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On independence and minimal generating set in semigroups and countable systems of semigroups

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Abstract

This paper studies independence in semigroup by exploring the relationship between the concepts of "minimal generating set" as compared to "minimum generating set" using comparability of elements, as induced by the orderings on the semigroup. It is shown that minimal generating set implies independence and vice versa. Two concluding results which are algorithms respectively for determining minimal generating sets for countable systems of semigroups and for any given semigroups, are given.

Keywords: minimal; maximal; minimum; maximum; generating set; semigroup; ordering; independence; countable systems

1 Introduction

The task of determining the minimal generating systems of algebraic structures can sometimes be made easier by utilizing the concept of independence. This paper is a continuation of studies towards characterizing independence and minimality of generators of semigroups as given in articles such as [7], [8] and [9]. As we shall see in the paper, both concepts ('Minimality' and 'Independence') are equivalent and so finding minimal generating set is as good as finding the independent set. The work is concluded with our focus on determining minimal generating sets of some semigroups by giving algorithms for obtaining this from any given generating set. We begin with some preliminary definitions. Definition of terms not defined but used in this paper can be found in [5], [8] and [9]. Some results in this paper strengthen more, the answers to some questions raised by (Sampson, 2023).

1.1 Definition (Semigroup)

Let S be a set and $*: S\times S\rightarrow S$ be a binary operation that maps each ordered pair (x, y) of S into an element of S. S is a Semigroup if $\forall\, x, y, z \in S$,

$$(x * y) * z = x * (y * z)$$

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1.2 Definition (Subsemigroup)

Let $\{S,*\}$ be an ordered pair, where S is a semigroup and $*$ is the binary operation in S. A subset $U \subset S$ with the same operation $*$ on it is called a subsemigroup of S.

1.3 Notation

We will write S instead of $\{S,*\}$ except where very necessary.

1.4 Definition (Finite Semigroup)

Let S be a Semigroup. S is a finite semigroup if it has only a finitely many elements.

1.5 Definition (Monoid)

Let $(S, *)$ be an ordered pair where $*$ is an associative binary operation. $(S, *)$ is called a monoid, if there exists an element $e \in S$ such that $e * x = x * e = x \in S$.

1.6 Definition (Group)

Given the ordered pair $(G,*)$, where G is a set and $*$ is a binary operation in G. $(G, *)$ is called a Group if

(i) $(x * y) * z = x * (y * z)\ \forall x, y, z \in G$.

(ii) $\exists e \in G$ such that $\forall x \in G, e * x = x * e = x$.

(iii) For each $x \in G, \exists\, x' \in G$, such that $x * x' = x' * x = e$, where e is the identity element.

1.7 Definition (Subgroup)

Let G, H be sets. Let $*$ form a group with the same operation $*: H \times H \rightarrow H$ on G. The subset $H \subset G$ is a subgroup of G.

1.8 Definition (Generating system of Semigroup)

Let S be a semigroup and let $H \subset S$ be a nonempty subset of S.
Given

$\langle H \rangle := \left\{\prod_{j=1}^{k} a_j \mid \forall \{a_j\}_{j=1}^{k} \subset H, 1 \leq k < \infty\right\}$,

where $\langle H \rangle$ is called the subsemigroup generated by H. H is a generating set of S if $S = \langle H \rangle$.

1.9 Definition (Generating Set)

Let $G \subset S$ be a subset of S. A nonempty subset $G \subset S$ is a generating set of S if $Cl(G) = S$. (by $\mathrm{Cl}(G)$ we mean the smallest closed set or the intersection of all the closed sets, containing G).

1.10 Example ([4])

Consider the infinite non abelian matrix semigroup $(M,*)$ with respect to the usual matrix multiplication $*$:

$$M = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a = \pm 1, b \in \mathbb{Z}\right\}$$

Notice that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$
$$\vdots$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b$$

So, for $X = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b \Rightarrow X = A^b \Rightarrow A$ is an element of the generating set. Notice also that $a = \pm 1$. We have considered the case for $a = 1$. For $a = -1$, we have

$\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ combine to generate

$Y = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$. That is; $Y = A^b * B$.

M is finitely generated by the generating set $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$, a $2-element$ subsemigroup.

1.11 Definition (Independent Element of a Semigroup)

An element $a\ in\ S$ is independent from a subset $U\ in\ S$ if $a \notin \langle U \rangle$.

1.12 Definition (Independent Sets of semigroup)

Let S be a semigroup. A subset $U\ of\ S$ is independent if, for every $u\ in\ U$, the element u does not belong to the subsemigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of U. Mathematically, we simply say if

$$\forall u \in U, u \notin \langle U \setminus \{u\} \rangle,$$

then U is independent.

1.13 Definition (Dependent Sets of Semigroup)

Let S be a finite semigroup. A subset $U\ of\ S$ is dependent if, for every $u\ in\ U$, the element u belongs to the subsemigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of U.

1.14 Remark

A subset U of a semigroup S is dependent if it is not independent.

1.1.16. **Definition** (Maximal Independent Sets of semigroup)

A Maximal Independent subset of a semigroup S is that subset $U \in S$ in which including any other element in the set makes it dependent.

1.15 Definition (Basis)

A subset $B \subset S$ is a basis in S if B is a minimal generating system in S;

1.16 Equivalences

One important tool for obtaining generators of algebraic systems is the use of relations that enable partitioning of the systems into smaller units. In the next section, we introduce these tools with definitions and some useful results.

1.17 Definitions (Relation on a set)

(i) Given the sets A and B, a relation from A to B is a subset $R \subseteq A \times B$. If $(a, b) \in R$, we say a is related to b by R. A is the domain of R. B is the codomain of R and the ordered pair (a, b) is a relation on R.

1.18 Definition (Binary Relation)

Let A be a set and let R be a relation $R \subseteq A \times A$. Then R is called a binary relation on the set A.

1.19 Definition (Composition of relation)

Let $\rho_1 \subset A \times B$ and let $\rho_2 \subset A \times B$ be relations. Then their composition $\rho_1 \circ \rho_2$ is defined by
$\rho_1 \circ \rho_2 = \{(x, y) | x \in A, \exists y \in B \ s.t. (x, y) \in \rho_1 \ and \ \exists z \in C \ s.t. (y, z) \in \rho_2\} \subset A \times C$.

1.20 Definition (Set of all Relations on a set)

Let $B(X)$ be the set of all relations on X. $B(X)$ is the set of all relations on the set defined by
$\rho \circ \sim = \{(x, z) | \exists y \in X, (x, y) \in \sim \ and \ \exists z \in X, \ (y, z) \in \rho\}$.

1.21 Definition (Identity Relation)

Let $B(X)$ be the set of all relations on X. The identity element of $B(X)$ is the identity relation $\iota = \iota_X = \{(x, x) \mid x \in X\}$. (It is the identity function on X).

1.22 Notations

If $(a, b) \in \rho$, then we write $a\rho b$. If $(a, b) \notin \rho$, then we write $a\rho' b$. For $A \times A$, we have that $(a, b) \in \rho \Leftrightarrow a\rho b$.

1.23 Definition (Equivalence Relation)

Let $B(X)$ be the set of all relations on X. A relation $\sim \in B(X)$ is an equivalence relation, if it is

(i) $\forall x \in X, \ x \sim x;$ (reflexive),

(ii) $\forall x, y \in X, \ x \sim y \Rightarrow y \sim x$ (symmetric)

(iii) $\forall x, y, z \in X, \ x \sim y \ and \ y \sim z \Rightarrow x \sim z$ (transitive) ,

1.24 Notations

The sets $x\sim$ are called equivalence classes. They form a partition of the set of union of all equivalences: X: $X = \bigcup_{x\in X} x\sim$, where $x\sim$ are equivalence classes
and (as we shall later see, if the intersection of two or more partitions are empty, it implies that they are equal)
$x\sim \cap\, y\sim \neq \emptyset \iff x\sim = y\sim$.

1.25 Definitions (Order Relation)

Let $\mathcal{A}$ be a set. A relation $\leq$ on $\mathcal{A}$ is an Order relation if
i) $\forall\, x \in \mathcal{A}, x \preccurlyeq x$ ($\preccurlyeq$ is reflexive),
ii) $\forall\, x, y, z \in \mathcal{A}, if\ x \preccurlyeq\ y\ and\ y \preccurlyeq\ z$, then $x \preccurlyeq z$ ($\leq$ is transitive),
iii) $\forall\, x, y \in \mathcal{A}\ , if\ x \preccurlyeq\ y\ and\ y \preccurlyeq\ x,$ then $x = y$, ($\preccurlyeq$ is anti-symmetric).

1.26 Definition (Partially Ordered Set)

The pair $(X, \preccurlyeq)$ is called a Partially Ordered Set (or, for short, Poset) if the relation $\preccurlyeq$ is an order relation and there exist $a, b \in X$ such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a.$

1.27 Definition (Total Ordering)

Let A be a set and then the order relation $\preccurlyeq \subseteq A \times A$ on A is a Total Ordering if the relation $\preccurlyeq$ is an order relation and $\forall a, b \in \mathcal{A},\ a \preccurlyeq b$ or $b \preccurlyeq a$ (that is; any two elements of the totally ordered set are comparable).

1.28 Remarks

A totally ordered set is also called a Chain. A totally ordered set is also called a Linearly ordered Set. Comparability of elements is not a necessary requirement for Partially Ordered Set.

1.29 Definition (Well Ordering)

Let $\mathcal{A}$ be a set and $\preccurlyeq$ a relation on $\mathcal{A}$. The relation $\preccurlyeq$ is a well ordering of $\mathcal{A}$ if every non-empty subset S of $\mathcal{A}$ has a least element.

1.30 Remark

The theorem that follows is a significant requirement for one of the main results in the fourth chapter of this thesis.

Theorem 1: If $\preccurlyeq$ is a total ordering on $\mathcal{A}$, then every non-empty finite subset S of $\mathcal{A}$ has a least element and a greatest element.

Proof. We show there is a least element and leave the rest to an exercise. By induction. Let $P(n)$ be the statement "every subset of $\mathcal{A}$ having n elements has a least element." Clearly $P(1)$ is true. Suppose $P(n)$ is true. If $S\ has\ n + 1$ elements, let $S = S' \cup \{y\}$, for some $y \in S$ and S' with n elements (namely, $S' = S \setminus \{y\}$, or S with y removed). By induction, S' has a least element, call it x.

Since $\preccurlyeq$ is a total ordering, either $x \preccurlyeq y$ or $y \preccurlyeq x$. In the first case, x is a least element of S. On the other hand, if $y \preccurlyeq r$ case, $y \preccurlyeq z$, as desired.

1.31 Remark

Any countable set can be well-ordered (this fact is equivalent to the popular axiom of choice). In fact, by Zermelo–Fraenkel Axiom of Choice (ZFC), this is true for any set and not just for countable set.

Theorem 2:

If $\preccurlyeq$ is a well ordering of $\mathcal{A}$ then it is a total ordering.

Proof. Suppose $x, y \in \mathcal{A}$. Let $S = \{x, y\} \neq \emptyset$. By hypothesis, S has a smallest element. If it is x, then $x \preccurlyeq y$, and if it is y, then $y \preccurlyeq x$.

1.32 Remark

A better appreciation of the major tasks of this paper will require a clearer understanding of the concept of minimal and maximal subsets as compared to that of minimum and maximum subsets of a semigroup. Scholars sometimes confuse one for the other. We will therefore begin with some explanations to clear confusions in usage. We will also mention some of the implications to the work.

2 Maximum/maximal and minimum/minimal

A vector space is ordered if there is an order relation which is additively translation invariant and multiplicatively "translation" invariant for non-negative factors:

Let V be a vector space over R (may be an ordered field!) and let $\preccurlyeq \subseteq V \times V$ be an order relation (see section 1.1.27.). It is compatible with the vector structure if the following hold:

(a) $\forall a, b, c \in V,\ a \preccurlyeq b \Longrightarrow a + c \preccurlyeq b + c$;

(b) $\forall a, b \in V and\ 0 \leq c \in R, a \preccurlyeq b \Longrightarrow a \times c \preccurlyeq b \times c$.

Theorem 3:

A binary relation $\preccurlyeq \subseteq V \times V$ is an order relation compatible with the vector structure if and only if $C_{\preccurlyeq,+} = \{x \mid 0 \preccurlyeq x \in V\}$ is a convex cone fulfilling the relation $C_{\preccurlyeq,+} \cap -C_{\preccurlyeq,+} = \{0\}$ and such that $\forall a, b \in V,\ a \preccurlyeq b \Leftrightarrow b - a \in C_{\preccurlyeq,+}$.

2.1 Corollaries

(a) If V is one dimensional, then the cone $C_{\preccurlyeq,+}$ is a half straight line $[0, \infty)$ hence

$$R = C_{\preccurlyeq,+} \cup (-C_{\preccurlyeq,+}) = [0, \infty) \cup (-[0, \infty))$$

and the ordering is complete ordering hence or otherwise $\forall a, b \in V$ either $a \preccurlyeq b$ or $b \preccurlyeq a$ holds. Since any two elements of V are comparable, the largest element $max(A) =: M_A \in A$ of a set $A \subset V$ is greater than or equal to all the elements of the set A, and the smallest element $min(A) =: m_A \in A$ is less than or equal to any element of the set A.

These are called maximum and minimum respectively, and they are comparable with each other and any element of the set.

A set may have one maximum and one minimum. If $a \in A$ is not maximum then there exists at least one $b \in A$ such that $a \preccurlyeq b$ and $a \neq b$. Similar for minimum, If $a \in A$ is not minimum then there exists at least one $b \in A$ such that $b \preccurlyeq a$ and $a \neq b$. Any two maximums are equal. The same holds for minimum.

(b) If the dimension of the vector space is two or more, then
$C_{\preccurlyeq,+} \cup (-C_{\preccurlyeq,+}) \neq V$ hence $V \backslash \left(C_{\preccurlyeq,+} \cup -C_{\preccurlyeq,+}\right) \neq \emptyset$.
This means that there are non-comparable pairs, the ordering is not complete.
This brings radical changes: Definitions of maximum and minimum change from that of a vector space with complete ordering (one dimension).

2.2 Definition (Minimum and Maximum)

$M_A \in A \subset V$ is a maximum of A if $x \preccurlyeq M_A,\ \forall x \in A.$ $m_A \in A \subset V$ is a minimum of A if $m_a \preccurlyeq x,\ \forall x \in A$.

2.3 Definition (Minimal and Maximal)

$M_A \in A \subset V$ is a maximal element of A if $\nexists x \in A$ such that $x \neq M_A \preccurlyeq x$.
$m_A \in A \subset V$ is a minimal element of A if $\nexists x \in A$ such that $x \preccurlyeq m_A \neq x$.

2.4 Remarks

Obviously, from 2.1(a) and 1.1(b) the concept of minimum and maximum cannot be upheld when V is not completely ordered. This necessitates the change to Minimal and Maximal as given in section 2.3.

One subtlety in the definition (of 2.1.3) is that maximal elements are not necessarily above everything in the partial order, but rather nothing in the partial order is above them. Compare this definition with "maximum": an element is called maximum in A if it is larger than all the other elements while an element $M_A \in A$ is called a maximal element of A, if there is no larger (comparable) element in A. Similarly, minimal elements are not necessarily lower than everything in the partial order, but rather nothing in the partial order is below them. A partial order can have many maximal elements as well as many minimal elements.

In other words, these definitions uphold the motive that an element $M_A \in A$ is a maximal element of A, if there is no larger (comparable) element in A. An element $m_A \in A$ is minimal element of A if there is no smaller element (comparable) in A.
To both minimal and maximal elements there may exists an element in A such that the maximal is not greater than or equal to it and the minimal is not less than or equal to it (when they are not comparable). Maximal elements of a set A may not be equal.

2.5 Example

Let B be an infinite set, let$A := \{U \mid \emptyset \neq U \subset B \neq U\}$, and let the order relation be $\preccurlyeq V \Leftrightarrow U \subset V$. All singleton members of A are minimal, not any two of them is comparable and all of them are different. $\forall x \in B$ the set $W = B \setminus \{x\}$ is maximal, and any two of them are non-comparable and different.

2.6 Remark

The worlds of maximum/minimum and maximal/minimal are completely different. Their existence is mostly compactness for maximum/minimum, Zorn's lemma stands for the existence of maximal/minimal. elements of a set.

2.7 Minimal generating sets Implies independence

Let us use the order relation described in the above example for subsets of a set.
Let A be a set, then the order relation is $U \preccurlyeq V \Leftrightarrow U \subset V,\ \forall U, V \subset A$.
Using the definition 2.1.3. for minimal elements.

2.8 Definition (A Minimal generating set)

A generating set G of a semigroup is minimal if $\forall H \preccurlyeq G,\ H \neq G, H$ is not a generating set.

Theorem 4 (Minimality of a Generating Set Implies the Independence of the set):
A generating set G of a structure is minimal if G is independent (for any structure like vector spaces, groups, semi-groups, monoids etc.).

Proof:
A generating set G is minimal$\Rightarrow$ it is independent: Let $x \in G$ be any element. If $x \in \langle G \setminus \{x\}\rangle$ then
$$(G \setminus \{x\}) \cup \{x\}) \subset \langle G \setminus \{x\}\rangle \Longrightarrow S = \langle G \rangle = \langle (G \setminus \{x\}) \cup \{x\}) \rangle \subset \langle \langle G \setminus \{x\} \rangle \rangle$$
which contradicts with the minimality of G.

2.9 Example (minimal generating set implies independence: a case study using a group)

We adapt an example in (page 4 of [4]) to have the following.
Let $\{p_i\}_{i=1}^{m}$, $n = \prod_{i=1}^{m} p_i$ and $\{Z, +\}$ be a group. Let
$$G_m := \left\{\frac{n}{p_i} \middle| 1 \leq i \leq m\right\}$$
Let $\frac{n}{p_i} \in G_m$ be selected:
$$\frac{n}{p_i} = p_1 \times p_2 \times \ldots . \times p_{i-1} \times p_{i+1} \times \cdots \times p_m.$$
Since p_i is not in the product, p_i is not its factor.
Consider $G_m \setminus \left\{\frac{n}{p_i}\right\}$. Then $\forall z \in G_m \setminus \left\{\frac{n}{p_i}\right\},\ p_i/z \Longrightarrow \forall u \in \langle G_m \setminus \left\{\frac{n}{p_i}\right\} \rangle,\ p_i | u$ while $p_i \nmid \frac{n}{p_i}$
In summary, $\frac{n}{p_i} \notin \langle G_m \setminus \left\{\frac{n}{p_i}\right\} \rangle$ meaning that $\left\{\frac{n}{p_i}\right\}$ is independent from $G_m \setminus \left\{\frac{n}{p_i}\right\}$.

Theorem 5 (Independence of a set implies minimal generating set):
A generating set G is independent $\Longrightarrow G$ is minimal.
Proof: Let $x \in G$ be any element. Then $x \notin \langle G \setminus \{x\} \rangle$. Hence $\langle G \setminus \{x\} \rangle$ is not a generating set. Let $\emptyset \neq H \subset G \neq H$. Then there is at least one element $x \in G \setminus H \Longrightarrow G \setminus \{x\} \supset H$. Since $G \setminus \{x\}$ is not a generating set, its subset H is also not a generating set.

Theorem 6:
A generating set G is independent if and only if G is minimal
Proof. The proof follows directly as implication of 2.5 and 2.9

2.11 Remarks

(i) The above theorem is actually true in all structures: vector spaces, groups, semigroups etc. Maximum number of independent elements is influenced by the constructing powers of operations.
(ii) In vector spaces given a basis, we can consider the linear subspace generated by a basis element. All members of this space can be obtained with multiplication by scalar which is a member of a field. Hence knowing one element, multiplication by non-zero factor gives access to any other non-zero member of the subspace by the multiplication being invertible.
(iii) Groups: Given a minimal generating set. Take any member of this set, it generates a subgroup. Using the inverse one can generate any member of this subgroup hence the constructive power does not allow the generation of further independent elements.
(iv) Semigroups: Take a one element generated cyclic group. If you consider the prime powers of the one element generator, these will be independent in an infinite cyclic group since there is multiplication only, no other way to generate any further semigroup element.
(v) From this follows that maximal independent system is not a good idea to minimal generating set: One element generated subsemigroup may have an infinite system of independents members.

2.12 Further comments on minimal generating sets

In free semigroups a minimal generating set is the generator members of the free semigroup. Its cardinality gives the cardinality of the minimal generating set.

3 Algorithms: determining minimal generating set for any system of semigroup

For illustration purpose, in Figure 3.1 below, the sets B and C are required to be subsets of a well-ordered index set $\mathcal{A}$, so we can find some least elements of the sets with which the minimal generating set can be obtained.

3.1 Remarks (implications of the nature of ordering on determination of generating systems)

The concept of "Well ordering" enables us to find generating sets in any semigroup and not just in countable systems like $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$ in general. For countable systems, we do not necessarily require the Well ordering. A direct implication of the Well Ordering Principle (Bukh, 2019; Ivan, 2018) for well ordered sets requires that there exists a least element. We shall see that for every generating set G, there is a minimal generating set that is comparable with G.

3.2 Algorithms: determining minimal generating set for any system of semigroup

For illustration purpose, in Figure 3.1 below, the sets B and C are required to be subsets of a well ordered index set $\mathcal{A}$, so we can find some least elements of the sets with which the minimal generating set can be obtained.

Let $\mathcal{A}$ be a well ordered index set. Let $B, C \subset \mathcal{A}$. If $G = \{e_t\}_{t \subset B \subset \mathcal{A}}$ is a semigroup generating set, then the flow chart below gives an algorithm to create the minimal generating set:

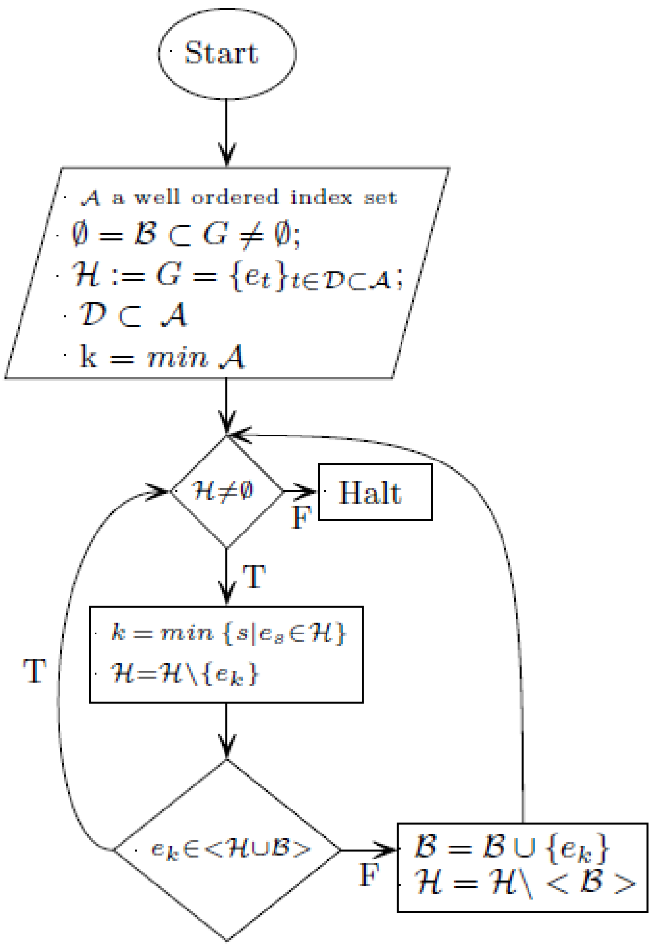

Figure 3.1. Algorithm to generate a minimal (independent) generating set B from a generating set G.

Theorem 7:
Let S be a semigroup and let $G \subset S$ be a generating set. Then the algorithm in Figure 3.1 selects a minimal generating set B from G.

Proof:

1. B is initialized to be empty set, $H \subset G$ since it is initialized by G and each step we remove some elements from it and add some of them to B. Hence $B \subset G$ in each step.

2. An $e_k \subset H$ is added to B only when $e_k \notin \langle H \cup B \rangle$ hence e_k is independent from all selected basis elements and those in H which will be selected into the final B. Hence the set B at the final step when $H = \emptyset$ is a set of independent elements.

3. In each step $\langle H \cup B \rangle \subset S$ is a generating set and when $H = \emptyset$; then B is a generating set.

3.3 Algorithm determining minimal generating set for countable system

In figure 3.2 below, the set $\mathcal{A}$ is the set of natural numbers and is well ordered, so we can determine a minimal generator of the set (as implied by the theorem of section 1.1.26).

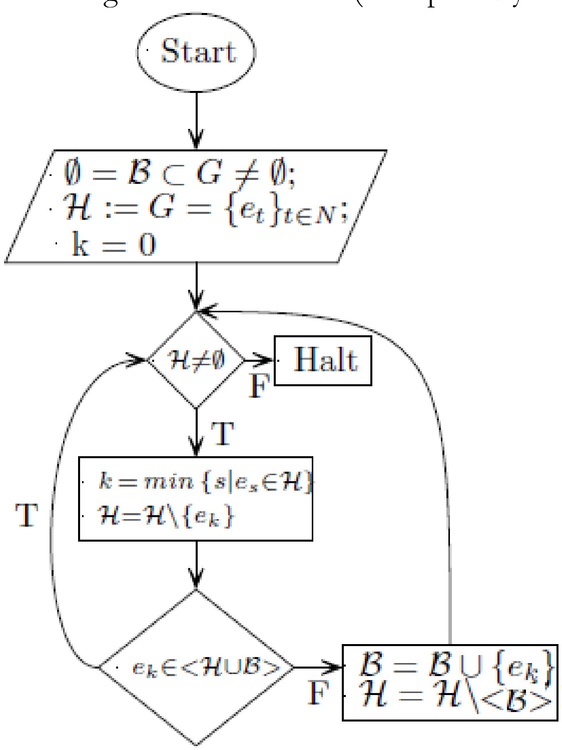

Figure 3.2. Algorithm to generate a minimal (independent) generating set B from a generating set G – Countable Systems.

Corollary

Let S be a semigroup and let $G \subset S$ be a generating set and G is a countable system. Then the algorithm in Figure 3.2 selects a countable minimal generating subset $B \subset G$.

Proof:

1. B is initialized to be empty set, $H \subset G$ since it is initialized by G and each step we remove some elements from it and add some of them to B. Hence $B \subset G$ in each step.

2. An $e_k \subset H$ is added to B only when $e_k \notin \langle H \cup B \rangle$ hence e_k is independent from all selected basis elements and those in H which will be selected into the final B. Hence the set B at the final step when $H = \emptyset$ is a set of independent elements.

3. In each step $\langle H \cup B \rangle \subset S$ is a generating set and when $H = \emptyset$ then B is a generating set.

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