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Solution of fractional order integro-differential equations by two semi-analytical methods

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Abstract

This paper is concerned with the solution of fractional order integro-differential equations where the fraction order, η is in the interval (3, 4]. Two semi-analytical methods; Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) are implemented for the solution of the class of problems considered. Few examples are solved and the results obtained compared favorably with the existing results in the literature. All codes used in solving the problems are written in Mathematica 11.3

Keywords and phrases: integro-differential equation; semi-analytical method; fractional order; caputo derivative; correction functional

1 Introduction

Recently, serious interest has been developed in fractional differential equation due to their many applications in physics and engineering. Absence of analytical solutions in most differential equations lead to application of approximation and numerical methods. A Chinese mathematician was the first to propose the Variational Iteration Method (VIM) [9] and Homotopy Perturbation Method (HPM) [10] to solve the linear and non-linear problems. Variational iteration method is based on Lagrange multiplier and homotopy perturbation method is a coupling of traditional perturbation method and homotopy in topology. In ascertaining the solutions of functional equations, which occur in engineering and scientific problems, authors apply these methods successfully.

The Adomain decomposition method presents solutions of functional equations, but some difficulties arise during the computation of Adomian polynomials and variational iteration method (VIM) and homotopy perturbation method (HPM) overcome the difficulties [3,14]. The Integro-differential Equation (IDE) is an equation that involves both integral and derivatives of an unknown function [1]. In recent years, several numerical methods such as spline functions, collocation method, etc, have been used to solve integro-differential equations. The purpose of this research is to apply both variational iteration method (VIM) [9,14,18] and homotopy perturbation method (HPM) [7,8,10] to solve fractional order integro-differential equations.

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2 Preliminaries

2.1 Fractional order derivative

Riemann – Liouville gave a very good definition of fractional order derivative, but with a draw back. The latter definition given by Caputo in 1967 accounted for the draw back and offered a more reliable definition that does not depend on the availability of initial conditions of the model being proffered solution to [2,15,18,19,20]. The superiority of Caputo derivative was established by Ford and Simpson in 2001.

2.2 Caputo Derivatives

Definition: Let $\beta \in \mathbb{R}_+$ and $\eta = \lceil \beta \rceil$. The operator D_{*a}^{β} defined by

$$D_{*a}^{\beta} f(t) = J_a^{\eta-\beta} D^{\eta} f(t) = \frac{1}{\Gamma(\eta-\beta)} \int_a^t (t-\tau)^{\eta-\beta-1} \left(\frac{d}{d\tau}\right)^{\eta} f(\tau) d\tau, \quad (1)$$

for $a \leq t \leq b$, is called the Caputo differential operator of order β. The definition can equally be expressed as

$$D^{\beta} t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\beta+1)} t^{\eta-\beta}. \quad (2)$$

2.3 Statement of problem

$$D^{\eta} u(x) = f(x) + \mu u(x) + \int_o^x \left[g(t)u(t) + h(t)F\big(u(t)\big)\right] dt, \quad (3)$$

$$0 < x < b, \quad 3 < \eta \leq 4,$$

subject to the following boundary conditions

$$u(0) = \mu_{0,} \quad u''(0) = \mu_2$$

$$u(b) = \beta_0, \quad u''(b) = \beta_2.$$

2.4 Implementation of Variational Iteration Method Algorithm

To illustrate (3) using VIM, we construct the correction functional as

$$u_{k+1}(x) = u_k(x) + J^{\eta}\left[\lambda\big(D^{\eta} u_k(x) - f(x) - \mu u_k - \int_0^x g(t)u_k(t) + h(t)F(u_k(t))dt\big)\right],$$

where J^{β} is the Reimann-Liouville fractional integral operator of order $\beta = \eta + 1 - m,$ λ is a general lagrange multiplier and u_k denotes restricted variation, $\lambda(s) = \frac{(-1)^m}{(m-1)!}(s-x)^{m-1}$, where m is the order of the differential equation.

If $m = 4$

$$\lambda(s) = \frac{1}{3!}(s-x)^3$$
$$= \frac{1}{6}(s-x)^3$$

$$J^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(t) dt, \quad \eta > 0$$

$$u_{k+1}(x) = u_k(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \lambda(s) \left(D^{\eta} u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t) u_k(t) + h(t) F(u_k(t)) dt\right).$$

But $\beta = \eta + 1 - m$, if $m = 4$,

$$\beta = \eta + 1 - 4$$
$$\beta = \eta - 1.$$

Also,

$$\beta - 1 = \eta - 3 - 1$$
$$\beta - 1 = \eta - 4$$

$$u_{k+1}(x) = u_k(x) + \frac{1}{6\Gamma(\eta-3)} \int_0^x (x-s)^{\eta-4} (s-x)^3 \left(D^{\eta} u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t) u_k(t) + h(t) F(u_k(t)) dt\right).$$

But

$$(x-s)^{\eta-4}(s-x)^3 = -(x-s)^3 (x-s)^{\eta-4} = -(x-s)^{\eta-1}.$$

Also,

$$\frac{1}{6\Gamma(\eta-3)} = \frac{(\eta-1)(\eta-2)(\eta-3)}{6\Gamma(\eta)}.$$

So,

$$u_{k+1}(x) = u_k(x) + \frac{(\eta-1)(\eta-2)(\eta-3)}{6\Gamma(\eta)} \int_0^x \left(-(x-s)^{\eta-1}\right) \left(D^{\eta} u_k(x) - f(x) - \mu u_k(x) - \int_0^x g(t) u_k(t) + h(t) F(u_k(t)) dt\right)$$

$$u_0(x) = u(0) + x u'(0) + \frac{x^2}{2!} u''(0) + \frac{x^3}{3!} u'''(0),$$

since $u_0 = \mu_0, u''(0) = \mu_2$. Then,

$$u_0(x) = \mu_0 + \mu_1 x + \frac{x^2}{2} \mu_2 + \frac{x^3}{6} \mu_3,$$

where $u'(0) = \mu_1$ and $u'''(0) = \mu_3$.

When $k = 0$:

$$u_1(x) = u_0(x) + \frac{-(\eta-3)(\eta-2)(\eta-1)}{6\Gamma(\eta)} \int_0^x \left(-(x-s)^{\eta-1}\right)\left(D^{\eta} u_0(x) - f(x) - \mu u_0(x) - \int_0^x g(t) u_0(t) + h(t) F(u_0(t)) dt\right).$$

But $J^{\eta}=\frac{1}{\Gamma(\eta)}\int_0^x (x-s)^{\eta-1}ds$

$u_1(x)=u_0(x)+\frac{-(\eta-3)(\eta-2)(\eta-1)}{6}J^{\eta}(D^{\eta}u_0(x)-f(x)-\mu u_0(x)-\int_0^x g(t)u_0(t)+h(t)F(u_0(t))dt.$

3 Numerical examples on fractional order integro-differential equation using VIM

Solutions of fractional order integro-differential equation by using VIM are considered here. The following problems are solved using VIM

Problem 1:

Consider the following non-linear integro-differential equation.
$D^{\eta}(x)=e^{x}-\frac{1}{2}e^{2x}+\frac{1}{2}-\int_0^x u(t)u''(t)dt, \qquad 0<x<1,\ \ 3<\eta\le 4,$

with the boundary conditions

$$u(0)=1,\ \ u'(0)=1$$
$$u(1)=e,\ \ u''(0)=1.$$

The correction functional is

$u_{k+1}(x)=u_k(x)-\frac{(\eta-3)(\eta-2)(\eta-1)}{6}J^{\eta}(D^{\eta}u_k-e^{x}+\frac{1}{2}e^{2x}-\frac{1}{2}-\int_o^x u_k(t)u_k''(t)dt).$

We take the truncated Taylor expansion for e^x and e^{2x} as $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$ And $e^{2x}=1+2x+2x^2+\frac{4x^3}{3}$.

Also,

$$u_0(x)=u(0)+xu'(0)+\frac{x^2}{2!}u''(0)+\frac{x^3}{3!}u'''(0)$$
$$u(0)=1, \qquad u'(0)=1, \qquad u''(0)=1$$
$$\therefore u_0(x)=1+x+\frac{x^2}{2}+\frac{Ax^3}{6}$$

where $A=u'''(0)$.

When $k=0$,

$$u_1(x)=u_0(x)-\frac{(\eta-3)(\eta-2)(\eta-1)}{6}J^{\eta}(D^{\eta}u_0(x)-e^{x}+\frac{1}{2}e^{2x}-\frac{1}{2}-\int_0^x u_0(t)u_0''(t)dt)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta (D^\eta \left(1 + x + \frac{x^2}{2} + \frac{Ax^3}{6}\right) - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}) + \frac{1}{2}(1 + 2x + 2x^2 + \frac{4x^3}{3}) - \frac{1}{2} - \int_0^x (1 + t + \frac{t^2}{2} + \frac{At^3}{6})(2 + At)dt)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(D^\eta \left(1 + x + \frac{x^2}{2} + \frac{Ax^3}{6}\right) - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{2} + x + x^2 + \frac{2x^3}{3} - \frac{1}{2} + 2x + x^2 + \frac{Ax^2}{2} + \frac{x^3}{6} + \frac{Ax^3}{3} + \frac{7Ax^4}{48} + \frac{A^2x^5}{30}\right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(0 - 1 + 2x + \frac{1}{2}(3 + A)x^2 + \frac{1}{3}(2 + A)x^3 + \frac{7Ax^4}{48} + \frac{A^2x^5}{30}\right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^\eta \left(-1 + 2x + \frac{1}{2}(3 + A)x^2 + \frac{1}{3}(2 + A)x^3 + \frac{7Ax^4}{48} + \frac{A^2x^5}{30}\right)$$

$$u_1(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{x^\eta}{\Gamma(1+\eta)} + \frac{2x^{\eta+1}}{\Gamma(2+\eta)} + \frac{(3+A)x^{\eta+2}}{\Gamma(3+\eta)} + \frac{(4+A)x^{\eta+3}}{\Gamma(4+\eta)} + \frac{7Ax^{\eta+4}}{2\Gamma(5+\eta)} + \frac{4A^2x^{\eta+5}}{\Gamma(6+\eta)}\right).$$

Imposing the boundary condition $u(1) = 3$,

$$u_1(x) = \frac{5}{2} + \frac{A}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} \left(-\frac{1}{\Gamma(1+\eta)} + \frac{2}{\Gamma(2+\eta)} + \frac{(3+A)}{\Gamma(3+\eta)} + \frac{(4+A)}{\Gamma(4+\eta)} + \frac{7A}{2\Gamma(5+\eta)} + \frac{4A^2}{\Gamma(6+\eta)}\right) = e\,.$$

Solving for A in $u(1)$, we have:

At $\eta = 4.00$, A = 1.20370
At $\eta = 3.75$, A = 1.22641
At $\eta = 3.50$, A = 1.25565

At η = 3.25, A = 1.28589.

Substituting the values of A in $u_1(x)$, we have:

at η = 4.00, A = 1.203699334

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.200617x^3 + 0.041667x^4 - 0.166667x^5 - 0.00583847x^6 - 0.00127131x^7 - 0.000208976x^8 - 0.000015971x^9$$

at η = 3.75, A = 1.22641

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.204402x^3 + 0.0362688x^{3.75} - 0.015297x^{4.75} - 0.0056123x^{5.75} - 0.00126946x^{6.75} - 0.000217922x^{7.75} - 0.0000174538x^{8.75}$$

at η = 3.50, A = 1.25565

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0.209275x^3 + 0.0268662x^{3.5} - 0.0119405x^{4.5} - 0.00461952x^{5.5} - 0.00108739x^{6.5} - 0.000195714x^{7.5} - 0.0000165209x^{8.5}$$

at η = 3.25, A = 1.28589

$$u_1(x) = 1 + x + \frac{x^2}{2} + 0{:}\,214315x^3 + 0.014144x^{3.25} - 0.00665618x^{4.25} - 0.00271692x^{5.25} - 0.00066656x^{6.25} - 0.00125927x^{7.25} - 0.00001121x^{7.25}$$

Table 1: Varying the value of x from 0 to 1, at $\eta = 4$, $u(x)$

x	Exact Solution	VIM	Error
0.0	1.0000000	1.0000000	0.0000000
0.1	1.1051709	1.1052046	0.0000337
0.2	1.2214028	1.2216659	0.0002631
0.3	1.3498588	1.3507091	0.0008503
0.4	1.4918247	1.4937093	0.0018846
0.5	1.6487213	1.6520584	0.0033371
0.6	1.8221188	1.8271257	0.0050069
0.7	2.0137527	2.0202108	0.0064581
0.8	2.2255409	2.2324886	0.0069477
0.9	2.2324886	3.4596031	0.2271151
1.0	2.7182818	2.7182818	0.0000000

Problem 2:

Consider the following non-linear integro-differential equation.

$$D^{\eta}u(x) = e^{-x} + e^{-3x} - 1 + 3\int_0^x u^3(t)dt, \qquad (4)$$

with the boundary conditions

$$u(0) = 1, \qquad u'(0) = -1$$
$$u''(0) = 1, \qquad u'''(0) = 1.$$

The correction functional is

$$u_{k+1}(x) = u_k(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^{\eta}(D^{\eta}u_k(x) - e^{-x} - e^{-3x} + 1 - 3\int_o^x u_k^3(t)u_k''(t)dt).$$

We take the truncated Taylor expansion for e^{-x} and e^{-3x} as $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ and $e^{-3x} = 1 - 3x + \frac{9x^3}{2} + \frac{9x^3}{2}$.

Also,

$$u_0(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0)$$
$$u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1 \quad u'''(0) = 1$$
$$\therefore u_0(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6}.$$

When $k = 0$,

$$u_1(x) = u_0(x) - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^{\eta}(D^{\eta}u_0(x) - e^{-x} - e^{-3x} + 1 - 3\int_o^x u_0^3(t)dt)$$

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^{\eta}\left(D^{\eta}\left(1 - x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(1 - x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) - \left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2}\right) + 1 - 3\int_0^x \left(1 - t + \frac{t^2}{2} + \frac{t^3}{6}\right)^3 dt\right)$$

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^{\eta}\left(0 - 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - 1 + 3x - \frac{9x^2}{2} + \frac{9x^3}{2} + 1 - 3\int_0^x \left(1 - t + \frac{t^2}{2} + \frac{t^3}{6}\right)^3 dt\right)$$

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6} J^{\eta}\left(-1 + x - \frac{x^2}{2} + \frac{13x^3}{6} + \frac{21x^4}{8} - \frac{3x^5}{4} - \frac{x^6}{8} + \frac{x^7}{8} - \frac{x^8}{64} - \frac{x^9}{72} - \frac{x^{10}}{720}\right)$$

$$\therefore u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6}\left(-\frac{x^{\eta}}{\Gamma(1+\eta)} + \frac{x^{\eta+1}}{\Gamma(2+\eta)} - \frac{x^{\eta+2}}{\Gamma(3+\eta)} - \frac{13x^{\eta+3}}{\Gamma(4+\eta)} + \frac{63x^{\eta+4}}{\Gamma(5+\eta)} - \frac{90x^{\eta+5}}{\Gamma(6+\eta)} - \frac{90x^{\eta+6}}{\Gamma(7+\eta)} + \frac{630x^{\eta+7}}{\Gamma(8+\eta)} - \frac{630x^{\eta+8}}{\Gamma(9+\eta)} + \frac{630x^{\eta+9}}{\Gamma(10+\eta)} - \frac{5040x^{\eta+10}}{\Gamma(11+\eta)}\right),$$

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{(\eta-3)(\eta-2)(\eta-1)}{6}\left(-\frac{1}{\Gamma(1+\eta)} + \frac{1}{\Gamma(2+\eta)} - \frac{1}{\Gamma(3+\eta)} - \frac{13}{\Gamma(4+\eta)} + \frac{63}{\Gamma(5+\eta)} - \frac{90}{\Gamma(6+\eta)} - \frac{90}{\Gamma(7+\eta)} + \frac{630}{\Gamma(8+\eta)} - \frac{630}{\Gamma(9+\eta)} + \frac{630}{\Gamma(10+\eta)} - \frac{5040}{\Gamma(11+\eta)}\right).$$

Varying the values of η in $u_1(x)$ we have:

at $\eta = 4.00$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \frac{13x^7}{5040} - \frac{x^8}{640} + \frac{x^9}{4032} + \frac{x^{10}}{40320} - \frac{x^{11}}{63360} + \frac{x^{12}}{760320} + \frac{x^{13}}{9884160} + \frac{x^{14}}{17297280}$$

At $\eta = 3.75$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0362688x^{3.75} - 0.00763555x^{4.75} + 0{:}00132792x^{5.75} + 0.00255748x^{6.75} - 0.00159922x^{7.75} + 0.000261097x^{8.75} + 0.0000267791x^{9.75} - 0.0000174376x^{10.75} + 1.48405\times 10^{-6}x^{11.75} + 1.16396\times 10^{-7}x^{12.75} + 6.77213\times 10^{-8}x^{13.75}$$

At $\eta = 3.50$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0268662x^{3.5} - 0.00597026x^{4.5} + 0.0010855x^{5.5} + 0.002171x^{6.5} - 0.0014028x^{7.5} + 0.000235765x^{8.5} + 0.0000248174x^{9.5} - 0.000016544x^{10.5} + 1.43869\times 10^{-6}x^{11.5} + 1.15095\times 10^{-7}x^{12.5} + 6.82045\times 10^{-8}x^{13.5}$$

At $\eta = 3.25$,

$$u_1(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.014144x^{3.25} - 0.00332809x^{4.25} + 0.000633922x^{5.25} + 0.00131856x^{6.25} - 0.000881371x^{7.25} + 0.000152618x^{8.25} + 0.0000164998x^{9.25} - 0.0000112678x^{10.25} + 1.00158\times 10^{-6}x^{11.5} + 8.17618\times 10^{-7}x^{12.5} + 4.93656\times 10^{-8}x^{13.5}.$$

Table 2: Varying the values of $x = 0(0.04)0.36$, at $\eta = 4$, $u(x)$

x	Exact Solution	VIM	Error
0.00	1.0000000	1.0000001	0.0000000
0.04	0.9607894	0.9608108	0.0000214
0.08	0.9231163	0.9232870	0.0001707
0.12	0.8869204	0.8874964	0.0005760
0.16	0.8521438	0.8535091	0.0013653
0.20	0.8187308	0.8213975	0.0026667
0.24	0.7866279	0.7912360	0.0004608
0.28	0.7557837	0.7631014	0.0073177
0.32	0.7261490	0.7370725	0.0109235
0.36	0.6976763	0.7132301	0.0155538

Statement of Problem

$$D^{\eta}u(x) = f(x) + \mu u(x) + \int_0^x \left[g(t)u(t) + h(t)F\big(u(t)\big)\right]dt \qquad (5)$$

$$0 < x < b, \quad 3 < \eta \leq 4,$$

subject to the following boundary conditions

$$u(0) = \mu_0, \qquad u''(0) = \mu_2$$
$$u(b) = \beta_0, \qquad u''(b) = \beta_2.$$

4 Implementation of Homotopy Perturbation Method Algorithm

To illustrate equation (5) using HPM, we construct the homotopy equation

$$(1-p)D^{\eta}u(x) = ph(D^{\eta}u(x) - f(x) - \mu u(x) - \textstyle\int_0^x \left[g(t)u(t) + h(t)F\big(u(t)\big)dt\right].$$

Putting $h = -1$,

$$D^{\eta}u(x) - pD^{\eta}u(x) = -(pD^{\eta}u(x) - pf(x) - p\mu u(x) - p\int_0^x \left[g(t)u(t) + h(t)F\big(u(t)\big)dt\right])$$

$$D^{\eta}u(x) - pf(x) - p\mu u(x) + \int_0^x \left[g(t)u(t) + h(t)F\big(u(t)\big)dt\right] = 0$$

$$D^{\eta}u(x) = pf(x) - p\mu u(x) + \int_0^x \left[g(t)u(t) + h(t)F\big(u(t)\big)dt\right]$$

$$u(x) = \sum_{n=0}^{\infty} y_n(x)p^n$$

$$D^{\eta}\sum_{n=0}^{\infty} y_n(x)p^n$$
$$= pf(x) - p\mu\sum_{n=0}^{\infty} y_n(x)p^n$$
$$+ \int_0^x \left[g(t)\sum_{n=0}^{\infty} y_n(x)p^n + h(t)F\left(u(t)\sum_{n=0}^{\infty} y_n(x)p^n\right)dt\right] = 0$$

$$D^{\eta}(u_0(x) + u_1(x)p + u_2(x)p^2 + y_3(x)p^3 + \cdots)$$
$$= p(f(x) + \mu u_0(x) + \mu u_1(x)p + \mu u_2(x)p^2 + \mu y_3(x)p^3 + \cdots$$
$$+ \int_0^x [g(t)u_0(t) + g\,(t)y_1(t)p + g(t)y_2(t)p^2 + \cdots h(t)F(u_0(t)$$
$$+ h(t)F(y_1(t)p) + h(t)F(y_2(t)p^2) + h(t)F(y_3(t)p^3)\cdots]dt$$

$$P^0 : D^{\eta}\big(u_0(x)\big) = 0$$

$$P^1 : D^{\eta}(u_1(x)) = f(x) + \mu u_0(x) + \int_0^x [g(t)u_0(t) + h(t)F\big(u_0(t)\big)dt]$$

$$P^2: D^\eta(u_2(x)) = \mu u_1(x) + \int_0^x [g(t)y_1(t) + h(t)F(y_1(t))dt]$$
$$P^3: D^\eta(u_3(x)) = \mu u_2(x) + \int_0^x [g(t)y_2(t) + h(t)F(y_2(t))dt].$$

5 Numerical examples on fractional order integro-differential using HPM

Solutions of fractional order integro-differential equation by using HPM are considered here. The following problems are solved using HPM.

Problem 1:
Consider the following non-linear integro-differential equation:

$$D^\eta u(x) = e^x - \frac{1}{2}e^{2x}\frac{1}{2} - \int_0^x u(t)u''(t)dt, \qquad 0 < x < 1, \qquad 3 < \eta \leq 4$$

with the boundary conditions

$$u(0) = 1, \quad u(1) = e$$
$$u'(0) = 1, \quad u''(0) = 1.$$

The homotopy equation is

$$D^\eta \sum_{k=0}^{\infty} u_k(x)P^k = P\left\{e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x \left(\sum_{k=0}^{\infty} u_k(x)P^k\right)\left(\sum_{k=0}^{\infty} u_k''(x)P^k\right)dt\right\}$$

$$\begin{aligned} D^\eta(u_0(x) + Pu_1(x) + P^2u_2(x) + \cdots) &= P\left\{e^x - \frac{1}{2}e^{2x} + \frac{1}{2}\right. \\ &\left.- \int_0^x (u_0(x) + Pu_1(x) + P^2u_2(x) + \cdots)(u_0''(x) + Pu_1''(x) + P^2u_2(x) + \cdots)\right\} \end{aligned}$$

$$P^0: D^\eta u_0(x) = 0$$
$$u_0(x) = 0$$
$$P^1: D^\eta u_1(x) = e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x u_0(t)u_0''(t)dt$$
$$P^2: D^\eta u_2(x) = -\int_0^x u_1(t)u_1''(t)dt$$
$$P^3: D^\eta u_3(x) = -\int_0^x u_2(t)u_2''(t)dt$$
$$u_0(x) = u_0 + xu_0' + \frac{x^2}{2!}u''_0 + A\frac{x^3}{3!}u_0''' \Rightarrow 1 + x + \frac{x^2}{2} + \frac{Ax^3}{5}$$

$u_0(x) = 1$
$u_1 = D^{-\eta}\left[e^x - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x (u_0(t))(u_0''(t))\right]dt.$

Let the truncated e^x and e^{2x} be

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ and $e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3}$,
also $D^{-\eta} = J^\eta$

$$u_1(x) = J^\eta \left\{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} - \frac{1}{2}\left(1 + 2x + 2x^2 + \frac{4x^3}{3}\right) + \frac{1}{2} - \int_0^x (1)(0)\right\}$$

$$u_1(x) = J^\eta \left\{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} - \frac{1}{2} - x - x^2 + \frac{2x^3}{3} + \frac{1}{2}\right\}$$

$$u_1(x) = J^\eta \left\{1 - \frac{x^2}{2} - \frac{x^3}{2}\right\}$$

$$u(x) = u_0(x) + u\_1(x)$$

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{Ax^3}{6} + \frac{x^\eta}{\Gamma(\eta + 1)} - \frac{x^{\eta+2}}{\Gamma(\eta + 3)} - \frac{2x^{\eta+3}}{\Gamma(\eta + 4)}$$

$$u(1) = \frac{5}{2} + \frac{A}{6} + \frac{1}{\Gamma(\eta+1)} - \frac{1}{\Gamma(\eta+3)} - \frac{2}{\Gamma(\eta+4)} = e.$$

Solving for A in $u(1)$, we have:

at $\eta = 4.00$
$A = 1.070405256$,
at $\eta = 3.75$
$A = 0.965114$,
at $\eta = 3.50$
$A = 0.821115$,
at $\eta = 3.25$

$$A = 0.628341.$$

Substituting the values of A in $u(x)$ we have:

at $\eta = 4.00$, $A = 1.070405256$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.178401x^3 + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^7}{2520}$$

at $\eta = 3.75$, $A = 0.965114$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.160852x^3 + 0.0602911x^{3.75} - 0.00220745x^{5.75} - 0.00065406x^{6.75}$$

at $\eta = 3.50$, $A = 0.821115$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.136852x^3 + 0.0859717x^{3.5} - 0.00347361x^{5.5} - 0.0010688x^{6.5}$$

at $\eta = 3.25$, $A = 0.628341$

$$u(x) = 1 + x + \frac{x^2}{2} + 0.104724x^3 + 0.120699x^{3.25} - 0.00540947x^{5.25} - 0.00173103x^{6.25}.$$

Table 3: Varying the values of x from 0 to 1, at $\eta = 4$, $u(x)$

x	Exact Solution	HPM	Error
0.0	1.0010000	1.0000000	0.0000000
0.1	1.1051709	1.1051826	0.0000117
0.2	1.2214028	1.2214938	0.0000910
0.3	1.3498588	1.3501532	0.0002944
0.4	1.4918247	1.4924710	0.0006463
0.5	1.6487213	1.6498795	0.001158
0.6	1.8221188	1.8238587	0.0017382
0.7	2.0137527	2.0159996	0.0022469
0.8	2.2255409	2.2279606	0.0024197
0.9	2.4596031	2.4614638	0.0018607
1.0	2.7182818	2.7182818	0.0000000

Problem 2:
Consider the following non-linear integro-differential equation:

$$D^{\eta}u(x) = e^{-x} + e^{-3x} - 1 + 3\int_0^x u^3(t)dt$$
$$0 < x < 1, \quad 3 < \eta \leq 4$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = -1$$
$$u''(0) = 1, \quad u'''(0) = 1.$$

The homotopy equation is

$$D^{\eta}\sum_{k=0}^{\infty} u_k(x)\,P^k = P\{e^{-x} + e^{-3x} - 1 + 3\int_0^x \sum_{k=0}^{\infty} u_k^3(t)P^k\,dt\}$$

$$D^{\eta}(u_0(x) + \,Pu_1(x) + \,P^2u_2(x) + \cdots)$$
$$= \,P\left\{e^{-x} + e^{-3x} - 1 + 3\int_0^x (u_0(t) + Pu_1(t) + P^2u_2(t) + \cdots)^3\,dt\right\}$$

$$D^{\eta}(u_0(x) + \,Pu_1(x) + \,P^2u_2(x) + \cdots)$$
$$= \,P\left\{e^{-x} + e^{-3x} - 1 \right.$$
$$+ 3\int_0^x \left(u_0^3(t) + 3Pu_1(t)u_0^2(t) + 3P^2u_0^2(t)u_2(t) + u_1^2(t)u_0(t)\right)$$
$$\left. + p^3(6u_0(t)u_1(t)u_2(t) + u_1^3(t) + \cdots)\,dt\right\}$$

$$P^0: D^{\eta}u_0(x) = 0$$
$$P^1: D^{\eta}u_1(x) = e^{-x} - e^{-3x} - 1 + 3\int_0^x (u_0^3(t))dt$$
$$P^2: D^{\eta}u_2(x) = \,3\int_0^x \left(3u_1(t)u_0^2(t)\right)dt$$

$$= 9\int_0^x (u_1(t)u_0^2(t))dt$$
$$P^3: D^\eta u_3(x) = -\int_0^x 3(u_0^2(t)u_2(t) + u_1^2(t)u_0(t)dt$$
$$= 9\int_0^x \left(\left(u_0^2(t)u_2(t) + u_1^2(t)u_0(t)\right)\right)dt.$$

We take the truncated Taylor expression for e^{-x} and e^{-3x} as

$$e^{-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{6!}$$
$$e^{-3x} = 1 - 3x + \frac{9x^3}{2} - \frac{9x^3}{2}$$
$$u_0(x) = u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0)$$
$$u_0(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6}$$
$$u_0(x) = 1$$
$$u_1(x) = D^{-\eta}\left\{e^{-x} + e^{-3x} - 1 + 3\int_0^x \left(u_0^3(t)\right)dt\right\}$$
$$D^{-\eta} = J^\eta$$
$$u_1(x) = J^\eta\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} - 1 + 3\int_0^x (1)^2\right\}$$
$$u_1(x) = J^\eta\left\{1 - x + 5x^2 - \frac{14x^3}{3}\right\}$$
$$u_1(x) = -x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^\eta}{\Gamma(\eta+1)} - \frac{x^{\eta+1}}{\Gamma(\eta+2)} + \frac{10x^{\eta+2}}{\Gamma(\eta+3)} - \frac{28x^{\eta+3}}{\Gamma(\eta+4)}$$
$$u_2(x) = D^\eta\left\{9\int_0^x \left(u_1(t)u_0^2(t)\right)dt\right\}$$
$$D^{-\eta} = J^\eta$$
$$u_2(x) = J^\eta\left\{9\int_0^x (-t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^\eta}{\Gamma(\eta+1)} - \frac{t^{\eta+1}}{\Gamma(\eta+2)} + \frac{10t^{\eta+2}}{\Gamma(\eta+3)} - \frac{28t^{\eta+3}}{\Gamma(\eta+4)})(1)^2dt\right\}$$
$$u_2(x) = J^\eta\left\{\frac{-9x^2}{2} + \frac{3x^2}{2} + \frac{3x^4}{8} + \frac{216x^{\eta+1}}{\Gamma(\eta+5)} - \frac{108x^{\eta+2}}{\Gamma(\eta+5)} + \cdots\right\}$$
$$u(x) = \frac{-9x^{\eta+2}}{\Gamma(\eta+3)} + \frac{9x^{\eta+3}}{\Gamma(\eta+4\,)} + \frac{9x^{\eta+4}}{\Gamma(\eta+5)} + \cdots$$
$$u(x) = u_0(x) + u_1(x) + u_2(x)$$
$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^\eta}{\Gamma(\eta+1)} - \frac{x^{\eta+1}}{\Gamma(\eta+2)} + \frac{10x^{\eta+2}}{\Gamma(\eta+3)} - \frac{28x^{\eta+3}}{\Gamma(\eta+4)} - \frac{9x^{\eta+2}}{\Gamma(\eta+3)} + \frac{9x^{\eta+3}}{\Gamma(\eta+4)} + \frac{9x^{\eta+4}}{\Gamma(\eta+5)} + \cdots$$

Varying the values of η in $u(x)$, we have:

At $\eta = 4.00$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{1008} + \frac{x^8}{4480} + \cdots$$

At $\eta = 3.75$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0602911x^{3.75} - 0.0126929x^{4.75} + 0.00220745x^{5.75} - 0.00163515x^{6.75} + 0.000379777x^{7.75}$$

At $\eta = 3.50$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.0859717x^{3.5} - 0.0191048x^{4.5} + 0.00347361x^{5.5} - 0.002672x^{6.5} + 0.000641281x^{7.5}$$

At $\eta = 3.25$

$$u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + 0.120699x^{3.25} - 0.0283997x^{4.25} + 0.00540947x^{5.25} - 0.00432758x^{6.25} + 0.00107443x^{7.25}$$

Table 4: Varying the values of $x = 0(0.04)0.36$, at $\eta = 4$, $u(x)$

x	Exact Solution	HPM	Absolute Error
0.0	1.0010000	1.0000000	0.0000000
0.04	0.9607894	0.9608108	0.0000214
0.08	0.9231163	0.9232870	0.0001707
0.12	0.8869204	0.8874964	0.0005762
0.16	0.8521438	0.8535091	0.0013653
0.2	0.8187308	0.8213974	0.0026666
0.24	0.7866279	0.7912358	0.0046079
0.28	0.75578374	0.7631010	0.0187122
0.32	0.7261490	0.7370715	0.0109225
0.36	0.6976763	0.7132278	0.0155515

6 Discussion

Tables 1 to 4 present results of problems 1 to 4 where the first two are for the implementation VIM, while HPM is implemented for the results in the last two tables. In results of VIM and HPM are evaluated for $\eta = 4$ at different step-sizes. It is obvious that the results for both VIM and HPM improve with reduction in step length.

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