v^4 + 604800uv^2 + 705600v^2 \\ A^2 + B^2 &= u^8 + 4u^6v^2 + 4u^2v^6 + v^8 + 40u^7 + 120u^5v^2 + 40uv^6 + 760u^6 + 1560u^4v^2 \\ &+ 840u^2v^4 + 40v^6 + 8880u^5 + 11040u^3v^2 + 2160uv^4 + 69360u^4 + \\ &4460u^2v^2 + \\ &2160v^4 + 369600u^3 + 100800uv^2 + 1310400u^4 + 100800v^2 + 2822400u + 2822400 \end{aligned}$$

$$\Psi(\bar{h}) = 1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4.$$

Set $\bar{h} = u + iv$ where $i^2 = -1$

Let $\Psi(u, v) = C(u, v) + iD(u, v)$

$$\Rightarrow \Psi(u, v) = 1680 - 840(u + iv) + 180(u + iv)^2 - 20(u + iv)^3 + (u + iv)^4$$

$$\Psi(u, v) = 1680 - 840(u + iv) + 180(u^2 + 2uvi + v^2i^2) - 20(u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3) + (u^4 + 4u^3vi + 6u^2v^2i^2 + 4uv^3i^3 + v^4i^4).$$

On expansion, we get

$$\|\Psi(u, v)\| = C^2(u, v) + D^2(u, v)$$

$$C(u, v) = u^4 - 6u^2v^2 + v^4 - 20u^3 + 60uv^2 + 180u^2 - 180v^2 - 840u + 1680$$

$$\begin{aligned} C^2 = & u^8 - 12u^6v^2 + 38u^4v^4 - 12u^2v^6 + v^8 - 40u^7 + 360u^5v^2 - 760u^3v^4 + 120uv^6 \\ & + 760u^6 - 4920u^4v^2 + 6120u^2v^4 - 360v^6 - 8880u^5 + 38880u^3v^2 - 23280uv^4 + \\ & 69360u^4 - 185760u^2v^2 + 35760v^4 - 369600u^3 + 504000uv^2 + 1310400u^2 - \\ & 604800v^2 - 2822400u + 2822400 \end{aligned}$$

$$D(u, v) = 4u^3v - 4uv^3 - 60u^2v + 20v^3 + 360uv - 840v$$

$$\begin{aligned} D^2 = & 16u^6v^2 - 32u^4v^4 + 16u^2v^6 - 480u^5v^2 - 640u^3v^4 - 160uv^6 + 6480u^4v^2 \\ & - 5280u^2v^4 \end{aligned}$$

$$+ 400v^6 - 49920u^3v^2 + 21120uv^4 + 230400u^2v^2 - 33600v^4 - 604800uv^2 + 705600v^2$$

$$C^2 + D^2 = u^8 + 4u^6v^2 + 6u^4v^4 + 4u^2v^6 + v^8 - 40u^7 - 120u^5v^2 - 120u^3v^4 - 40uv^6$$

$$\begin{aligned} & + 760u^6 \\ & + 1560u^4v^2 + 840u^2v^4 + 40v^6 - 8880u^5 - 11040u^3v^2 - 2160uv^4 + \\ & 69360u^4 + \\ & 4460u^2v^2 + 2160v^4 - 369600u^3 - 100800uv^2 + 1310400u^4 + 100800v^2 - 2822400u \\ & + 2822400 \end{aligned}$$

$$||\zeta(\bar{h})|| \leq 1 \Leftrightarrow \|\phi(u, v)\| - \|\Psi(u, v)\| \leq 0$$

$$\Leftrightarrow A(u, v)^2 + B(u, v)^2 - C(u, v)^2 - D(u, v)^2 \leq 0$$

$$\begin{aligned} A^2 + B^2 - (C^2 + D^2) = & 80u^7 + 240u^5v^2 + 240u^3v^4 + 80uv^6 + 17760u^6 \\ & + 22080u^3v^2 + 4320uv^4 + 739200u^3 + 201600uv^2 + 5644800u \\ & \leq 0 \end{aligned}$$

which shows that the integrator is A – Stable if and only if $u < 0$.

To obtain our Region of Absolute Stability (RAS) we shall employ the polar form

where $u = R\cos\theta$, and $v = R\sin\theta$

$$\begin{aligned} & 80R^7\cos(\theta)^7 + 240R^7\cos(\theta)^7\sin(\theta)^2 + 240R^7\cos(\theta)^7\sin(\theta)^4 + 80R^7\cos(\theta)\sin(\theta)^6 \\ & + 17760R^5\cos(\theta)^5 + 22080R^5\cos(\theta)^3\sin(\theta)^2 + 4320R^5\cos(\theta)\sin(\theta)^4 \\ & + 739200R^3\cos(\theta)^3 \end{aligned}$$

$$\begin{aligned}
 &+201600R^3 \cos(\theta) \sin(\theta)^2 + 5644800R \cos(\theta) \\
 &\Rightarrow 80R^7 \cos(\theta)^7 + 240R^7 \cos(\theta)^7 \cos\theta(1 - \cos^2\theta) + 240R^7 \cos(\theta)^7 \cos\theta(1 - \cos^2\theta)^2 \\
 &+ 80R^7 \cos(\theta) \cos\theta(1 - \cos^2\theta)^3 + 17760R^5 \cos(\theta)^5 + 22080R^5 \cos(\theta)^3 \cos\theta(1 - \cos^2\theta) \\
 &+ 4320R^5 \cos\theta(1 - \cos^2\theta)^2 + 739200R^3 \cos(\theta)^3 + 201600R^3 \cos(\theta) \cos\theta(1 - \cos^2\theta)^2 \\
 &+ 5644800R \cos(\theta) = 0 \\
 &\Rightarrow 80R^7 \cos(\theta)^7 - 20800R^5 \cos(\theta)^5 + 13440R^5 \cos(\theta)^3 + 22080R^5 \cos(\theta)^7 + 4320R^5 \cos\theta \\
 &\quad + 3360000R^3 \cos(\theta)^3 + 201600R^3 \cos\theta + 2016000R^3 \cos(\theta)^5 + 5644800R \cos(\theta)
 \end{aligned}$$

6 The numerical analysis of the stability curve $m = 4$

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

We take various degrees of θ from 0° to 360° and get a corresponding values of R.

Table 1.

θ°	R	θ°	R
0	25.0431937230128	195	-23.5878310205734
15	23.5878310205734	210	-20.2973921906952
30	20.2973921906952	225	-17.1429824862624
45	17.1429824862624	240	-15.9025813811573
60	15.9025813811578	255	-12.2683823503038
75	12.2683823503038	270	0.000000000000000
90	0.000000000000000	285	12.2683823503038
105	-12.2683823503038	300	15.90258138115783
120	-15.9025813811573	315	17.14298248626244
135	-17.1429824862624	330	20.29739219069527
150	-20.2973921906952	345	23.5878310205734
165	-23.5878310205734	360	25.0431937230128
180	-25.0431937230128		

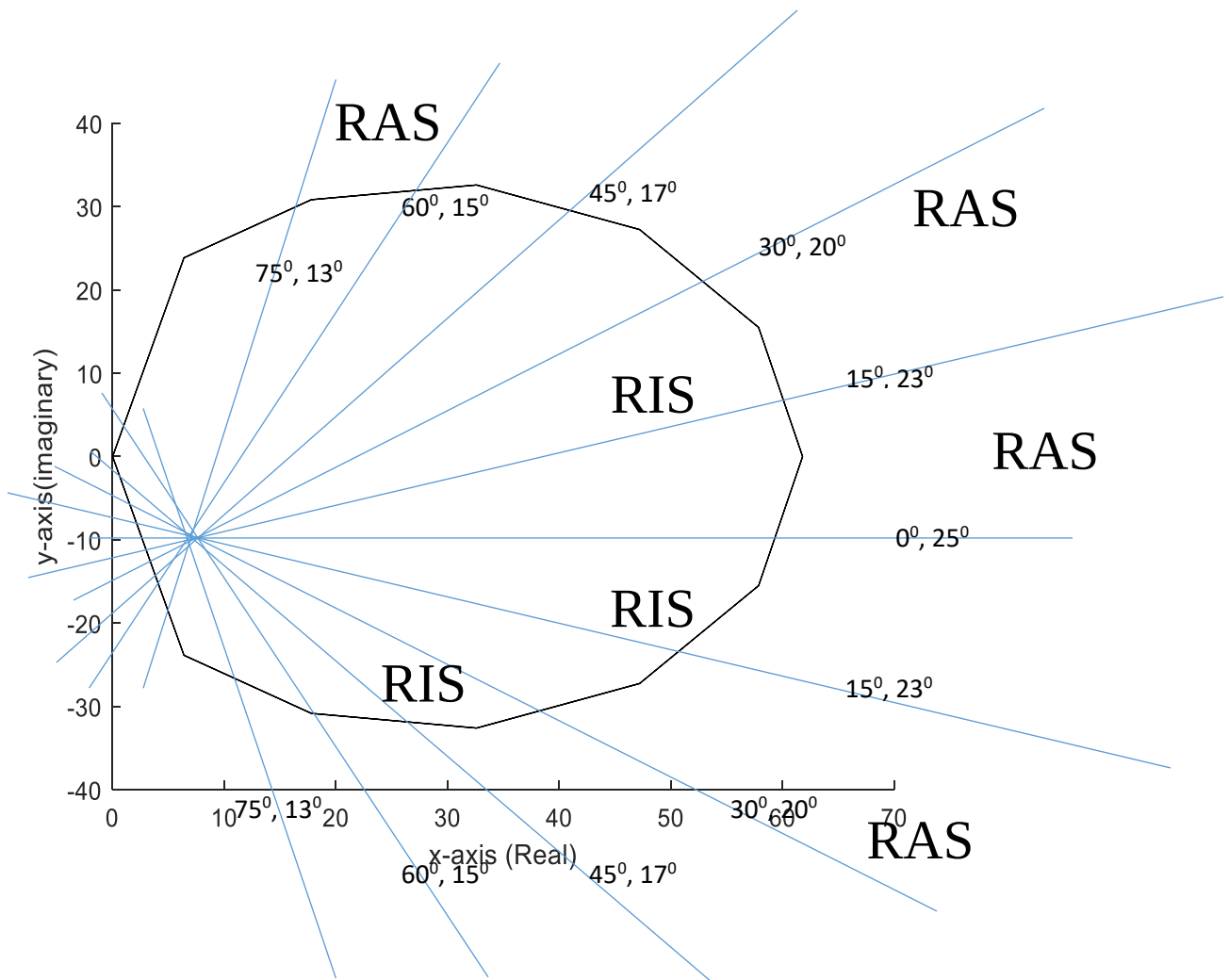


Figure 1.

Hence the integrator is A - stable so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

7 Numerical experiments

Problem 1: [10]: Consider the Riccati initial value problem

$y' = 1 + y^2$, $y(0) = 0$ with exact solution $y(x) = \tan x$, the step size $h = 0.1$.

Table 2

N	x_n	$y(x_n)$	Error in T_{n+1}	New method
1.0	0.000	1.0000000e+00	0.0000000e+00	0.0000000e+00
2.0	0.010	1.00359940e-01	2.5266803e-05	3.4086356e-05
3.0	0.020	2.02755448e-01	4.5446459e-05	4.0498433e-04
4.0	0.030	3.0939959e-01	6.3336523e-05	5.2880846e-04
5.0	0.040	4.2287746e-01	8.4243633e-05	7.4066631e-04
6.0	0.050	5.4642406e-01	1.2156595e-04	1.4395093e-04
7.0	0.060	6.8434715e-01	2.1034559e-04	2.4142834e-04
8.0	0.070	8.4273164e-01	4.4325944e-04	3.3508012e-04
9.0	0.080	1.0307117e-00	1.0731557e-04	2.2639649e-04
10.0	0.090	1.2629892e-00	2.8310182e-04	4.1641662e-04
11.0	0.100	1.5654375e-00	8.0297304e-03	7.0589025e-04

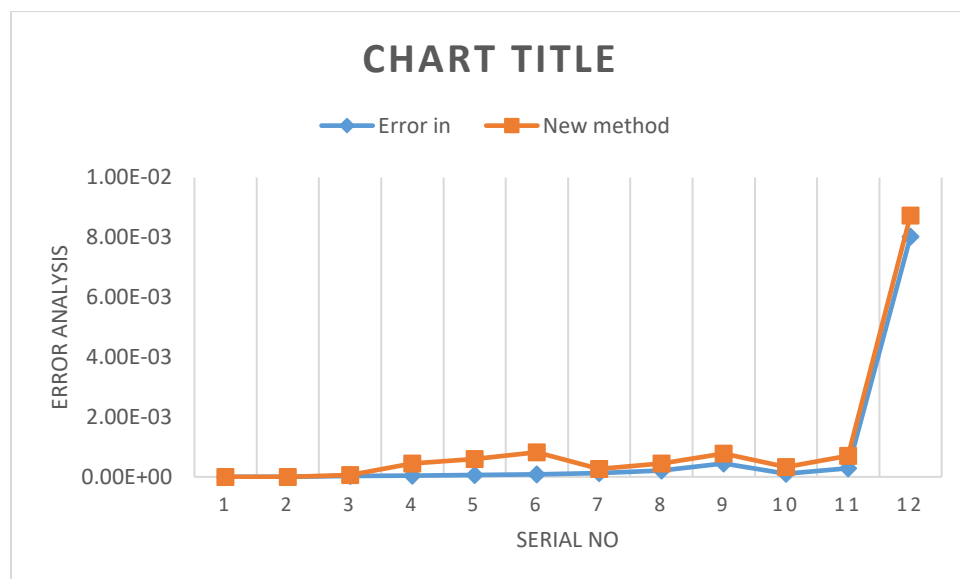


Figure 2

Remark: Table 2 and Figure 2 shows the performance of our new numerical integrator. Our new integrator competes favorably well with the development of a new one step scheme for the solution of initial value problem in ordinary differential equations of [10] with a very high rate of convergence.

Problem 2: [3]: Root Mean Square Runge-Kutta (RMS-RK4)
 $y' = y^2 - y$, $y(0) = 2$, $0 \leq x \leq 1$ and $h = 0.1$.

Table 3

N	TSOL	$y(x_n)$	Error in $RMS - RKM$	New method
0.1	2.2351E+00	2.2351E+00	1.4016E-05	3.40352e-05
0.2	2.5687E+00	2.5688E+00	5.6286E-05	3.08843e-05
0.3	3.0763E+00	3.0765E+00	1.9805E-04	3.76084e-05
0.4	3.9356E+00	3.9364E+00	7.8005E-04	2.49631e-05
0.5	5.6935E+00	5.6976E+00	4.0994E-03	2.49509e-04
0.6	1.1243E+01	1.1268E+01	2.4043E-02	2.91428e-04
0.7	-1.4543E+02	1.4498E+02	2.9041E+02	2.5608e-02
0.8	-8.8676E+00	4.8510E+15	4.8510E+15	2.2939e-01
0.9	-4.3516E+00	0.4406E+232	0.4406E+232	3.3641e+10
1.0	-2.7844E+00	NAN	NAN	NAN

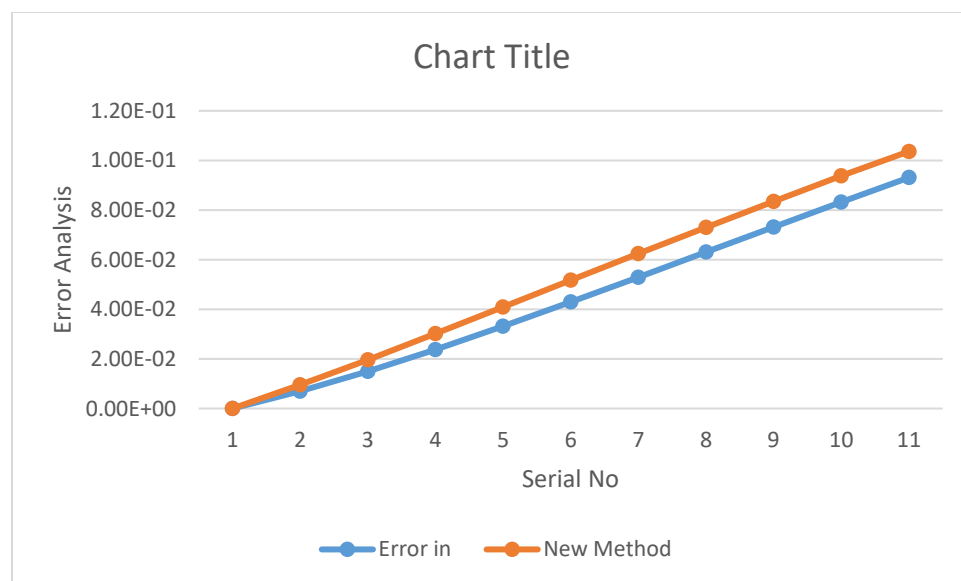


Figure 3.

Table 3 and Figure 3 shows the performance of our new numerical integrators. Our new integrator competes favorably well Implicit 4th Order RKF of [3], with a very high rate of convergence.

Problem 3: [12]

$$y' = -\sin x - 200(y - \cos x); \quad y(0) = 0; \quad h = 0.001.$$

The exact solution is $y(x) = \cos x - e^{-200x}$.

Table 4

XN	TSOL	Sunday, et al (2013)	New method
1.00e-03	1.81268e-01	1.8270e-07	4.1388e-09
2.00e-03	3.29677e-01	1.4085e-07	5.5915e-09
3.00e-03	4.51183e-01	5.5609e-07	5.5869e-09
4.00e-03	5.50663e-01	3.9270e-07	4.8676e-09
5.00e-03	6.32108e-01	2.2589e-07	3.8687e-09
6.00e-03	6.98787e-01	1.8561e-07	2.8328e-09
7.00e-03	7.53378e-01	1.5193e-07	1.8852e-09
8.00e-03	7.98071e-01	1.2601e-07	1.0808e-09
9.00e-03	8.34660e-01	1.1594e-07	4.3446e-10
10.0e+03	8.64614e-01	1.6619e-07	6.1043e-11

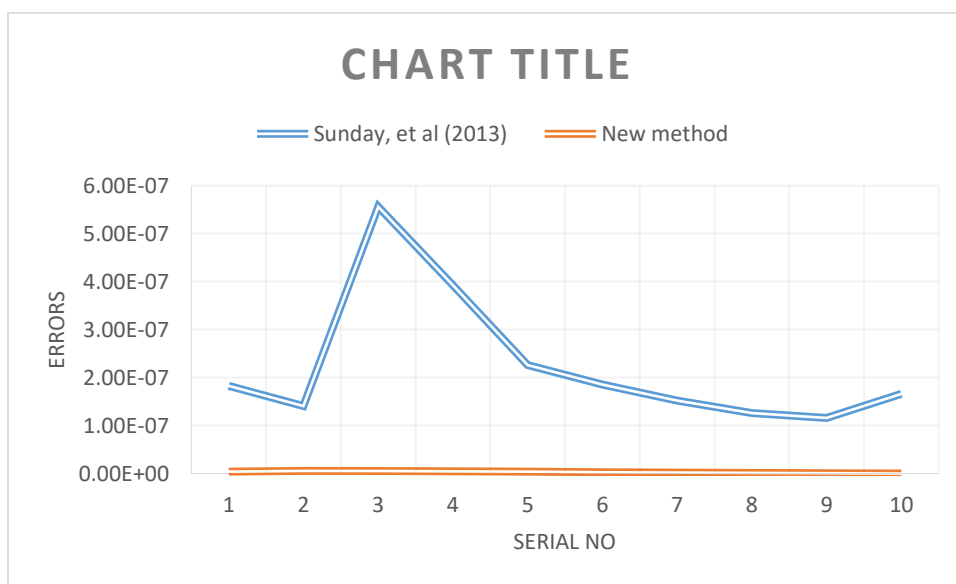


Figure 4.

Table 4 and Figure 4 shows the performance of our new numerical integrator. The trend shows that for a given mesh point, the errors in numerical integrator decreases as m increases. Our new integrator competes fairly well with the Extended Block Integrator method of [12].

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Numerical analysis on broken home and its causes

C. Nkutura^{*} and C.K. Paago[†]

Abstract

Non-linear $I_H B_H$ mathematical model with XYZ sub-compartment was used to study the dynamics of broken home and marriage repair. The existence and stability of the broken home free and endemic equilibria were discussed. The broken home free equilibrium was locally asymptotically stable with $R_0 < 1$ and unstable with $R_0 > 1$. Numerical simulations were conducted to confirm the analytic results. The findings were that, reducing the contact rate between the intact marriage and broken home, increases the number of marriage that go into separation and educating separators to refrain from divorce can be useful in combating the broken home. The study also shows that from the broken home free equilibrium when all parameters are fixed and 50% variation of beta and k, the intact marriage increases monotonically while separated and divorce are monotonically decreased. Again, from the Endemic equilibrium when all parameters are fixed and 50% variation of beta and k, the intact marriage fluctuates at the initial stage and later increases, the separated and divorce monotonically decreased. In further studies there should be a development of various related topics that will help in analyzing other aspects that are closely related to broken home in order to get further understanding and avoid the problem of broken home that happen to us and to give us information about the effects of broken home.

Keywords and phrases: mathematical model; intact home; broken home; separated; divorce; causes of broken home

1 Introduction

Broken home connotes the division or the separation of parents who have been together and suddenly divorce or terminated their relationship. The institution of marriage is as old as the creation of man. It is the coming together of two persons in love with consent of parents, guardians and witnesses for the purpose of procreation and companionship. It is an institution that play very important-role in the upbringing and the development of human beings. Marriage is the joining together of two matured people who agree to spend their lifetime together. Thus taking joint care or responsibility over their children who are the fruits of the union. [1,16, 24].

Marriage is expected to be an affair of intimacy and compromise, where spouses complement each other. It is also expected to be a life time companionship or unionism, but the current trend in the marriage dyad departs from these traditional assertions, as the marital union is today mere associations where rancor, conflict, savage mutual attacks are endemic. Today the couples tend to dwell together only when things are rosy and prefer to leave or separate when misunderstandings

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occur or when a conflictual situation arises. Despite the threats of the disintegration in marriages, there are still model marriages whose dispositions give hope for badly managed ones. Conflict is said to be good, especially when the tension so generated leads to change which may be thought aggressive, yet ultimately ends in convincingly peaceful resolution. Thus, though conflict is disruptive, it helps to create awareness among different people, groups and society with respect to the existence of boundaries of common interest.

Marital conflict breeds unhealthy relationship in the couples themselves, the children and dependants. For revitalisation of convivial living, counselling must come into play. Some tips on couple's interaction must be introduced as a combing device for couples in conflict [9; 10] Couples require some skills to aid them in the management of crisis point at any likely breaking point caused by pressure generated by conflict as they cohabit.

Globally many marriages are crashing because of faulty foundations. Most broken homes occur as a result of illegal marriages, improper way of paying dowries, in most cases when the bride price is not duly paid, the marital vows are not well maintained this leads to broken home. The incessant rate of the broken home in our society has become very alarming, and this call for an urgent attention from our parents and the government to rehabilitate the collapsed area in most homes. This may have future implication on our children in all spheres of their lives.

Marriage relationships are said to have become generally strained in recent past, especially among literate couples, as indicated by scholars Efforts have, however, been made by scholars, churches, professional organisations, etc. to curb this menace. Yet, the incidence of marital conflict and disharmony appears endemic as the stabilising of personality role of marriage and the family wanes at alarming rates. Scholars, teachers, churches, etc. are at a loss on the options available for identifying the causes of this situation in their attempt to proffer solutions.

This study is therefore hinged on the incessant occurrence of marital conflict among the couples. The couples are by virtue of their education expected to detect conflict signals and hence employ conflicts managing tactics to effectively control the situation without disrupting the marital dyad and cohesion. The existence of this phenomenon (marital conflict) amongst the couples points to poor marital conflict management skills.

The purpose of this research work is geared at achieving the following:

1. To discover the level of marital conflict to broken home amongst couples.
2. To explore the causes of broken home amongst the couples.

2 Review of related literature

The Concept of Marriage and Family

The terms "marriage" and "family", though difficult to separate, are not exactly the same. [15] posits that it is necessary to dichotomise the two words for the purpose of simplicity and understanding. [9] argues that there could be confusion in literature among the authorities due to the snail-shell relationship which exists between marriage and family. He further contends that the arrival of a child in a marriage makes the marriage and family two social systems operating under the roof at the same time as a subsystem of marriage and marriage as a subsystem of family. Though the two terms (concepts) seem inseparable and are often used interchangeably, they possess features that mark them out as different. This is because the features of marriage as a heterosexual union of consenting adults cannot be said to be the same with family, which is basically a social grouping that has people who are either consequently or conjugally related or both ways related. [18] contends that the concept of marriage was instituted in the Garden of Eden by God.

The work in [8] defines the term "marriage" as a legal union of a man and a woman as husband and wife. This, which represents the English definition of marriage, has been mirrored outside culture and custom, hence the definition is purely a linguistic one devoid of ethical connotations. The concept of marriage and family attracts many experts from all walks of life. [2]

views marriage as a cohabitational union of man and woman. It appears he views marriage as a social system, which operates under one roof. In the same vein, [21] sees marriage as a statutory expectation of a young man and woman who are supposed to pledge their love for each other to live as husband and wife under marriage ordinance. Marriage is something deeper than the mere union of a man and a woman under the same roof. [15] perceives marriage as a sacred and a permanent contract which is assumed to be enacted when two people decide on their own accord and in the presence of at least two witnesses to exchange the formal consent to live a life of vocation of love and sharing for the purpose of promoting their mutual growth and welfare as persons in their journey together through life.

In [6], contends that in a marriage contract, a certificate is issued to the man and woman who are involved in this contract of marriage union. [6] must be referring to the Church and Court marriages alone, ignoring the traditional customary marriage where certificates are not issued before the contract is made between the spouses among the elders and families of the spouses. In the African context, marriage is one of the most important occasions in one's life. The other occasions are one's birthday and one's day of demise. Of course, a new-born baby does not know what is happening nor is the dead person aware of anything. The most exciting day in African context is the very day one marries. Marriage, therefore, is not a private affair. It involves the families, the kith and kin of the intending couples, including the community at large. [17] points out that marriage is so exciting that the road cannot be all roses just as a coin has two phases. In the garden of love, as one adage describes the situation of the newly married couples, the other side of the coin is not envisaged. [10] posit that the marriage of young people may not be strictly accepted as marriage; rather it could be co-habiting since certain rites and or ceremonies that legitimize marriage are absent. This is to say that not all unions are marriages. In the study area, such "marriages" are not uncommon and are often caused by pregnancy as young people get themselves involved in love relationships. This situation has serious implications for the adolescent parents whose education is often truncated, especially the girl who would have to remain at home to give birth and take care of the child. Some marriages are therefore not properly constituted and may not be considered marriage in its strictest sense. Some marriages procreate while others do not; hence while some unions end up only as marriages, others become families [23:151] also contends that "Marriage, after all, continues well beyond one hundred game trials; for it is a relationship; one of the very few in our culture that is supposed to last". Taking marriages as a game of which only a few are meant to last is contrary to the religious and cultural views of marriage in terms of permanency. Marriage is supposed to be in existence till death separates the spouses and not a game of trial or temporary situations, especially in the African and Christian traditions.

The work in [7] opines that marriage is a compromising union of man and woman which poses developmental life cycle tasks that require individuals to address issues of intimacy and reciprocity. One may share the idea of compromising. The married people have to compromise as they continue to reciprocate each other in the cohabitation.

Family could be defined as two or more people who are related by birth, marriage, or adoption, and residing in the same housing unit. Although sociologists define as a group related to one another by bonds of blood, marriage, or adoption and who live together to form a unit.

Causes of Broken Homes

Family is said to be the first institution where one starts to equip oneself to grow. But differences have been found to each family. Having healthy and happy family is what everyone's dream. But many could not succeed it. Due to different problems, the healthy families have been broken up as well. When unwanted things are coming up, many families have been broken, and the members are separated. This is a tragic happening out of unrestricted reasons as well as unfortunate reasons.

When we say broken family, it is not distinct and strange thing, rather common and experiencing problem in and around us. Family life is in a crisis when some problems aroused and which causes separation of the members in the families, which is simply called broken family. It is very controversial that whether to claim every splitting up is broken family, while they still run the family well. There are many families without father, mother, and other members but still conditionally and systematically running. They may not like to call them broken, of course they are not broken, rather just some members left away.

Death

It is obvious that death reduces the family strength. More adversely death sometime leads to broken family. Like other reason all death in the family do not necessarily bring any broken, but by the death of mother or father children become mother/father less. When children realize that the parent will never return, they get emotionally disturbed. Moreover if the left partner live a drastic life in search of happiness after losing his/her partner, the results in the family could not be hale and hearty for the family life, then it starts to be broken.

In early life, loss of mother is more damaging than loss of the father. The mother plays a very important role in nurturing the child during the formative years of life. But as the children grow older, loss of the father is often more serious than loss of the mother, especially for boys. Sometimes mother may lack the time and energy to give children the care they need as she has double burden of homemaking and bread-earning. But if children loss both parents, the effects are doubly serious.

Misconception between family members

Apart from divorce parents and death, there can be a family problem caused by misconception between the members in the home. The reason may be differ from each other. However the most common reasons are drugs, abusing, drinking alcohol, misbehavior manner and doing other intoxicant things by the father, sons or any other members. Then there comes misconception, ruining and anger, which in turn caused breaking the relationship of himself and herself from the family.

Unconditional administration

Failure in administration in the family is one cause for the breaking up of family. If the father or mother or any other heads fail to administer, there will be problems among the members of the family. They will be free to do whatever and however they like in the home and outside. Then the unsystematic and improper living will affect the remaining relationship in the family. If the relationship has been worsening, there will definitely be losing of caring which can lead to broken in the family.

Friends influence

Another reason why a family has broken up is because of parental or friends influence. When a third party has involved itself in the matters inside the home, bad-mouthing will just likely to happen and misunderstanding will only grow worse. Everyone has friends; we used to share our problems with them. But there can be negative impact. To whom we thought helps will come, another unexpected advice may come. Many a time, broken family, particularly splitting up of parents took place due to the influences made by our friends.

In-Law's role in marriage

In-laws in marriage relationship refers to the relatives one acquires through marriage. These in-laws are relatives of either the husband or wife. Those who constitute in-laws in this context are parents and siblings of the couples. In a large group, members of extended family, for example, nieces, nephews, cousins, and grandparents are also regarded as in-laws.

The desire to have a home makes a young man and a young woman to come together as husband and wife. Evans-Pritchard (1973) notes that among the Nvers of the Northern Sudan, young man talks of a home with reference to his family of procreation. The moment one leaves home to marry, something happens to the bond that holds parents and child, because a child remains emotionally attached to his or her parents for life. [3; 20] point out that parents often get involved in arranging their children's marriages to help the children overcome this attachment. The fact that parents are involved in the marriage negotiation and the subsequent innate desire to help their children settle down quickly, lead to intrusion into the privacy of the couple, which could occasionally cause misunderstanding. [22] opines that: although parents' intrusion may cause strife in the home, these in-laws are often unaware that their presence is causing someone pain, some may claim objectivity in their intrusion in the couple's marital life feeling that it is their legitimate rights to interact with their children [20]. As the in-laws interact with the young couples, problems such as jealousy, bad socio-cultural practices and negative attitudes to infertility and all-female children could emanate or be aggravated.

Jealousy

It is always very rare for a mother in-law and daughter in-law to have a cordial relationship. [19] feels that the cause of the problem is accusations and counter-accusations between the two women. The mother in-law often sees her daughter in-law as a rival. She picks on everything that the daughter in-law does, for instance, good family meals, children's clothing and the wife's dressing apparels are condemned outright as acts of extravagance by the mother in-law. The young wife is accused of increasing the poverty level in the family. If the husband listens to his wife and neglects his parents' wish, the wife is accused of using love potion on their son. Once the daughter-in-law becomes aware of the negative attitude towards her, she also develops resentment for her in-laws. She sees the mother-in-law as being jealous, authoritative and attempting to take over her husband and the running of her home. The sister-in-laws of marriageable age are more desperate in their attacks since they feel that the sister-in-law is depriving them of their rights. With the consent of their mother, they keep their brothers' wife uncomfortable in her home. This ill feeling often leads to character assassination and calumny which breeds' tension, conflicts and contempt that results in exchanging of abusive language and continual physical assaults. The sour mother-in-law - daughter-in-law relationship has often ended in the break-up of marriages since the man is placed in a precarious position of not knowing whom to support. Parents could do their children well if they show love and affection to those who are married.

Infertility

When a man gets married, the anticipation by family members is having children in the nearest possible time. As the marriage progresses if there is no sign of pregnancy, worries set in for the couples; the period of waiting for pregnancy is a time of anxiety and unsolicited advice and suggestions from relatives and friends. The woman who is unable to become pregnant is the focus of discussion by the in-laws. The in-laws never care to know who is the cause of the barrenness but would simply accuse the woman either of infidelity, teenage abortion or witchcraft. The woman is called names and the man would be asked to look for another wife for he was "marrying a man". Society appreciates fecundity; anyone who falls short of achieving this aim encounters enormous difficulties in marriage. The problem of barrenness is especially excruciating in polygamous homes. Hannah the wife of Elkana in the Bible suffered most humiliating experience annually at Shiloh [18]. [11] is of the view that a childless woman is not only just tormented by co-wife/wives but also by the extended family members, especially the in-laws.

All-female children

This is a related problem to barrenness as it concerns in-laws. This is simply the inability of a wife to give birth to male children. Even literate families still blame lack of male children on the wife. The female children and their mother are often castigated for their sex. Mother-in-laws create opportunities to nag the daughter-in-law and develop disaffection for the children. Daughters are made to understand that they are not liked and are of less value than boys. In extreme cases the husband is put under pressure to keep concubines in a bid to raise male children who would sustain the family name. The desire to please parents often leads to misunderstanding. The love existing in the couples wanes when negative attributes of the woman are magnified at the expense of virtues.

3 Mathematical model formulation

Broken home in this study is considered as a spreading disease. Hence the broken home model is an epidemic model, which is liken to be a predator-prey model. Thus the predator-prey model is the building block of the bio and ecosystem as biomasses are grown out of their resource masses [12; 13; 14]. A mathematical model was formulated and the population was divided into two major compartments: such as Intact Home (I_H) i.e $X(t)$ and the Broken Home (B_H). Under the broken, we have two sub-compartments which are Separated $Y(t)$ and Divorce $Z(t)$. The marriage dynamics between the intact home and the broken home (separated and divorce) are shown in the figure below.

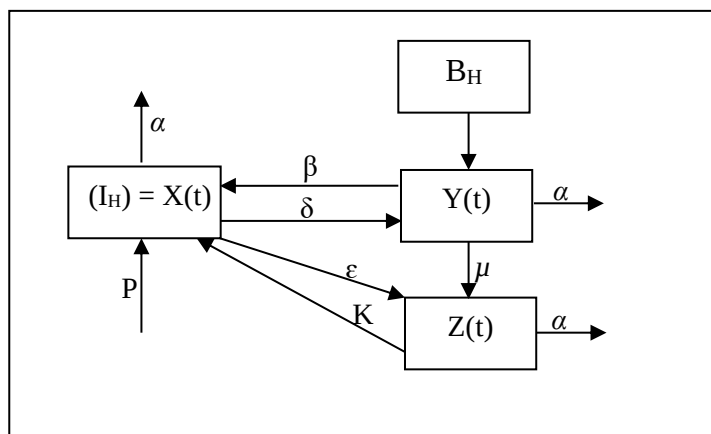


Figure 1: A schematic diagram showing intact and broken home model

From the above schematic diagram, the assumptions were: i). broken home occurs in a closed environment; ii). Factors such as sex, religion, race social and economic do not affect the occurrence of broken home; iii). There is homogeneous mixture of members. With the initial conditions $X(0) \geq 0$, $Z(0) \geq 0$ and $Y(0) \geq 0$, where $X=1$, $Y=1$ and $Z=1$.

Thus, the model for this study is given as:

$$\begin{aligned}\frac{dX}{dt} &= P - aX - \delta XY - \varepsilon XZ + \beta Y + KZ, \\ \frac{dY}{dt} &= \delta XY - \beta Y - \mu Y - \alpha Y, \\ \frac{dZ}{dt} &= \mu Y + \varepsilon XZ - KZ - \alpha Z.\end{aligned}\tag{3.1}$$

4 Numerical results

In this section, we use numerical simulations to show the dynamical behaviour of our results, by assuming that our total population (X, Y, Z) is 100% and X = 0.55, Z = 0.30 and Y = 0.15. The other parameters that would be used in this section are displayed in Table 1 and Table 2 respectively.

Table 4.1: Model parameters at broken home free equilibrium (adopted from [4])

Parameter	$\mathbf{p}$	$\mathbf{\varepsilon}$	K	$\mathbf{\alpha}$	$\mathbf{\beta}$	$\mathbf{\delta}$	$\mathbf{\mu}$
Value	0.4	0.5	0.4	0.25	0.09	0.4	0.2

Table 4.1.1: Broken home free equilibrium when all parameters are fixed.

Variation of parameter values for Broken home fr

equilibrium for $\beta = 0.09, k = 0.4$		
intact married	separated	divorced
100.0000	10.0000	10.0000
0.583902	37.34403	47.29659
0.590813	27.52676	36.18501
0.597033	20.34375	27.43093
0.603165	15.07138	20.67858
0.609364	11.19300	15.54092
0.616910	8.335290	11.66648
0.625493	6.227279	8.762274
0.636285	4.670479	6.593532
0.649441	3.519618	4.977930
0.665523	2.667771	3.775773
0.684693	2.036316	2.881469
0.707074	1.567303	2.215697
0.732801	1.217968	1.719154
0.761559	0.956892	1.347844
0.792909	0.760941	1.069157
0.826571	0.613004	0.858895
0.861677	0.500667	0.699398
0.897441	0.414784	0.577617
0.933351	0.348579	0.483916

Table 4.1.1 shows that while the intact marriage is monotonically increasing, the divorce and separated are decreasing monotonically. Intact married ranging from 0.583902 to 0.933351, Separated from 37.34403 to 0.348579 and Divorce ranging from 47.29659 to 0.483916.

Table 4.1.2: Broken home free equilibrium had $R_0 < 1$ when all parameters are fixed.

$p=0.4, \epsilon=0.5, k=0.4, \alpha=0.25, \beta=0.09, \delta=0.4, \mu=0.2$

50% variation of beta and k on broken home free equilibrium

Variation of parameter values for Broken home free equilibrium for $\beta = 0.0045, k=0.2$

intact married	separated	divorced
100	10	10
0.305518	35.00337	50.52793
0.317347	24.16905	40.4812
0.327993	16.76335	31.62099
0.338231	11.67504	24.25778
0.348734	8.164877	18.36593
0.360619	5.735388	13.77242
0.374239	4.049228	10.25798
0.39066	2.875951	7.606568
0.410697	2.057508	5.627506
0.434955	1.485099	4.162342
0.46446	1.083472	3.084192

0.499256	0.80068	2.294391
0.539482	0.600645	1.717475
0.584733	0.458363	1.29668
0.63411	0.356503	0.989801
0.686624	0.282992	0.765718
0.740959	0.229476	0.601729
0.795242	0.190214	0.481307
0.848756	0.161071	0.392536

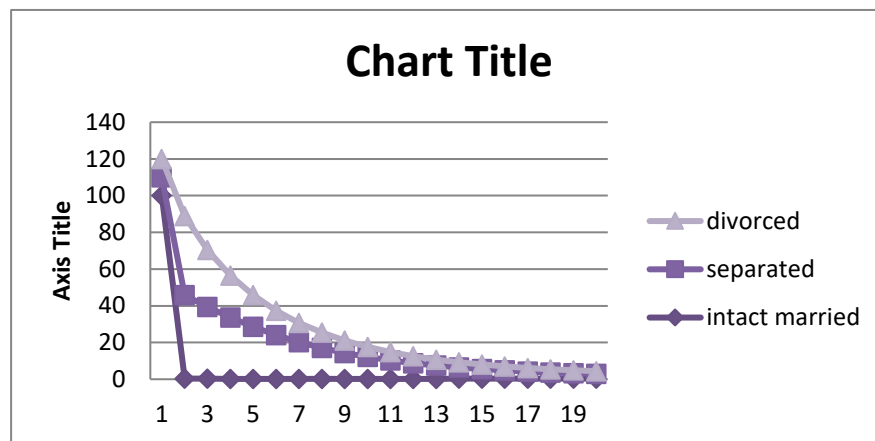


Table 4.1.2 shows that while the intact marriage is monotonically increasing, separated and divorce are decreasing monotonically. Intact married ranging from 0.305518 to 0.848756, separated is decreasing from 35.00337 to 0.161071 and divorce from 50.52793 to 0.392536

150% variation of beta and k on broken home free equilibrium.

Variation of parameter values for Broken home free equilibrium for $\beta = 0.135, k = 0.6$

Table 4.1.3

Intact marriage	Separated	Divorced
100	10	10
0.847075	39.7166	44.76104
0.840412	31.00387	33.60959
0.837469	24.15817	25.61454
0.837081	18.81323	19.70075
0.838764	14.65487	15.24149
0.842132	11.42604	11.84031
0.846472	8.921769	9.228954
0.851962	6.980373	7.216047
0.859068	5.475341	5.660489
0.867734	4.308399	4.456437
0.878295	3.403076	3.523111
0.89046	2.700307	2.798882

0.904698	2.154025	2.235908
0.92098	1.728717	1.797481
0.939134	1.396946	1.455314
0.958824	1.13755	1.187653
0.980298	0.933943	0.977342
1.003107	0.773556	0.811465
1.026733	0.646721	0.680154

Table 4.1.3 shows that while the intact marriage decreases at the initial stage and later increases while separated and divorce are decreasing monotonically. Intact married ranging from 0.847075 to 1.026733, separated is decreasing from 39.7166 to 0.646721 and divorce from 44.76104 to 0.680154.

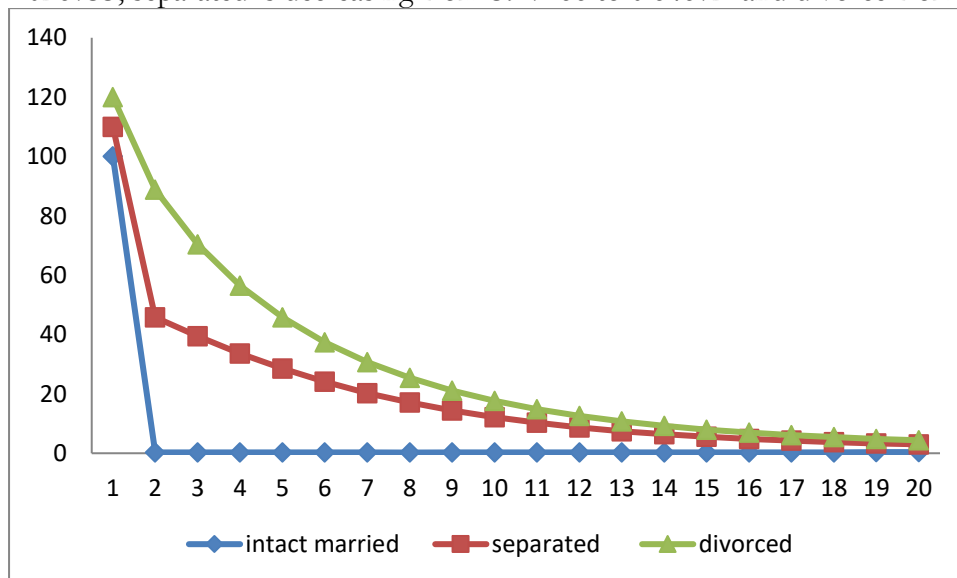


Table 4.2: Model parameters at endemic equilibrium (adopted from Gambrah & Adzadu 2018)

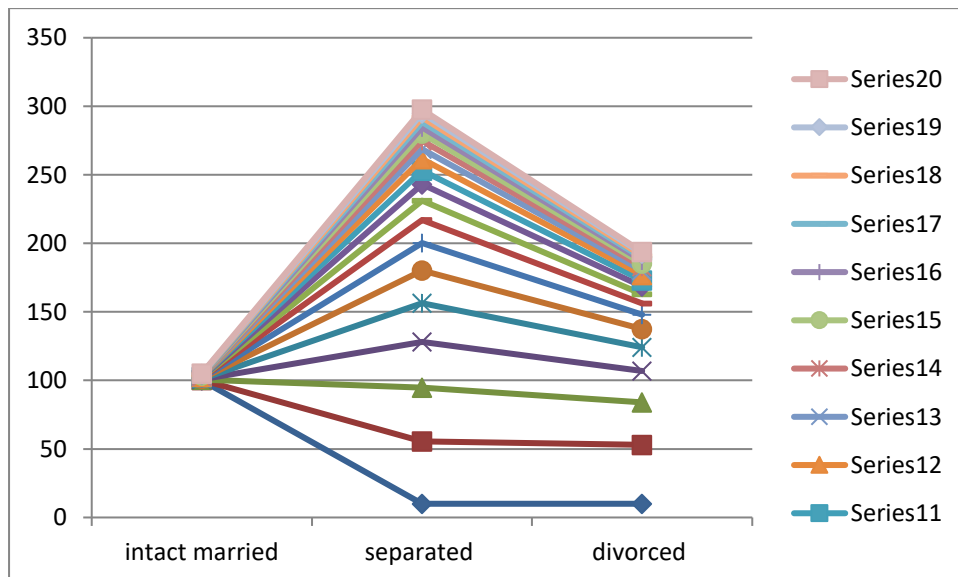
Parameter	$\mathbf{p}$	$\boldsymbol{\varepsilon}$	$\mathbf{K}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\delta}$	$\boldsymbol{\mu}$
Value	0.4	0.6	0.2	0.25	0.01	0.5	0.025

Endemic equilibrium when all parameters are fixed

intact married	separated	divorced
100	10	10
0.19994	43.72628	44.71117
0.193056	36.27589	33.0635
0.187854	30.00386	24.75885
0.184272	24.76185	18.7672
0.182605	20.40814	14.39543
0.182442	16.81196	11.17208
0.184037	13.85503	8.771953
0.187631	11.43286	6.968237

0.192961	9.454973	5.601202	Variation of parameter values for endemic equilibrium $K=0.2, \beta=0.01$ Table 4.2.1
0.200031	7.843754	4.556744	
0.20904	6.533393	3.752412	
0.2198	5.46892	3.128368	
0.232261	4.604553	2.640539	
0.246329	3.902514	2.256277	
0.261624	3.331962	1.951405	
0.27807	2.867575	1.707585	
0.295438	2.488865	1.511059	
0.31338	2.17933	1.351488	
0.331645	1.925638	1.220951	

Table 4.2.1 shows that while value of the intact marriage fluctuates at the initial stage and later increases, the separated and divorce are decreasing monotonically. Intact marriage ranging from 0.19994 to 0.331645, separated is decreasing from 43.72628 to 1.925638 and divorce from 44.71117 to 1.220951.



A graph showing endemic equilibrium, $p=0.4, \epsilon=0.6, k=0.2, \alpha=0.25, \beta=0.01, \delta=0.5, \mu=0.025$

50% variation of beta and k on endemic equilibrium

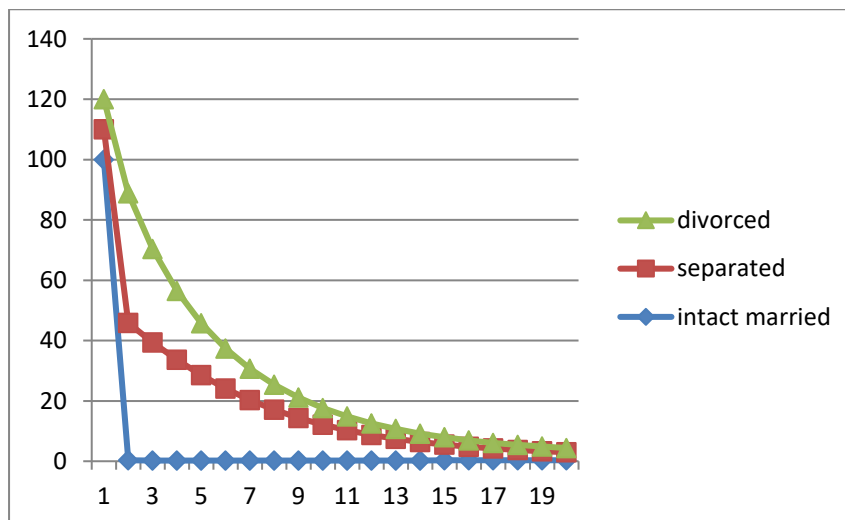
Variation of parameter values for endemic equilibrium $K=0.1, \beta=0.005$

Table 4.2.2

intact married	separated	divorced
100	10	10
0.107206	41.91423	46.62013
0.107732	33.42582	35.66491
0.108926	26.66816	27.3891

0.110991	21.29392	21.12127
0.114014	17.02398	16.3625
0.118094	13.63441	12.74088
0.123491	10.9457	9.978236
0.130457	8.814208	7.865928
0.139014	7.125261	6.247147
0.149339	5.787273	5.00354
0.161630	4.72722	4.045576
0.175879	3.88705	3.30548
0.192101	3.220595	2.731745
0.210027	2.691333	2.285329
0.22963	2.270215	1.936351
0.250425	1.934469	1.662275
0.272152	1.666034	1.445807
0.294438	1.450744	1.273823
0.316916	1.277472	1.136324

Table 4.2.2 shows that while the intact marriage decreases at the initial stage and later increases while separated and divorce are decreasing monotonically. Intact married ranging from 0.107206 to 0.316916, separated is decreasing from 41.91423 to 1.277472 and divorce from 46.62013 to 1.136324.



150% variation of beta and k on endemic equilibrium.

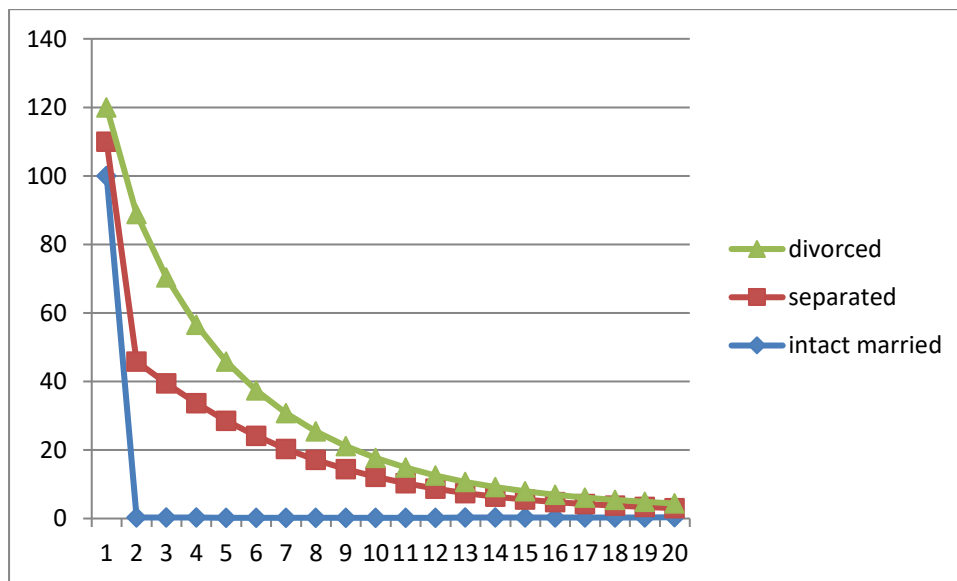
Variation of parameter values for endemic equilibrium $K=0.3$, $\beta=0.015$

Table 4.2.3

intact married	separated	divorced
100	10	10
0.286725	45.54859	42.9998
0.268082	39.14688	30.97636
0.25365	33.36716	22.87582
0.243229	28.26395	17.27472

0.235949	23.83644	13.30807
0.231901	20.04665	10.43677
0.230461	16.83651	8.31746
0.231508	14.13884	6.725685
0.234452	11.88561	5.511912
0.239556	10.01172	4.573544
0.24635	8.458418	3.839599
0.25487	7.173508	3.259355
0.264826	6.112024	2.796378
0.27613	5.235518	2.423776
0.288798	4.511521	2.121417
0.302252	3.913254	1.874546
0.316634	3.418097	1.671414
0.331689	3.007597	1.503231
0.347109	2.666628	1.363231

Table 4.2.3 shows that while value of the intact marriage fluctuates at the initial stage and later increases, the separated and divorce are decreasing monotonically. Intact married ranging from 0.286725 to 0.347109, separated is decreasing from 45.54859 to 2.666628 and divorce from 42.9998 to 1.363231.



4.1 Sensitivity analysis of the basic reproductive numbers

We investigate the nature of the model by conducting sensitivity analysis of the reproductive number (R_0).

(a) At the broken home -free equilibrium $\varepsilon = 0.5, K = 0.4$ and $\alpha = 0.25, R_0 = 0.76923 < 1$. If the value of ε is increased to any figure greater than 0.65 and the values of K and α are maintained $R_0 > 1$.

(b) At the endemic equilibrium, $\varepsilon = 0.6$, $K = 0.2$ and $\alpha = 0.25$, $R_0 = 1.333 > 1$. If ε is reduced to 0.44 and α , K are maintained the same, $R_0 < 1$.

We use XYZ model to study the dynamics of broken home as an epidemic. We discussed the existence and stability of divorce free and endemic equilibrium and performed the sensitivity analysis of the reproductive numbers. Based on the data in Table 1, the basic reproductive number of the divorce free equilibrium is estimated to be $R_0 = 0.76923 < 1$. This implies that only marriage population is present and the separated and broken home populations reduces to zero ($Y = 0$, $Z = 0$). This means that the model is asymptotically stable at $R_0 < 1$ and therefore satisfies Theorem 1. This has been verified numerically in Fig. 2. In the stability analysis of the broken home free equilibrium, the eigenvalues are $\lambda_1 = -0.25$, $\lambda_2 = -0.05$ and $\lambda_3 = -0.15$. This also indicates that the broken home free equilibrium is asymptotically stable.

Considering the situation when $R_0 > 1$, the reproductive number of the endemic equilibrium is estimated to be $R_0 = 1.333 > 1$ using the data in Table 2. This shows the situation in which the marriage, separated and broken home coexist in the population $(X^*, Y^*, Z^*) = (0.75, 0.204, 0.85)$

This indicates the existence of divorce problem in the population. People with divorce problem will continue to transform more Marriages into divorced and the broken home free equilibrium becomes unstable at R_0 . This is in line with our analytical results and has also been verified numerically.

5 Discussion

The analytical solution of the system of inferential equations shows that at the first initial condition, $X = 0$, $Y = 0$, $Z = 0$; $X(t)$ is equal to the rate of entering into marriage is the same or equal to the rate of leaving the marriage which could be by any means (death). This also means that at onset there was no marriage, no separation nor divorce. This depicts that there was no marriage between the two persons meaning that there was no concrete consent of either family involved.

Based on the zero initial boundary condition, $Y(t)$ and $Z(t)$ analytical solution show that there was no marriage existing between the two persons, this means that they do not have any objective reality even though they were cohabiting.

However, the second initial boundary conditions at $X = Y = Z = 1$; the analytical results show that where there is marriage, then separation and divorce is likely to occur if not well managed. These solutions shows that both the intact home ($X(t)$), separated ($Y(t)$) and divorced ($Z(t)$) exist and grows in an exponential rate with respect to time (t). The result showcase that for separation and divorce not to occur, then marriage should be avoided completely.

The analytical result shows that for broken home to be avoided there must be a good mutual understanding in the intact home and that will lead it to grow exponentially with love bond and without cracks or breakages.

6 Conclusion

By being human beings we all have limitations and shortcomings, due to these we used to make problems for other for some time. Family filled with mutual love and care is the ideal for all of us. But because of such human incapacity, we cannot succeed it. Then conflicts aroused which caused broken family. There can be many reasons.

Family filled with mutual love and care is the ideal for all of us. But because of such human incapacity, we cannot succeed it. Then conflicts aroused which caused broken family. There can be many reasons. Broken family also can be resolved at the same time, it could be prevented. The model shows that, broken home epidemic cannot only be controlled by reducing the contact rate

between the married and the divorced but also increasing the number of marriages that go into separation and educating separated to refrain from divorce can be useful in combating the epidemic.

Recommendations

1. In further studies there should be a development of various related topics that will help in analyzing other aspects that are closely related to broken home in order to get further understanding and avoid the problem of broken home that happen to us and to give us information about the effects of broken home.
2. The Ministry of Education and Vocational Training (MEVT) in collaboration with Universities and religious organizations should develop training programme on love, marriage and care for parents in order to educate the society on the importance of love and family care for healthy families, communities and society in general.

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On independence and minimal generating set in semigroups and countable systems of semigroups

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Abstract

This paper studies independence in semigroup by exploring the relationship between the concepts of “minimal generating set” as compared to “minimum generating set” using comparability of elements, as induced by the orderings on the semigroup. It is shown that minimal generating set implies independence and vice versa. Two concluding results which are algorithms respectively for determining minimal generating sets for countable systems of semigroups and for any given semigroups, are given.

Keywords: minimal; maximal; minimum; maximum; generating set; semigroup; ordering; independence; countable systems

1 Introduction

The task of determining the minimal generating systems of algebraic structures can sometimes be made easier by utilizing the concept of independence. This paper is a continuation of studies towards characterizing independence and minimality of generators of semigroups as given in articles such as [7], [8] and [9]. As we shall see in the paper, both concepts (‘Minimality’ and ‘Independence’) are equivalent and so finding minimal generating set is as good as finding the independent set. The work is concluded with our focus on determining minimal generating sets of some semigroups by giving algorithms for obtaining this from any given generating set. We begin with some preliminary definitions. Definition of terms not defined but used in this paper can be found in [5], [8] and [9]. Some results in this paper strengthen more, the answers to some questions raised by (Sampson, 2023).

1.1 Definition (Semigroup)

Let S be a set and $*$: $S \times S \rightarrow S$ be a binary operation that maps each ordered pair (x, y) of S into an element of S . S is a Semigroup if $\forall x, y, z \in S$,

$$(x * y) * z = x * (y * z)$$

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1.2 Definition (Subsemigroup)

Let $\{S, *\}$ be an ordered pair, where S is a semigroup and $*$ is the binary operation in S . A subset $U \subset S$ with the same operation $*$ on it is called a subsemigroup of S .

1.3 Notation

We will write S instead of $\{S, *\}$ except where very necessary.

1.4 Definition (Finite Semigroup)

Let S be a Semigroup. S is a finite semigroup if it has only a finitely many elements.

1.5 Definition (Monoid)

Let $(S, *)$ be an ordered pair where $*$ is an associative binary operation. $(S, *)$ is called a monoid, if there exists an element $e \in S$ such that $e * x = x * e = x \in S$.

1.6 Definition (Group)

Given the ordered pair $(G, *)$, where G is a set and $*$ is a binary operation in G . $(G, *)$ is called a Group if

- (i) $(x * y) * z = x * (y * z) \quad \forall x, y, z \in G$.
- (ii) $\exists e \in G$ such that $\forall x \in G, e * x = x * e = x$.
- (iii) For each $x \in G, \exists x' \in G$, such that $x * x' = x' * x = e$, where e is the identity element.

1.7 Definition (Subgroup)

Let G, H be sets. Let $*$ form a group with the same operation $* : H \times H \rightarrow H$ on G . The subset $H \subset G$ is a subgroup of G .

1.8 Definition (Generating system of Semigroup)

Let S be a semigroup and let $H \subset S$ be a nonempty subset of S .

Given

$$\langle H \rangle := \left\{ \prod_{j=1}^k a_j \mid \forall \{a_j\}_{j=1}^k \subset H, 1 \leq k < \infty \right\},$$

where $\langle H \rangle$ is called the subsemigroup generated by H . H is a generating set of S if $S = \langle H \rangle$.

1.9 Definition (Generating Set)

Let $G \subset S$ be a subset of S . A nonempty subset $G \subset S$ is a generating set of S if $Cl(G) = S$. (by $Cl(G)$ we mean the smallest closed set or the intersection of all the closed sets, containing G).

1.10 Example ([4])

Consider the infinite non abelian matrix semigroup $(M, *)$ with respect to the usual matrix multiplication $*$:

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a = \pm 1, b \in \mathbb{Z} \right\}$$

Notice that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ &\vdots \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b$$

So, for $X = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b \Rightarrow X = A^b \Rightarrow A$ is an element of the generating set. Notice also that $a = \pm 1$. We have considered the case for $a = 1$. For $a = -1$, we have

$$\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ combine to generate}$$

$$Y = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}. \text{ That is; } Y = A^b * B.$$

M is finitely generated by the generating set $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, a 2 – *element* subsemigroup.

1.11 Definition (Independent Element of a Semigroup)

An element a in S is independent from a subset U in S if $a \notin \langle U \rangle$.

1.12 Definition (Independent Sets of semigroup)

Let S be a semigroup. A subset U of S is independent if, for every u in U , the element u does not belong to the subsemigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of U . Mathematically, we simply say if

$$\forall u \in U, u \notin \langle U \setminus \{u\} \rangle,$$

then U is independent.

1.13 Definition (Dependent Sets of Semigroup)

Let S be a finite semigroup. A subset U of S is dependent if, for every u in U , the element u belongs to the subsemigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of U .

1.14 Remark

A subset U of a semigroup S is dependent if it is not independent.

1.1.16. Definition (Maximal Independent Sets of semigroup)

A Maximal Independent subset of a semigroup S is that subset $U \in S$ in which including any other element in the set makes it dependent.

1.15 Definition (Basis)

A subset $B \subset S$ is a basis in S if B is a minimal generating system in S ;

1.16 Equivalences

One important tool for obtaining generators of algebraic systems is the use of relations that enable partitioning of the systems into smaller units. In the next section, we introduce these tools with definitions and some useful results.

1.17 Definitions (Relation on a set)

(i) Given the sets A and B , a relation from A to B is a subset $R \subseteq A \times B$. If $(a, b) \in R$, we say a is related to b by R . A is the domain of R . B is the codomain of R and the ordered pair (a, b) is a relation on R .

1.18 Definition (Binary Relation)

Let A be a set and let R be a relation $R \subseteq A \times A$. Then R is called a binary relation on the set A .

1.19 Definition (Composition of relation)

Let $\rho_1 \subset A \times B$ and let $\rho_2 \subset A \times B$ be relations. Then their composition $\rho_1 \circ \rho_2$ is defined by $\rho_1 \circ \rho_2 = \{(x, y) | x \in A, \exists y \in B \text{ s.t. } (x, y) \in \rho_1 \text{ and } \exists z \in C \text{ s.t. } (y, z) \in \rho_2\} \subset A \times C$.

1.20 Definition (Set of all Relations on a set)

Let $B(X)$ be the set of all relations on X . $B(X)$ is the set of all relations on the set defined by $\rho \circ \sim = \{(x, z) | \exists y \in X, (x, y) \in \sim \text{ and } \exists z \in X, (y, z) \in \rho\}$.

1.21 Definition (Identity Relation)

Let $B(X)$ be the set of all relations on X . The identity element of $B(X)$ is the identity relation $\iota = \iota_X = \{(x, x) | x \in X\}$. (It is the identity function on X).

1.22 Notations

If $(a, b) \in \rho$, then we write apb . If $(a, b) \notin \rho$, then we write $ap'b$. For $A \times A$, we have that $(a, b) \in \rho \Leftrightarrow apb$.

1.23 Definition (Equivalence Relation)

Let $B(X)$ be the set of all relations on X . A relation $\sim \in B(X)$ is an equivalence relation, if it is

- (i) $\forall x \in X, x \sim x$; (reflexive),
- (ii) $\forall x, y \in X, x \sim y \Rightarrow y \sim x$ (symmetric)
- (iii) $\forall x, y, z \in X, x \sim y \text{ and } y \sim z \Rightarrow x \sim z$ (transitive),

1.24 Notations

The sets $x\sim$ are called equivalence classes. They form a partition of the set of union of all equivalences: $X: X = \bigcup_{x \in X} x\sim$, where $x\sim$ are equivalence classes and (as we shall later see, if the intersection of two or more partitions are empty, it implies that they are equal)
 $x\sim \cap y\sim \neq \emptyset \Leftrightarrow x\sim = y\sim$.

1.25 Definitions (Order Relation)

Let $\mathcal{A}$ be a set. A relation $\leq$ on $\mathcal{A}$ is an Order relation if

- i) $\forall x \in \mathcal{A}, x \leq x$ ($\leq$ is reflexive),
- ii) $\forall x, y, z \in \mathcal{A}$, if $x \leq y$ and $y \leq z$, then $x \leq z$ ($\leq$ is transitive),
- iii) $\forall x, y \in \mathcal{A}$, if $x \leq y$ and $y \leq x$, then $x = y$, ($\leq$ is anti-symmetric).

1.26 Definition (Partially Ordered Set)

The pair $(X, \leq)$ is called a Partially Ordered Set (or, for short, Poset) if the relation $\leq$ is an order relation and there exist $a, b \in X$ such that neither $a \leq b$ nor $b \leq a$.

1.27 Definition (Total Ordering)

Let A be a set and then the order relation $\leq \subseteq A \times A$ on A is a Total Ordering if the relation $\leq$ is an order relation and $\forall a, b \in A$, $a \leq b$ or $b \leq a$ (that is; any two elements of the totally ordered set are comparable).

1.28 Remarks

A totally ordered set is also called a Chain. A totally ordered set is also called a Linearly ordered Set. Comparability of elements is not a necessary requirement for Partially Ordered Set.

1.29 Definition (Well Ordering)

Let $\mathcal{A}$ be a set and $\leq$ a relation on $\mathcal{A}$. The relation $\leq$ is a well ordering of $\mathcal{A}$ if every non-empty subset S of $\mathcal{A}$ has a least element.

1.30 Remark

The theorem that follows is a significant requirement for one of the main results in the fourth chapter of this thesis.

Theorem 1: If $\leq$ is a total ordering on $\mathcal{A}$, then every non-empty finite subset S of $\mathcal{A}$ has a least element and a greatest element.

Proof. We show there is a least element and leave the rest to an exercise. By induction. Let $P(n)$ be the statement "every subset of $\mathcal{A}$ having n elements has a least element." Clearly $P(1)$ is true. Suppose $P(n)$ is true. If S has $n + 1$ elements, let $S = S' \cup \{y\}$, for some $y \in S$ and S' with n elements (namely, $S' = S \setminus \{y\}$, or S with y removed). By induction, S' has a least element, call it x .

Since $\leq$ is a total ordering, either $x \leq y$ or $y \leq x$. In the first case, x is a least element of S . On the other hand, if $y \leq r$ case, $y \leq z$, as desired.

1.31 Remark

Any countable set can be well-ordered (this fact is equivalent to the popular axiom of choice). In fact, by Zermelo–Fraenkel Axiom of Choice (ZFC), this is true for any set and not just for countable set.

Theorem 2:

If $\leq$ is a well ordering of $\mathcal{A}$ then it is a total ordering.

Proof. Suppose $x, y \in \mathcal{A}$. Let $S = \{x, y\} \neq \emptyset$. By hypothesis, S has a smallest element. If it is x , then $x \leq y$, and if it is y , then $y \leq x$.

1.32 Remark

A better appreciation of the major tasks of this paper will require a clearer understanding of the concept of minimal and maximal subsets as compared to that of minimum and maximum subsets of a semigroup. Scholars sometimes confuse one for the other. We will therefore begin with some explanations to clear confusions in usage. We will also mention some of the implications to the work.

2 Maximum/maximal and minimum/minimal

A vector space is ordered if there is an order relation which is additively translation invariant and multiplicatively “translation” invariant for non-negative factors:

Let V be a vector space over R (may be an ordered field!) and let $\leq \subseteq V \times V$ be an order relation (see section 1.1.27.). It is compatible with the vector structure if the following hold:

- (a) $\forall a, b, c \in V, a \leq b \Rightarrow a + c \leq b + c$;
- (b) $\forall a, b \in V \text{ and } 0 \leq c \in R, a \leq b \Rightarrow a \times c \leq b \times c$.

Theorem 3:

A binary relation $\leq \subseteq V \times V$ is an order relation compatible with the vector structure if and only if $C_{\leq,+} = \{x \mid 0 \leq x \in V\}$ is a convex cone fulfilling the relation $C_{\leq,+} \cap -C_{\leq,+} = \{0\}$ and such that $\forall a, b \in V, a \leq b \Leftrightarrow b - a \in C_{\leq,+}$.

2.1 Corollaries

- (a) If V is one dimensional, then the cone $C_{\leq,+}$ is a half straight line $[0, \infty)$ hence

$$R = C_{\leq,+} \cup (-C_{\leq,+}) = [0, \infty) \cup (-[0, \infty))$$

and the ordering is complete ordering hence or otherwise $\forall a, b \in V$ either $a \leq b$ or $b \leq a$ holds.

Since any two elements of V are comparable, the largest element $\max(A) =: M_A \in A$ of a set $A \subset V$ is greater than or equal to all the elements of the set A , and the smallest element $\min(A) =: m_A \in A$ is less than or equal to any element of the set A .

These are called maximum and minimum respectively, and they are comparable with each other and any element of the set.

A set may have one maximum and one minimum. If $a \in A$ is not maximum then there exists at least one $b \in A$ such that $a \leq b$ and $a \neq b$. Similar for minimum, If $a \in A$ is not minimum then there exists at least one $b \in A$ such that $b \leq a$ and $a \neq b$. Any two maximums are equal. The same holds for minimum.

(b) If the dimension of the vector space is two or more, then

$$C_{\leq,+} \cup (-C_{\leq,+}) \neq V \text{ hence } V \setminus (C_{\leq,+} \cup -C_{\leq,+}) \neq \emptyset.$$

This means that there are non-comparable pairs, the ordering is not complete.

This brings radical changes: Definitions of maximum and minimum change from that of a vector space with complete ordering (one dimension).

2.2 Definition (Minimum and Maximum)

$M_A \in A \subset V$ is a maximum of A if $x \leq M_A, \forall x \in A$. $m_A \in A \subset V$ is a minimum of A if $m_A \leq x, \forall x \in A$.

2.3 Definition (Minimal and Maximal)

$M_A \in A \subset V$ is a maximal element of A if $\nexists x \in A$ such that $x \neq M_A \leq x$.

$m_A \in A \subset V$ is a minimal element of A if $\nexists x \in A$ such that $x \leq m_A \neq x$.

2.4 Remarks

Obviously, from 2.1(a) and 1.1(b) the concept of minimum and maximum cannot be upheld when V is not completely ordered. This necessitates the change to Minimal and Maximal as given in section 2.3.

One subtlety in the definition (of 2.1.3) is that maximal elements are not necessarily above everything in the partial order, but rather nothing in the partial order is above them. Compare this definition with “maximum”: an element is called maximum in A if it is larger than all the other elements while an element $M_A \in A$ is called a maximal element of A , if there is no larger (comparable) element in A . Similarly, minimal elements are not necessarily lower than everything in the partial order, but rather nothing in the partial order is below them. A partial order can have many maximal elements as well as many minimal elements.

In other words, these definitions uphold the motive that an element $M_A \in A$ is a maximal element of A , if there is no larger (comparable) element in A . An element $m_A \in A$ is minimal element of A if there is no smaller element (comparable) in A .

To both minimal and maximal elements there may exist an element in A such that the maximal is not greater than or equal to it and the minimal is not less than or equal to it (when they are not comparable). Maximal elements of a set A may not be equal.

2.5 Example

Let B be an infinite set, let $A := \{U \mid \emptyset \neq U \subset B \neq U\}$, and let the order relation be $U \leq V \Leftrightarrow U \subset V$. All singleton members of A are minimal, not any two of them is comparable and all of them are different. $\forall x \in B$ the set $W = B \setminus \{x\}$ is maximal, and any two of them are non-comparable and different.

2.6 Remark

The worlds of maximum/minimum and maximal/minimal are completely different. Their existence is mostly compactness for maximum/minimum, Zorn's lemma stands for the existence of maximal/minimal elements of a set.

2.7 Minimal generating sets Implies independence

Let us use the order relation described in the above example for subsets of a set.

Let A be a set, then the order relation is $U \leq V \Leftrightarrow U \subset V, \forall U, V \subset A$.

Using the definition 2.1.3. for minimal elements.

2.8 Definition (A Minimal generating set)

A generating set G of a semigroup is minimal if $\forall H \leq G, H \neq G, H$ is not a generating set.

Theorem 4 (Minimality of a Generating Set Implies the Independence of the set):

A generating set G of a structure is minimal if G is independent (for any structure like vector spaces, groups, semi-groups, monoids etc.).

Proof:

A generating set G is minimal $\Rightarrow$ it is independent: Let $x \in G$ be any element. If $x \in \langle G \setminus \{x\} \rangle$ then

$$(G \setminus \{x\}) \cup \{x\} \subset \langle G \setminus \{x\} \rangle \Rightarrow S = \langle G \rangle = \langle (G \setminus \{x\}) \cup \{x\} \rangle \subset \langle \langle G \setminus \{x\} \rangle \rangle$$

which contradicts with the minimality of G .

2.9 Example (minimal generating set implies independence: a case study using a group)

We adapt an example in (page 4 of [4]) to have the following.

Let $\{p_i\}_{i=1}^m, n = \prod_{i=1}^m p_i$ and $\{Z, +\}$ be a group. Let

$$G_m := \left\{ \frac{n}{p_i} \mid 1 \leq i \leq m \right\}$$

Let $\frac{n}{p_i} \in G_m$ be selected:

$$\frac{n}{p_i} = p_1 \times p_2 \times \dots \times p_{i-1} \times p_{i+1} \times \dots \times p_m.$$

Since p_i is not in the product, p_i is not its factor.

Consider $G_m \setminus \left\{ \frac{n}{p_i} \right\}$. Then $\forall z \in G_m \setminus \left\{ \frac{n}{p_i} \right\}, p_i \nmid z \Rightarrow \forall u \in \langle G_m \setminus \left\{ \frac{n}{p_i} \right\} \rangle, p_i \nmid u$ while $p_i \nmid \frac{n}{p_i}$

In summary, $\frac{n}{p_i} \notin \langle G_m \setminus \left\{ \frac{n}{p_i} \right\} \rangle$ meaning that $\left\{ \frac{n}{p_i} \right\}$ is independent from $G_m \setminus \left\{ \frac{n}{p_i} \right\}$.

Theorem 5 (Independence of a set implies minimal generating set):

A generating set G is independent $\Rightarrow G$ is minimal.

Proof: Let $x \in G$ be any element. Then $x \notin \langle G \setminus \{x\} \rangle$. Hence $\langle G \setminus \{x\} \rangle$ is not a generating set. Let $\emptyset \neq H \subset G \neq H$. Then there is at least one element $x \in G \setminus H \Rightarrow G \setminus \{x\} \supset H$. Since $G \setminus \{x\}$ is not a generating set, its subset H is also not a generating set.

Theorem 6:

A generating set G is independent if and only if G is minimal

Proof. The proof follows directly as implication of 2.5 and 2.9

2.11 Remarks

(i) The above theorem is actually true in all structures: vector spaces, groups, semigroups etc. Maximum number of independent elements is influenced by the constructing powers of operations.

(ii) In vector spaces given a basis, we can consider the linear subspace generated by a basis element. All members of this space can be obtained with multiplication by scalar which is a member of a field. Hence knowing one element, multiplication by non-zero factor gives access to any other non-zero member of the subspace by the multiplication being invertible.

(iii) Groups: Given a minimal generating set. Take any member of this set, it generates a subgroup. Using the inverse one can generate any member of this subgroup hence the constructive power does not allow the generation of further independent elements.

(iv) Semigroups: Take a one element generated cyclic group. If you consider the prime powers of the one element generator, these will be independent in an infinite cyclic group since there is multiplication only, no other way to generate any further semigroup element.

(v) From this follows that maximal independent system is not a good idea to minimal generating set: One element generated subsemigroup may have an infinite system of independent members.

2.12 Further comments on minimal generating sets

In free semigroups a minimal generating set is the generator members of the free semigroup. Its cardinality gives the cardinality of the minimal generating set.

3 Algorithms: determining minimal generating set for any system of semigroup

For illustration purpose, in Figure 3.1 below, the sets B and C are required to be subsets of a well-ordered index set $\mathcal{A}$, so we can find some least elements of the sets with which the minimal generating set can be obtained.

3.1 Remarks (implications of the nature of ordering on determination of generating systems)

The concept of “Well ordering” enables us to find generating sets in any semigroup and not just in countable systems like $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ in general. For countable systems, we do not necessarily require the Well ordering. A direct implication of the Well Ordering Principle (Bukh, 2019; Ivan, 2018) for well ordered

sets requires that there exists a least element. We shall see that for every generating set G , there is a minimal generating set that is comparable with G .

3.2 Algorithms: determining minimal generating set for any system of semigroup

For illustration purpose, in Figure 3.1 below, the sets B and C are required to be subsets of a well ordered index set $\mathcal{A}$, so we can find some least elements of the sets with which the minimal generating set can be obtained.

Let $\mathcal{A}$ be a well ordered index set. Let $B, C \subset \mathcal{A}$. If $G = \{e_t\}_{t \in B \cup C}$ is a semigroup generating set, then the flow chart below gives an algorithm to create the minimal generating set:

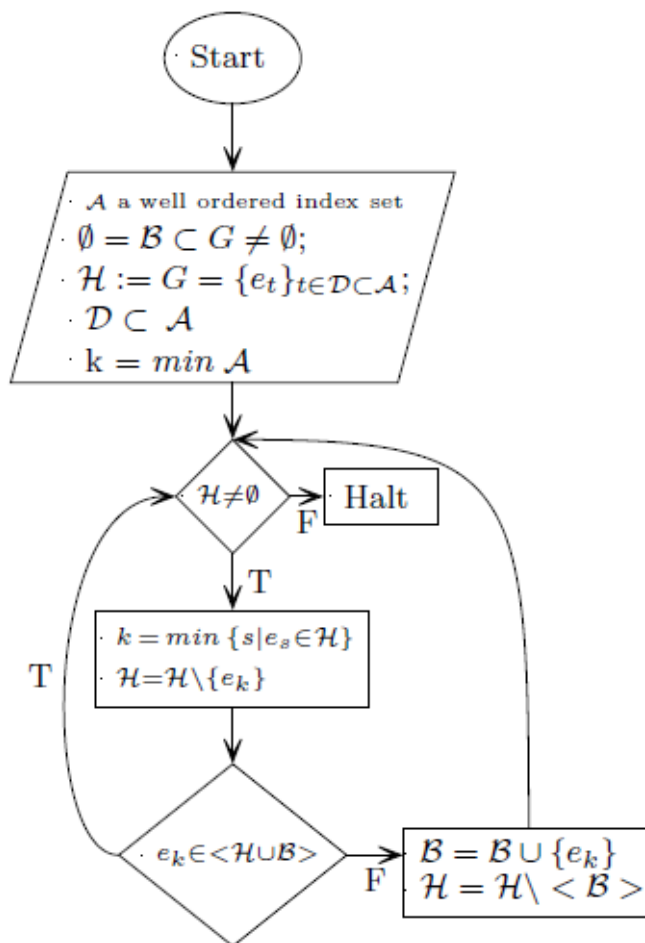


Figure 3.1. Algorithm to generate a minimal (independent) generating set B from a generating set G .

Theorem 7:

Let S be a semigroup and let $G \subset S$ be a generating set. Then the algorithm in Figure 3.1 selects a minimal generating set B from G .

Proof:

1. B is initialized to be empty set, $H \subset G$ since it is initialized by G and each step we remove some elements from it and add some of them to B . Hence $B \subset G$ in each step.
2. An $e_k \in H$ is added to B only when $e_k \notin \langle H \cup B \rangle$ hence e_k is independent from all selected basis elements and those in H which will be selected into the final B . Hence the set B at the final step when $H = \emptyset$ is a set of independent elements.
3. In each step $\langle H \cup B \rangle \subset S$ is a generating set and when $H = \emptyset$; then B is a generating set.

3.3 Algorithm determining minimal generating set for countable system

In figure 3.2 below, the set $\mathcal{A}$ is the set of natural numbers and is well ordered, so we can determine a minimal generator of the set (as implied by the theorem of section 1.1.26).

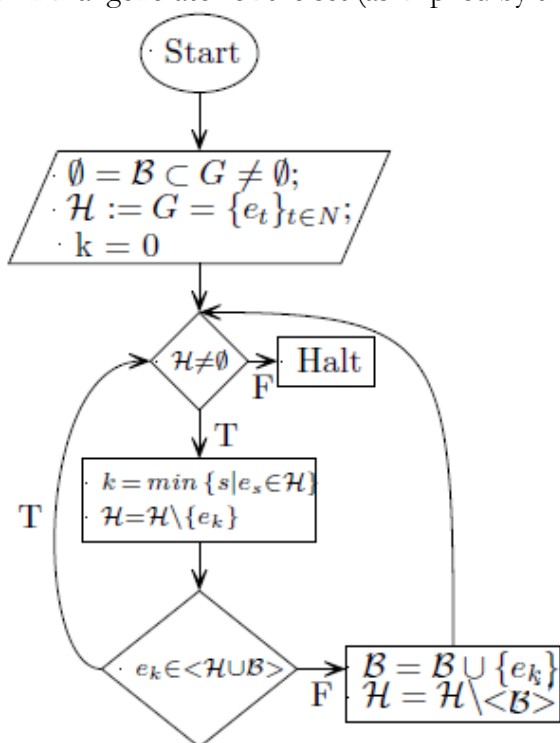


Figure 3.2. Algorithm to generate a minimal (independent) generating set B from a generating set G – Countable Systems.

Corollary

Let S be a semigroup and let $G \subset S$ be a generating set and G is a countable system. Then the algorithm in Figure 3.2 selects a countable minimal generating subset $B \subset G$.

Proof:

1. B is initialized to be empty set, $H \subset G$ since it is initialized by G and each step we remove some elements from it and add some of them to B . Hence $B \subset G$ in each step.

2. An $e_k \in H$ is added to B only when $e_k \notin \langle H \cup B \rangle$ hence e_k is independent from all selected basis elements and those in H which will be selected into the final B . Hence the set B at the final step when $H = \emptyset$ is a set of independent elements.

3. In each step $\langle H \cup B \rangle \subset S$ is a generating set and when $H = \emptyset$ then B is a generating set.

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Analysis of investment portfolio with 2×2 positive definite matrix and mortality risk under exponential utility

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Abstract

This paper investigates a pension plan member's (PPM) portfolio in a defined contributory (DC) pension plan with return clause and charge on balance exhibiting constant absolute risk averse (CARA). A portfolio comprising of a risk-free asset and two risky assets modelled by the geometric Brownian motion (GBM) process is considered, where the instantaneous volatilities of the two risky assets form a 2×2 matrix $k = \{k_{a,b}\}_{2 \times 2}$ such that kk^T is positive definite. The Abraham De Moivre mortality force function is used to determine the mortality rate of PPMs during accumulation phase and an optimization problem is obtained from the Hamilton Jacobi Bellman (HJB) equation using dynamic programming approach. Using Legendre transformation and dual theory with variable change technique, the optimal value function (OVF), optimal control plan (OCP) and the optimal fund size (OFS) are obtained under exponential utility function. Furthermore, some numerical analysis of some sensitive parameters such as risk free interest rate, risk averse coefficient, entry age of PPM, charge on balance and OFS are presented to explain their impact on the OCP.

Keywords and phrases: Abraham De Moivre mortality force function; optimal control plan; Legendre transform and dual theory; GBM process; exponential utility.

1 Introduction

In the defined contributory pension scheme, PPMs are mandated to contribute a certain proportion of their income into their retirement savings account (RSA) for the purpose of investment and to plan for the retirement needs of PPMs by the pension fund administrators (PFA) [6]. In order for the PFAs to maximize the PPM's wealth at the time of retirement, there is need to understand how investments are done for maximum returns with minimal risk; this has led to the study of OCP. A good number of works have been done in this regards under different assumption [1, 5, 7, 8, 9, 10, 12, 17, 22, 23, 26, 28, 29].

One of the major challenges faced by most PFAs in the DC pension scheme is the determination of investment strategy under mortality risk [11]. This is so, because they will be faced with the challenges of paying off the contributions due to PPMs who die during the accumulation period and at the same time managed the portfolio of the remaining PPMs. In achieving this, two mortality force functions have been used by some researchers to model the mortality rate of PPM in the DC pension scheme during the accumulation phase which include Weibull mortality force function and Abraham De Moivre force function [4, 11, 16, 19, 21, 24, 25]. However, a number of research works have been done in this regard. They include; [11], who studied OCP with return of premium clauses under mean variance utility. In their research work, they consider a PPM with a portfolio consisting of one risk free asset and a risky asset modelled by GBM and used the game theoretic approach to determine the optimal investment strategy and the efficient frontier of the PPM. [24], investigated studied the optimal investment problem with return clause for a PPM having a portfolio comprising of one risk free asset and a risky asset modelled by Heston volatility model; they also used the mean variance utility and

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the same method in [11] to determine the OCP and the efficient frontier. In [19], the default risk was introduced into the model in [11], where pension funds are allowed to be invested in a risk-free asset, a defaultable bond and a risky asset whose price process follows the CEV model. They used the variational method to obtain the OCP and the efficient frontier. [4] studied the OCP with return clause; in their work, they considered investment in three different assets comprising of fixed deposit, stock and loan and used the game theoretic approach to determine the OCP and the efficient frontier under mean variance utility. [25], studied the same problem in [11] but used the Jump diffusion process to model the risky asset and determine the OCP and efficient frontier. Also, [2, 3] investigated the OCP with return of premium with predetermined interest; in their research work, the returned premium is with interest from the invested fund in risk free asset. They went further to determine the OCP and the efficient frontier of the PPM. In all the research work mentioned above, the utility function used was the mean variance utility developed by Markowitz [20] and the mortality rate of the death PPM were determined using Abraham De Moivre force function.

Very recently, [16] studied the OCP with return clause under power utility function; in their work, they considered a PPM with a portfolio consisting of one risk free asset and a risky asset which follows the constant elasticity of variance model. Also, they used the Weibull mortality force function to model the mortality rate of members who died during the accumulation phase and determine the OCP of the PPM. In [21], the OCP with return clause was studied under Weibull mortality force function. They took into consideration tax on the invested funds.

The main contribution of this paper is that we choose a portfolio consisting of three different assets, one risk free asset and two risky asset which forms a 2×2 matrix which is positive definite similar to one in [18]. Also, the Abraham De Moivre force function is used to determine the mortality rate of the PPM during the accumulation phase and the charge on balance on the PPM portfolio is also considered. The Legendre transformation method and dual theory is applied to solve for the OCP, the OVF and OFS which are not captured in the literature for a PPM with return of premium exhibiting constant absolute risk averse. Finally, we study the impact of some sensitive parameters on OCP. More precisely, the impact of PPM's entry age on the optimal control plan and optimal fund size for investment are studied and the impact of the charge on balance on the OCP.

2 The investment model

In this section, we shall consider a financial market which is continuously open for an interval $t \in [0, T]$ where $T > 0$, is the expiration date of the investment. Also, we shall consider a portfolio with one risk free asset and two risky assets. Let $\{\mathcal{W}_1(t), \mathcal{W}_2(t), t = 0\}$ be two standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a real space and $\mathcal{P}$ is a probability measure and $\mathcal{F}$ is the filtration which represents the information generated by the two Brownian motions.

Let $\mathcal{N}_t^0(t)$ denote the price of the risk free asset at time t and the model is given as follows

$$d\mathcal{N}_t^0(t) = r\mathcal{N}_t^0(t) dt, \mathcal{N}_t^0(t) = n_0 > 0 \quad (2.1)$$

where $r(t)$ is the predetermined interest such that $r > 0$

Let $\mathcal{N}_t^1(t)$ and $\mathcal{N}_t^2(t)$ be the prices of stock 1 and stock 2 respectively which are modelled by geometric Brownian motions model and the dynamics of the stock market prices are given by stochastic differential equations at $t \geq 0$ as follows

$$\begin{pmatrix} \frac{d\mathcal{N}_t^1(t)}{\mathcal{N}_t^1(t)} \\ \frac{d\mathcal{N}_t^2(t)}{\mathcal{N}_t^2(t)} \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} dt + \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} d\mathcal{W}_1 \\ d\mathcal{W}_2 \end{pmatrix} \quad (2.2)$$

where ℓ_1 and ℓ_2 represent the expected appreciation rate of stock 1 and 2 respectively. $m_{11}, m_{12}, m_{21}, m_{22}$ represent the instantaneous volatilities and form a 2×2 matrix $m = \{m_{ab}\}_{2 \times 2}$ such that mm^T give a positive definite matrix [18].

3 Optimization problem

3.1 Wealth formulations and HJB equation

Let $k = (k_0, k_1, k_2)$ represents be the optimal control plan (OCP), k_1 is the proportion of investment in risk free asset and k_1 and k_2 are proportion of investment in the two risky assets. Next, we define the utility attained by the investor from a given state h at time t as

$$\mathcal{F}_k(t, h) = E_k U(\mathcal{H}(T)) | \mathcal{H}(t) = h, \quad (3.1)$$

where t is the time and h is the wealth of the pension plan member. Our interest here is to obtain the OCP and the optimal value function of the PPM given by

$$k^* \text{ and } \mathcal{F}(t, h) = \sup_k \mathcal{F}_k(t, h) \quad (3.2)$$

Respectively such that

$$\mathcal{F}_{k^*}(t, h) = \mathcal{F}(t, h). \quad (3.3)$$

Since our interest is to maximize $\mathcal{F}_k(t, h)$ subject to the PPM's wealth denoted by $\mathcal{H}(t)$, we proceed to formulate our wealth function as follows

Let c be the mandatory monthly contribution paid by the PPM at time t , e_0 the initial age of accumulation phase, T the accumulation phase period such that $e_0 + T$ is the terminal age of the PPM. $\aleph_{\frac{1}{i}, e_0+t}$ is the mortality rate from time t to $t + \frac{1}{i}$, tc is the accumulated contributions at time t , $tc\aleph_{\frac{1}{i}, e_0+t}$ is the returned contributions to the next of kin during the accumulation period.

Corresponding to the OCP $k = (k_0, k_1, k_2)$, where $k_0 = 1 - k_1 - k_2$ and the accumulation phase period $[t, t + \frac{1}{i}]$, the stochastic differential form for the wealth function similar to [11] is given as:

$$\mathcal{H}\left(t + \frac{1}{i}\right) = \left[\mathcal{H}(t) \left(k_0 \frac{\mathcal{N}_{t+\frac{1}{i}}^0}{\mathcal{N}_t^0} + k_1 \frac{\mathcal{N}_{t+\frac{1}{i}}^1}{\mathcal{N}_t^1} + k_2 \frac{\mathcal{N}_{t+\frac{1}{i}}^2}{\mathcal{N}_t^2} \right) + c \frac{1}{i} - tc\aleph_{\frac{1}{i}, e_0+t} \right] \left(\frac{1}{1 - \aleph_{\frac{1}{i}, e_0+t}} \right) \quad (3.4)$$

$$\mathcal{H}\left(t + \frac{1}{i}\right) - \mathcal{H}(t) = \left[\mathcal{H}(t) \left((1 - k_1 - k_2) \left(\frac{\mathcal{N}_{t+\frac{1}{i}}^0 - \mathcal{N}_t^0}{\mathcal{N}_t^0} \right) + k_1 \left(\frac{\mathcal{N}_{t+\frac{1}{i}}^1 - \mathcal{N}_t^1}{\mathcal{N}_t^1} \right) + k_2 \left(\frac{\mathcal{N}_{t+\frac{1}{i}}^2 - \mathcal{N}_t^2}{\mathcal{N}_t^2} \right) + c \frac{1}{i} - tc\aleph_{\frac{1}{i}, e_0+t} \right) \right] \left(1 + \frac{\aleph_{\frac{1}{i}, e_0+t}}{1 - \aleph_{\frac{1}{i}, e_0+t}} \right) \quad (3.5)$$

From [11],

$$\begin{cases} c \frac{1}{i} \rightarrow cdt, \frac{\mathcal{N}_{t+\frac{1}{i}}^0 - \mathcal{N}_t^0}{\mathcal{N}_t^0} \rightarrow \frac{d\mathcal{N}_t^0}{\mathcal{N}_t^0}, \frac{\mathcal{N}_{t+\frac{1}{i}}^1 - \mathcal{N}_t^1}{\mathcal{N}_t^1} \rightarrow \frac{d\mathcal{N}_t^1}{\mathcal{N}_t^1}, \frac{\mathcal{N}_{t+\frac{1}{i}}^2 - \mathcal{N}_t^2}{\mathcal{N}_t^2} \rightarrow \frac{d\mathcal{N}_t^2}{\mathcal{N}_t^2}, \\ \frac{1}{i} \rightarrow 0 \text{ as } i \rightarrow \infty, \aleph_{\frac{1}{i}, e_0+t} = \emptyset(e_0 + t) dt, \frac{\aleph_{\frac{1}{i}, e_0+t}}{1 - \aleph_{\frac{1}{i}, e_0+t}} = \emptyset(e_0 + t) dt, \end{cases} \quad (3.6)$$

Substituting (3.6) into (3.5), we have

$$d\mathcal{H}(t) = \left[\mathcal{H}(t) \left(k_1 \frac{d\mathcal{N}_t^1}{\mathcal{N}_t^1} + k_2 \frac{d\mathcal{N}_t^2}{\mathcal{N}_t^2} + (1 - k_1 - k_2) \frac{d\mathcal{N}_t^0}{\mathcal{N}_t^0} \right) + cdt - tc\emptyset(e_0 + t) dt \right] (1 + \emptyset(e_0 + t) dt) \quad (3.7)$$

Substituting (2.1) and (2.2) into (3.7), we have

$$d\mathcal{H}(t) = \left[\mathcal{H}(t) \left([k_1(\ell_1 - r) + k_2(\ell_2 - r) + (r - \vartheta) + \emptyset(e_0 + t)] dt + (k_1 m_{11} + k_2 m_{21}) d\mathcal{W}_1 + (k_1 m_{12} + k_2 m_{22}) d\mathcal{W}_2 + c(1 - t\emptyset(e_0 + t)) dt \right) \right] \quad (3.8)$$

Since $\emptyset(t)$ is the force function and e is the maximal age of the life table. From [11] [15], the Abraham De Moivre mortality force function formula is given as

$$\emptyset(t) = \frac{1}{(e-t)} \quad 0 \leq t < e. \quad (3.9)$$

This implies that

$$\emptyset(e_0 + t) = \frac{1}{(e - e_0 - t)} \quad (3.10)$$

Substituting (3.10) into (3.8) and simplifying it, we have

$$d\mathcal{H}(t) = \left[\begin{array}{l} \mathcal{H}(t) \left(\left[k_1(\ell_1 - r) + k_2(\ell_2 - r) + \left(r - \vartheta + \frac{1}{e - e_0 - t} \right) \right] dt \right. \\ \left. + (k_1 m_{11} + k_2 m_{21}) d\mathcal{W}_1 + (k_1 m_{12} + k_2 m_{22}) d\mathcal{W}_2 \right) \\ \left. + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) dt \right] \\ \mathcal{H}(0) = h_0 \end{array} \right] \quad (3.11)$$

From [13], we apply the Ito's lemma and maximum principle theorem to derive the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (3.11). This is obtained by maximizing (3.1) subject to (3.11) as follows

$$\left\{ \begin{array}{l} \mathcal{F}_t + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) h + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] \mathcal{F}_h \\ + \sup_{k_1, k_2} \left[\left(\frac{1}{2} (m_{11}^2 + m_{12}^2) k_1^2 + (m_{11} m_{21} + m_{12} m_{22}) k_1 k_2 \right. \right. \\ \left. \left. + \frac{1}{2} (m_{21}^2 + m_{22}^2) k_2^2 \right) \mathcal{F}_{hh} \right. \\ \left. + (k_1(\ell_1 - r) + k_2(\ell_2 - r)) \mathcal{F}_h \right] \end{array} \right\} = 0 \quad (3.12)$$

If $m_{11}^2 + m_{12}^2 = \mathcal{P}$, $m_{21}^2 + m_{22}^2 = \mathcal{Q}$ and $m_{11} m_{21} + m_{12} m_{22} = \mathcal{R}$, then we (3.12) becomes

$$\left\{ \begin{array}{l} \mathcal{F}_t + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) h + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] \mathcal{F}_h \\ + \sup_{k_1, k_2} \left[\left(\frac{1}{2} k_1^2 \mathcal{P} + k_1 k_2 \mathcal{R} + \frac{1}{2} k_2^2 \mathcal{Q} \right) \mathcal{F}_{hh} \right. \\ \left. + (k_1(\ell_1 - r) + k_2(\ell_2 - r)) \mathcal{F}_h \right] \end{array} \right\} = 0 \quad (3.13)$$

Next, we differentiate (3.13) with respect to k_1 and k_2 , to obtain the first order maximizing condition for equation (3.13) as

$$k_1^* = - \frac{\mathcal{R}(\ell_2 - r) - \mathcal{Q}(\ell_1 - r)}{h(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \frac{\mathcal{F}_h}{\mathcal{F}_{hh}} \quad (3.14)$$

$$k_2^* = - \frac{\mathcal{R}(\ell_1 - r) - \mathcal{P}(\ell_2 - r)}{h(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \frac{\mathcal{F}_h}{\mathcal{F}_{hh}} \quad (3.15)$$

Substituting (3.14) and (3.15) into (3.13), we have

$$\left\{ \mathcal{F}_t + \frac{1}{2} (\mathcal{C} - \mathcal{D} - \mathcal{E}) \frac{\mathcal{F}_h^2}{\mathcal{F}_{hh}} + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) h + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] \mathcal{F}_h \right\} = 0 \quad (3.16)$$

where

$$\left\{ \mathcal{C} = \frac{2\mathcal{R}(\mu_1 - r)(\mu_2 - r)}{(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)}, \mathcal{D} = \frac{\mathcal{Q}(\mu_1 - r)^2}{(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)}, \mathcal{E} = \frac{\mathcal{P}(\mu_2 - r)^2}{(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \right\} \quad (3.17)$$

3.2 Legendre transformation and dual theory

We can observed that (3.16) is a non linear PDE and is some worth difficult to solve. This section, intend to introduce the Legendre transformation and dual theory and apply it to transform the non linear PDE to a linear PDE.

Theorem 3.1. Let $g : R^n \rightarrow R$ be a convex function for $z > 0$, define the Legendre transform

$$\mathcal{M}(x) = \max_h \{g(h) - zh\}, \quad (3.18)$$

where $\mathcal{M}(x)$ is the Legendre dual of $y(h)$.

Since $y(h)$ is convex, then from theorem 3.1 and [14, 27], the Legendre transform for the value function $\mathcal{F}(t, h)$ is defined thus

$$\widehat{\mathcal{F}}(t, x) = \sup \mathcal{F}(t, h) - xh \mid 0 < h < \infty \quad 0 < t < T \quad (3.19)$$

where $\widehat{\mathcal{F}}$ is the dual of $\mathcal{F}$ and $x > 0$ is the dual variable of x .

The value of x where this optimum is achieved is represented by $y(t, z)$, such that

$$y(t, x) = \inf h \mid \mathcal{F}(t, h) \geq xy + \widehat{\mathcal{F}} \quad 0 < t < T. \quad (3.20)$$

From (3.20), the function y and $\widehat{\mathcal{F}}$ are very much related and can be refers to as the dual of $\mathcal{F}$ and are related thus

$$\widehat{\mathcal{F}}(t, x) = \mathcal{F}(t, y) - xy. \quad (3.21)$$

where

$$y(t, x) = h, \mathcal{F}_h = x, y = -\widehat{\mathcal{F}}_x. \quad (3.22)$$

Differentiating (3.22) with respect to t and h , we have

$$\begin{cases} \mathcal{F}_t = \widehat{\mathcal{F}}_t, \mathcal{F}_h = x, \\ \mathcal{F}_{hh} = \frac{-1}{\widehat{\mathcal{F}}_{xx}}, \end{cases} \quad (3.23)$$

Substituting (3.23) into (3.14), (3.15) and (3.16), we have

$$k_1^* = -\frac{\mathcal{R}(\ell_2 - r) - \mathcal{Q}(\ell_1 - r)}{y(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} x \widehat{\mathcal{F}}_{xx} \quad (3.24)$$

$$k_2^* = -\frac{\mathcal{R}(\ell_1 - r) - \mathcal{P}(\ell_2 - r)}{y(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} x \widehat{\mathcal{F}}_{xx} \quad (3.25)$$

$$\left\{ \mathcal{F}_t - \frac{1}{2} (\mathcal{C} - \mathcal{D} - \mathcal{E}) x^2 \widehat{\mathcal{F}}_{xx} + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) y + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] x \right\} = 0 \quad (3.26)$$

If we differentiate (3.24), (3.25) and (3.26) with respect to x , we will have

$$\left\{ y_t + \frac{1}{2} (\mathcal{C} - \mathcal{D} - \mathcal{E}) x^2 y_{xx} - \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) y + c \left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] \right. \\ \left. - \left(r - \vartheta + \frac{1}{e - e_0 - t} - \frac{1}{2} (\mathcal{C} - \mathcal{D} - \mathcal{E}) \right) x y_x \right\} = 0 \quad (3.27)$$

$$k_1^* = -\frac{\mathcal{R}(\ell_2 - r) - \mathcal{Q}(\ell_1 - r)}{y(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} x y_x \quad (3.28)$$

$$k_2^* = -\frac{\mathcal{R}(\ell_1 - r) - \mathcal{P}(\ell_2 - r)}{y(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} x y_x \quad (3.29)$$

Next, we proceed to solve (3.27) for y considering PPM with exponential utility using change of variable, after which we will use the solution of y to simplify the OCPs in (3.28) and (3.29).

4 OCP with return clause and charge on balance under exponential utility

In this section, we define our utility function in terms of the dual function. Let $U(h)$ be the original utility function given as

$$U(h) = \frac{1}{\gamma} e^{-\gamma h} \quad (4.1)$$

And the value function is given in terms of the utility as

$$\mathcal{F}(t, h) = U(h)$$

At terminal time T , we define the dual utility in terms of the original utility function $U(h)$ as

$$\hat{U}(x) = \sup \{ U(h) - hx \mid 0 < x < \infty \},$$

and

$$G(x) = \sup \{ h \mid U(h) \geq xh + \hat{U}(x) \}.$$

As a result $\hat{\mathcal{F}}(t, x) = \mathcal{F}(t, y) - xy$.

$$G(x) = (U')^{-1}(x), \quad (4.2)$$

where G is the inverse of the marginal utility U

At terminal time T , we can define

$$y(T, x) = \inf_{x>0} h \mid U(h) \geq xh + \hat{\mathcal{F}}(t, x) \text{ and } \hat{\mathcal{F}}(t, x) = \sup_{x>0} \{ U(x) - xh \}$$

so that

$$y(T, x) = (U')^{-1}(x). \quad (4.3)$$

Proposition 4.1. *The optimal value function, optimal fund size and the optimal control plan are given as*

$$y(t, x) = \left(-\frac{1}{\gamma} \left[\left(\frac{e-e_0-T}{e-e_0-t} \right) \text{Exp}[r-\vartheta](t-T) \left(\begin{array}{c} \ln x \\ + (r-\vartheta - \frac{1}{2}(\mathcal{E} - \mathcal{D} - \mathcal{E}))(t-T) \\ + \ln \left(\frac{e-e_0-T}{e-e_0-t} \right) \end{array} \right) \right] \right. \\ \left. + \frac{2(t-T)}{(r-\vartheta)(e-e_0-t)} \text{Exp}[r-\vartheta]t + \frac{2}{(r-\vartheta)^2(e-e_0-t)} [\text{Exp}[r-\vartheta](t-T) - 1] \right) \quad (4.4)$$

$$\mathcal{H}(t) = \left(\begin{array}{c} \frac{(2\mathcal{R}(\ell_1-r)(\ell_2-r) - \mathcal{P}(\ell_2-r)^2 - \mathcal{Q}(\ell_1-r)^2)}{\gamma(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \left[\frac{t \text{Exp}[r-\vartheta](t-T)}{e-e_0-t} \right. \\ \left. - \frac{e-e_0-T}{e-e_0-t} \right] \\ + c \left(\frac{t-T}{(e-e_0-t)(e-e_0-T)} \right) \left(\frac{2r-2\vartheta+e-e_0}{(r-\vartheta)^2} \right) + h_0 \end{array} \right) \quad (4.5)$$

$$k_1^* = \frac{\mathcal{R}(\ell_2-r) - \mathcal{Q}(\ell_1-r)}{\gamma h_0(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \left(\frac{e-e_0-T}{e-e_0-t} \right) \text{Exp}[r-\vartheta](t-T) \quad (4.6)$$

$$k_2^* = \frac{\mathcal{R}(\ell_1-r) - \mathcal{P}(\ell_2-r)}{\gamma h_0(\mathcal{P}\mathcal{Q} - \mathcal{R}^2)} \left(\frac{e-e_0-T}{e-e_0-t} \right) \text{Exp}[r-\vartheta](t-T) \quad (4.7)$$

Proof. From (4.1),

$$y(T, x) = (U')^{-1}(x) = -\frac{1}{\gamma} \ln x \quad (4.8)$$

Next, we conjecture a solution to (3.27) similar to the one in [5, 30] with the form:

$$\begin{cases} y(t, x) = -\frac{1}{\gamma} [u(t)(\ln x + v(t))] + w(t) \\ u(T) = 1, v(T) = 0, w(T) = 0, \end{cases} \quad (4.9)$$

$$\left. \begin{aligned} y_t &= -\frac{1}{\gamma} [u_t (\ln x + v(t)) + u(t) v_t] + w_t, \\ y_x &= -\frac{1}{x^\gamma} u(t), \\ y_{xx} &= \frac{1}{x^{2\gamma}} u(t) \end{aligned} \right\} \quad (4.10)$$

Substituting (4.10) into (3.27), we have

$$\left\{ \begin{aligned} & \left[u_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) u(t) \right] \ln x \\ & - \left[w_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) w(t) - c \left(\frac{e-e_0-2t}{e-e_0-t} \right) \right] \gamma = 0 \\ & - \frac{u(t)}{\gamma} \left[v_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) + \frac{1}{2} (\mathcal{E} - \mathcal{D} - \mathcal{E}) \right] \end{aligned} \right\} \quad (4.11)$$

Splitting (4.11) we have

$$\left\{ \begin{aligned} u_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) u(t) &= 0 \\ u(T) &= 1 \end{aligned} \right. \quad (4.12)$$

$$\left\{ \begin{aligned} w_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) w(t) - c \left(\frac{e-e_0-2t}{e-e_0-t} \right) &= 0 \\ w(T) &= 0 \end{aligned} \right. \quad (4.13)$$

$$\left\{ \begin{aligned} v_t - \left(r - \vartheta + \frac{1}{e-e_0-t} \right) + \frac{1}{2} (\mathcal{E} - \mathcal{D} - \mathcal{E}) &= 0 \\ v(T) &= 0 \end{aligned} \right. \quad (4.14)$$

Solving equation (4.12), (4.13) and (4.14) we have,

$$u(t) = \left(\frac{e - e_0 - T}{e - e_0 - t} \right) \text{Exp}[r - \vartheta](t - T) \quad (4.15)$$

$$w(t) = \frac{2(t - T)}{(r - \vartheta)(e - e_0 - t)} \text{Exp}[r - \vartheta]t + \frac{2}{(r - \vartheta)^2(e - e_0 - t)} [\text{Exp}[r - \vartheta](t - T) - 1] \quad (4.16)$$

$$v(t) = \left(r - \vartheta - \frac{1}{2} (\mathcal{E} - \mathcal{D} - \mathcal{E}) \right) (t - T) + \ln \left(\frac{e - e_0 - T}{e - e_0 - t} \right) \quad (4.17)$$

Substituting (4.11), (4.12) and (4.13) into (4.9), we obtain (4.4)

Also, from (4.10), we have

$$y_x = -\frac{1}{x^\gamma} \left(\frac{e - e_0 - T}{e - e_0 - t} \right) \text{Exp}[r - \vartheta](t - T) \quad (4.18)$$

Substituting (4.18) in (3.28) and (3.29), we obtain (4.6) and (4.7)

Substituting (4.6) and (4.7) into (3.11) and solve the resultant differential equation, we obtain (4.5).

5 Numerical analysis

In this section, some numerical simulations are presented to analyze the impact of some sensitive parameters on the optimal control plans under exponential utility. To achieve this, the following data will be used unless otherwise stated; $r(0) = 0.07$, $T = 20$, $h_0 = 1$, $\vartheta = 0.02$, $m_{11} = 1$, $m_{12} = 0.85$, $m_{21} = 0.95$, $m_{22} = 0.8$, $\ell_1 = 0.12$, $\ell_1 = 1.0$, $\gamma = 0.05$, $e = 100$, $e_0 = 20$.

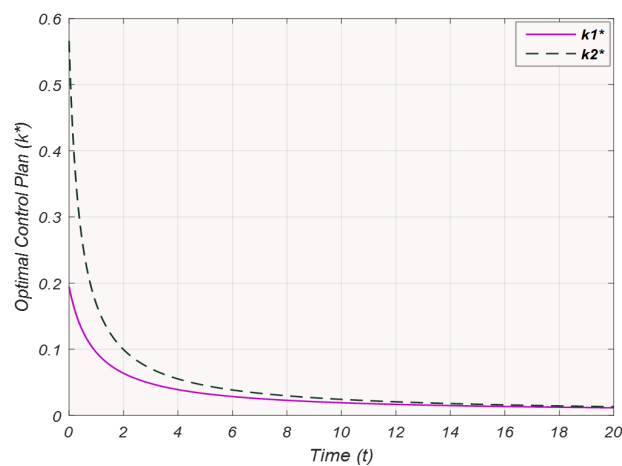


Figure 1: Optimal control plan for the two risky assets

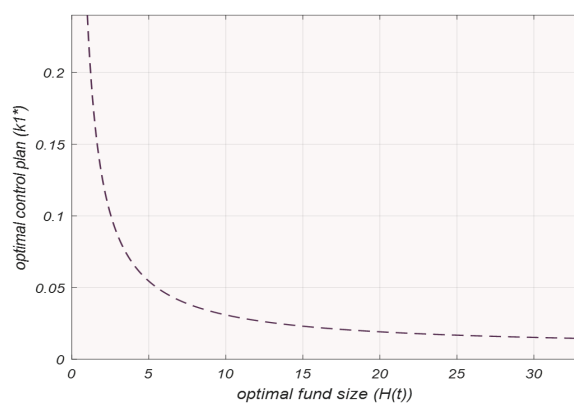


Figure 2: Impact of optimal fund size on optimal control plan k_1^*

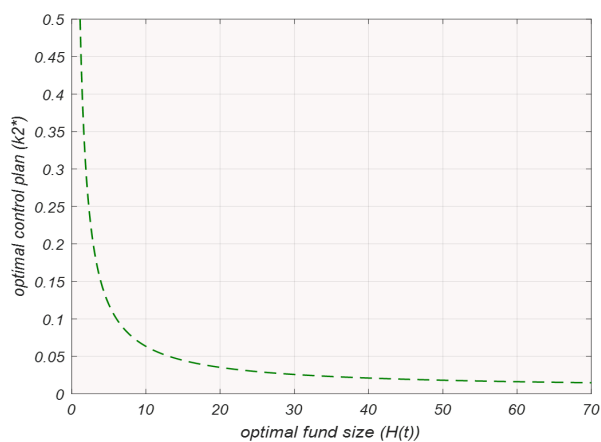


Figure 3: Impact of optimal fund size on optimal control plan k_2^*

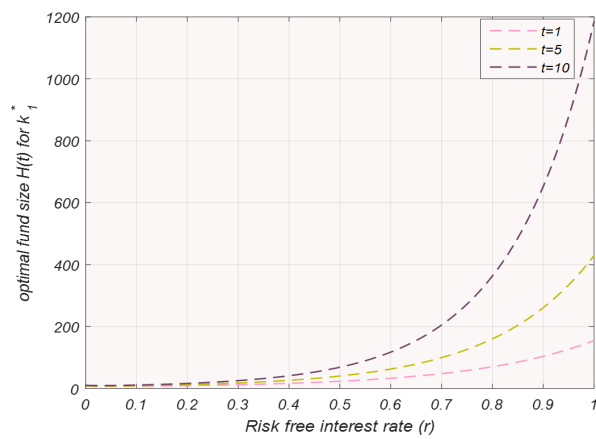


Figure 4: Impact of risk free interest on optimal control plan k_1^*

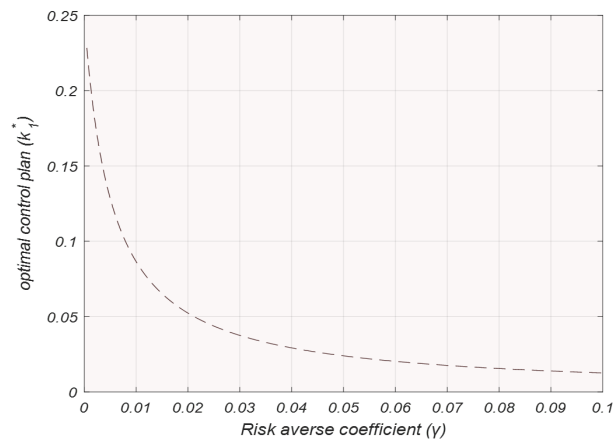


Figure 5: Impact of risk averse coefficient on optimal control plan k_1^*

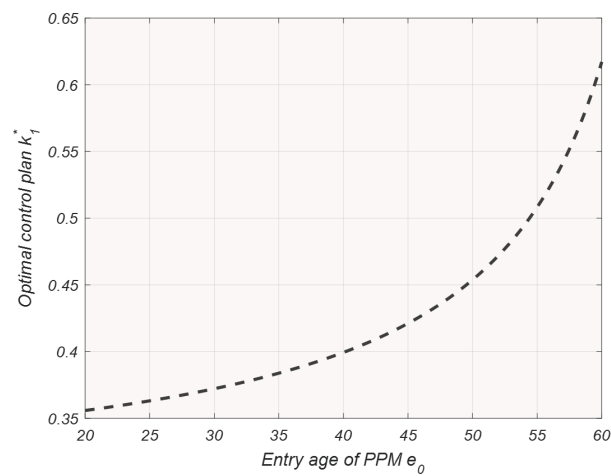
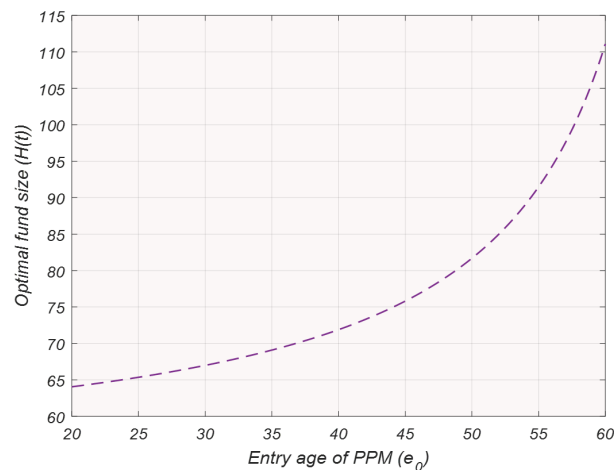
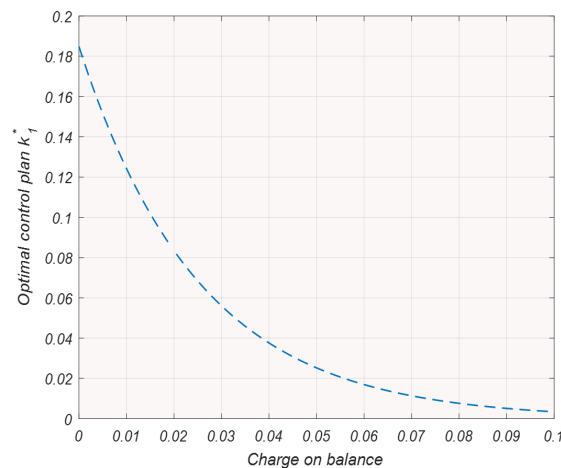


Figure 6: Impact of entry age of PPM on optimal control plan k_1^*

Figure 7: Impact of entry age of PPM on optimal fund size $H(t)$ Figure 8: Impact of charge on balance on optimal control plan k_1^*

6 Discussion and conclusion

In this section, the impacts of some parameters on the optimal control plan are discussed. In Figure 1, the graph of optimal control plans for the two risky assets are presented as a function of time; the graph shows that, the proportion of investment in the two risky assets decreases with time and gets smaller and smaller as the PPM approaches retirement age. In Figure 2, the graph of optimal control plan for stock 1 against the optimal fund size is presented; the graph shows that, optimal control plan for the stock 1 decrease as the optimal fund size increases. The implication of this graph is that members with more funds may prefer to invest in less in the risky asset as retirement age approaches and vice versa. Figure 3 is the graph of optimal control plan for the stock 2 against the optimal fund size; the graph shows that, optimal control plan for the stock 2 decreases as the optimal fund size increases. This is similar to figure 2.

In figure 4, the graph of the optimal fund size against the risk free interest rate is presented; the graph shows that the optimal fund size is an increasing function of the risk free interest rate. The implication of the graph is that members with large funds prefers to invest where there is less risk since they may not want to lose what they have gathered already. In figure 5, the graph of optimal control plan against the risk averse coefficient is presented; the graph shows that the optimal control plan is inversely proportional to the risk averse coefficient. This implies that the more fearful the PPM is to investment in risky asset, the more likelihood that he will reduce the proportion of his wealth to be

invested in risky asset.

In figure 6, the graph illustrates the impact of PPM's entry age on the optimal control plan. It is observed that the optimal control plan increases as the entry age of the PPM increases. The implication is that PPM with late entry in to the pension scheme, tends to increase the proportion of their wealth to be invested in the risky asset. This can be confirm from figure 7 which shows that the optimal fund size of a PPM is directly proportional to the entry age of the PPM. In figure 8, the relationship between the optimal control plan and the charge on balance is presented. It is observe that the optimal control plan developed by the pension fund administrators (PFA) is a decreasing function of the charge on balance. The implication in figure 7 is that, the higher the charge on balance on investment of the risky asset, the more likelihood that PPM of the scheme will be discouraged from investing a large proportion of their wealth in risky asset and vice versa.

In conclusion, the PPMs portfolios were studied under mortality risk which exhibit constant absolute risk averse. The mortality rate was determined De Moivre mortality force function and a portfolio comprising of two risky assets which forms a 2×2 positive definite matrix was considered to build the PPM's portfolio. The Legendre transformation and dual theory was used to solve for the optimal control plan, value function and the optimal fund size. It was observed that the OCP is a decreasing function of the optimal fund size, risk averse, predetermined interest rate and charge on balance. Also, we observed that the optimal control plan is an increasing function of the PPM's entry age.

Conflict of Interests

The authors declare that there is no conflict of interests regarding this paper.

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Mathematical model of terrorism population dynamics: a case of a community under the attack of two terrorists groups

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Abstract

We formulated in this work, a mathematical model which is made up of system of non-linear ordinary differential equations that is analyzed so to study the dynamics of terrorism, a case of a community under the attack of two terrorists groups. The basic terrorism multiplication number (M_0) was computed after which we investigated the existence of Terrorism Free Equilibrium point (TFE) which was investigated to be locally and globally asymptotically stable whenever (M_0) < 1 and (M_0) > 1 respectively. A unique equilibrium point known as, Terrorism Persistent Equilibrium point (TPE) was also investigated to exist whenever (M_0) > 1. Sensitivity analysis of the basic terrorism multiplication number (M_0) was then carried out to see the impacts of some parameters to the growth of terrorism in a population. Results show that, the rate at which individuals are exposed to terrorists group two and the rate at which terrorists in group one are recruited into terrorists group two represented by (ω) and (θ_1) respectively are the most sensitive parameters that increase terrorism in a population, as an increase in the parameter values reduces the value of the basic terrorism multiplication number (M_0) to a value less than unity. The obtained mathematical result implies that, the terrorists group two (T_2) has a mighty negative effect on the growth of terrorism in a population. Finally, numerical simulation was performed which results were in conformity with our analytical results.

Keywords and phrases: terrorism; community; terrorists; groups; population; stability.

1 Introduction

Terrorism simply refers to the application of criminal violence to cause a state of fear or terror, majorly with the aim to achieve a political or religious motives. Terrorism in this context primarily refers to purposeful violence during peacetime or during war against civilians and non-combatants in most cases [1, 2]. Bruce defined terrorism in [1] as a deliberate killing of innocent individuals at random, to generate fear through a whole population and thereby forcing the hands of its political leaders. The term is mostly used with implication of something that is wrong morally. Non governmental and governmental societies make use of terrorism to abuse opposing groups [3-6].

Terrorism originated during the French revolution of the late 18th century but became globally used and got a universal attention in the 1970s during the fights in the Northern Ireland [3, 7, 8, 9]. Terrorism is grouped into six categories namely; civil disorder terrorism, political disorder terrorism, Non-political terrorism and limited political terrorism [8]. Groups choose terrorism as a tactic because it can; act as a form of asymmetric warfare so as to directly force a government to agree to demands, intimidate a group of people into capitulating to the demands in order to avoid future injury, get attention and thus political support for a cause, directly inspire more people to the cause such as revolutionary acts [10-12].

Nigeria as a country has been under a serious siege created by terrorists in the past 10 years [13]. Currently in Nigeria, the activities of a group known as Boko Haram over the years has caused alot

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of damages to lives and properties. Research reveals that Boko Haram group is responsible for more than 80% of terrorism activities in the country with more than 27000 deaths as at 2018 as provided in [13]. The negative impacts of terrorism in Nigeria and the world at large over the years can not be over-emphasized as their activities has tempered economic growth and developments, to this end, the entire world requires urgently an effective counter terrorism strategies so as to bring the activities of these deadly groups to an end.

This paper provides both the theoretical and mathematical understanding so as to enable us gain insight into terrorism dynamics; a case of a community or a population under the attack of two terrorists groups. Mathematical model is regarded as a vital tool used in solving real life problems like that of terrorism which currently affects almost every part of the entire world. Many real life problems like disease outbreaks have been studied extensively via mathematical modeling in [14-18].

Udoh and Oladejo presented in [19] a simple mathematical model of terrorist evolutionary dynamics represented by sets of ordinary differential equations. Their model provides a long-term solution of allocating resources towards eradicating terrorism in a population. In 2016, Lopes et al presented in [20], a terrorism global stability dynamical model to study the concepts of terrorism for the period of 1970-2014. Several mathematical tools and techniques were used like the Least Square Methods, Fractal dimensions for their model analysis. Udawadia et al presented also in [21], a dynamical mathematical model of terrorism, where they considered a long term evolutionary operations of terrorists. In [13] similarly, Okoye et al studied terrorism dynamics using mathematical approach. They investigated the impact of counter-terrorism measures in tackling terrorism.

After searching literatures, we discovered that most terrorism mathematical models developed over the years did not cover a special case where a community or population is under an attack of two different terrorists groups. In this present work, we consider and formulate a mathematical model of terrorism population dynamics that incorporates a case of a community under the attack of two terrorists groups in the presence of a sustained military intelligence operations to eliminate and take over the camp of terrorists.

2 Model Formulation and Procedures

We consider a population size of $N(t)$ which is divided into five sub-classes namely; Susceptible Individuals (S), Exposed Individuals (E), Terrorists Group One (T_1), Terrorists Group Two (T_2) and Removed Individuals (R). Where; $N(t) = S(t) + E(t) + T_1(t) + T_2(t) + R(t)$.

Individuals are recruited into the susceptible compartment at a rate (Λ) and the susceptible individuals are exposed to the two terrorists groups (T_1) and (T_2) at the rates of (β_1) and (β_2) respectively. The exposed individuals (E) become full terrorists and therefore moved to either terrorist group (T_1) or terrorist group (T_2) at the rates of (α) or (ω).

We equally assume that those in terrorists group one (T_1) can join terrorists group two (T_2) and terrorist group two (T_2) joining terrorist group one (T_1) at the rates of (θ_1) and (θ_2) respectively. Both terrorists groups can be eliminated by a sustained military intelligence operations in a bid to take over terrorists camps at rates (k_1) and (k_2) respectively. The two terrorists groups due to the risk nature of their operations can also die in a bid to attack a community at terrorism induced death rates of (μ_1) and (μ_2) respectively. The two groups due to terrorism induced deaths and continuous elimination of their members by effective military operations can renounce memberships of both groups at rates of (γ_1) and (γ_2) respectively, they repent and become removed fully from the population.

We assume that the sustained military operation to eliminate terrorists camps is 100% effective so as the renounced terrorists do not go back to terrorism again. Finally, individuals in all the five classes can die naturally at rate (μ). Based on these assumptions, the model flow diagram is shown in Fig.1 below;

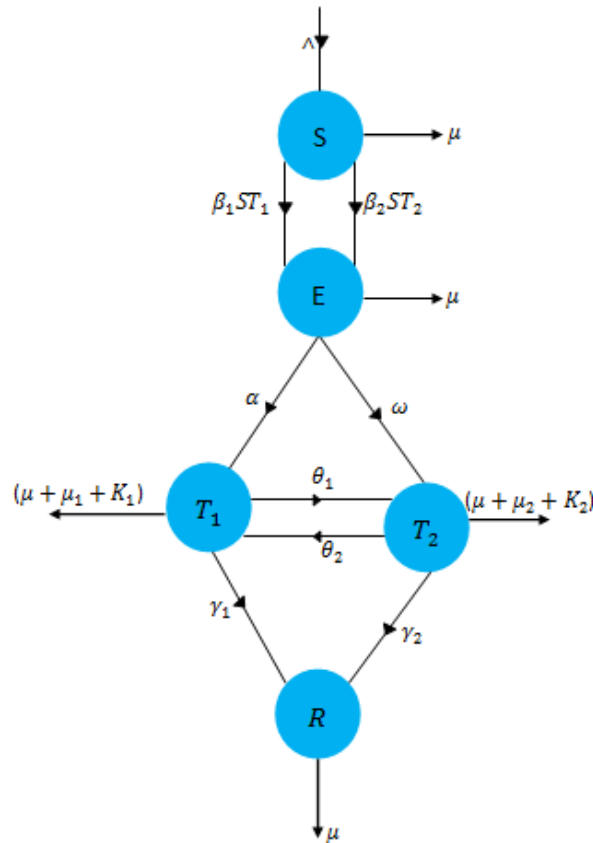


Fig. 1 Model flow diagram for terrorism dynamics.

The associated equations corresponding to the model flow diagram in Fig. 1 is given by;

$$\begin{cases} \frac{dS}{dt} = \Lambda - (\beta_1 T_1 + \beta_2 T_2)S - \mu S \\ \frac{dE}{dt} = (\beta_1 T_1 + \beta_2 T_2)S - (\alpha + \omega + \mu)E \\ \frac{dT_1}{dt} = \alpha E + \theta_2 T_2 - (\theta_1 + \gamma_1 + \mu + \mu_1 + k_1)T_1 \\ \frac{dT_2}{dt} = \omega E + \theta_1 T_1 - (\theta_2 + \gamma_2 + \mu + \mu_2 + k_2)T_2 \\ \frac{dR}{dt} = \gamma_1 T_1 + \gamma_2 T_2 - \mu R. \end{cases} \quad (1)$$

The meaning of variables(parameters) used in model (1) with their various units is shown below.
Table 1. Descriptions of variables and parameters used in model (1) with their various units.

Now without of loss of generality, from model (1), we let; $m_0 = (\alpha + \omega + \mu)$, $m_1 = (\theta_1 + \gamma_1 + \mu + \mu_1 + k_1)$ and $m_2 = (\theta_2 + \gamma_2 + \mu + \mu_2 + k_2)$.

Variables/Parameters	Descriptions	Units
S	Susceptible Individuals	Individuals km^{-2} months $^{-1}$
E	Exposed Individuals	Individuals km^{-2} months $^{-1}$
T_1	Terrorists Group One	Individuals km^{-2} months $^{-1}$
T_2	Terrorists Group Two	Individuals km^{-2} months $^{-1}$
R	Removed Individuals	Individuals km^{-2} months $^{-1}$
Λ	Recruitment rate	Months $^{-1}$
β_1	Exposure rate of (S) to (T_1)	Months $^{-1}$
β_2	Exposure rate of (S) to (T_2)	Months $^{-1}$
α	Rate at which (E) moves to (T_1)	Months $^{-1}$
ω	Rate at which (E) moves to (T_2)	Months $^{-1}$
θ_1	Rate at which (T_1) moves to (T_2)	Months $^{-1}$
θ_2	Rate at which (T_2) moves to (T_1)	Months $^{-1}$
γ_1	Rate at which (T_1) moves to (R)	Months $^{-1}$
γ_2	Rate at which (T_2) moves to (R)	Months $^{-1}$
μ_1	Terrorist induced death rate for (T_1)	Months $^{-1}$
μ_2	Terrorist induced death rate for (T_2)	Months $^{-1}$
k_1	Rate at which military eliminates (T_1)	Months $^{-1}$
k_2	Rate at which military eliminates (T_2)	Months $^{-1}$
μ	Natural death rate for all classes	Months $^{-1}$

Model (1) can therefore be written as:

$$\left\{ \begin{array}{l} \frac{dS}{dt} = \Lambda - (\beta_1 T_1 + \beta_2 T_2)S - \mu S \\ \frac{dE}{dt} = (\beta_1 T_1 + \beta_2 T_2)S - m_0 E \\ \frac{dT_1}{dt} = \alpha E + \theta_2 T_2 - m_1 T_1 \\ \frac{dT_2}{dt} = \omega E + \theta_1 T_1 - m_2 T_2 \\ \frac{dR}{dt} = \gamma_1 T_1 + \gamma_2 T_2 - \mu R \end{array} \right. \quad (2)$$

Taking a critical look at equation (2.2), the equation involving removed individuals (R) is independent of other equations since (R) does not appear in the remaining equations, we therefore decoupled (R) from the rest and consider the model presented in equation (2.3) below as a submodel.

$$\left\{ \begin{array}{l} \frac{dS}{dt} = \Lambda - (\beta_1 T_1 + \beta_2 T_2)S - \mu S \\ \frac{dE}{dt} = (\beta_1 T_1 + \beta_2 T_2)S - m_0 E \\ \frac{dT_1}{dt} = \alpha E + \theta_2 T_2 - m_1 T_1 \\ \frac{dT_2}{dt} = \omega E + \theta_1 T_1 - m_2 T_2 \end{array} \right. \quad (3)$$

2.1 Positivity of Solutions

For us to show that our model is mathematically meaningful, we therefore show that all the variables of model (2.2) are positive for all time $t > 0$.

Lemma 2.1 We let the initial data be :

$$S(0) \geq 0, E(0) \geq 0, T_1(0) \geq 0, T_2(0) \geq 0, R(0) \geq 0,$$

The solutions $S(t), E(t), T_1(t), T_2(t), R(t)$ of the model (2.2) are non-negative for all $t > 0$.

Proof

If $S(0) \geq 0, E(0) \geq 0, T_1(0) \geq 0, T_2(0) \geq 0, R(0) \geq 0$, then from the first equation of model (2.2), we have that;

$$\frac{dS(t)}{dt} = \Lambda - (\beta_1 T_1 + \beta_2 T_2)S - \mu S \quad (4)$$

which can be written as;

$$\begin{aligned} \frac{dS(t)}{dt} \exp\left[\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k))dk + \mu t\right] + S(t)(\beta_1 T_1(k) + \beta_2 T_2(k) + \mu) \exp\left[\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k))dk + \mu t\right] \\ = \Lambda \exp\left[\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k) + \mu t)\right] \end{aligned}$$

Therefore;

$$\frac{d}{dt}(S(t) \exp\left[\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k))dk + \mu t\right]) = \Lambda \exp\left[\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k))dk + \mu t\right]$$

We therefore have that;

$$\begin{aligned} S(0) = \exp\left[-\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k) - \mu t)\right] + \exp\left[-\int_0^t (\beta_1 T_1(k) + \beta_2 T_2(k))dk - \mu t\right] \left[\int_0^t \Lambda \exp\left[\int_0^t (\beta_1 T_1(n) \right. \right. \\ \left. \left. + \beta_2 T_2(n))dn + \mu k\right]dk \right] \end{aligned}$$

In the same way, it can also be shown that $E(t) > 0, T_1(t) > 0, T_2(t) > 0$ and $R(t) > 0$. We therefore conclude that the solutions of $S(t), E(t), T_1(t), T_2(t)$ and $R(t)$ of model (2.2) are non-negative for all $t > 0$.

2.2 Invariant Region

For our model to be relevant mathematically, we need to show that all model variables and parameters are positive for all time $t \geq 0$ in a feasible region (Ω) .

Lemma 2.2

The model feasible region (Ω) is defined by;

$$\Omega = [(S, E, T_1, T_2, R)] \in R_+^5 : S + E + T_1 + T_2 + R \leq \frac{\Lambda}{\mu}$$

with initial condition $S(0) \geq 0, E(0) \geq 0, T_1(0) \geq 0, T_2(0) \geq 0, R(0) \geq 0$ is positively invariant for model (2.2).

Proof

From equation (2.2); we have that;

$$\frac{dN}{dt} = \Lambda - \mu N - m_1 T_1 - m_2 T_2 \leq \Lambda - \mu N \quad (5)$$

therefore, $0 \leq N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t}$, where $N(0)$ represents the initial values of the total population.

Therefore $0 \leq N(t) \leq \frac{\Lambda}{\mu}$ as t tends to infinity.

We can then say that the region;

$\Omega = [(S, E, T_1, T_2, R)] \in R_+^5 : S + E + T_1 + T_2 + R \leq \frac{\Lambda}{\mu}$ is positively invariant set for model (2.2).

To make our model analysis simple, we shall therefore consider the dynamics of the submodels (2.3).

3 Terrorism Model Analysis

3.1 Terrorism Free Equilibrium (TFE)

This is the equilibrium point where terrorism does not exit in the population. The state is represented as;

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right) \quad (6)$$

3.2 Basic Terrorism Multiplication Number(M_0)

The Basic terrorism multiplication number (M_0) is the same as the basic reproduction number (R_0) in epidemiological models which is also obtained using similar approach, that is the well known next generation matrix method [15, 18, 22, 23].

The basic terrorism multiplication number is therefore defined as the average number of individuals a terrorist can recruit into terrorism throughout his/her life time in the absence of counter-terrorism measure [13].

Now, at the terrorism free equilibrium point, we obtain the basic terrorism multiplication number (M_0) using the next generation matrix.

Let $y = (E, T_1, T_2, S)^T$, then our model (2.3) can be expressed as;

$$\frac{dy}{dt} = F(y) - V(y) \quad (7)$$

where;

$$F(y) = \begin{bmatrix} (\beta_1 T_1 + \beta_2 T_2) S \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

and

$$V(y) = \begin{bmatrix} m_0 E \\ m_1 T_1 - \alpha E - \theta_2 T_2 \\ m_2 T_2 - \omega E - \theta_1 T_1 \\ \mu S + (\beta_1 T_1 + \beta_2 T_2) S - \Lambda \end{bmatrix}$$

where;

$$F = \begin{bmatrix} 0 & \frac{\beta_1 \Lambda}{\mu} & \frac{\beta_2 \Lambda}{\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (9)$$

$$V = \begin{bmatrix} m_0 & 0 & 0 \\ -\alpha & m_1 & -\theta_2 \\ -\omega & -\theta_1 & m_2 \end{bmatrix}$$

But $M_0 = FV^{-1}$ and is obtained as;

$$M_0 = \frac{\Lambda \beta_1 (m_2 \alpha + \theta_2 \omega) + \beta_2 (\theta_1 \alpha + m_1 \omega)}{\mu m_0 (m_1 m_2 - \theta_1 \theta_2)} \quad (10)$$

Following the theorem 3 adopted in [18], we have the following theorem.

Theorem 3.2

The terrorism free equilibrium point (E_0) is locally asymptotically stable for $M_0 < 1$ and unstable for $M_0 > 1$.

3.3 Global Stability of the Terrorism Free Equilibrium

Theorem 3.3

The model (2.3) terrorism free equilibrium point is globally asymptotically stable provided $M_0 > 1$.

Proof

Considering the Lyapunov function;

$$P = M_0 E + \frac{\Lambda(\beta_1 m_2 + \beta_2 \theta_1) T_1}{\mu(m_1 m_2 - \theta_1 \theta_2)} + \frac{\Lambda(\beta_2 m_1 + \beta_1 \theta_2) T_2}{\mu(m_1 m_2 - \theta_1 \theta_2)}$$

which its derivative along the solutions to our model (2.3) is;

$$\begin{aligned} P^1 &= M_0(\beta_1 T_1 + \beta_2 T_2) S - (\beta_1 T_1 + \beta_2 T_2) \frac{\Lambda}{\mu} \leq M_0(\beta_1 T_1 + \beta_2 T_2) \frac{\Lambda}{\mu} - (\beta_1 T_1 + \beta_2 T_2) \frac{\Lambda}{\mu} \\ &= (\beta_1 T_1 + \beta_2 T_2) \frac{\Lambda}{\mu} (M_0 - 1) \leq 0 \end{aligned}$$

Then $P^1 = 0$ only if $E = T_1 = T_2 = 0$ or if $M_0 = 1$. Then the maximum invariant set in $[(S, E, T_1, T_2) : P^1 = 0]$ is the singleton (E_0) . The global stability of (E_0) is now obtained when $M_0 < 1$ by Lasalle's Invariance principle.

3.4 Terrorism Persistent Equilibrium (TPE)

At this stage, terrorism is present in the population. From our model (2.3), we have that;

$$S = \frac{\Lambda - M_0 E}{\mu}, T_1 = \frac{(m_2 \alpha + \theta_2 \omega) E}{m_1 m_2 - \theta_1 \theta_2}, T_2 = \frac{(\theta_1 \alpha + m_1 \omega) E}{m_1 m_2 - \theta_1 \theta_2} \quad (11)$$

where $m_0 = (\alpha + \omega + \mu)$, $m_1 = (\theta_1 + \gamma_1 + \mu + \mu_1 + k_1)$ and $m_2 = (\theta_2 + \gamma_2 + \mu + \mu_2 + k_2)$

Now by substituting equation (3.6) into $(\beta_1 T_1 + \beta_2 T_2) S - m_0 E = 0$, the Exposed class (E) satisfies the following equation;

$$\frac{(\beta_1 m_2 + \theta_2 \omega)}{(m_1 m_2 - \theta_1 \theta_2)} + \frac{(\beta_2 \theta_1 \alpha + m_1 \omega)(\Lambda - m_0 E)}{(m_1 m_2 - \theta_1 \theta_2) \mu} - m_0 = 0$$

when E is not equal to zero, we therefore have that;

$$Q(E) = \frac{(\beta_1 m_2 + \theta_2 \omega)}{(m_1 m_2 - \theta_1 \theta_2)} + \frac{(\beta_2 \theta_1 \alpha + m_1 \omega)(\Lambda - m_0 E)}{(m_1 m_2 - \theta_1 \theta_2) \mu} - m_0 \quad (12)$$

equation (3.7) implies that;

$$Q(E) = \frac{m_0 \beta_1 (m_2 + \theta_2 \omega) + \beta_2 (\theta_1 \alpha + m_1 \omega)}{(m_1 m_2 - \theta_1 \theta_2) \mu} < 0 \quad (13)$$

Equation (3.8) confirms that $Q(E)$ decreases since;

$$Q(E) = \frac{\Lambda \beta_1 (m_2 + \theta_2 \omega) + \beta_2 (\theta_1 \alpha + m_1 \omega)}{(m_1 m_2 - \theta_1 \theta_2) \mu} - m_0 = m_0 (M_0 - 1)$$

provided that the basic terrorism multiplication number $(M_0) > 1$, then $Q(0)$ is non-negative and that;

$$Q\left(\frac{\Lambda}{m_0}\right) = -m_0 < 0 \quad (14)$$

We therefore say that, by monotonicity of $Q(E)$, we have a special non-negative root of which the result is presented in theorem (3.4) below.

Theorem 3.4

If $M_0 > 1$, model (2.3) has a special endemic equilibrium point $E_1^*(S^*, E^*, T_1^*, T_2^*)$ where;

$$S^* = \frac{\Lambda - M_0 E^*}{\mu}, T_1^* = \frac{(m_2 \alpha + \theta_2 \omega) E^*}{m_1 m_2 - \theta_1 \theta_2}, T_2^* = \frac{(\theta_1 \alpha + m_1 \omega) E^*}{m_1 m_2 - \theta_1 \theta_2} \quad (15)$$

where E^* represents the special non-negative root of $Q(E) = 0$

4 Sensitivity Analysis

We performed sensitivity analysis of the basic terrorism multiplication number (M_0) to see the impacts of some parameters to the persistence of terrorism in a population.

We recall that, the basic terrorism multiplication number;

$$M_0 = \frac{\Lambda\beta_1(m_2\alpha + \theta_2\omega) + \beta_2(\theta_1\alpha + m_1\omega)}{\mu m_0(m_1m_2 - \theta_1\theta_2)}$$

By substituting the expressions for m_0, m_1 and m_2 , we have that;

$$M_0 = \frac{\Lambda\beta_1((\theta_2 + \gamma_2 + \mu + \mu_2 + k_2)\alpha + \theta_2\omega) + \beta_2(\theta_1\alpha + (\theta_1 + \gamma_1 + \mu + \mu_1 + k_1)\omega)}{\mu(\alpha + \omega + \mu)((\theta_1 + \gamma_1 + \mu + \mu_1 + k_1)(\theta_2 + \gamma_2 + \mu + \mu_2 + k_2) - \theta_1\theta_2)}$$

then;

$$\begin{aligned} \frac{\partial M_0}{\partial \alpha} &= \frac{\Lambda(\beta_1m_2 + \beta_2\theta_1)(\alpha + \omega + \mu) - (\beta_1m_2 + \beta_2\theta_1)\alpha - (\beta_1\theta_2 + \beta_2m_1)\omega}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \\ &= \frac{\Lambda(\beta_1m_2 + \beta_2\theta_1)(\omega + \mu) - (\beta_1\theta_2 + \beta_2m_1)\omega}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \\ &\geq \frac{\Lambda\omega[(\beta_1m_2 + \beta_2\theta_1) - (\beta_1\theta_2 + \beta_2m_1)]}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \end{aligned}$$

similarly,

$$\begin{aligned} \frac{\partial M_0}{\partial \omega} &= \frac{\Lambda(\beta_1\theta_2 + \beta_2m_1)(\alpha + \omega + \mu) - (\beta_1m_2 + \beta_2\theta_1)\alpha - (\beta_1\theta_2 + \beta_2m_1)\omega}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \\ &= \frac{\Lambda(\beta_1\theta_2 + \beta_2m_1)(\alpha + \mu) - (\beta_1m_2 + \beta_2\theta_1)\alpha}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \\ &\geq \frac{\Lambda\alpha[(\beta_1\theta_2 + \beta_2m_1) - (\beta_1m_2 + \beta_2\theta_1)]}{\mu(\alpha + \omega + \mu)^2(m_1m_2 - \theta_1\theta_2)} \end{aligned}$$

when $(\beta_1m_2 + \beta_2\theta_1) > (\beta_1\theta_2 + \beta_2m_1)$, we have that; $\frac{\partial M_0}{\partial \alpha} > 0$ and $\frac{\partial M_0}{\partial \omega} < 0$.

This implies that the basic terrorism multiplication number (M_0) increases when (α) increases and decreases when (ω) .

Similarly

$$\frac{\partial M_0}{\partial \theta_1} = \frac{-\Lambda\beta_2(m_1\omega + \theta_1\alpha) + \beta_1(m_2\alpha + \theta_2\omega)(\mu + \mu_2 + k_2) + \beta_2(\alpha + \omega)(\theta_1\theta_2 - m_1m_2)}{\mu m_0(m_1m_2 - \theta_1\theta_2)^2}$$

also in the same way;

$$\frac{\partial M_0}{\partial \theta_2} = \frac{-\Lambda\beta_2(m_1\omega + \theta_1\alpha) + \beta_1(m_2\alpha + \theta_2\omega)(\mu + \mu_1 + k_1) + \beta_2(\alpha + \omega)(\theta_1\theta_2 - m_1m_2)}{\mu m_0(m_1m_2 - \theta_1\theta_2)^2}$$

when, $\beta_2(m_1\omega + \theta_1\alpha) + \beta_1(m_2\alpha + \theta_2\omega)(\mu + \mu_2 + k_2) + \beta_2(\alpha + \omega)(\theta_1\theta_2 - m_1m_2) > 0$,

we have that $\frac{\partial M_0}{\partial \theta_1} < 0$.

Also when, $\beta_2(m_1\omega + \theta_1\alpha) + \beta_1(m_2\alpha + \theta_2\omega)(\mu + \mu_1 + k_1) + \beta_1(\alpha + \omega)(\theta_1\theta_2 - m_1m_2) < 0$,

we have that $\frac{\partial M_0}{\partial \theta_2} > 0$.

This implies that the basic terrorism multiplication number (M_0) increases when (θ_2) increases and decreases when (θ_1) increases. The mathematical interpretations of this result is that, the terrorist group two has a mighty bad effect on the increase rate of terrorism in a population under some specific conditions. Figure (8) confirms this claim.

Variables/Parameters	Descriptions	Units
S	10000	[13]
E	5000	[Assumed]
T_1	200	[13]
T_2	100	[Assumed]
R	100	[Assumed]
Λ	0.08	[Assumed]
β_1	0.11	[13]
β_2	0.08	[Assumed]
α	0.09	[Assumed]
ω	0.87	[Assumed]
θ_1	0.92	[Assumed]
θ_2	0.069	[Assumed]
γ_1	0.2	[Assumed]
γ_2	0.3	[19]
μ_1	0.002	[21]
μ_2	0.003	[Assumed]
k_1	0.1	[Assumed]
k_2	0.15	[Assumed]
μ	0.0012	[19]

4.1 Numerical Simulation and Discussion of Results

Numerical simulation is performed to support our analytical results. Using the value of variables and parameters tabulated in Table 2 we performed the numerical simulation of our model.

Table 2. Value of variables and parameters used for model simulation.

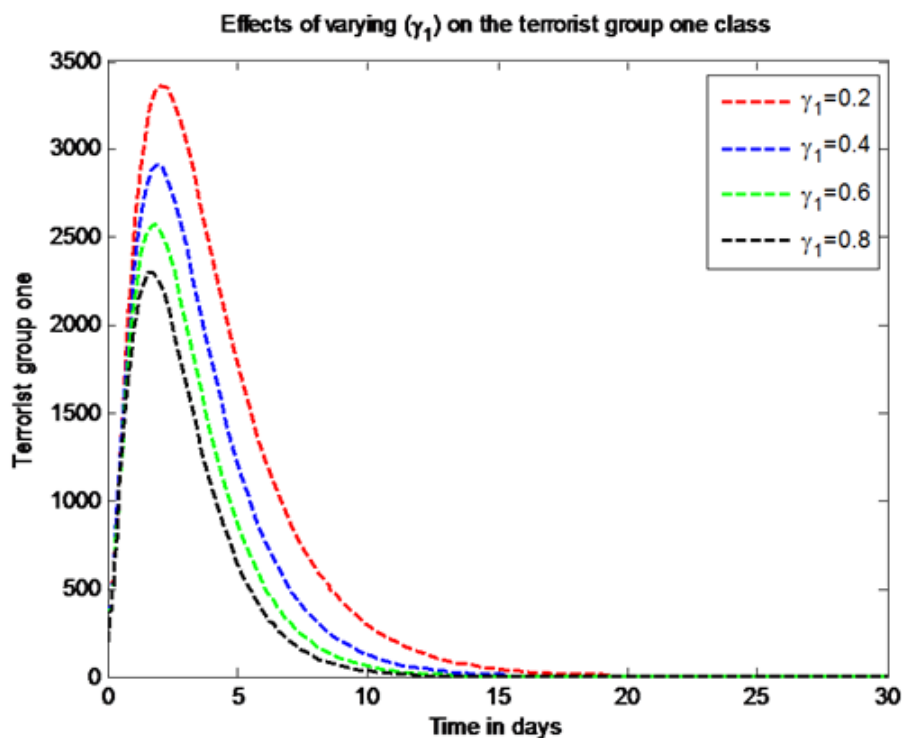


Fig. 2 Effects of varying (γ_1) on the terrorists group one population.

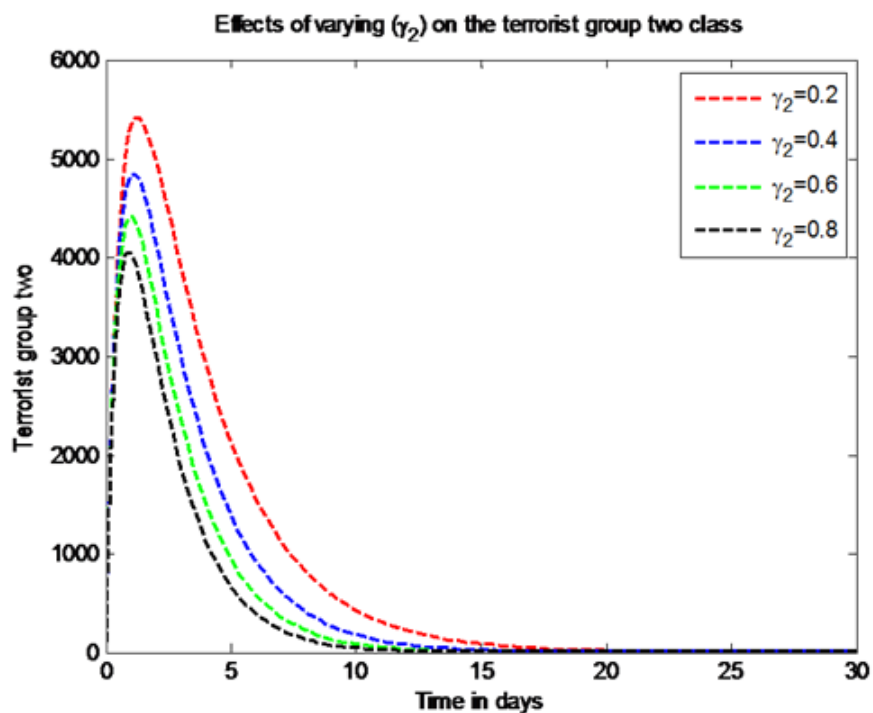


Fig. 3 Effects of varying (γ_2) on the terrorists group two population.

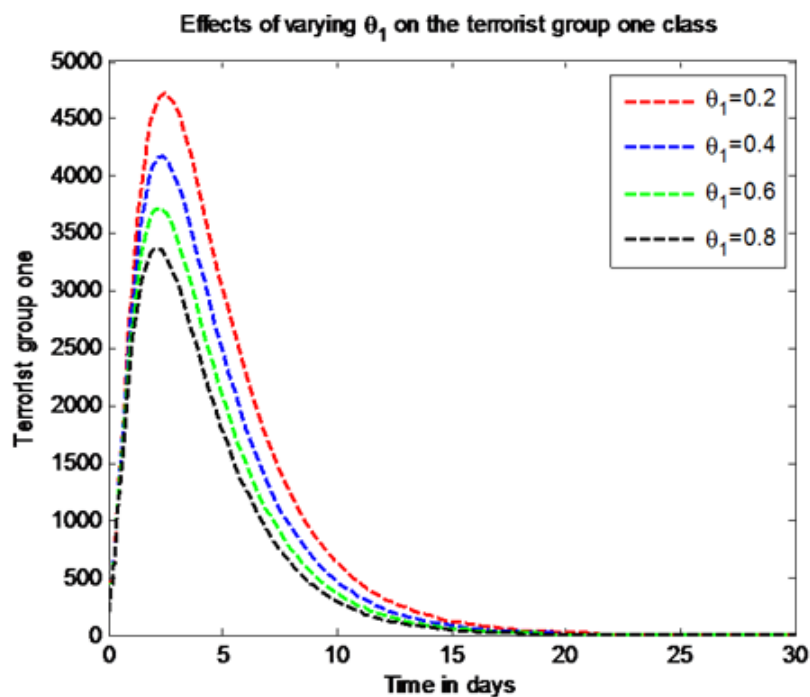


Fig. 4 Effects of varying (θ_1) on the terrorists group one population.

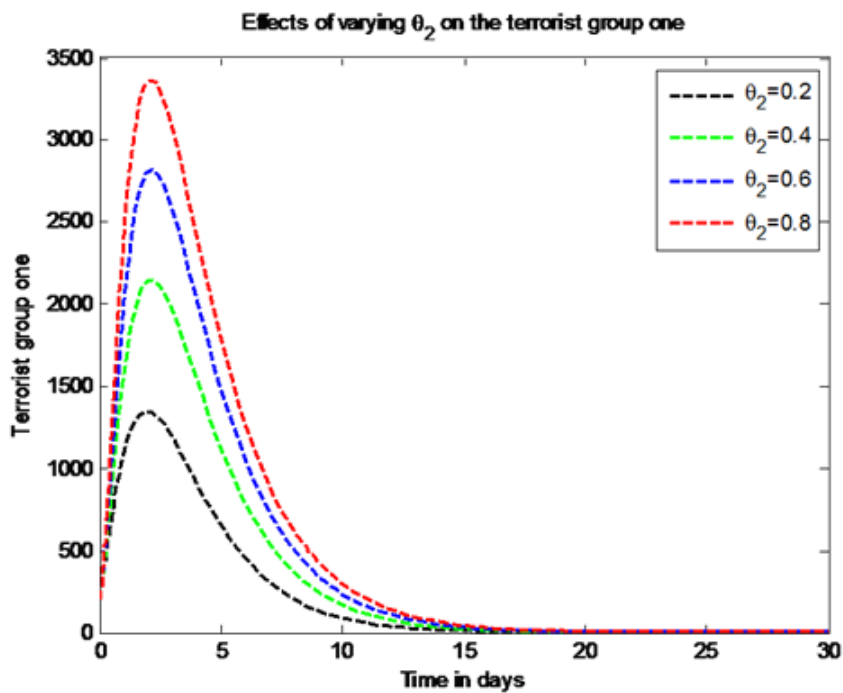


Fig. 5 Effects of varying (θ_2) on the terrorists group one population.

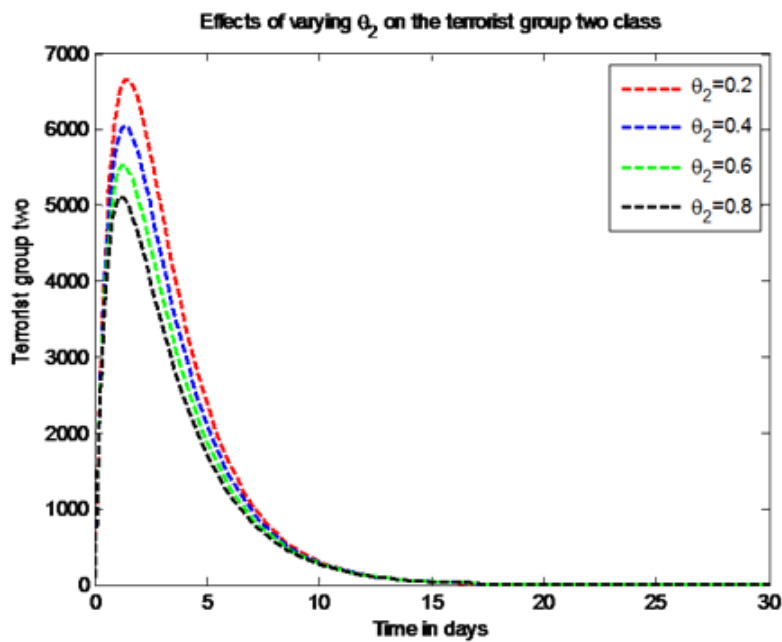


Fig. 6 Effects of varying (θ_2) on the terrorists group two population.

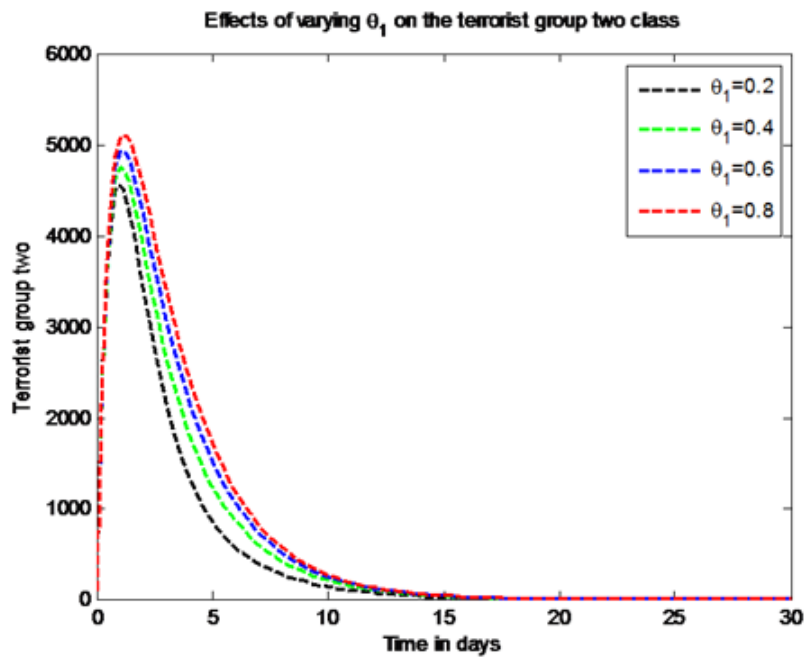


Fig. 7 Effects of varying (θ_1) on the terrorists group two population.

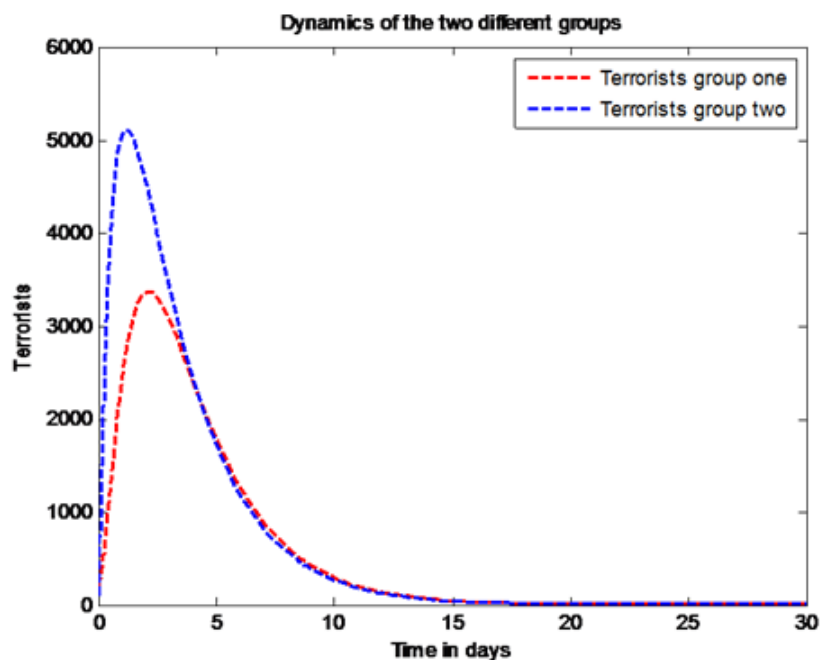


Fig. 8 Dynamics of the two different terrorism groups.

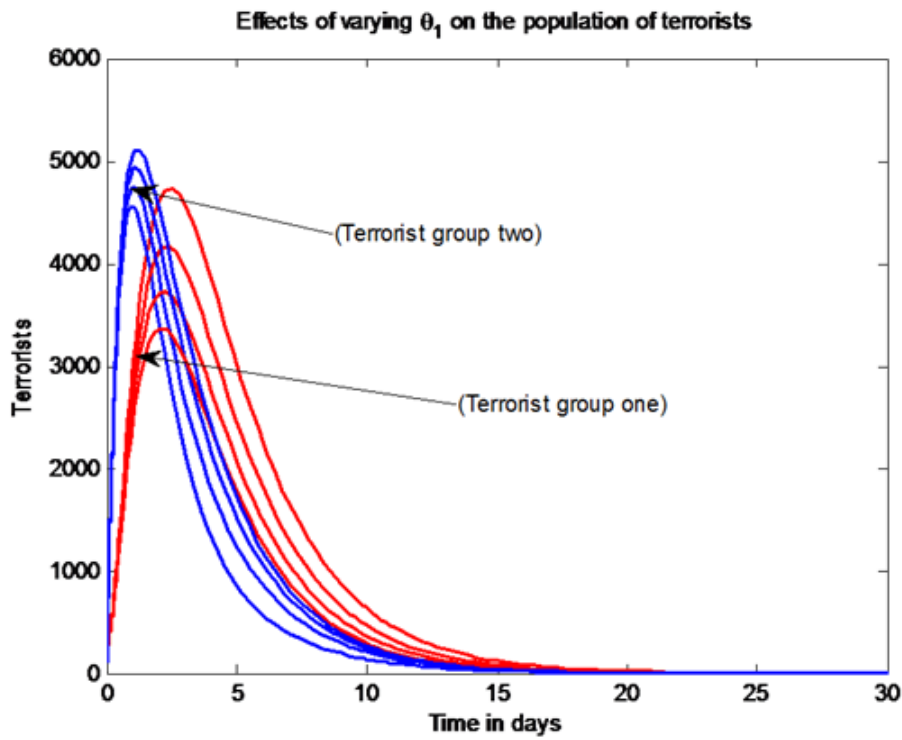


Fig. 9 Effects of varying (θ_1) on terrorists population.

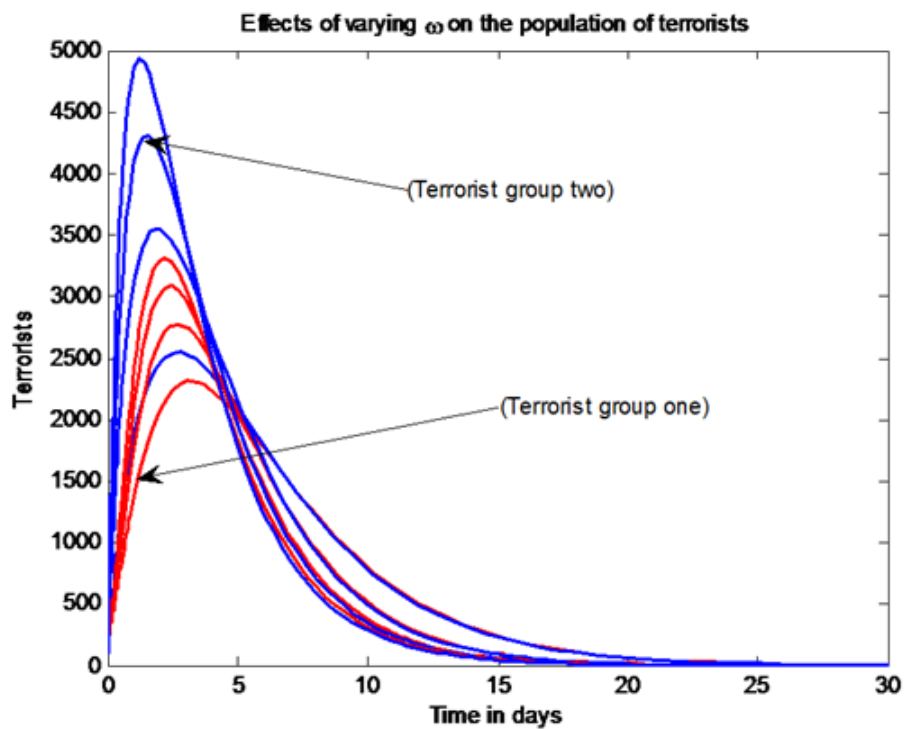


Fig. 10 Effects of varying (ω) on terrorists population.

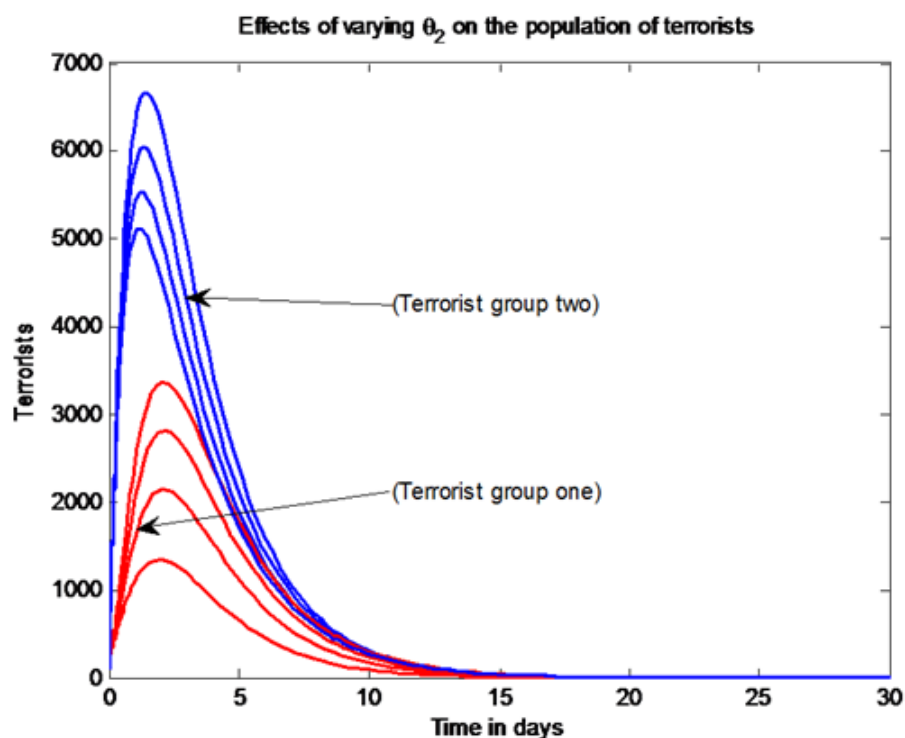


Fig. 11 Effects of varying (θ_2) on terrorists population.

Fig. 2 and Fig. 3 show the effects of varying (γ_1) on the two terrorists groups, where an increase in the value of this parameter leads to a decrease in the population on terrorists. This result implies that, as the military sustained intelligence operation to eliminate and take over terrorists camps increases, their population reduces drastically as many terrorists will desert their respective camps and many will equally die.

Fig.4 and Fig.5 also show the effects of varying (θ_1) and (θ_2) on the terrorists group one population dynamics, where an increase in the value of these parameters lead to a decrease in the population of terrorists in group one class. Similar dynamics was obtained in Fig. 6, where an increase in the value of (θ_2) reduces the population of terrorists in group two.

It is also evident from Fig.7, that as terrorists in group one join terrorists in group two, those in group two reinforce and dominate the population as their population increases, the side effect of this reinforcement is seen in Fig. 8 where terrorists in group become so powerfull and then dominates the terrorists population and as a result, terrorism activities increase in such population.

Fig 9-11, shows the impact of vaying (θ_1), (ω) and (θ_2) on the population of both terrorists groups. We observed that the activities of terrorists in group two dominates the activities of those in terrorist group one.

5 Conclusion

A mathematical model of terrorism population dynamics; a case of a community under attack of two terrorists groups is presented in this paper. As evident from our model analysis, the continious use of millitary intelligence operations to take over terrorists camps reduces terrorism in a population, as many terrorists will be eliminated and intending members will be discouraged in joining terrorism and if this practice is sustained with time activities of terrorists will be reduced significantly.

Finally, our obtained mathematical results agree with what is obtainable in real life and as such the obtained results can be applied to reduce terrorism activities in a population.

Conflict of interest:

The Authors declare no conflict of interest in this research work.

Data Availability:

The data supporting this work are previously published articles and they have been duly cited in this paper.

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Modelling 3D stokes flows in one-sided bumpy curved channels

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Abstract

The mathematical models of 3D flows through curved channels with one-sided bumps are developed. Two channel configurations are considered, namely, the top bumpy wall-smooth wall (T-S) configuration and the smooth wall-bottom bumpy wall (S-B) configuration. The analytical solutions of the complex flow models are given. The dependence of the flows on the parameters of the problems are shown explicitly through the analytical solutions. The present work gives a fundamental insight to describing and understanding flows in channels with protrusions.

Keywords and phrases: bumps; radius of curvature; wavenumber; phase shift; flow rate

1 Introduction

For decades, flow in channels have been investigated extensively by researchers in the scientific community. Since the pioneering work of Poiseuille, several analyses of pressure driven flows have emerged with significant practical applications in science and engineering. In particular, due to the manifestations of rough channels in various aspects of science and engineering, flows in channels enclosed or bounded by rough walls have been investigated to understand the effects of both two-dimensional (2D) and three-dimensional (3D) surface roughness on the flow characteristics in conduits with uneven walls. The flow naturally depends on the morphology of the surface roughness and can be considered to be a perturbation of the primary flow, i.e., the flow in a smooth conduit devoid of uneven walls. Wang [1,2] presented a perturbation analysis for flows in straight channels with corrugations. The analysis of Stokes flow in a corrugated pipe was given by Phan-Thien [3]. Flows in annuli with longitudinal grooves were presented by Moradi and Floryan [4]. The enhancement of mass transfer through a sinusoidal-wavy walled channel was examined by Nishimara and Kojima [5], for an unsteady flow. Convective heat transfer in periodic wavy passages was studied by Wang and Vanka [6]. Poiseuille flow of a Bingham fluid in a channel with a superhydrophobic groovy wall was

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recently presented by Rahmani, and Taghari [7]. The works in Refs. [1]-[7] demonstrate that perturbed boundaries have significant impacts on the flow behaviours, and can be used to achieve flow augmentation to a certain extent depending on physical and geometric parameters associated with the flow problem. Dean's problem [8] has also been extended to include the effects of surface roughness, to examine the change in the flow characteristics in curved channels with rough boundaries. Peterson [9] studied steady flow through a curved tube with wavy walls perturbing from the Deans problem. Okechi and Asghar [10,11] investigated steady Stokes flows in corrugated curved channels perturbing from the primary flow in a smooth curved channel without corrugations.

The theoretical models of 3D flows in straight channels and tubes with 3D bumps have been developed for both pressure and non-pressure driven flows. Stokes flow in bumpy tube and channel were reported by Wang [12,13]. The problems were extended by Yu and Wang [14,15] to include the analyses of flows in bumpy channel and tube enclosing a porous medium to determine the effects of the bumps subject to the permeability of the medium. For flows generated by electroosmotic mechanism, the works of Chang et al. [16] and lei et al. [17] can be referred for bumpy channel and tube, respectively. However, the study of 3D flow in curved channel with 3D bumps has not be considered so far.

In the present work, the models of 3D inertia free flow of a Newtonian fluid in curved channels are given. The essence of this work is to expand the current literature on channel flows by developing mathematical models for flows in bumpy curved channels, which is a salient advancement in the area of bumpy channel flows. Two distinct cases of the flow geometry are considered, including, a curved channel with bumpy top wall and a smooth bottom wall as well as a curved channel with smooth top wall and a bumpy bottom wall. The analytical solutions for the Newtonian flow are derived via boundary perturbation analysis, for each case, with the relative amplitude of the bumps as the perturbation parameter. In the subsequent sections, the 3D mathematical models are given, followed by the analytical solutions and the results are expressed explicitly as functions of the inherent parameters of the problems.

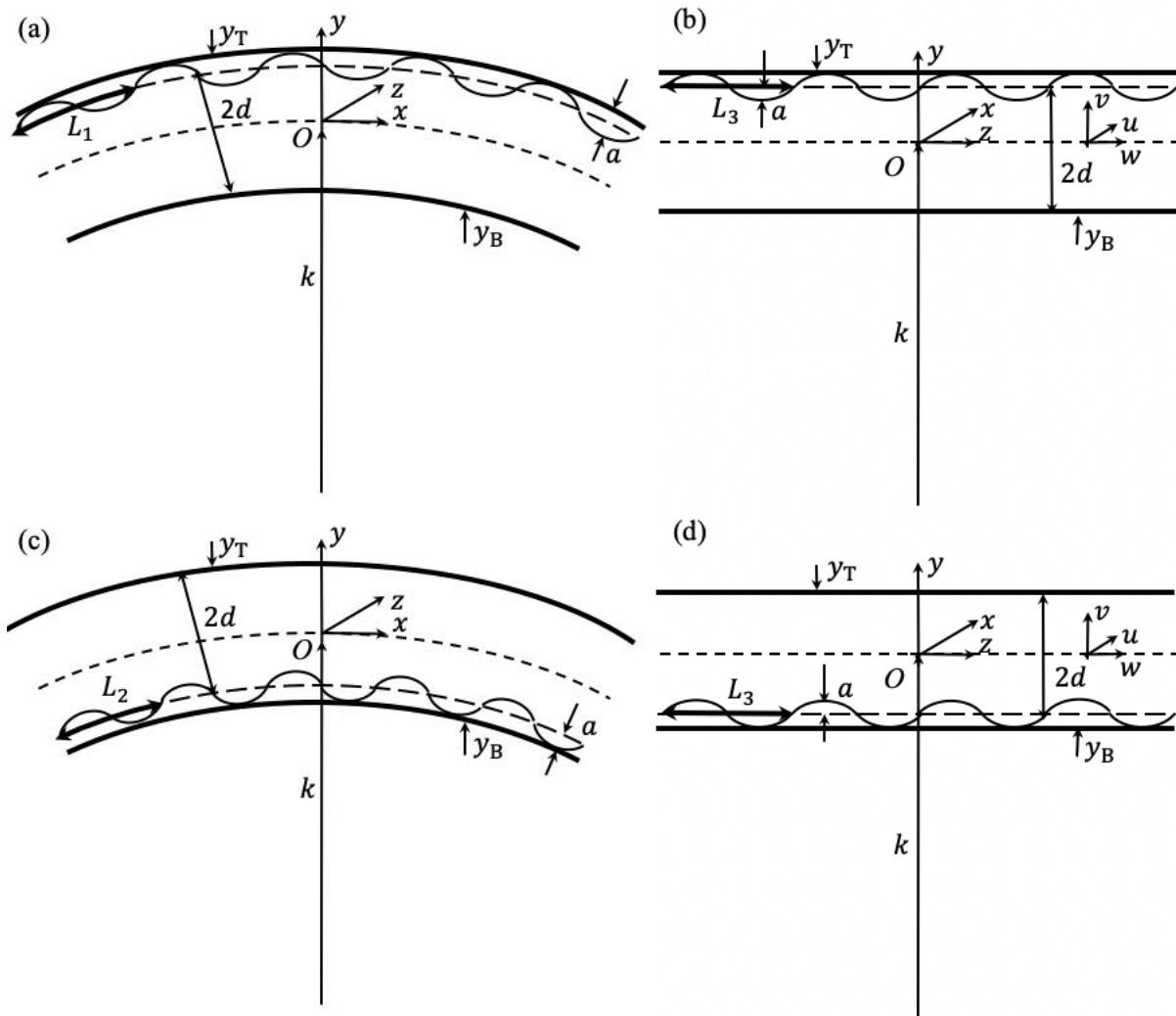


Fig. 1. Flow schematic. (a) and (b) are the axial and cross sections, respectively, of the T-S configuration, while (c) and (d) are the axial and cross sections, respectively, of the S-B configuration. The flow is pressure driven, along the curvilinear x -direction.

2 Mathematical model

Consider a flow through a mildly curved channel of radius of curvature k . Let (x, y, z) denote the curvilinear coordinates for the flow geometry. The flow is assumed to be incompressible and generated by a mean pressure gradient applied in the x -direction. Two distinct flow cases are of interest: A case where the flow is enclosed within a bumpy top wall and a smooth bottom wall and a case where the flow bounded by a smooth top wall and a bumpy bottom wall. The walls are set apart by a mean

distance $2d$, the wavelength of the bumps of amplitude a in the x direction are taken to be L_1 and L_2 as shown in Fig. 1, whereas L_2 is the wavelength in the z direction.

The equations governing three-dimensional incompressible steady flow are given by

$$\begin{aligned}\nabla \cdot \mathbf{U} &= 0, \\ \mu \nabla^2 \mathbf{U} - \nabla p &= \mathbf{0},\end{aligned}\tag{1}$$

The Eq. (1) is the continuity equation, whereas Eq. (2) is the momentum equation. The vector $\mathbf{U} = (u, v, w)$ is the velocity in the curvilinear coordinates (x, y, z) , μ is the dynamic viscosity of the fluid and p is the pressure. The flow is assumed to be dominated by the viscous forces, thus the Stokes approximation is used. Suppose G is the mean pressure gradient driving the flow, we can normalize (x, y, z) , (u, v, w) and pressure gradient in Eqs. (1) and (2) by $d, Gd^2/\mu$, and G , respectively. Thus, we have the following equations governing the 3D flow

$$\begin{aligned}\partial_y((y+k)v) + k\partial_x u + (y+k)\partial_z w &= 0, \\ (y+k)^2\partial_y((y+k)^{-1}\partial_y((y+k)u)) + 2k\partial_x v + k^2\partial_{xx}u + (y+k)^2\partial_{zz}u &= k(y+k)\partial_x p, \\ (y+k)^2\partial_y((y+k)^{-1}\partial_y((y+k)v)) - 2k\partial_x u + k^2\partial_{xx}v + (y+k)^2\partial_{zz}v &= (y+k)^2\partial_y p, \\ (y+k)\partial_y((y+k)\partial_y w) + k^2\partial_{xx}w + (y+k)^2\partial_{zz}w &= (y+k)^2\partial_z p,\end{aligned}\tag{3}$$

where $\partial_j(\cdot)$ is the derivative with respect to j . Let the bumps be sinusoidal, then the no-slip boundary conditions for the top bumpy wall-smooth wall (T-S) configuration is given by

$$u = v = w = 0 \text{ at } y = \begin{cases} y_T = 1 + \varepsilon \sin(\alpha x) \sin(\vartheta z) \\ y_B = -1 \end{cases}\tag{7}$$

And that of the smooth wall- bottom bumpy wall (S-B) read

$$u = v = w = 0 \text{ at } y = \begin{cases} y_T = 1 \\ y_B = -1 + \varepsilon \sin(\alpha x) \sin(\vartheta z) \end{cases}.\tag{8}$$

Here, $\varepsilon = a/d$, $\alpha = 2\pi d/L_1 \approx 2\pi d/L_2$ and $\vartheta = 2\pi d/L_3$ are the relative amplitude, longitudinal wavenumber and transverse wavenumber, respectively.

3 Analytical method and results

We apply perturbation method to obtain the solutions to Eqs. (3)-(8). The velocity and the pressure can be written in the following regular perturbation expansions for small ε , given by

$$u = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} u_{\beta},\tag{9}$$

$$v = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} v_{\beta},\tag{10}$$

$$w = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} w_{\beta},\tag{11}$$

$$\begin{aligned} w &= \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} w_{\beta}, \\ p &= \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} p_{\beta}. \end{aligned} \quad (12)$$

The boundary conditions in Eqs. (7) and (8) can be expanded as

$$u|_{y=y_T} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^{1-j} \partial_{y^{\beta}} u \Big|_{y=1} = 0, \quad (13)$$

$$v|_{y=y_T} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^{1-j} \partial_{y^{\beta}} v \Big|_{y=1} = 0, \quad (14)$$

$$w|_{y=y_T} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^{1-j} \partial_{y^{\beta}} w \Big|_{y=1} = 0, \quad (15)$$

$$u|_{y=y_B} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^j \partial_{y^{\beta}} u \Big|_{y=-1} = 0, \quad (16)$$

$$v|_{y=y_B} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^j \partial_{y^{\beta}} v \Big|_{y=-1} = 0, \quad (17)$$

$$w|_{y=y_B} = \sum_{\beta=0}^{\sigma} \varepsilon^{\beta} (\sin^{\beta}(\alpha x) \sin^{\beta}(\vartheta z) / \beta!)^j \partial_{y^{\beta}} w \Big|_{y=-1} = 0. \quad (18)$$

For $j = 0$ we get the T-S configuration with corresponding boundary conditions, while for $j = 1$ we have the B-S configuration.

The notation $\partial_{j^{\beta}}(\cdot)$ is β th the derivative with respect to j . The differential equations at each perturbation order are solved subject to the corresponding boundary conditions. We stop at the second-order of perturbation, i.e., $\sigma = 2$, since the terms of $\sigma > 2$ are small for $\varepsilon \ll 1$.

O(1) problem

At the zeroth-order, the curved channel is smooth and free of bumps, hence, the velocity vector is given by $\mathbf{U} = (u_0^j, 0, 0)$. The velocity components v_0^j and w_0^j in y and z directions, respectively, are zero when $\varepsilon = 0$. The pressure $p_0^j = -x$ generates the one-dimensional flow. Therefore, Eqs. (3)-(8) with Eqs. (9)-(12) and Eqs. (13)-(18) read

$$\begin{aligned} (y+k)^2 \partial_y \left((y+k)^{-1} \partial_y \left((y+k) u_0^j \right) \right) &= k(y+k), \\ u_0^j \Big|_{y=1} &= u_0^j \Big|_{y=-1} = 0. \end{aligned} \quad (19)$$

The solution of Eq. (19) is expressed as

$$u_0^j = a_0^j(y+k) + b_0^j(y+k)^{-1} + k(y+k)[1 - 2\ln(y+k)]/4, \quad (20)$$

where

$$a_0^j = (k+1)^2 \ln(k+1)/8 - (k-1)^2 \ln(k-1)/8 - k/4, \quad (21)$$

and

$$b_0^j = (k+1)^2(k-1)^2 \ln(k-1/k+1)/8. \quad (22)$$

$O(\varepsilon)$ problem

The velocity vector for the first-order equation is given by $\mathbf{U} = (u_1^j, v_1^j, w_1^j)$. At second-order, the flow is 3D and the pressure gradients along (x, y, z) are non-zero due to the bumpy walls. Eqs. (3)-(8) with Eqs. (9)-(12) and Eqs. (13)-(18) allow the solutions

$$u_1^j = U_1^j \sin(\alpha x) \sin(\vartheta z), \quad (23)$$

$$v_1^j = V_1^j \cos(\alpha x) \sin(\vartheta z), \quad (24)$$

$$w_1^j = W_1^j \cos(\alpha x) \cos(\vartheta z), \quad (25)$$

$$p_1^j = P_1^j \cos(\alpha x) \sin(\vartheta z), \quad (26)$$

using Eqs. (23)-(26), we have

$$\partial_y \left((y+k) V_1^j \right) + \alpha k U_1^j - \vartheta (y+k) W_1^j = 0, \quad (27)$$

$$(y+k) \partial_y \left((y+k) \partial_y U_1^j \right) - 2\alpha k V_1^j - (\vartheta^2 (y+k)^2 + \alpha^2 k^2 + 1) U_1^j = -\alpha k (y+k) P_1^j, \quad (28)$$

$$(y+k) \partial_y \left((y+k) \partial_y V_1^j \right) - 2\alpha k U_1^j - (\vartheta^2 (y+k)^2 + \alpha^2 k^2 + 1) V_1^j = (y+k)^2 \partial_y P_1^j, \quad (29)$$

$$(y+k) \partial_y \left((y+k) \partial_y W_1^j \right) - (\alpha^2 k^2 + \vartheta^2 (y+k)^2) W_1^j = \vartheta (y+k)^2 P_1^j, \quad (30)$$

with boundary conditions

$$U_1^j|_{y=1} = -(1-j) \partial_y u_0^j|_{y=1}, \quad (31)$$

$$U_1^j|_{y=-1} = -j \partial_y u_0^j|_{y=-1}, \quad (32)$$

$$V_1^j|_{y=1} = V_1^j|_{y=-1} = 0, \quad (33)$$

$$W_1^j|_{y=1} = W_1^j|_{y=-1} = 0. \quad (34)$$

The solutions of the 3D coupled system Eqs. (27)-(30) are obtained as

$$U_1^j = \frac{1}{2} \left(a_1^j I_{\alpha k+1}(\vartheta(y+k)) + b_1^j K_{\alpha k+1}(\vartheta(y+k)) + c_1^j I_{\alpha k-1}(\vartheta(y+k)) + d_1^j K_{\alpha k-1}(\vartheta(y+k)) \right), \quad (35)$$

$$V_1^j = \frac{1}{2} \left(a_1^j I_{\alpha k+1}(\vartheta(y+k)) + b_1^j K_{\alpha k+1}(\vartheta(y+k)) - c_1^j I_{\alpha k-1}(\vartheta(y+k)) - d_1^j K_{\alpha k-1}(\vartheta(y+k)) + e_1^j (y+k) I_{\alpha k}(\vartheta(y+k)) + f_1^j (y+k) K_{\alpha k}(\vartheta(y+k)) \right), \quad (36)$$

$$W_1^j = \frac{1}{2\vartheta} \left(e_1^j \vartheta(y+k) I_{\alpha k+1}(\vartheta(y+k)) - f_1^j \vartheta(y+k) K_{\alpha k+1}(\vartheta(y+k)) \right) \quad (37)$$

$$+ \left(\vartheta(a_1^j - c_1^j) + f_1^j(\alpha k + 2) \right) I_{\alpha k}(\vartheta(y+k)) \quad (38)$$

$$+ \left(\vartheta(d_1^j - b_1^j) + e_1^j(\alpha k + 2) \right) K_{\alpha k}(\vartheta(y+k)),$$

$$P_1^j = e_1^j I_{\alpha k}(\vartheta(y+k)) + f_1^j K_{\alpha k}(\vartheta(y+k)).$$

where I_{ϖ} and K_{ϖ} are order ϖ modified Bessel functions of the first and second kind, respectively.

The functions $a_1^j, b_1^j, c_1^j, d_1^j, e_1^j$ and f_1^j can be obtained algebraically from the boundary conditions in Eqs. (31)-(34).

$O(\varepsilon^2)$ problem

To proceed, we introduce an overbar which represents the average with respect to $\varkappa$. That is to say,

$$\bar{m} = \frac{\vartheta}{2\pi} \int_0^{2\pi/\vartheta} m \, dz. \quad (39)$$

This procedure is required as only the $\varkappa$ -averaged second-order terms are needed to evaluate the flow rate [11-16]. The first-order solution is periodic in the y -direction, such that the net flow rate through the channel is unaffected by the terms of the first-order. Taking Eq. (39) into consideration, Eqs. (3)-(6) with Eqs. (9)-(12) become

$$\partial_y \left((y+k) \bar{v}_2^j \right) + k \partial_x \bar{u}_2^j = 0, \quad (40)$$

$$(y+k)^2 \partial_y \left((y+k)^{-1} \partial_y \left((y+k) \bar{u}_2^j \right) \right) + 2k \partial_x \bar{v}_2^j + k^2 \partial_{xx} \bar{u}_2^j = k(y+k) \partial_x \bar{p}_2^j, \quad (41)$$

$$(y+k)^2 \partial_y \left((y+k)^{-1} \partial_y \left((y+k) \bar{v}_2^j \right) \right) - 2k \partial_x \bar{u}_2^j + k^2 \partial_{xx} \bar{v}_2^j = (y+k)^2 \partial_y \bar{p}_2^j, \quad (42)$$

$$(y+k) \partial_y \left((y+k) \partial_y \bar{w}_2^j \right) + k^2 \partial_{xx} \bar{w}_2^j = 0, \quad (43)$$

with the corresponding boundary conditions

$$\bar{u}_2^j|_{y=1} = -\frac{1}{4}(1-j)(1-\cos(2\alpha x))(\partial_y U_1^j - \partial_{yy} u_0^j)|_{y=1}, \quad (44)$$

$$\bar{u}_2^j|_{y=-1} = -\frac{1}{4}j(1-\cos(2\alpha x))(\partial_y U_1^j - \partial_{yy} u_0^j)|_{y=-1}, \quad (45)$$

$$\bar{v}_2^j|_{y=1} = -\frac{1}{4}(1-j)\sin(2\alpha x)\partial_y V_1^j|_{y=1}, \quad (46)$$

$$\bar{v}_2^j|_{y=-1} = -\frac{1}{4}j\sin(2\alpha x)\partial_y V_1^j|_{y=-1}, \quad (47)$$

$$\bar{w}_2^j|_{y=1} = \bar{w}_2^j|_{y=-1} = 0. \quad (48)$$

By Eqs. (44)-(48), the solutions of Eqs. (40)-(43) take the forms

$$u_2^j = U_2^j + U_3^j \cos(2\alpha x), \quad (49)$$

$$v_2^j = V_2^j \sin(2\alpha x), \quad (50)$$

$$w_2^j = 0, \quad (51)$$

$$p_2^j = P_2^j \sin(2\alpha x). \quad (52)$$

Therefore, we have

$$\partial_y \left((y+k)V_2^j \right) + 2\alpha k U_3^j = 0, \quad (53)$$

$$(y+k)\partial_y \left((y+k)\partial_y U_2^j \right) - U_2^j = 0, \quad (54)$$

$$(y+k)\partial_y \left((y+k)^{-1}\partial_y \left((y+k)U_3^j \right) \right) + 4\alpha k (V_2^j - \alpha k U_3^j) = -2\alpha k (y+k)P_2^j, \quad (55)$$

$$(y+k)\partial_y \left((y+k)^{-1}\partial_y \left((y+k)V_2^j \right) \right) + 4\alpha k (V_2^j - \alpha k U_3^j) = (y+k)^2 \partial_y P_2^j, \quad (56)$$

and the boundary conditions

$$U_2^j|_{y=1} = -\frac{1}{4}(1-j)(\partial_y U_1^j + \partial_{yy} u_0^j)|_{y=1}, \quad (57)$$

$$U_2^j|_{y=-1} = -\frac{1}{4}j(\partial_y U_1^j + \partial_{yy} u_0^j)|_{y=-1}, \quad (58)$$

$$U_3^j|_{y=1} = \frac{1}{4}(1-j)(\partial_y U_1^j + \partial_{yy} u_0^j)|_{y=1}, \quad (59)$$

$$U_3^j|_{y=-1} = \frac{1}{4}j(\partial_y U_1^j + \partial_{yy} u_0^j)|_{y=-1}, \quad (60)$$

$$V_2^j|_{y=1} = -\frac{1}{4}(1-j)\partial_y V_1^j|_{y=1}, \quad (61)$$

$$V_2^j|_{y=-1} = -\frac{1}{4}j\partial_y V_1^j|_{y=-1}. \quad (62)$$

The solutions of second-order are found to be

$$U_2^j = a_2^j(y+k) + b_2^j/(y+k), \quad (63)$$

$$U_3^j = c_2^j(y+k)^{2\alpha k-1} - d_2^j(y+k)^{-2\alpha k-1} + \frac{1}{4}\{e_2^j[(2\alpha k-1)^{-1}-1](y+k)^{1-2\alpha k} + f_2^j[(2\alpha k+1)^{-1}-1](y+k)^{1+2\alpha k}\}, \quad (64)$$

$$V_2^j = c_2^j(y+k)^{2\alpha k-1} + d_2^j(y+k)^{-2\alpha k-1} \quad (65)$$

$$+ \frac{\alpha k}{2}[e_2^j(2\alpha k-1)^{-1}(y+k)^{1-2\alpha k} + f_2^j(2\alpha k+1)^{-1}(y+k)^{1+2\alpha k}], \quad (66)$$

$$P_2^\pm = e_2^j(y+k)^{-2\alpha k} + f_2^j(y+k)^{2\alpha k}.$$

The coefficients $a_2^j, b_2^j, c_2^j, d_2^j, e_2^j$ and f_2^j are readily obtained algebraically using the boundary condition in Eqs. (57)-(62).

The total volumetric flow rate per unit area in the x -direction evaluated at $x = 0$, where the cross section is simpler [11-16] reads

$$Q^j = \frac{1}{2} \int_{-1}^1 \bar{u}^j|_{x=0} dy = \frac{1}{2} \int_{-1}^1 (\bar{u}_0^j + \varepsilon \bar{u}_2^j + \dots)|_{x=0} dy. \quad (67)$$

Eq. (67) can be written in the form

$$Q^j = q^j(1 - \varepsilon^2 \chi^j + O(\varepsilon^4)). \quad (68)$$

Here

$$q^j = ka_0^j + b_0^j \ln((k+1)/2(k-1)) - k((k+1)^2 \ln(k+1) - (k-1)^2 \ln(k-1))/8 \quad (69)$$

is the flow rate of the curved channel with smooth walls and

$$\begin{aligned} \chi^j = & 2a_2^j k/q^j + b_2^j \ln((k+1)/q(k-1)) \\ & + (c_2^j((k+1)^{2\alpha k} - (k-1)^{2\alpha k}) + d_2^j((k+1)^{-2\alpha k} - (k-1)^{-2\alpha k}))/2\alpha k q^j \\ & + e_2^j((2\alpha k - 1)^{-1} - 1)((k+1)^{-2\alpha k+2} - (k-1)^{-2\alpha k+2})/16\alpha k q^j(1 - \alpha k) \\ & + f_2^j((2\alpha k + 1)^{-1} - 1)((k+1)^{2\alpha k+2} - (k-1)^{2\alpha k+2})/16\alpha k q^j(1 + \alpha k). \end{aligned} \quad (70)$$

The function χ^j imposes effect of the bumpy curved walls on the flow rate, for both T-S and S-B situations, such that for $\chi^j > 0, Q < q^j$ and for $\chi^j < 0, Q > q^j$, whenever $0 < \varepsilon \ll 1$.

4 Conclusion

Mathematical models describing 3D viscous flows of a Newtonian fluid in one-sided bumpy curved channels is presented. Two configurations are considered including the T-S and S-B configurations. A regular perturbation solution for the 3D complex configurations is derived from perturbation analysis with the relative amplitude as the perturbation parameter. The results show that irrespective of the position of the bumpy wall, the flow rate is a function of χ^j and the amplitude of the bumps. For a fixed pressure gradient, $\varepsilon^2 \chi^j$ represents the percentage decrease in the total flow rate through the curved channels due to the flow resistance imposed by the bumps. The increase in the flow resistance is determined by the function χ^j . Maximum and minimum values of χ^j would result in maximal and minimal reductions in the total flow, respectively (see Eq. (68)). The given mathematical models give the descriptions of flows in a curvilinear geometry with sinusoidal 3D surface roughness and can be further extended to include other morphologies of roughness.

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- [2] Castillo-Chavez C. and Song B. (2004). Dynamical models of tuberculosis and their applications, Math. Biosci. Eng., 1: 361-404.

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