Journal of the Nigerian Society for Mathematical Biology
Volume 2 (2019), pp. 1 - 13

Theoretical analysis of a mathematical model for the dynamics of corruption: a guide from epidemiological modelling

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Abstract

A new deterministic model for the spread of corruption is designed and used to qualitatively assess the role of convictions and loss of 'immunity' to corruption on the transmission process of corruption in a society. It is shown that loss of 'immunity' to corruption and the rate at which previously convicted and jailed corrupt individuals revert to being susceptible to corruption could induce the phenomenon of backward bifurcation when the associated corruption reproduction number is less than unity. This will make eradicating corruption from the community difficult. However, when the cause of the backward bifurcation phenomena is removed, it is shown that the corruption-free equilibrium of the model is globally-asymptotically stable whenever the corruption reproduction number is less than unity. We further show that there exists a unique corruption endemic equilibrium which is shown to be locally-asymptotically stable when the the associated corruption reproduction number is greater than unity. The qualitative analysis suggest that effort should be geared towards providing policies that could prevent previously convicted (and jailed) individuals from becoming susceptible to corruption after their release from prison. Furthermore, individuals who become 'self recovered' from corruption due to some intervention programmes are to be assisted to remain resistant to corruption, if corruption is to be stamped out of the target population.

Keywords: mathematical model; backward bifurcation; global stability; convictions;

1 Introduction

Corruption is the abuse of public office or entrusted power for private gain [7, 14]. It can be distinguished in various forms, such as political and bureaucratic corruption [7]. Corruption has been blamed for the failures of some developing countries and, in fact, attributed to their 'developing' status [14]. Studies such as the ones found in [10, 18] confirm the link between higher perceived corruption and lower investment and growth [14].

Several reasons have been propounded for the causes of corruption in any corrupt society. As stated in [14], political scientists and economists have suggested a variety of characteristics of countries' economic, political, and social systems that might affect expected costs, benefits, or both. Also empirical evidence suggests that corruption tends to become prevalent in countries undergoing political and economic transition [7]. Transparency International (TI) have, over the years, prepared the TI index to assess the level of corruption in the countries of the world, with a mission to stop corruption and promote transparency, accountability and integrity at all levels and across all sectors of society [19].

The major cost of corruption is the risk of getting caught and punished, convictions and jailing. However, the probability of getting caught depends strongly on the effectiveness of the

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countries' legal and security system [14]. According to [8], poverty, slow economic growth, and unequal wealth and income distribution are prevalent in most African countries. In fact, the continent that has made the least progress in improving on the living standards of its populace is Africa, with widespread corruption seen as major cause of poor standard of living therein [8].

One may wish to see the detailed articles on corruption, its causes and effects as well as efforts to fight corruption in [7, 14].

With the existence of a large body of literature on the political, social and economic aspects of corruption and attempts to provide policies to combat corruption [6], several attempts have been made to use mathematical models to study the dynamics of corruption in a population. The first model formulated to understand the dynamics of corrupt structures was by Rose-Ackerman [13]. Waykar [16, 17] used simple growth equations (a first order differential equation) to study the growth of corruption in a population; consideration was given to corruption in educational institutes and government offices in India as case studies. The work by Hathroubi [6] studied corruption as a disease epidemic process and used the Kermack and McKendrick (1927) classical SIR epidemiological mathematical model for modelling the spread of corruption; he considered the population as made up of individuals susceptible to corruption (S), corrupt individuals (I)and individuals who recover and are immune to corruption (R), where the notations for the state variables were adjusted differently in Hathroubi [6]. Rastogi and Mathur [12] examined the impact of chaos theory in social systems afflicted with corruption

This study presents a new deterministic model that studies the population dynamics of corruption in a population. The model treats corruption as a disease (as was done by Hathroubi [6]); hence the model could fit well for an epidemiological setting. Instead of having just the three classes used by Hathroubi [6], we propose dividing the population into five classes, which includes individuals who are convicted and jailed for corruption and individuals who already are resistant to corruption. Also our model allows for changes in human behaviour where someone who was previously immune to corruption becomes susceptible to corruption again.

The paper is organized as follows. The model is formulated in Section 2 and analysed in Sections 3 and 4. Section 5 provides some discussions and conclusions.

2 Model Formulation

The total population at time t, denoted by $N(t)$, is subdivided into the mutually-exclusive compartments of susceptible individuals to corruption ($S(t)$); corrupt individuals ($C(t)$); individuals who 'recover' from corruption ($R(t)$); corrupt individuals who are convicted and jailed ($V(t)$); and individuals who are resistant to corruption ($T(t)$), so that

$$N(t) = S(t) + C(t) + R(t) + V(t) + T(t).$$

Corrupt interactions in the population is modelled using a standard incidence function.

The population of individuals who are susceptible to corruption ($S(t)$) is increased by the recruitment of a proportion p $(0 \leq p \leq 1)$ of immigrants (who are susceptible) into the population (at a rate Λ), 'recovered' individuals who become susceptible to corruption again (at a rate π) and convicted individuals who, after their release, become susceptible to corruption again (at the rate ξ). The susceptible population is decreased by corruption, at a rate λ (the corruption force of 'infection'), where

$$\lambda = \frac{\beta C}{N - V},$$

and β is the corruption transmission rate. The susceptible population is further decreased by natural death (at a rate μ; natural removal occurs in all compartments at this rate), so that

$$\frac{dS}{dt} = p\Lambda - \lambda S - \mu S + \pi R + \xi V.$$

The population of corrupt individuals ($C(t)$) is increased by the corrupt interaction that affects and subsequently 'infects' susceptible individuals (at the rate λ). This population is decreased by trial, conviction and subsequent jailing, which leads to removal from the society (at the rate γ), 'recovery' (at the rate α) and natural death. This gives

$$\frac{dC}{dt} = \lambda S - (\gamma + \alpha + \mu)C.$$

The population of corrupt individuals who were not caught and convicted but 'recover', following counselling and public enlightenment programmes on the need to shun corruption (recall that we are dealing with an imperfect society where not all persons who are corrupt faces convictions) ($R(t)$) is increased by corrupt individuals who abandon a corrupt lifestyle (at a rate α). This population is decreased by developing resistance to corruption (at a rate ϕ), becoming susceptible to corruption again after some time (at a rate π) and natural death, so that we have

$$\frac{dR}{dt} = \alpha C - (\phi + \pi + \mu)R.$$

The population of individuals who are resistant to corruption ($T(t)$) is increased by a proportion $(1-p)$ of immigrants coming into the population who are resistant to corruption (at the rate Λ) and 'recovered' and previously convicted individuals who have developed resistance to corruption (at the rates ϕ and ψ, respectively). This population is decreased only by natural death. Hence we have

$$\frac{dT}{dt} = (1-p)\Lambda + \phi R + \psi V - \mu T.$$

The population of convicted individuals (in a form of isolation and with no interaction with the outside world) is increased by corrupt individuals who are caught and convicted (at the rate γ) and decreased by such individuals who later become susceptible to corruption after their release, considering the period spent in jail (at the rate ξ), convicted individuals who, after serving their time, become resistant to corruption (at the rate ψ) and natural death. This gives

$$\frac{dV}{dt} = \gamma C - (\xi + \psi + \mu)V.$$

Based on the above formulations and assumptions, the model for the spread of corruption in a population consists of the following system of nonlinear (deterministic) differential equations (Table 1 describes the associated state variables and parameters in the model).

$$\begin{aligned}
\frac{dS}{dt} &= p\Lambda - \lambda S - \mu S + \pi R + \xi V,\\
\frac{dC}{dt} &= \lambda S - (\gamma + \alpha + \mu)C,\\
\frac{dR}{dt} &= \alpha C - (\phi + \pi + \mu)R,\\
\frac{dT}{dt} &= (1-p)\Lambda + \phi R + \psi V - \mu T,\\
\frac{dV}{dt} &= \gamma C - (\xi + \psi + \mu)V,
\end{aligned} \tag{1}$$

with initial conditions

$$S(0) = S_0, C(0) = C_0, R(0) = R_0, T(0) = T_0, V(0) = V_0.$$

Some of the main assumptions and features of the model are itemized below.

i We assume that we are dealing with an imperfect society whereby not all corrupt individuals are caught and convicted. Hence we make room for corrupt individuals who, likely due to counselling and enlightenment campaigns, abandon their corrupt lifestyle and 'recover' from corrupt tendencies;

ii Incorporating a class of convicted individuals who are jailed and in complete isolation from the outside.

Variable	Interpretation
S	Population of individuals susceptible to corruption
C	Population of corrupt individuals
R	Population of individuals who 'recover' from corruption
T	Population of individuals who are resistant to corruption
V	Population of convicted individuals who are jailed (isolated)
Parameter	**Interpretation**
μ	Natural mortality rate
Λ	Recruitment rate
β	Corruption transmission rate
p	Fraction of immigrants who are susceptible to corruption
α	corruption 'recovery' rate
γ	Conviction rate
ϕ	Rate at which 'recovered' individuals become resistant to corruption
π	Rate at which the 'recovered' individuals become susceptible to corruption
ψ	Rate at which the convicted individuals become resistant to corruption after they leave jail
ξ	Rate at which the convicted individuals become susceptible to corruption after their release from jail

Table 1: Description of variables and parameters in the corruption model (1).

2.1 Basic Properties

For the model (1) to be meaningful, it is important to prove that all its state variables are non-negative for all time (t). In other words, the solutions of the model (1) with positive initial data will remain positive for all $t \geq 0$.

2.1.1 Positivity of solutions and well-posedness of model

Since the model (1) monitors human population, all the state variables and parameters of the model are non-negative. Consider the feasible region

$$\mathcal{D} = \{S, C, R, T, V \in \mathbb{R}^5_+ : 0 \leq N \leq \Lambda/\mu\}$$

It can be shown that the set $\mathcal{D}$ is a positively invariant set and a global attractor of this system. That is, any phase trajectory initiated anywhere in the non-negative region $\mathbb{R}^5_+$ of the phase space eventually enters the region $\mathcal{D}$ and remains in $\mathcal{D}$ thereafter.

Let $S_0, C_0, R_0, T_0, V_0 \geq 0$, $S_0 + C_0 + R_0 + T_0 + V_0 = N_0$. Then there exists solutions $S(t), C(t), R(t), T(t), V(t)$ for the dynamical system (1), with initial data S_0, C_0, R_0, T_0, V_0 at time $t = 0$, that are defined for all time $t \geq 0$. In fact, $S(t), C(t), R(t), T(t), V(t)$ are nonnegative and $S(t) + C(t) + R(t) + T(t) + V(t) = N(t)$ for all t. If $C_0 = 0, R_0 = 0, V_0 = 0$, then $C(t) \equiv 0$ and $V(t) \equiv 0$.

Since $N = S + C + R + T + V$, it follows that the rate of change of the total population is given by

$$\frac{dN}{dt} = \Lambda - \mu N. \tag{2}$$

and, therefore, that $N(t) \to \Lambda/\mu$ as $t \to \infty$. That is the total population is asymptotically constant. The well-posedness of the model follows a straight forward application of the classical theory [11].

Lemma 2.1 *The region D is positively-invariant for the model* (1) *and, if $C_0 > 0$, then $S(t) > 0$, $R(0) > 0$,$T(0) > 0$ and $C(0) > 0$ for all $t > 0$ and V is bounded by $\hat{V} = max\{V_0, \frac{\gamma}{\mu+\xi+\psi}\}$.*

Proof. The right hand side of system (1) is continuously differentiable and hence it is locally Lipschitz, and therefore there exists a unique solution $S(t), C(t), R(t), T(t), V(t)$ to system (1) with the initial data S_0, C_0, R_0, T_0, V_0 that is defined on a maximal forward interval of existence [11]. Consider the set $\mathcal{D} \subset \mathbb{R}^5_+$, we have that since $C(0) \geq 0$ from system (1) we have that

$$\begin{aligned}
C(t) &\geq C_0 \text{exp} \int_0^{\infty} \big[\frac{\beta S}{N - V} - K_1\big] dt \\
R(t) &= \big[R_0 + \int_0^t \alpha C(\zeta) e^{K_2(\zeta - t_0)} d\zeta\big] e^{K_2(t_0 - t)} \\
V(t) &= \big[V_0 + \int_0^t \gamma C(\zeta) e^{K_3(\zeta - t_0)} d\zeta\big] e^{K_3(t_0 - t)} \\
T(t) &= \big[T_0 + (1-p)\Lambda t + \int_0^t (\phi R(\zeta) + \psi V(\zeta)) e^{\mu(\zeta - t_0)} d\zeta\big] e^{\mu(t_0 - t)}
\end{aligned}$$

so that $S(t) \geq 0$, $C(t) \geq 0$, $R(t) \geq 0$, $T(t) \geq 0$, $V(t) \geq 0$, $\forall \quad t > 0$, where $K_1 = \alpha + \gamma + \mu$, $K_2 = \phi + \pi + \mu$ and $K_3 = \xi + \psi + \mu$.

To show that V is bounded by $\hat{V} = \max\{V_0, \frac{\gamma}{\mu+\xi+\psi}\}$, all we need do is to show whether $V(t) \leq \omega$, $\forall\ t \geq 0$ and that $\omega \geq \frac{\gamma}{K_3}$ if $V_0 \leq \omega$. Suppose that the above inequalities do not hold, then there exists a time, t_1, with $V'(t_1) > 0$ and $V'(t) > \omega$. Using the equation for V in (1), we have

$$\frac{dV(t_1)}{dt} = \gamma C(t_1) - K_3 V(t_1) \leq \gamma(C(t_1) - 1) \leq 0,$$

since $(C(t_1) - 1) < 0$ and $\gamma > 0$. This is a contradiction which implies that $V(t) < \omega, \forall\, t \geq 0$. Suppose now that $V(0) > \omega \geq \frac{\gamma}{K_3}$. In order to show that $V(t) \leq V(0)\ \forall\, t \geq 0$, we assume that the last inequality does not hold. Hence there exists a time $t_2 > 0$ such that $V(t_2) \geq V(0)$ and $V'(t_2) > 0$.

However since $V(t_2) > \frac{\gamma}{K_3}$, then

$$\frac{dV(t_2)}{dt} = \gamma C(t_2) - K_3 V(t_2) \leq \gamma(C(t_2) - 1) \leq 0,$$

but $V'(t_2) > 0$. Hence we reach a contradiction and $V(t)$ is bounded from above by $\hat{V}$, where $\hat{V} = \max\{V_0, \frac{\gamma}{K_3}\}$. □

Lemma 2.1 guarantees the positivity of the different groups of individuals making up the total population, S, C, R, T, and V, for all time $t \geq 0$.

3 Asymptotic Stability of the Corruption-Free Equilibrium (CFE)

The corruption-free equilibrium of the model (1) is given by

$$\mathcal{E}_0 = (S^*, C^*, R^*, V^*, T) = \left(\frac{p\Lambda}{\mu}, 0, 0, 0, \frac{(1-p)\Lambda}{\mu}\right).$$

The linear stability of $\mathcal{E}_0$ can be established using the next generation operator method on the system (1) [4, 15]. Using the notations in [15], it follows that F and V, which stands for the new corruption infection terms and remaining transition terms, respectively, are given by $F = p\beta$ and $V = \alpha + \mu + \gamma$.

It follows that the effective corruption reproduction number of the model (1), denoted by $\mathcal{R}_0$, is given by

$$\mathcal{R}_0 = FV^{-1} = \frac{p\beta}{\alpha + \mu + \gamma}.$$

The result below follows from Theorem 2 in [15].

Lemma 3.1 *The CFE (ξ_0) of the model* (1) *is locally-asymptotically stable (LAS) if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

The threshold quantity, $\mathcal{R}_0$, is the effective corruption reproduction number. It represents the average number of secondary cases of corruption generated by a typical corrupt individual in a population that is completely susceptible to corruption (using the typical language in epidemiology [15]). The implication of Lemma 3.1 is that when $\mathcal{R}_0$ is less than unity (i.e. $\mathcal{R}_0 < 1$), corruption can be eliminated from the population if the initial sizes of the sub-population of the model are in the basin of attraction of the corruption-free equilibrium ($\mathcal{E}_0$). Hence, a small influx of corruption-infected individuals into the community will not generate large 'outbreaks' of corruption, and corruption dies out in time.

3.1 Backward Bifurcation Analysis

It is instructive to characterize the type of bifurcation the model (1) may undergo. This will go a long way in determining factors that could hinder efforts in tackling corruption in the population. We claim the following result.

Theorem 3.1 *The model* (1) *exhibits backward bifurcation at $\mathcal{R}_0 = 1$ whenever a bifurcation coefficient, denoted by a (and given below) is positive.*

Proof. Let

$$\mathcal{E}_1 = (S^{**}, C^{**}, R^{**}, T^{**}, V^{**})$$

represents any arbitrary endemic equilibrium of the model(1) (that is, an equilibrium in which at least one of the infected components is non-zero). The existence of backward bifurcation will be explored using the Center Manifold theory [3, 1]. To apply this theory, it is convenient to carry out the following change of variables. Let $S = x_1$, $C = x_2$, $R = x_3$, $T = x_4$, $V = x_5$. Further, by using the vector notation $X = (x_1, x_2, ..., x_5)^T$, the model (1) can be written in the form $\frac{dX}{dt} = F(X)$, with $F = (f_1, f_2, ..., f_5)^T$, as follows

$$\begin{aligned}
\dot{x_1} &\equiv f_1 = p\Lambda - \lambda x_1 - \mu x_1 + \pi x_3 + \xi x_5,\\
\dot{x_2} &\equiv f_2 = \lambda x_1 - K_1 x_2,\\
\dot{x_3} &\equiv f_3 = \alpha x_2 - K_2 x_3,\\
\dot{x_4} &\equiv f_4 = \phi x_3 + (1-p)\Lambda - \mu x_4 + \psi x_5,\\
\dot{x_5} &\equiv f_5 = \gamma x_3 - K_3 x_5,
\end{aligned} \tag{3}$$

where $K_1 = \alpha + \mu + \gamma$, $K_2 = \phi + \mu + \pi$ and $K_3 = \mu + \xi + \psi$ and the force of infection given by

$$\lambda = \frac{\beta x_2}{N - x_5} \quad \text{and} \quad N = x_1 + x_2 + x_3 + x_4 + x_5.$$

Let us choose $\beta = \beta^*$ as a bifurcation parameter. Solving for $\beta = \beta^*$ from $\mathcal{R}_0 = 1$ gives

$$\beta = \beta^* = \frac{K_1}{p}.$$

The Jacobian of the transformed system (3), evaluated at the corruption-free equilibrium ($\mathcal{E}_0$) with $\beta = \beta^*$, is given by

$$J^* = J(\mathcal{E}_0)|_{\beta=\beta^*} = \begin{pmatrix} -\mu & -p\beta^* & \pi & 0 & \xi \\ 0 & p\beta^* - K_1 & 0 & 0 & 0 \\ 0 & \alpha & -K_2 & 0 & 0 \\ 0 & 0 & \phi & -\mu & \psi \\ 0 & \gamma & 0 & 0 & -K_3 \end{pmatrix}$$

The matrix J^* has a right eigenvector given by $\mathbf{w} = (\omega_1, \omega_2, ..., \omega_5)^T$, where

$$\begin{aligned}
\omega_1 &= \frac{-p\beta^*\omega_2 + \pi\omega_3 + \xi\omega_5}{\mu},\\
\omega_2 &= \omega_2 > 0,\\
\omega_3 &= \frac{\alpha}{K_2}\omega_2 > 0,\\
\omega_4 &= \frac{\phi\alpha}{\mu K_2} + \frac{\psi\gamma}{\mu K_3}\omega_2 > 0,\\
\omega_5 &= \frac{\gamma}{K_3}\omega_2 > 0.
\end{aligned} \tag{4}$$

Furthermore, J^* has a left eigenvector $v = (\nu_1, \nu_2, ..., \nu_5)$ satisfying $\mathbf{v.w = 1}$, with

$$\nu_1 = \nu_3 = \nu_4 = \nu_5 = 0 \quad \text{and} \quad \nu_2 = \nu_2 > 0. \tag{5}$$

It follows from Theorem 4.1 in [1], by computing the associated non-zero partial derivatives of F(x) (evaluated at the disease-free equilibrium, $\mathcal{E}_0$), that the associated bifurcation coefficients, a and b, defined by

$$a = \sum_{k,i,j=1}^{5} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0),$$

and

$$b = \sum_{k,i=1}^{5} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta^*}(0,0),$$

are computed to be

$$a = \frac{2\beta^* \mu \nu_2 \omega_2^2}{\Lambda} \left[\frac{1-p}{\mu} \left(\frac{\pi \alpha K_3 + \xi \gamma K_2}{K_2 K_3} \right) - \left(\frac{(1-p)K_1}{\mu} + \frac{\alpha}{K_2} + \frac{\phi \alpha}{\mu K_2} + \frac{\psi \gamma}{\mu K_3} + 1 \right) \right], \tag{6}$$

and

$$b = \nu_2 \omega_2 p > 0. \tag{7}$$

Since the bifurcation coefficient b is positive, it follows from Theorem 4.1 in [1] that the model (1), or the transformed model (3), will undergo a backward bifurcation if the backward bifurcation coefficient a, given by (6), is positive. □

The phenomenon of backward bifurcation, which has been observed in numerous disease transmission dynamics (for example see [1, 5]), is typically characterized by, in this case, the co-existence of a stable corruption-free equilibrium and a stable corruption endemic equilibrium when the associated corruption reproduction number of the model is less than unity. The societal implication of the backward bifurcation phenomenon of the model (1) is that the classical requirement of having the corruption reproduction number ($\mathcal{R}_0$) to be less than unity, while necessary, is no longer sufficient for the control (eradication) of corruption in the population. It is worth stating that, setting $\pi = \xi = 0$ in the expression for a in (6) (and noting that all parameters of the model (1) are positive and $\beta^* > 0$), the bifurcation parameter a reduces to (where ν_2 and ω_2 are defined above)

$$a = -\frac{2\beta^* \mu \nu_2 \omega_2^2}{\Lambda} \left[\frac{(1-p)K_1}{\mu} + \frac{\alpha}{K_2} + \frac{\phi \alpha}{\mu K_2} + \frac{\psi \gamma}{\mu K_3} + 1 \right] < 0, \tag{8}$$

with $\nu_2 > 0$ and $\omega_2 > 0$.

Thus, it follows from Theorem 4.1 in [1] that the model (1) does not undergo backward bifurcation if $\pi = \xi = 0$. Hence, this study shows that the rates at which individuals who 'recover' from corrupt practices due to certain interventions like public enlightenment, counselling e.t.c and previously convicted individuals become susceptible to corruption ($\pi \neq 0$ and $\xi \neq 0$, respectively), are the causes of backward bifurcation in the corruption transmission model.

Hence, even when the parameters suggests that corruption can be controlled and even eradicated in the community, using the corruption reproduction number as basis for control, this may prove difficult when these causes of the backward bifurcation are present in the dynamics of corruption in the population.

Simply stated, the mathematical analysis in this section shows that having previously convicted and 'recovered' individuals becoming susceptible to corruption (again) will make it difficult to have corruption eradicated from the population.

3.2 Global Asymptotic Stability: Special Case $\pi = \xi = 0$.

Consider the model (1) with $\pi = \xi = 0$. We claim the following:

Theorem 3.2 *The CFE of the model* (1) *with* $\pi = \xi = 0$ *is globally-asymptotically stable (GAS) in* $\mathcal{D}$ *whenever* $\mathcal{R}_0 \leq 1$.

Proof. Consider the model (1) with $\pi = \psi = 0$. Further, consider the following linear Lyapunov function

$$\mathcal{F} = pC(t),$$

with Lyapunov derivative (where a dot represents differentiation with respect to t)

$$\begin{aligned}\dot{\mathcal{F}} &= p\dot{C},\\ &= \left[\frac{p\beta SC}{N-V} - (\alpha+\mu+\gamma)\right]C,\\ &= \frac{C}{(\alpha+\mu+\gamma)}\left[\frac{p\beta}{\alpha+\mu+\gamma}\frac{S}{N-V} - 1\right].\end{aligned} \tag{9}$$

It follows from (9), noting that $S(t) \le N(t) - V(t)$ in $\mathcal{D}$ for all $t > 0$, that

$$\dot{\mathcal{F}} \le \frac{1}{\alpha + mu + \gamma}(\mathcal{R}_0 - 1)C.$$

Hence, $\dot{\mathcal{F}} \le 0$ if $\mathcal{R}_0 \le 1$ with $\dot{\mathcal{F}} = 0$ if and only if C=0. Therefore $\mathcal{F}$ is a Lyapunov function in $\mathcal{D}$ and it follows from the LaSalle's Invariance Principle [9], that every solution to the equations in (1) (with $\pi = \xi = 0$) with initial conditions in $\mathcal{D}$ converges to $\mathcal{E}_0$ as $t \to \infty$. i.e

$$(C(t), R(t), V(t)) \to (0, 0, 0)$$

as $t \to \infty$. Substituting $C = R = V = 0$ into the first equation of (1) gives $S(t) \to \frac{p\Lambda}{\mu}$ as $t \to \infty$.

Thus $(S, C, R, T, V) \to (\frac{p\Lambda}{\mu}, 0, 0, \frac{(1-p)\Lambda}{\mu})$ as $t \to \infty$ for $\mathcal{R}_0 \le 1$, so that the corruption-free equilibrium, $\mathcal{E}_0$, is GAS in $\mathcal{D}$ if $\mathcal{R}_0 \le 1$ for the case $\pi = \xi = 0$. □

The significance of the above result on the society is that, in the absence of 'recovered' individuals and persons previously convicted (and jailed) for corruption becoming susceptible to corruption again, corruption will be eliminated from the population if the corruption reproduction threshold $\mathcal{R}_0 < 1$.

4 Existence and Stability of the Corruption Endemic Equilibrium

To establish the existence of the corruption endemic equilibrium of the model (1), let

$$\mathcal{E}_1 = \{S^{**}, C^{**}, R^{**}, T^{**}, V^{**}\}$$

represents any arbitrary corruption endemic equilibrium of the model (1).

The equations in (1) are solved in terms of the corruption force of infection at steady state to give

$$\begin{aligned}S^{**} &= \frac{1}{\lambda^{**}+\mu}\left[\frac{K_3\alpha\pi C^{**} + K_2\xi\gamma C^{**} + K_2K_3p\Lambda}{K_2K_3}\right],\\ C^{**} &= \frac{\lambda^{**}K_2K_3p\Lambda}{K_1K_2K_3(\lambda^{**}+\mu) - \lambda^{**}(K_3\alpha\pi + K_2\xi\gamma)},\\ R^{**} &= \frac{\alpha C^{**}}{K_2},\quad T^{**} = \frac{1}{\mu}\left[\frac{K_3\phi\alpha C^{**} + \psi\gamma K_2 C^{**} + K_2K_3(1-p)\Lambda}{K_2K_3}\right],\quad V^{**} = \frac{\gamma C^{**}}{K_3}.\end{aligned} \tag{10}$$

Note that the corruption force of infection at steady state, λ^{**}, is expressed as

$$\lambda^{**} = \frac{\beta C^{**}}{S^{**} + C^{**} + R^{**} + T^{**}}. \tag{11}$$

Substituting the expressions in (10) into (11) give

$$H\lambda^{**2} - \lambda^{**}(p\beta\mu K_2K_3 - \mu K_1K_2K_3) = 0, \tag{12}$$

where $H = \alpha[(p-1)\pi + p(\mu+\phi)]K_3 + [\gamma((p-1)\xi + p\psi) + (p\mu + (1-p)K_1)K_3]K_2$, and further reduces to

$$H\lambda^{**} - \mu K_1K_2K_3(\mathcal{R}_0 - 1) = 0,$$

which finally leads to

$$\lambda^{**} = \frac{\mu K_1K_2K_3}{H}(\mathcal{R}_0 - 1). \tag{13}$$

Hence, a unique corruption endemic equilibrium exist (i.e. $\lambda^{**} > 0$) when $\mathcal{R}_0 > 1$ as far as $H > 0$. Clearly, if $\pi = \xi = 0$, then $H > 0$ and there exists a unique corruption endemic equilibrium for this case. Thus, the following result is established.

Theorem 4.1 *The model* (1) *has a unique endemic(positive) equilibrium when* $\pi = \xi = 0$ *whenever* $\mathcal{R}_0 > 1$.

The possibility of the existence of a corruption endemic equilibria exists even when $\mathcal{R}_0 < 1$. This occurs when $H < 0$ which is possible when $\pi \neq 0$ and $\xi \neq 0$ since $0 \leq p \leq 1$. This further proves the possibility of a backward bifurcation when $\mathcal{R}_0 < 1$ whereby the corruption free equilibrium will co-exist with a corruption endemic equilibrium. Therefore, the societal implication of this is as stated before: individuals who 'recover' from corruption after an intervention programme and previously convicted individuals should strive not to become susceptible to corruption, again. The relevant government agencies may determine and formulate policies that could help with preventive strategies.

4.1 Local Asymptotic Stability of the Corruption Endemic Equilibrium

Since the total population is asymptotically constant, the result in [2] guarantee that the following system

$$\begin{aligned}
\frac{dS}{dt} &= p\Lambda - \lambda S - \mu S + \pi R + \xi V,\\
\frac{dC}{dt} &= \lambda S - (\gamma + \alpha + \mu)C,\\
\frac{dR}{dt} &= \alpha C - (\phi + \pi + \mu)R,\\
\frac{dV}{dt} &= \gamma C - (\xi + \psi + \mu)V,
\end{aligned} \tag{14}$$

where $\lambda = \frac{\beta C}{N-V}$, with $N = \frac{\Lambda}{\mu}$, has the same asymptotic qualitative dynamics as those of system (1).

The Jacobian matrix for system (14) at the endemic equilibrium, $\mathcal{E}_1$, is

$$J(\mathcal{E}_1) = \begin{pmatrix} -A-\mu & -B & \pi & -D+\xi \\ A & B-K_1 & 0 & D \\ 0 & \alpha & -K_2 & 0 \\ 0 & \gamma & 0 & -K_3 \end{pmatrix}$$

where

$$A = \frac{\beta C^{**}}{N - V^{**}}, \quad B = \frac{\beta S^{**}}{N - V^{**}}, \quad \text{and} \quad D = -\frac{\beta S^{**}C^{**}}{(N - V^{**})^2}.$$

The characteristic polynomial, associated with the local stability of $J(\mathcal{E}_1)$, is

$$A_4(\vartheta)\varrho^4 + A_3(\vartheta)\varrho^3 + A_2(\vartheta)\varrho^2 + A_1(\vartheta)\varrho + A_0 = 0, \tag{15}$$

where

$$
\begin{aligned}
A_4 &= 1,\\
A_3 &= A - B + \mu + K_1 + K_2 + K_3,\\
A_2 &= -D\gamma - B\mu + (A - B + \mu)K_3 + K_2(A - B + \mu + K_3)\\
&\quad + K_1(A + \mu + K_2 + K_3),\\
A_1 &= -D\gamma\mu - A\gamma\xi - A\alpha\pi + (-B\mu + (A + \mu)K_1)K_3\\
&\quad + K_2(-D\gamma - B\mu + (A - B + \mu)K_3 + K_1(A + \mu + K_3),\\
A_0 &= -A\alpha\pi K_3 - K_2(\gamma(D\mu + A\xi) + (B\mu - (A + \mu)K_1)K_3)
\end{aligned}
$$

and ϱ_i represents the eigenvalues of the Jacobian with ϑ denoting the model's parameter vector.

The local stability of the corruption endemic equilibrium is tied to the roots of (15). The corruption endemic equilibrium point is locally asymptotically stable when the polynomial in (15) have negative real roots.

5 Discussions and Conclusions

In this work, we designed and discussed the asymptotic behaviour of a new deterministic model for the dynamics of corruption in a population. The model (1) has a locally-asymptotically stable corruption-free equilibrium whenever the associated corruption reproduction number ($\mathcal{R}_0$) is less than unity. This model undergoes the phenomenon of backward bifurcation, where the stable corruption-free equilibrium co-exists with a stable corruption endemic equilibrium. The analysis showed that this phenomenon is caused by the rates at which individuals who previously recovered from corruption, following some intervention programme, and individuals who previously were convicted and jailed for corruption, revert to being susceptible (again) to corruption.

It is also shown that the model has a globally-asymptotically stable corruption-free equilibrium (DFE) whenever the associated corruption reproduction number is less than unity when individuals who recover from corruption and those previously convicted and jailed for corruption never become susceptible to corruption but have developed total resistance to corruption.

We also showed that the unique corruption endemic equilibrium is locally- asymptotically stable whenever the associated corruption reproduction number is greater than unity (as far as the cause of the backward bifurcation phenomena, explained above, is no longer part of the dynamics of the system).

This work has provided a theoretical basis for tackling corruption. It suggests that efforts should be geared toward monitoring individuals who claimed that they had been helped by some intervention programmes and seems resistant to corruption, only for such individuals to return to being susceptible to corruption once more. Also persons who were previously convicted and jailed for corruption should be closely monitored after their release as their being susceptible to corruption (once more) could hinder efforts in eradicating corruption. Due to lack of data on the state variables and parameters in the corruption model (1), it is not possible for us to carry out meaningful and practical simulations on the model. However, this work has thrown up important parameters (for example π and ξ) that could be gathered by the relevant government agencies for better understanding of the burden of corruption in the society.

The assumption of having some individuals 'recover' from corruption due to, say enlightenment programmes, and never tried and convicted is based on the fact that we live in an imperfect society where not all corrupt individuals get caught but some could, on there own, change from their corrupt attitude and become better citizens in the society. However, it can be said that monitoring such individuals in the population could prove difficult as they may not be moved to report themselves to the relevant security agencies. However, through anonymous questionnaire systems, it may be possible to obtain an insight into peoples general attitude to corruption and their honest appraisal as to their current stand on the issue and their involvements, if any.

It is imperative to examine two critical issues that could assist the society in tackling corruption: self recovered individuals and previously convicted and jailed individuals becoming susceptible to corruption (again). While the former individuals may be difficult to track and monitor, the latter individuals can be monitored after their release from prison. This is left to the relevant government agencies to think about.

Acknowledgement

D. O. will like to thank C.G. Agbo, Angela A. Ekwonu, A.A. Alani, O.T. Ikpuri and N.A. Opara (Department of Mathematics, University of Benin, Benin City, Nigeria, during the 2013-2014 academic session) for providing some of the articles on corruption used in this work, their valuable opinions on corruption and the associated classification of individuals in the population.

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