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Flow propensities in biological vessels and valves

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Abstract

Flows through the vascular appurtenances and biological valves are in a deep sense analogous to those obtained in other pipes and valves of similar configuration. However, the pulsatile nature of the cardiovascular flow, and the intrinsic coherent perturbation associated with the physiological and anatomical complications that reside in the biological structures, precisely the arteries and veins, and the heart valves, strike a remarkable distinction between them. The flow through the vascular structures was treated as a cylindrical pipe flow with characteristic coherent perturbation; the sound associated with or resulting from flow was considered, and the relationship between pressure and flow in open heart valves was treated, all of which were presented with the relevant equations. The equation involving the open heart valves is as good as furnishing the volume of blood in a physiological transmitral flow.

***Keywords and phrases*:** circulatory system; acoustic wave; turbulence; coherent perturbation; boundary conditions

1 Introduction

Blood vessels are conduits through which blood circulates, and valves are structures in a hollow organ, such as the heart, each with a flap to insure a unidirectional flow of fluid (blood) through it. In describing the flow of blood in a vessel, the driving pressure is the pressure difference between any two chosen points along a specified length of the vessel. However, this does not define the flow of blood in an organ in which the pressure difference is expressed as the variation between arterial pressure (P_A) and venous pressure (P_V). In the kidney, the renal artery pressure, renal vein pressure, and renal vascular resistance determine the flow of blood. Pressure difference at points along the vessel or across a heart valve is crucial in determining the pathophysiology of an organ and the cardiovascular parameters. For example, a pressure difference of at least 30mmHg is a baseline for turbulent flow between heart chambers [7]. The circulatory system is virtually marked by laminar blood flow, with concentric layers of blood flowing in parallel down the length of a blood vessel. However, the laminar flow may be disrupted and become turbulent under conditions of high flow velocity such that the critical Reynolds number (Re) is exceeded. When this occurs, blood does not flow linearly and smoothly in adjacent layers instead, the flow can be described as being chaotic. In the ascending aorta, laminar flow may transit to turbulent in the event of extremely high flow velocity [12]. Flow in the vicinity of the valve resembles a pulsed jet through a nonaxisymmetric orifice, as pulsatile cardiac blood flow causes the heart valve leaflets to open and close, inducing a dynamically changing area. In effect, shear layers, regions of recirculation, and flow instabilities that could eventually lead to a

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transition to turbulence develop. Heart valves serve as governors or control devices and are the major sources of noise in piping systems. They perform the following essential functions: (a) modulating or throttling flow, (b) regulating pressure, (c) getting rid of excess pressure, and (d) preventing backflow. The complex flow patterns are intrinsically valve-and-patient-specific [22]. Sabbah and Stein [20] corroborated studies in humans that show that turbulent flow occurs above the aortic valve under some conditions in normal humans, and it occurs characteristically in the presence of aortic valvular disease. They disclosed that intra-arterial ejection murmurs were at all times associated with turbulent or highly disturbed flow in the ascending aorta. The vibrating cardio-vascular structures, distressed regions of turbulent flow, mixing of flows of diverse temperatures, and their similitudes generate pressure fluctuations that are communicated through a fluid as *sound*. This sound is the progression of compressions and rarefactions propagated away from the acoustic source vicinity. In the vasculature as well as the cardiac chambers, murmurs are generated by turbulent flow. Heart murmurs are well described concerning the site on the thoracic cage, pitch and volume, and phase of the cardiac cycle [23]. Vascular bruit turbulence outside the heart can be located along the carotid artery, which supplies blood to the brain. Jung and Rajat [10] were of the view that the primary source of arterial bruits is perturbation initiated by vortex flow in the near post-stenotic region. Turbulence has its mixed blessings; Wei *et al.* [25] suggested that flow turbulization is a major way of heat transfer enhancement; accordingly, special tubulising elements can be placed into tubes and channels to enhance heat transfer. Turbulence may create some physiological cardiac murmurs, but murmurs are most often considered pathologic [10, 21]. In their work, Farzad *et al.* [4] sought to evaluate the performance of modern acoustic detection of murmurs by acoustic signals taken from patients undergoing coronary angiography. According to them, acoustic signal patterns may identify intracoronary murmurs generated by hemodynamically substantial coronary artery stenosis in all major vessels.

A large amount of investigation has been carried out about flows with free boundaries that include [9, 14]. The question of where turbulence develops and how the concomitant acoustic wave propagates through the arterial vessels requires elucidation. Laufer [13] was driven to investigate the structure of turbulent flow in fully developed pipe flow. Elsewhere [17] the noise generated by turbulent pipe flow was studied. Both cases [9, 14] are tenable in arterial flow [20] and therefore may use used for investigating such flow in arteries and valves.

2 Flow in pipes

In what follows, attention is given to a cylindrical pipe in line with the arterial/venous structure.

2.1 Reynolds equation of pipe flow

Consider a section of the artery with incompressible mean flow. The continuity equation and Reynolds equation in cylindrical polar coordinates read:

$$\frac{\partial U}{\partial x}+\frac{1}{r}\left(\frac{\partial_r V}{\partial r}+\frac{\partial W}{\partial \varphi}\right)=0 \tag{1}$$

$$\left.\begin{aligned}
&U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial r}+\frac{W}{r}\frac{\partial U}{\partial \varphi}=-\frac{1}{\rho}\frac{\partial P}{\partial x}-\left(\frac{\partial}{\partial x}\overline{u^2}+\frac{1}{r}\frac{\partial}{\partial r}r\overline{uv}+\frac{1}{r}\frac{\partial}{\partial \varphi}\overline{uw}\right)+\upsilon\nabla^2 U\\
&U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial r}+\frac{W}{r}\frac{\partial V}{\partial \varphi}-\frac{W^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}-\left(\frac{\partial}{\partial x}\overline{uv}+\frac{1}{r}\frac{\partial}{\partial r}r\overline{v^2}+\frac{1}{r}\frac{\partial}{\partial \varphi}\overline{vw}-\frac{\overline{W^2}}{r}\right)+\upsilon\left(\nabla^2 V-\frac{V}{r^3}-\frac{2}{r^2}\frac{1}{r}\frac{\partial W}{\partial \varphi}\right)\\
&U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial r}+\frac{W}{r}\frac{\partial W}{\partial \varphi}+\frac{VW}{r}=-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}-\left(\frac{\partial}{\partial x}\overline{uw}+\frac{\partial}{\partial r}\overline{vw}+\frac{1}{r}\frac{\partial}{\partial \varphi}\overline{w^2}-\frac{2\overline{vw}}{r}\right)+\upsilon\left(\nabla^2 W-\frac{2}{r^2}\frac{\partial W}{\partial \varphi}-\frac{W}{r^2}\right)
\end{aligned}\right\} \tag{2.a, b, c.}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$, and U and P encode the mean velocity and pressure respectively that are measured at any point in the artery or valve; u, v, w represent the instantaneous velocity fluctuations in x (axial), r (radial), and φ (azimuthal) directions respectively; $\overline{u}, \overline{v}, \overline{w}$ are the average velocity in x, r and φ directions respectively; V and W are the mean velocities in radial and azimuthal directions respectively, $\mathbf{\nu}$ is the viscosity, and ϱ, density. As a consequence of a fully developed flow, the mean velocities in radial and azimuthal directions vanish. The least velocity (V= 0) is observed along the vessel wall. We find $V = W = 0$; since pipe flow is homogeneous in the axial and azimuthal directions, $\frac{\partial}{\partial \varphi} = 0$, and $\frac{\partial}{\partial x} = 0$ except at total pressure. Moreover, the velocity field does not depend on the axial coordinate, x. With these, equation (2) reads

$$\left.\begin{aligned} &\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{1}{r}\frac{d}{dr}r\overline{uv} + \nu\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{dU}{dr}\right) \\ &\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{1}{r}\frac{d}{dr}r\overline{v^2} + \frac{\overline{w^2}}{r} \\ &-\frac{d}{dr}\overline{vw} - \frac{2\overline{vw}}{r} = 0 \end{aligned}\right\} . \tag{3. a, b, c}$$

Applying the boundary condition $\overline{vw} = 0$ at $r = r_1$ shows that $\overline{vw} = 0$ for each value of r. Therefore the turbulent pipe flow is

$$\left.\begin{aligned} &\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{1}{r}\frac{d}{dr}r + \left(\overline{uv} - \nu\frac{dU}{dr}\right) \\ &\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{1}{r}\frac{d}{dr}r\overline{v^2} + \frac{\overline{w^2}}{r} \end{aligned}\right\} \tag{4. a, b}$$

Note that, $\overline{vw}$ must be zero everywhere save at the pipe centre; but if it is continuously dependent on r, then it will vanish at the pipe centre. With suitable radial boundary conditions the shearing stress, $\overline{uv}$, is given in terms of the mean velocity by

$$\overline{uv} = \nu\frac{dU}{dr} + \frac{r}{r_1}U_*^2 \;, \tag{5}$$

and the normal stresses $\overline{v^2}$ and $\overline{w^2}$ relate to mean pressure by

$$\overline{v^2} + \int_{r_1}^{r} \frac{\overline{v^2} - \overline{w^2}}{r}\, dr = -\frac{P}{\rho} - \frac{2}{r_1}U_*^2 x. \tag{6}$$

where r_1 is pipe radius, U_* is friction velocity. The above equation, excluding the second term on the right hand side, may be rewritten as

$$\frac{1}{\rho}\int\left(\frac{\partial P}{\partial r}\right)dr = -\int d\overline{v^2} + \int \frac{\overline{w^2} - \overline{v^2}}{r}dr \tag{7}$$

Carrying out the indicated integration between the limits r_1 and r and noting at $r = r_1$, $\overline{v^2} = 0$ one gets

$$\frac{P_{r,x}-P_{r_1,x}}{\rho}=-\overline{v^2}+\int_{r_1}^{r}\frac{\overline{w^2}-\overline{v^2}}{r}dr \tag{8}$$

where, $P_{r,x}$ = mean pressure at r, x; $P_{r1,x}$ = mean pressure at r_1, x. At an arbitrary point, x_1 one finds

$$\frac{P_{r,x}-P_{r,x_1}}{\rho}=-\frac{2U_*^2 x}{r_1}. \tag{9}$$

Thus, adding (9) to (8) gives

$$\frac{P_{r,x}-P_{r_1,x_1}}{\rho}=-\overline{v^2}+\int_{r_1}^{r}\frac{\overline{w^2}-\overline{v^2}}{r}dr-\frac{2U_*^2 x}{r_1}. \tag{10}$$

The equations so far are given in terms of the radial coordinate of the vessel, r. The energy equation takes the form, (see Laufer [13] for the derivation):

$$\overline{uv}\frac{dU}{dr}+\frac{1}{r}\frac{d}{dr}r\overline{v\left(\frac{u^2+v^2+w^2}{2}+\frac{p}{\rho}\right)}-\frac{v}{r}\frac{d}{dr}r\frac{d}{dr}\overline{\left(\frac{u^2+v^2+w^2}{2}\right)}+v\overline{\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}=0. \tag{11}$$

In the equation above, the first term encodes the rate of production of turbulent energy due to shearing stresses. The second term is the diffusion term, the third term may represent a gradient form of energy diffusion which is of interest near the wall. The fourth term indicates viscosity-induced energy dissipation into heat.

2.2 Perturbation equations in cylindrical coordinates

The fundamental Navier-Stokes equations that govern flow include the continuity equation, the momentum equations, and the energy equation. The continuity equation reads:

$$\frac{\partial\tilde{\rho}}{\partial t}+\bar{u}_i\frac{\partial\tilde{\rho}}{\partial x_i}+\rho_0\frac{\partial\tilde{u}_i}{\partial x_i}=0, \tag{12}$$

where ρ is the fluid density, and u_i is the velocity.

The momentum equation for the coherent **perturbations reads:**

$$\left(\frac{\partial\tilde{u}_i}{\partial t}+\tilde{u}_j\frac{\partial\bar{u}_i}{\partial x_j}+\bar{u}_j\frac{\partial\tilde{u}_i}{\partial x_j}+\frac{\tilde{\rho}\bar{u}_j}{\rho_0}\frac{\partial\bar{u}_i}{\partial x_j}\right)=-\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial x_i}+v\left[\frac{\partial^2\tilde{u}_i}{\partial x_j\partial x_j}+\left(\frac{1}{3}+\frac{\mu_B}{\mu}\right)\frac{\partial^2\tilde{u}_j}{\partial x_j\partial x_i}\right]-\frac{\partial\tilde{r}_{ij}}{\partial x_j}, \tag{13}$$

where μ is the dynamic viscosity, and μ_B is the bulk viscosity (see **Pierce** [16]) which is assumed constant here. The perturbation Reynolds stress reads

$$\tilde{r}_{ij}\equiv\left\langle u_i'u_j'\right\rangle-\overline{u_i'u_j'}. \tag{14}$$

The quantity $-\ \tilde{r}_{ij}$ may be construed as the oscillation of the background Reynolds stress as a result of the passage of the coherent perturbations [2]. Equations (13, 14) govern the motion of a Newtonian fluid, together with the motion of coherent perturbations, such as sound wave propagation. In cardiovascular flows it is often advisable to examine the flow and the coherent perturbation quantities separately, essentially by some splitting technique (see Morfey [15]). The following decomposition ensues:

$$H(\mathrm{x},t) = \bar{H}(\mathrm{x}) + \tilde{H}(\mathrm{x},t) + \hat{H}(\mathrm{x},t), \tag{15}$$

where $\bar{H}$ encodes the mean (time-averaged) contribution, $\tilde{H}$ is the coherent perturbation (periodic wave) driven by the pulsatile structure of flow, and $\hat{H}$ represents to the turbulent fluctuation. Here H denotes any one of the variables: ρ, u_i, T, or p, as the case may be. The time-averaged mean defined as

$$\bar{H}(\mathrm{x,t}) = \lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} H(\mathrm{x,t'})\mathrm{dt'}, \tag{16}$$

and the phase average is

$$\langle H(\mathrm{x},t)\rangle = \lim_{N\to\infty} \frac{1}{N_p}\sum_{\xi=0}^{N_p} H\left(\mathrm{x},\, t+\frac{\xi}{f}\right), \tag{17}$$

where f is the frequency of the coherent perturbation.

In a fully developed turbulent pipe flow, the there is a homogeneous stream-wise mean flow, whose velocity is $\bar{\boldsymbol{u}} = [\bar{u}_x(r), 0, 0]$. The mean flow is locally parallel in fully developed turbulent pipe flows, therefore the mean flow velocity $\bar{\boldsymbol{u}} = [\bar{u}_x(r), 0, 0]$ is parallel to the x-axis and fluctuating in the radial direction r. The perturbation equations in cylindrical coordinates are:

$$\left(\frac{\partial}{\partial t}+\bar{\varsigma}\right)\frac{\tilde{\rho}}{\rho_0}+\left(\frac{\partial \tilde{u}_r}{\partial r}+\frac{\tilde{u}_r}{r}+\frac{1}{r}\frac{\partial \tilde{u}_\theta}{\partial \theta}+\frac{\partial \tilde{u}_x}{\partial x}\right)=0 \qquad \text{(continuity equation)} \tag{18}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\bar{\varsigma}\right)\tilde{u}_x+\frac{\tilde{\rho}}{\rho_0}\bar{\varsigma}\bar{u}_x &= -\frac{1}{\rho_0}\frac{\partial\tilde{\rho}}{\partial x}+\nu\left(\frac{\partial^2\tilde{u}_x}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{u}_x}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde{u}_x}{\partial\theta^2}+\frac{\partial^2\tilde{u}_x}{\partial x^2}\right)\\ &\quad+\nu\left(\frac{1}{3}+\frac{\mu_B}{\mu}\right)\left(\frac{\partial^2\tilde{u}_r}{\partial x\partial r}+\frac{1}{r}\frac{\partial^2\tilde{u}_\theta}{\partial x\partial\theta}+\frac{1}{r}\frac{\partial^2\tilde{u}_r}{\partial x}+\frac{\partial^2\tilde{u}_x}{\partial x^2}\right)\\ &\quad-\left[\frac{\partial\tilde{r}_{rx}}{\partial r}+\frac{1}{r}\left(\frac{\partial\tilde{r}_{\theta x}}{\partial\theta}+\tilde{r}_{rx}\right)+\frac{\partial\tilde{r}_{xx}}{\partial x}\right]\end{aligned} \qquad \text{(x}-\text{momentum equation)} \tag{19}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\bar{\varsigma}\right)\tilde{u}_r+\frac{\tilde{\rho}}{\rho_0}\bar{\varsigma}\bar{u}_r-2\frac{\bar{u}_\theta\tilde{u}_\theta}{r} &= -\frac{1}{\rho_0}\frac{\partial\tilde{\rho}}{\partial r}+\nu\left(\frac{\partial^2\tilde{u}_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{u}_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial\tilde{u}_r}{\partial r}-\frac{\tilde{u}_r}{r^2}-\frac{2}{r^2}\frac{\partial\tilde{u}_\theta}{\partial\theta}+\frac{\partial^2\tilde{u}_r}{\partial x^2}\right)\\ &\quad+\nu\left(\frac{1}{3}+\frac{\mu_B}{\mu}\right)\left(\frac{\partial^2\tilde{u}_r}{\partial r^2}+\frac{1}{r}\frac{\partial^2\tilde{u}_\theta}{\partial\theta\partial r}+\frac{1}{r}\frac{\partial^2\tilde{u}_r}{\partial r}-\frac{1}{r^2}\frac{\partial\tilde{u}_\theta}{\partial\theta}-\frac{\tilde{u}_r}{r^2}+\frac{\partial^2\tilde{u}_x}{\partial r\partial x}\right)\\ &\quad-\left[\frac{\partial\tilde{r}_{rr}}{\partial r}+\frac{1}{r}\left(\frac{\partial\tilde{r}_{\theta x}}{\partial\theta}+(\tilde{r}_{rr}-\tilde{r}_{\theta\theta})\right)+\frac{\partial\tilde{r}_{xr}}{\partial x}\right]\end{aligned} \qquad \text{(r}-\text{momentum equation)} \tag{20}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\bar{\varsigma}+\frac{\bar{u}_r}{r}\right)\tilde{u}_\theta+\left(\frac{\tilde{\rho}}{\rho_0}\bar{\varsigma}+\frac{\tilde{u}_r}{r}\right)\bar{u}_\theta &= -\frac{1}{\rho_0 r}\frac{\partial\tilde{\rho}}{\partial\theta}+\nu\left(\frac{\partial^2\tilde{u}_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{u}_\theta}{\partial r}-\frac{\tilde{u}_\theta}{r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial\tilde{u}_\theta}{\partial\theta}+2\tilde{u}_r\right)+\frac{\partial^2\tilde{u}_\theta}{\partial x^2}\right)\\ &\quad+\nu\left(\frac{1}{3}+\frac{\mu_B}{\mu}\right)\left(\frac{1}{r}\frac{\partial^2\tilde{u}_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\tilde{u}_r+\frac{\partial\tilde{u}_\theta}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2\tilde{u}_x}{\partial x\partial\theta}\right)\\ &\quad-\left[\frac{\partial\tilde{r}_{r\theta}}{\partial r}+\frac{1}{r}\left(\frac{\partial\tilde{r}_{\theta\theta}}{\partial\theta}+(\tilde{r}_{r\theta}-\tilde{r}_{\theta r})\right)+\frac{\partial\tilde{r}_{x\theta}}{\partial x}\right]\end{aligned} \qquad (\theta-\text{momentum equation}) \tag{21}$$

$$\left(\frac{\partial}{\partial t}+\bar{\varsigma}\right)\tilde{T}+\frac{\tilde{\rho}}{\rho_0}\bar{\varsigma}\bar{T}=\frac{1}{c_{p\rho_0}}\left[\left(\frac{\partial}{\partial t}+\bar{\varsigma}\right)\tilde{p}+\tilde{\zeta}\,\bar{p}\right]+\lambda\left(\frac{\partial^2\tilde{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{T}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde{T}}{\partial\theta^2}+\frac{\partial^2\tilde{T}}{\partial x^2}\right) , \quad \textit{The energy equation,} \tag{22}$$
$$-\left[\frac{\partial\tilde{q}_r}{\partial r}+\frac{1}{r}\left(\frac{\partial\tilde{q}_\theta}{\partial\theta}+\tilde{q}_r\right)+\frac{\partial\tilde{q}_x}{\partial x}\right]+\frac{2\nu}{c_p}(\alpha+\beta)$$

where,

$$\bar{\varsigma}=\bar{u}_x\frac{\partial}{\partial x} \text{ and } \tilde{\zeta}=\tilde{u}_x\frac{\partial}{\partial x}, \quad \text{and} \quad \alpha=\frac{\partial\bar{u}_x}{\partial r}\frac{\partial\tilde{u}_x}{\partial r}, \ \ \beta=\frac{\partial\bar{u}_x}{\partial r}\frac{\partial\tilde{u}_r}{\partial x}.$$

3 Acoustic analogy and noise generation

Noise is inviolable component of turbulence. It is created by the Reynolds stresses. Velocity and pressure fluctuations do result in the development of acoustic sources which radiate sound with intensities and directionality that are typical of the particular nature and magnitude of the sources and the acoustic features of the surrounding media. Pressure is a parameter of much interest in the wave equation in the sense that ears and the auscultation devices, such as the stethoscope for measuring sound within the body, are sensitive to pressure fluctuations. Turbulent fluctuating pressures which are related to acoustical pressure are termed *sound pressure* or *sound*, and fluctuations of pressures which are related to sound are frequently referred to as *intra-arterial sound* [5]. Aerodynamically generated noise is essentially based on density fluctuations, which are frequently converted to pressure variations by means of a linear relationship. A good relationship is found between pressure and velocity because the acoustic velocity is frequently used as a boundary condition in calculations involving solid bodies. Acoustic velocity is the usual fluid dynamic boundary condition that must be satisfied [16]. Morfey and Wright [15] improved on the pioneering work of Lighthill [14] on acoustic analogy. Their formulation for bounded domains enables the application of pressure-based options to the fluid density as a wave variable.

Consider the windowed equations of motion for a fluid occupying a region *R*. The expressions for conservation of mass and momentum are

$$\underline{\frac{\partial}{\partial t}(\rho-\rho_0)}+\underline{\frac{\partial}{\partial x_i}(\rho u_i)}=0, \qquad \underline{\frac{\partial}{\partial t}(\rho u_i)}+\underline{\frac{\partial}{\partial t}(\rho u_i u_j+p_{ij})}=\underline{B_i}. \tag{23}$$

In the above, and subsequently, an underscored quantity or variable means that it is multiplied by some Heaviside functions G and Ξ. Thus, it is windowed in space and time; u_i is the fluid velocity in the x_i direction; $p_{ij}=P_{ij}-P_0\delta_{ij}$ where P is absolute pressure and P_{ij} is the compressive stress in the fluid; δ_{ij} is the Kronecker delta and B_i is an applied body force per unit volume. The Heaviside functions G and Ξ are such that:

$$G(\boldsymbol{n})=\begin{cases}1 & \text{in } \mathcal{R} \text{ and on } S\\ 0 & \text{in } \mathcal{R}' \text{ (prime indicates complement);}\end{cases} \tag{24}$$

$$\Xi(t)=\begin{cases}1 & \text{for } t\geq 0\\ 0 & \text{for } t<0\end{cases} , \tag{25}$$

In (24), **n** is a local normal coordinate defined for points in the neighbourhood of S; $\boldsymbol{n}=f$.**S** is a moving closed surface in an appropriate dimension, assumed smooth and $|\nabla f|$ is single-valued. The following derivatives apply from the property of the Heaviside function:

$$\frac{\partial \Xi}{\partial t} = \delta(t), \quad \frac{\partial \Xi}{\partial x_i} = 0, \quad \frac{\partial G}{\partial n} = \delta(n), \quad \frac{\partial G}{\partial x_i} = \mathbf{n}_i \delta(n), \tag{26}$$

where $\delta(.)$ is the Dirac delta function and $\boldsymbol{n}_i = \partial n / \partial x_i$ is the unit normal to S. For any field variable η,

$$\underline{\underline{\frac{\partial \eta}{\partial x_i}}} - \frac{\partial \eta}{\partial x_i} = -\eta \mathbf{n}_i \delta(n)\Xi, \qquad \underline{\underline{\frac{\partial \eta}{\partial t}}} - \frac{\partial \eta}{\partial t} = \eta\left[v_i \mathbf{n}_i \delta(n)\Xi - G\delta(t)\right] \tag{27}$$

where $v_i = v\boldsymbol{n}_i$, v is the normal velocity of the surface S directed into $\mathscr{R}$. The application of equation (27) to equation (23) yields

$$\underline{\frac{\partial}{\partial t}(\rho - \rho_0)} + \frac{\partial}{\partial x_i}\left(\underline{\rho u_i}\right) = (\rho - \rho_0)G\delta(t) + \left[\rho u_i - (\rho - \rho_0)v_i\right]\mathbf{n}_i \delta(n)\Xi \tag{28}$$

and

$$\frac{\partial}{\partial t}\left(\underline{\rho u_i}\right) + \frac{\partial}{\partial x_i}\left(\underline{\rho u_i u_j + p_{ij}}\right) = \underline{B_i} + \rho u_i G\delta(t) + \left[\rho u_i\left(u_j - v_j\right) + p_{ij}\right]\mathbf{n}_j \delta(n)\Xi. \tag{29}$$

The second time derivative of the perturbed density valid for all (x_i, t) may be obtained by eliminating $\underline{\rho u_i}$ from (28) and (29). Thus,

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\left(\underline{\rho - \rho_0}\right) = {} & \frac{\partial}{\partial t}[(\rho - \rho_0)G\delta(t)] - \frac{\partial}{\partial x_i}\left[\rho u_i G\delta(t)\right] + \frac{\partial}{\partial t}\left[\left[\rho u_i - (\rho - \rho_0)v_i\right]\mathbf{n}_i \delta(n)\Xi\right] \\ & - \frac{\partial}{\partial x_i}\left[\left[\rho u_i\left(u_j - v_j\right) + p_{ij}\right]\mathbf{n}_j \delta(n)\Xi\right] - \frac{\partial \underline{B_i}}{\partial x_i} + \frac{\partial}{\partial x_i \partial x_j}\left(\underline{\rho u_i u_j + p_{ij}}\right) \end{aligned} \tag{30}$$

The density form of the acoustic analogy is in the form (see Morfey and Wright [15])

$$\begin{aligned} \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[c_0^2\left(\underline{\rho - \rho_0}\right)\right] = {} & \frac{\partial}{\partial t}\left[(\rho - \rho_0)G\delta(t)\right] - \frac{\partial}{\partial x_i}\left[\rho u_i G\delta(t)\right] + \frac{\partial}{\partial t}\left[J_i \boldsymbol{n}_i \delta(n)\Xi\right] \\ & - \frac{\partial}{\partial x_i}\left[L_{ij}\boldsymbol{n}_j \delta(n)\Xi\right] - \frac{\partial \underline{\underline{B_i}}}{\partial t} + \frac{\partial^2 \underline{\underline{T_{ij}}}}{\partial x_i \partial x_j} \end{aligned} \tag{31}$$

Tij, Ji and *Lij* on the right of (31) encode: the Lighthill stress tensor

$$T_{ij} = \rho u_i u_j + p_{ij} - c_0^2\left(\rho - \rho_0\right)\delta_{ij}, \tag{33}$$

the surface mass flux vector

$$J_i = \rho u_i - \left(\rho - \rho_0\right)v_i \equiv \rho\left(u_i - v_i\right) + \rho_0 v_i, \tag{34}$$

and the surface momentum flux tensor

$$L_{ij} = \rho u_i\left(u_i - v_i\right) + p_{ij}. \tag{35}$$

The right members of (31) are the sources that represent the physical properties of the acoustic wave. The first two members encode the aggregate impulsive mass and momentum required to drive flow from its original reference state; the second and third members contain the so called Ffowcs Williams & Hawkings (FWH) formulation [5] of surface monopoles and dipoles, evinced by Ξ, and subsequently the volume source terms, with the body force B_i and the stress tensor T_{ij} encoded spatially and temporally by $G\ \Xi$. Reethuf [17] suggested that mass-flow fluctuations may be associated with monopole sources as compact fluctuating forces, momenta, or pressures in a flow field relate with dipole sources; and fluctuating shear stresses relate with quadrupole sources. The FWH equation (31) with no initial source terms furnishes an extrapolative bridge between the simulation domain and the acoustic far field. Using [18], the replacement of the local density ϱ by a new variable ϱ^* related to ϱ together with some arithmetic operations give the equation for $\partial^2\left(\rho^* - \rho_0\right)/\partial t^2$ in the form

$$\frac{\partial^2\left(\rho-\rho_0\right)}{\partial t^2} = \frac{\partial}{\partial t}[\left(\rho^* - \rho_0\right)G\delta(t)] - \frac{\partial}{\partial x_i}\left[\rho^* u_i G\delta(t)\right] + \frac{\partial}{\partial t}\left[J_i^* \mathbf{n}_i \delta(n)\Xi\right]$$
$$-\frac{\partial}{\partial x_i}\left[L_{ij}^* \mathbf{n}_j \delta(n)\Xi\right] + \frac{\partial \Psi^*}{\partial x_i} - \frac{\partial}{\partial x_i}\left[\Psi^* u_i - (\rho - \rho^*)\frac{Du_i}{Dt} + B_i\right] + \frac{\partial}{\partial x_i \partial x_j}\left(\rho^* u_i u_j + p_{ij}\right),$$

(36)

Where J_i^* and L_{ij}^* are similar to Ji and Lij when ρ^* replaces ϱ and

$$\Psi^* = \frac{D\rho^*}{Dt} + \rho^*\Delta\ . \tag{37}$$

Now, a general acoustic analogy may be written in the form

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[c_0^2\left(\rho^* - \rho_0\right)\right] = \frac{\partial}{\partial t}\left[(\rho^* - \rho_0)G\delta(t)\right] - \frac{\partial}{\partial x_i}\left[\rho^* u_i G\delta(t)\right] + \frac{\partial}{\partial t}\left[J_i^* \boldsymbol{n}_i \delta(n)\Xi\right]$$
$$-\frac{\partial}{\partial x_i}\left[L_{ij}\boldsymbol{n}_j \delta(n)\Xi\right] + \frac{\partial \Psi^*}{\partial t} - \frac{\partial}{\partial x_i}\left(\Psi^* u_i + \frac{\rho^*}{\rho}B_i + \frac{\rho - \rho^*}{\rho}\frac{\partial p_{ij}}{\partial x_j}\right) + \frac{\partial^2 T_{i\ j}^*}{\partial x_i \partial x_j} \tag{38}$$

where the penultimate term is gotten by setting the equation of conservation as

$$\frac{Du_i}{Dt} = \frac{B_i}{\rho} - \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}, \tag{39}$$

and T_{ij}^* is defined in the same way as T_{ij} when ρ^* replaces ϱ

3.1 Propagation of sound inside pipes (vessels) with flow

Turbulent noise sources are typically contained in rigid walled vessels, which are acoustically reflecting. Some vessel walls are not acoustically rigid and impermeable; they therefore permit the absorption of some of the incident sound and in so doing progressively reduce the propagated and radiated sound power. In arteriosclerosis the induration (pathological hardening or thickening) of the arterial wall may cause the vasculature to be treated as rigid walled vessels. Therefore such vascular tissues will be acoustically reflecting. Non-acoustically rigid (compliant) blood vessels reduce the propagated and radiated sound power by permitting the absorption of some of the incident sound. The compliance or capacitance is a measure of changes in volume in response to changes in pressure. If it is assumed that vessel length does not alter with transmural pressure, the arterial cross-sectional compliance C_c (see [18]) could be estimated by

$$C_c = \frac{dA}{dp}, \tag{36}$$

where A is the vessel's cross-sectional area. The vessel's distensibility D_v is given in terms of its compliance by

$$D_v = \frac{1}{A}\frac{dA}{dp} = \frac{1}{A}C_c. \tag{37}$$

Suppose the perturbation fields $\tilde{H}$ (see Eq. (15)) in the pipe can be characterised by modal solutions, and each mode can be written in cylindrical coordinate in the form

$$\tilde{H}(x,r,\theta,t)=\left[\hat{H}_+(r)e^{(-ik_+x)}+\hat{H}_-(r)e^{(ik_-x)}\right]e^{(i\omega t+im\theta)}, \tag{36}$$

where subscripts + and − indicate the downstream and upstream propagating waves corresponding to the mean flow, $\omega = 2\pi f$ is the angular frequency, and $k\pm$ are the corresponding wavenumbers. The phase shift and the attenuation of the sound waves can be determined from the wave number $k\pm$ when the modes which represent the sound waves are known. The dimensionless wavenumber

$$\Gamma_\pm = \frac{k_\pm}{k_0} \tag{37}$$

may be used to described the propagation, with $k_0 = \omega/c_0$ the wavenumber in free space. Γ consists of the real part, which specifies the phase shift of the waves within one reference wavelength $\lambda_0 = 2\pi c_0/\omega$, and the imaginary part which specifies the decay of the wave amplitudes within λ_0. The value of $\Gamma_\pm$ is largely affected by mean flow convection and refraction, viscothermal effects within the acoustic boundary layer, turbulent absorption, among other factors.

We consider a plane wave regime associated with acoustic waveguide theory [16]. Assume an inviscid and adiabatic fluid, and assume further that the mean velocity profile (MVP) is uniform over the pipe cross-section, then, by indication [1] the acoustic pressure in the pipe is

$$\tilde{p}_m = \hat{p}_m(r)e^{(i\omega t-ik_mx+im\theta)}, \tag{38}$$

where,

$$\hat{p}_m(r) = BJ_m(\Re_m), \qquad B \text{ is a constant,} \tag{39}$$

and

$$\Re_m = \sqrt{(k_0 - Mk_m)^2 - k_m^2} \tag{40}$$

represents the radial wavenumber, represents the azimuthal wavenumber, $k_0 = \omega/c_0$ is the wavenumber in free space, J_m is the m-th order Bessel function of the first kind, and M is the Mach number. The boundary condition is such that the wall-normal velocity $\hat{u}_r$ is zero at the duct wall.

For inviscid and adiabatic fluids this requires that

$$\hat{u}_r\big|_{r=R} = \frac{i}{\rho_0\omega}\frac{d\hat{p}_m}{dr}\bigg|_{r=R} = 0 \Rightarrow J'_m(\Re_m R) = \hat{u}_r\big|_{r=R} = \frac{i}{\rho_0\omega}\frac{d\hat{p}_m}{dr}\bigg|_{r=R} = 0 \Rightarrow J'_m(He_m^c) = 0, \tag{41}$$

where J'_m denotes the first derivative of J_m with respect to r. The n-th solution satisfying the boundary condition Eq. (41), $He_{m,n}^c$, and the corresponding acoustic mode is known as the (m, n)th mode. The axial wavenumber $k_{\pm,m,n}$ can be calculated from (40) as

$$k_{\pm,m,n} = -\frac{Mk_0}{1-M^2} \pm \frac{k_0\left(1-(H_{m,n}^c\sqrt{1-M^2}/He)^2\right)^{\frac{1}{2}}}{1-M^2} \tag{42}$$

where $He = k_0R$ is Helmholtz number.

There are two cases: $k_{\pm,m,n}$ is a purely real number if $He \geq He^{c}_{m,n}\sqrt{1-M^2}$, else $k_{\pm,m,n}$ contains imaginary part. The latter case is instrumental in the evanescence of sound wave since the (m, n)th mode decays exponentially in the direction of propagation. The cut-on frequency of the (m,n)th mode (i.e. the mode propagating without decaying) is $He^{c}_{m,n}\sqrt{1-M^2}$. Equations (38) and (42) presuppose that the fluid is inviscid and adiabatic, and therefore the wavenumbers of the cut-on waves are wholly real, i.e. the wave propagates without attenuation. As a rule however, the propagating waves would be unavoidably damped by the fluid or the flow.

4 Pressure and flow in open valves

The flow across a heart valve is analogous to that along a blood vessel; nevertheless, the two pressures on either side of the valve account for the pressure difference [12]. Accordingly, the pressure difference across the aortic valve that drive flow across the valve in the event of ventricular ejection is the variation between intraventricular pressure (P_{IV}) and the aortic pressure (P_{Ao}). The motion of the heart valves is usually determined via the Navier–Stokes equation wherein boundary conditions of the blood pressures, pericardial fluid, and external loading are used as the constraints. It is taken as a boundary condition in the Navier–Stokes equation in defining the fluid dynamics of blood ejection from the left and right ventricles into the aorta and the lung [8].

If the following conditions are assumed: there is conservation of inflow energy, there is stagnant region behind leaflets, there is conservation of outflow momentum, there is flat velocity profile, then the pressure drop, Δp, across an open heart valve is related to the flow rate, Q, through the valve by

$$a_1\frac{\partial Q}{\partial t} + a_2 Q^2 = \Delta p \tag{43}$$

where a_1 and a_2 are nonzero constants. For valves assumed to have a single degree of freedom (such as studied using the aortic and mitral valves [8] the pressure and flow relationship is given by the Euler equations

$$\frac{\partial p}{\partial x} + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0, \tag{44}$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0, \tag{45}$$

where, u = axial velocity, p = pressure, A^* = orifice area of valve. We get

$$A^*(x,t) = A_0^*\left(1 - [1-\Lambda(t)]\frac{x}{L}\right)^2 \tag{46}$$

where, $\Lambda^2(t) = \frac{A^*(L,t)}{A_0^*}$ for $\Lambda(t)$being 1d.f., and L the axial length of the valve. Now,

$$\int_0^L p(x,t)\frac{\partial A^*}{\partial x}dx = \left[A_0^* - A^*(L,t)\right]p(L,t). \tag{47}$$

Note that in the case of a heart valve, the *orifice area* of the opened valve is to be used since the opening is not circular. Therefore the radius has to be given in terms of the area. Since $A = \pi r^2$, therefore $r = \sqrt{(A/\pi)} = A^*$.

To elucidate the above relationship we consider, as in [23], the effect of pressure on flow and the effect of flow on chamber pressures. Suppose Q is the flow rate of blood through the mitral valve (transmitral flow). This transmitral flow (Q) is as good as the rate of change in ventricular volume (dV_v/dt) and the negative of the rate of change in left atrial volume ($-dV_a/dt$). The time rate of change of flow can be written in the form

$$\frac{dQ}{dt} = \frac{\left(p_a - p_v - R_c Q^2 - R_v Q\right)}{M}, \tag{48}$$

where, $p_a(t)$ = Left atrial pressure (mm Hg), $p_v(t)$ = Left ventricular pressure (mm Hg), R_c = convective resistance of mitral valve (g/cm^{-7}), and R_v = viscous resistance of mitral valve (g/cm^{-4}/s^{-1}), M is a distributed mass term, $\varrho l/A^*$ with units g/cm^{-4}. Here ϱ is blood density (1.05 g/cm^3).

To decipher how chamber pressures are affected by flow, assume that at any time, the left ventricular pressure depends on the left ventricular volume, $p_v(V_v)$ and, by the same token, left atrial pressure depends on the atrial volume, $p_a(V_a)$.

We require expressions that relate the rate of change in chamber pressure (dp/dt) to the flow into or out of that chamber. The equations

$$\frac{dp_a}{dt} = \frac{\left(dV_a\right)}{dt} / \frac{dV_a}{dp_a} = -\frac{Q}{C_a} \tag{49}$$

$$\frac{dp_v}{dt} = \frac{\left(dV_v\right)}{dt} / \frac{dV_v}{dp_v} = \frac{Q}{C_v} \tag{50}$$

are the temporal left arterial and ventricular rate of change of pressure given in terms of the left arterial compliance, C_a ,and left ventricular compliance, C_v respectively.(Note that $1/C_{a,v}$ indicates chamber stiffness in each case). While equation (48) orders how pressure affects flow, equations (49-50) orders how chamber pressure is affected by flow. The coupled system of equations (49) and (50) is one that may predict the time progression of mitral flow, and atrial and ventricular pressure, together with prescribed initial pressures ($p_a(0)$ and $p_v(0)$), mitral impedance (M, R_c, and R_v) and chamber compliance.

Reduce (48)-(50) to two coupled system. Let $\Delta p = p_a - p_v$, and $C_+ = \left(\frac{1}{C_a} + \frac{1}{C_v}\right)^{-1}$,

where Δp is the transmitral gradient and C_+ is the net compliance of the two chambers. Equation (48) now takes the form

$$\frac{dQ}{dt} = \frac{\left(\Delta p - R_v Q - R_c Q^2\right)}{M}, \tag{51}$$

and subtract (50) from (49) to get

$$\frac{d\Delta p}{dt} = \frac{-Q}{C_+} \tag{52}$$

From (51) and (52) the transmitral flow (Q) may be determined.

5 Summary and result

The flow propensities in the cardiovascular structures were studied. The circulatory system is marked by laminar blood flow wherein concentric layers of blood flow in parallel down the length of a blood vessel. Nevertheless, laminar flow may transit to a chaotic (turbulent) flow under conditions of high flow velocity in which the critical Reynolds number (Re) is exceeded. In describing the flow characteristics in the cardiovascular structures associated phenomena such as acoustic wave and noise generation, coherent perturbation, and valvular pressure and flow relationship were discussed.The motion of the heart valves is usually taken as a boundary condition in the Navier–Stokes equation in defining the fluid dynamics of blood ejection from the left and right ventricles into the aorta and the lung.

The excerpts from this work are:

1. The pulsatile structure of flow drive the coherent perturbations (periodic waves) in any vascular region. The waveforms are vital to the determination of the physiological state of the cardiovascular parameters.
2. If the pressure drop, Δp , across an open heart valve is known the flow, Q, through the valve can be determined.

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