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A two-step iterative algorithm for finding common fixed point of Bregman quasi strict pseudo-contraction mappings with applications

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Abstract

Motivated by some results of authors in this direction, our purpose in this paper is to construct a two-step iterative algorithm with inertial extrapolation term defined with respect to Bregman distance function for approximating common fixed points of a finite family of a closed Bregman quasi strictly pseudo-contraction mappings when the intersection of its set is assumed to be non empty. We prove a strong convergence theorem for it under flexible and relaxed conditions in real reflexive Banach space. Our algorithm is applied in solving finite convex feasibility and equilibrium problems in reflexive Banach space. Furthermore, we demonstrate numerical examples to justify the efficiency and implementability of our algorithms in a finite dimensional real Banach space. Our results improves, generalizes and complements other previously and recently cited results of some authors in the literature of our procedure. It is intended that our procedure complement our motivated result cited in the literature.

Keywords and Phrases: Bregman quasi strict pseudo-contraction mappings; common fixed point; strong convergence theorem; reflexive Banach space; Bregman distance function

1 Introduction

In this paper, all spaces are taking to be real $\mathbb{R}$, with $\mathbb{R}$ as the set of all real numbers and $\mathbb{N}$ the set of natural numbers. The extended real numbers is denoted by $\overline{\mathbb{R}}$. Let X represents reflexive real Banach space with its dual spaces denoted as X^*. Let $||.|| : X \to \mathbb{R}$ represents the norm function. Let $d_h : dom\, h \times int(dom\, h) \to \mathbb{R}^+$ represent a function induced by a convex function $h : X \to (-\infty, +\infty]$. Let $dom\, h := \{u \in X : h(x) < \infty\}$ represents the domain of the convex function h and $int(dom\, h)$ represents the interior domain of h.

A function $h^* : X^* \to (-\infty, +\infty]$ defined by

$$h^*(x^*) = sup\,\{\langle x, x^*\rangle - h(x), x \in X\}$$

is called the conjugate function of h. We see from the conjugate inequality that

$$h(x) \geq \langle x, x^*\rangle - h^*(x^*), \forall x \in X, x^* \in X^*.$$

A function h on X is coercive (see [26]), if the sublevel set of h is bounded, equivalently

$$\lim_{||x||\to\infty} h(x) = +\infty.$$

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It is said to be strongly coercive (see [26]), if

$$\lim_{\|x\|\to\infty} \frac{h(x)}{\|x\|} = +\infty.$$

If for any $u \in int(dom\, h)$ and $z \in X$, we have the right-hand directional derivative given by

$$h^o(u,z) := \lim_{s\to 0^+} \frac{h(u+sz)-h(u)}{s}.$$

The convex function $h : X \to (-\infty, +\infty]$ is said to be Gâteaux differentiable at a point u if

$$\lim_{s\to 0^+} \frac{h(u+sz)-h(u)}{s}$$

exists for any z element of X. By this definition, $h^o(u,z) := \nabla h(u)$ which is the gradient of a convex function h. The function h is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. The convex function h is said to be uniformly Fréchet differentiable on a subset K of X if the limit is attained uniformly for $x \in K$ and $\|y\| = 1$ [25].

Let h be a Gâteaux differentiable function at u, then the bi-function $d_h : domh \times int(domh) \to \mathbb{R}^+$ defined by

$$d_h(z,u) := h(z) - h(u) - \langle \nabla h(u), z-u \rangle \tag{1}$$

is the Bregman distance function induced by the convex function h [25], where $\langle ., . \rangle$ is the duality pairing of X and X^*. It is easy to see that (1) is not symmetric and does not satisfy the well-known triangle inequality associated with classical distance functions, but has the following nice properties (see [7, 10]):

Remark 1.1

$d_h(., u)$ is convex

P2. $d_h(z,u)$ *is positive if and only if* $z \neq u$

P3. $d_h(z,u) = d_h(z,v) + d_h(v,u) + \langle \nabla h(v), z-v \rangle - \langle \nabla h(u), z-v \rangle$

P4. $d_h(u,v) + d_h(v,u) := \langle \nabla h(u), u-v \rangle - \langle \nabla h(v), u-v \rangle$

P5. $d_h(u,v) \leq ||u||.||\nabla h(u) - \nabla h(v)|| + ||v||.||\nabla h(u) - \nabla h(v)||$

P6. $d_h(u,u) = 0.$

Let K represent a non-void, closed and convex subset of $int(domh)$. Let $T : K \to K$ represent a self-map. The self-map T on K is said to be nonexpansive if $||Tu - Tz|| \leq ||u - z||$, $\forall u \in K$, $z \in K$. Similarly, the self-map T on K is said to be quasi-nonexpansive if $||Tu - z^o|| \leq ||u - z^o||$, $\forall u \in K$, $z^o \in Fix(T)$, where $Fix(T) := \{z^o \in K : Tu = u\}$ is the fixed point set of the self-map T on K. A point u^* is called asymptotic fixed point of a self-map T on K if there exist $u_n \subset K$ which converges weakly to u^* ($u_n \rightharpoonup u^*$) so that $||u_n - Tu_n|| \to 0$ as $n \to \infty$. The asymptotic fixed point (see [11] and the references therein) is represented by $\hat{Fix}(T)$.

A map $T : K \to int(dom\, h)$ with respect to a convex function $h : X \to (-\infty, +\infty]$ is called

1. Bregman relatively nonexpansive (shortly,(BRNE))[26, 24] if the following conditions holds

$$d_h(z^o, Tu) \leq d_h(z^o, u), \forall u \in K, \forall z^o \in Fix(T) = \hat{Fix}(T)$$

2. Bregman quasi nonexpansive (shortly,(BQNE))[34] if the following conditions holds

$$d_h(z^o, Tu) \leq d_h(z^o, u), \forall u \in K, \forall z^o \in Fix(T)$$

3. Bregman quasi strict pseudo-contraction (shortly,(BQSPC))[27] if there exists a constant $\rho \in [0, 1)$ and $Fix(T) \neq \emptyset$ such that the following conditions holds

$$d_h(z^o, Tu) \leq d_h(z^o, u) + \rho d_h(u, Tu), \forall u \in K, \forall z^o \in Fix(T) \tag{2}$$

4. T is called closed if for any $\{u_n\} \subset K$ with $u_n \to u$ and $Tu_n \to z \in K$ as $n \to \infty$, then $Tu = z$.

Example 1.1 *(see [28],[29]). Let X be a smooth Banach space, $h(u) := ||u||^2, \forall u \in X$. Let $u_0 \neq 0$ be any element of X, let $T : X \to X$ be defined by*

$$T(u) := \begin{cases} \frac{2^{n+1}+2}{2(2^{n+1})} u_0 & \text{if } u = \frac{2^n+2}{2^{n+1}} u_0 \\ -u & \text{if } u \neq \frac{2^n+2}{2^{n+1}} u_0, \end{cases} \tag{3}$$

$\forall n \geq 1$. Then T is a Bregman quasi strict pseudo-contraction with $Fix(T) = 0$.

Example 1.2 *Let $X = \mathbb{R}$, $K = [-1, 1]$, $h(u) := \frac{2}{3}u^2, \forall u \in K$. Let $T : K \to K$ be defined by*

$$T(u) := \begin{cases} u^2 + u & \text{if } u \in [-1, 0] \\ u & \text{if } u \in (0, 1]. \end{cases} \tag{4}$$

Then T is a Bregman quasi-strict pseudo-contraction with $Fix(T) = [0, 1]$. Proof. It is obvious from the definition of fixed point set and the mapping above that $Fix(T) := [0, 1]$ $\forall u \in [-1, 1]$. Next, we prove that T is a BQSPC mapping. From the definition of Bregman function (see 1), we want to show that since $Fix(T) \neq \emptyset$, we can find some constant ρ where $0 \leq \rho < 1$ satisfying (2) Now, if $u \in [-1, 0]$, we know that $Fix(T) = 0 = z^o$, and we get

$$\begin{aligned} d_h(0, Tu) &= h(0) - h(u^2 + u) - \left\langle \nabla h(u^2 + u), 0 - (u^2 + u) \right\rangle \\ &= 0 - \frac{2}{3}(u^2 + u)^2 + \left\langle \frac{4}{3}(u^2 + u), u^2 + u \right\rangle \\ &= \frac{2}{3}(u^2 + u)^2, \qquad (5) \\ d_h(0, u) &= h(0) - h(u) - \langle \nabla h(u), 0 - u \rangle \\ &= 0 - \frac{2}{3}u^2 + \left\langle \frac{4}{3}(u), u \right\rangle \\ &= \frac{2}{3}u^2, \qquad (6) \\ d_h(u, Tu) &= h(u) - h(u^2 + u) - \left\langle \nabla h(u^2 + u), u - (u^2 + u) \right\rangle \\ &= \frac{2}{3}u^2 - \frac{2}{3}(u^2 + u)^2 + \left\langle \frac{4}{3}(u^2 + u)u^2 \right\rangle \\ &= \frac{2}{3}u^4. \qquad (7) \end{aligned}$$

Thus,

$$d_h(0, Tu) \leq d_h(0, u) + \rho d_h(u, Tu), \forall u \in [-1, 0]. \tag{8}$$

Similarly, if $u \in (0,1]$, *we know* $Fix(T) := (0,1]$, *and for any* $z^o \in (0,1]$ *we arrive at*

$$\begin{aligned} d_h(z^o, Tu) &= h(z^o) - h(u) - \langle \nabla h(u), z^o - u \rangle \\ &= \frac{2}{3}(z^o)^2 - \frac{2}{3}u^2 - \frac{4}{3}z^o u + \frac{4}{3}u^2 \\ &= \frac{2}{3}(u - z^o)^2, \qquad (9) \\ d_h(z^o, u) &= h(z^o) - h(u) - \langle \nabla h(u), z^o - u \rangle \\ &= \frac{2}{3}(z^o)^2 - \frac{2}{3}(u)^2 + \frac{4}{3}z^o u + \frac{4}{3}u^2 \\ &= \frac{2}{3}(u - z^o)^2, \qquad (10) \\ d_h(u, Tu) &= h(u) - h(u^2+u) - \left\langle \nabla h(u^2+u), u - (u^2+u) \right\rangle \\ &= \frac{2}{3}u^2 - \frac{2}{3}(u^2+u)^2 + \left\langle \frac{4}{3}(u^2+u)u^2 \right\rangle \\ &= \frac{2}{3}u^4. \qquad (11) \end{aligned}$$

Thus,

$$d_h(z^o, Tu) \leq d_h(z^o, u) + \rho d_h(u, Tu), \forall u \in (0,1]. \qquad (12)$$

Therefore, from (8) and (12), we conclude that

$$d_h(z^o, Tu) \leq d_h(z^o, u) + \rho d_h(u, Tu), \forall u \in [-1,1]. \qquad (13)$$

Thus, we have proved that T *is a BQSPC mapping.*

Let $P_K^h : int(dom\ ) \to K$ be a mapping such that $P_K^h(u) \in K$ satisfying

$$d_h(P_K^h(u), u) := inf \{d_h(z,u) : z \in K\}, \qquad (14)$$

is the Bregman Projection (see[25]) of $u \in int(dom\ h)$ onto a nonempty closed and convex set $K \subset dom\ h$.

Remark 1.2 *We remark here that, if* X *is a smooth and strictly convex Banach spaces and* $h(u) =: \|u\|^2$, $\forall u \in X$, *then we have that* $\nabla h(u) = 2Ju$, $\forall u \in X$, *where* J *is the normalized duality mapping. Clearly, we obtain that*

$$\begin{aligned} d_h(z, u) &:= h(z) - h(u) - \langle \nabla h(u), z - u \rangle \\ &= \|z\|^2 - \|u\|^2 - 2\langle z, Ju \rangle + 2\|u\|^2 \\ &= \|u\|^2 - 2\langle z, Ju \rangle + \|z\|^2 \\ &= \phi(z, u), \forall u, z \in X, \end{aligned}$$

which is the Lyapunov function introduced by [1] and has extensively been used by various authors (see for example [18, 12] and the references they contain). We clearly see that $P_K^h(u)$ *reduces to the generalized projection [1] given as*

$$\Pi_K(u) := argmin_{z \in K} \phi(z, u).$$

In addition, if X *coincides with* H, *in Hilbert space then* $J = I$ *which is the identity map and*

$$\begin{aligned} d_h(z, u) &:= h(z) - h(u) - \langle \nabla h(u), z - u \rangle \\ &= \|u\|^2 - \|z\|^2 - 2\langle u, z \rangle + 2\|z\|^2 \\ &= \|u\|^2 + \|z\|^2 - 2\langle u, z \rangle \\ &= \|u - z\|^2, \forall u, z \in X. \end{aligned}$$

Hence the Bregman Projection $P_K^h(x)$ *reduces to metric projection of* H *onto* K, $P_K(u)$.

A map $T : K \to K$ is called

1. ϕ relatively nonexpansive if the following conditions holds

$$\phi(z^o, Tu) \leq \phi(z^o, u), \forall u \in K, \forall z^o \in Fix(T) = \hat{Fix}(T)$$

2. ϕ quasi strict pseudo-contraction if there exists a constant $\rho \in [0, 1)$ and $Fix(T) \neq \emptyset$ such that the following conditions holds

$$\phi(z^o, Tu) \leq \phi(z^o, u) + \rho\phi(u, Tu), \forall u \in K, \forall z^o \in Fix(T) \tag{15}$$

The subdifferential of h at x is the convex set defined by

$$\partial h(x) = \{x^* \in X^* : h(x) + \langle x^*, z - x\rangle \leq h(z); \forall z \in X\}. \tag{16}$$

The mapping $J : X \to 2^{X^*}$ called normalized duality mapping is defined by the set

$$J(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||^2, \forall x \in X\right\}, \tag{17}$$

A function $h : X \to (-\infty, +\infty]$ is said to be a Legendre function [9], if it satisfies the following two conditions:

1. $int(dom\, h) \neq \emptyset$, h is Gâteaux differentiable on $int(dom\, h)$ and $dom\, f = int(dom\, h)$,
2. $int(dom\, h^*) \neq \emptyset$, h^* is Gâteaux differentiable on $int(dom\, h^*)$ and $dom\, h^* = int(dom\, h^*)$.

Remark 1.3 *(See for e.g [6, 4, 23, 8] Since X is reflexive, then we have that* $(\partial h^{-1}) := \partial h^*$ *and since h is Legendre, then* ∂h *is a bijection which satisfies* $\nabla h = (\nabla h^*)^{-1}$, $ran\nabla h = dom\nabla h^* = int(dom\ h^*)$ *and* $ran\nabla h^* = dom\nabla h = int(dom\ h)$. *h and* h^* *are strictly convex on their* $int(dom\, h)$. *If the subdifferential of h is single valued, it coincides with the gradient of h, that is* $\partial h := \nabla h$. *Example of a Legendre function is* $h(u) := \frac{1}{p}||u||^p$, $(1 < p < \infty)$. *If X is smooth and strictly convex Banach spaces, then in this case the gradient* ∇h *coincides with the generalised duality mapping of X, that is* $\nabla h = J_p$. *If the space is a Hilbert space, H then* $\nabla h = I$, *where I is the identity mapping in H. Throughout this paper, we assumed that h is Legendre.*

Let $h : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of h at $u \in int(dom\, h)$ is the function $V_h(u, .) : int(dom\, h) \times [0, +\infty) \to [0, +\infty)$ defined by

$$V_h(u, t) := inf\, \{d_h(z, u) : z \in dom\, h, \|z - u\| = t\}. \tag{18}$$

The function h is called totally convex (see [8] and the references it contains) at u if $V_h(u, t) > 0$ whenever $t > 0$. The function h is called totally convex if it is totally convex at any point $u \in int(dom\, h)$. The function is said to be totally convex on bounded sets if $V_h(B, t) > 0$ for any non-void bounded subset B of X and $t > 0$, where the modulus of total convexity of the function h on the set B is the function $V_h : int(dom\, h) \times [0, +\infty) \to [0, +\infty)$ defined by

$$V_h(B, t) := inf\, \{V_h(u, t) : u \in B \cap dom\, h\}. \tag{19}$$

Let $V_h : X \times X^* \to [0, +\infty)$ associated with h ([19]) be defined by

$$V_h(u, u^*) := h(u) - \langle u, u^*\rangle + h^*(u^*), \forall u \in X, u^* \in X^*. \tag{20}$$

We see that $V_h(,) \geq 0$ and the relation

$$V_h(u, u^*) := d_h(u, \nabla h^*(u^*)), \tag{21}$$

holds.

Vast research work has been done on construction of effective and implimentable algorithms for finding fixed points of the class of nonexpansive mappings and its natural extensions like pseudo-contraction mappings in the framework of real Hilbert spaces, (see for e.g [18], [33] and the references therein). Outside Hilbert space, some researchers have studied fixed points problems of these important nonlinear mappings. In particular is the great attention given to construction of algorithms for finding fixed points of Bregman nonexpansive-type mappings by the use of Bregman distance techniques (see [2],[3],[15],[16],[19],[30],[20],[35]).

Some well known and used iteration methods abound in the literature (see [21],[33] and the references they contain). Most notably is the Mann iteration method (see [17]) constructed as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{22}$$

for an appropriate chosen value of the scalar $\alpha_n \in [0,1]$. One limitation about this method is that it only guarantees weak convergence to the fixed points of the mapping T when

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{23}$$

In a bid to establish strong convergence of the generated sequences to the fixed points of the mappings (since strong convergence is better than weak convergence), some authors have over the years successfully modified Mann iteration methods (see for examples, [24],[27],[29] and the references contained in them). In addition to this fact, most of the results obtained in the literature only focused on a one-way iterative methods for the approximation fixed points of various mappings either in Hilbert spaces or general Banach spaces (see for examples [11, 24, 34, 27, 28, 29, 19, 33, 30, 20, 15, 31] and the references therein).

Now, to approximate common fixed points of Bregman quasi strictly pseudo-contraction mappings defined a closed and convex subset K of a reflexive Banach space X, Wang and Wei in [29], formulated and studied the following modified Mann iterative method:

$$\begin{cases} x_0 \in K, \text{ Chosen arbitraily },\\ C_0^i = K, i = 1,2,...,N,\\ C_0 = \cap_{i=1}^N C_o^i,\\ y_n^i = \nabla h^*(b_n \nabla h(x_n) + (1-b_n)\nabla h(T_i)z_n^i),\\ C_{n+1}^i = \{z \in C_n : d_h(z, y_n^i) \leq b_n d_h(z, x_n) + (1-b_n)d_h(z, z_n^i) + \frac{\rho_i}{1-\rho_i}\langle \nabla h(z_n^i) - z, z_n^i - \nabla h(T_i z_n^i)\rangle\},\\ C_{n+1} = \cap_{i=1}^N C_{n+1}^i,\\ x_{n+1} = P_{C_{n+1}}^h(x_0), n \geq 0, \end{cases} \tag{24}$$

where $z_n^i = x_n + e_n^i$, e_n^i is the sequence of errors in X satisfy $\lim_{n\to\infty} e_n^i = 0$ for each $i = 1,2,...,N$ and $b_n \subset [0,1]$ satisfying $\liminf_{n\to\infty}(1-b_n) > 0$ $\rho_i \in [0,1)$. They proved that the sequence $\{x_n\}$ generated by the algorithm (24) converges strongly to the Bregman projection of X onto the common fixed point set.

Very recently in [20], Ogbuisi studied the operator $Res_{\lambda B}^h A_\lambda^h$ which is a composition of the resolvent of a maximal operator B and the anti-resolvent of a Bregman inverse strongly monotone operator A with respect to $\lambda > 0$. They constructed an iterative method for approximating a common solution of a monotone inclusion problem and a fixed point problem involving a Bregman quasi strictly pseudo-contractive mapping in a reflexive Banach space. Below is their algorithm:

$$\begin{cases} x_0 \in K = C_0, \text{ Chosen arbitraily }, \\ y_n = \nabla h^*(\alpha_n \nabla h(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla h(x_n) + \gamma_n \nabla h(Tx_n)], \\ u_n = \nabla h^*(\beta_n \nabla h(y_n) + (1-\beta_n)\nabla h(Res^h_{\lambda B} A^h_\lambda y_n)), \\ C_{n+1} = \{z \in C_n : d_h(z, y_n) + d_h(z, u_n) \\ \leq \frac{1+\lambda}{1-\lambda}\langle \nabla h(x_n) - \nabla h(Tx_n), x_n - z\rangle + \langle \nabla h(Tx_n) - \nabla h(u_n), x_n - z\rangle\} \\ x_{n+1} = P^h_{C_{n+1}}(x_0), n \geq 0, \end{cases} \tag{25}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty}(1-\alpha_n)\gamma_n > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. They proved that the sequence $\{x_n\}$ generated by the algorithm (25)converges strongly to the Bregman projection of K onto the common solution set.

Furthermore, some authors have introduced a two-step iterative method with an inertial extrapolation term to improve the rate of convergence of their introduced algorithms (see [12, 14, 2, 16]). Besides, an inertial modified Krasnoselskii-Mann iteration which is an inexact scheme, was introduced and studied by [21] for approximating nonexpansive mapping in Hilbert space. They assumed that the sum of their positive scalar and control sequences in $(0,1)$ to be less than one, that is $b_n + c_n < 1$. The advantage of this assumption is that it bridges the gap between weak and strong convergence results.

Thus, inspired by the established results of the authors cited above, our purpose in this paper is to construct a two-step iterative algorithm with inertial extrapolation term defined with respect to Bregman distance function for approximating common fixed points of a finite family of a closed Bregman quasi strictly pseudo-contraction mappings when the intersection of its set is assumed to be non empty. We prove a strong convergence theorem for it under flexible and relaxed conditions in real reflexive Banach space. Our algorithm is applied in solving finite convex feasibility and equilibrium problems in reflexive Banach space. Furthermore, we demonstrate numerical examples to justify the efficiency and implementability of our algorithms in a finite dimensional real Banach space. Our results improves, generalizes and complements other previously and recently cited results of some authors in the literature.

The organization of this paper is as follows: section 2 is a collection of basic theoretical framework (lemmas); section 3 is devoted to our main result which involves the analysis of the theoretical convergence properties of the proposed algorithm and some of its consequences; section 4 gives the application of our algorithm with some example; section 5 presents the numerical examples and computations to discuss the convergence of our algorithm and section 6 is the conclusion of our paper.

2 Preliminaries

Lemma 2.1 *[8] The function $h : X \to (-\infty, +\infty]$ is totally convex on bounded sets if and only if given two sequences $\{u_n\}$ and $\{z_n\}$ in X such that either $\{u_n\}$ or $\{z_n\}$ is bounded, then*

$$\lim_{n\to\infty} d_h(z_n, u_n) = 0 \Rightarrow ||z_n - u_n|| = 0$$

Lemma 2.2 *[27] Let $h : X \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of X. Let K be a non-empty, closed and convex subset of X. Let $G : K \to K$ be a Bregman quasi-strict pseudo-contractive mapping with respect to $h : X \to (-\infty, +\infty]$. Then $Fix(T)$ is closed and convex.*

Lemma 2.3 *[32] Let X be a reflexive real Banach space. Let $h : X \to (-\infty, +\infty]$ be a continuous convex function which is super coercive. Then the following assertions are equivalent:*

1. *$h : X \to (-\infty, +\infty]$ is bounded on bounded subsets and uniformly smooth on bounded subsets of X*

2. *$h : X \to (-\infty, +\infty]$ is Fréchet differentiable and ∇h^* is uniformly norm-to-norm continuous on bounded subsets of X^**

3. *dom $h^* = X^*$, h^* is super coercive and uniformly convex on bounded subsets of X^**

Lemma 2.4 *[25] Let $h : X \to (-\infty, +\infty]$ be a differentiable function on int(dom h) such that ∇h^* is bounded on bounded subsets of dom h^*. Let $u_0 \in X$ and $\{u_n\}$ is a sequence in X. If $\{d_h(u_0, u_n)\}$ is bounded, then $\{u_n\}$ is bounded.*

Lemma 2.5 *[27] Let $h : X \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of X. Let K be a non-empty, closed and convex subset of X. Let $T : K \to K$ be a Bregman quasi strict pseudo-contractive mapping with respect to $h : X \to (-\infty, +\infty]$. Then for any $u \in K$, $q \in Fix(T)$ and $\rho \in [0, 1)$, the following hold*

$$d_h(u, Tu) \leq \frac{1}{1-\rho} \langle \nabla h(u) - \nabla h(Tu), u - q \rangle .$$

Lemma 2.6 *[32] Let K be anon-empty, closed and convex subsets of X. Let $h : X \to (-\infty, +\infty]$ be a differentiable and totally convex function and let $u \in X$. Then we have the following equivalent conditions:*

1. *$P_K^h(u) = u_0$ if and only if $\langle \nabla h(u) - \nabla h(u_0), z - u_0 \rangle \leq 0, \forall z \in K$*

2. *$d_h(z, P_K^h(u)) + d_h(P_K^h(u), u) \leq d_h(z, u), \forall z \in K$.*

3 Main results

In this section, we construct and study the following algorithm when $\cap_{i=1}^N Fix(T_i) \neq \emptyset$:

$$\begin{cases} x_0, x_1 \in C_1 = K \\ C_{1,i} = \cap_{i=1}^N C_{1,i}, \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_{n,i} = \nabla h^* \left(b_n \nabla h\left(x_n\right) + \left(1 - b_n - c_n\right) \nabla h\left(T_i z_n\right) + c_n \nabla h\left(z_n\right) \right), \\ C_{n+1,i} = \{u \in C_n : d_h\left(u, y_{n,i}\right) \leq d_h\left(u, x_n\right) + d_h\left(x_n, z_n\right) \\ +\langle \nabla h\left(z_n\right) - \nabla h\left(x_n\right), x_n - u\rangle + \frac{\rho_i}{1-\rho_i} \langle \nabla h\left(z_n\right) - \nabla h\left(T_i z_n\right), z_n - u\rangle\}, \\ C_{n+1} = \cap_i^N C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}^h(x_0), n \geq 1, \end{cases} \tag{26}$$

where ∇h and ∇h^* are called the gradient functions of h and h^* respectively. Our control sequences: $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are all in $(0, 1)$ such that

1. $c_n \in (0, (1 - \rho_i)(1 - b_n) \subset (0, 1)$, $1 \leq i \leq N$,

2. $\liminf_{\to\infty} (1 - b_n - c_n) > 0$.

Theorem 3.1 *Let K be a nonempty, closed and convex subset of X with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X. Let $T_i : K \to K, \forall 1 \leq i \leq N$ represent finite-family of Bregman quasi strictly pseudo-contraction (BQSPC) maps which is also closed such that $\Omega = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. Then for each choice of $x_1, x_0 \in K$ with $x_1 < x_0$, we have sequences $\{x_n\}$ and $\{z_n\}$ satisfying (26) which converges strongly to $\overline{x} = P_\Omega^h(x_0)$ nearest to x_0, as $n \to \infty$ and where P_Ω^h is known to be Bregman projection map of C_{n+1} onto Ω.*

Proof. This is broken into 6 ways.

Step 1: Show $\Omega = \cap_{i=1}^{N} Fix(T_i)$ to be closed,convex.

It is obvious from Lemma 2.2 that $Fix(T_i)$ is closed,convex for any $1 \leq i \leq N$. Consequently, since the intersection of two or more convex set is convex, we get $\Omega = \cap_{i=1}^{N} Fix(T_i)$ to be closed,convex.

Step 2: Demonstrate the half-set C_n to be closed,convex for each positive integer.

To realize this, note $C_1 = K$ to be closed convex. Again, we demonstrate C_n to be closed convex with $n > 1$. First we demonstrate C_n to be convex for some $n > 1$. We let $u_1, u_2 \in C_{n+1,i}$ be given. Consider $u = \lambda u_1 + (1-\lambda)u_2, \forall \lambda \in [0,1]$. We show that $\lambda u_1 + (1-\lambda)u_2 \in C_{n+1,i} \subset C_n$. From our scheme, $d_h(u, y_{n,i}) \leq d_h(u, x_n) + d_h(x_n, z_n) + \langle \nabla h(z_n) - \nabla h(x_n), x_n - u\rangle + \frac{\rho_i}{1-\rho_i}\langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - u\rangle$ is equivalent to

$h(z_n) - h(y_{n,i}) + \langle \nabla h(x_n), u - x_n\rangle - \langle \nabla h(y_{n,i}), u - y_n\rangle + \langle \nabla h(z_n), x_n - z_n\rangle - \langle \nabla h(z_n), x_n - u\rangle + \langle \nabla h(x_n), x_n - u\rangle - \frac{\rho_i}{\rho_i}\langle \nabla h(z_n), z_n - u\rangle + \frac{\rho_i}{1-\rho_i}\langle \nabla h(T_i z_n), z_n - u\rangle \leq 0.$

Thus, by an easy argument, C_{n+1} and hence C_n is convex with $n > 1$. Secondly, we demonstrate C_n to be closed, $n > 1$. We show that given a sequence $u_n \in C_{n+1}$ such that $u_n \to u$, then $u \in C_{n+1} \subset C_n$ for some $n > 1$. Following the same line of argument above, and taking limits as $n \to \infty$, we have shown C_n to be closed, $n > 1$. Therefore, we have shown C_n to be closed convex.

Step 3: Demonstrate that $\Omega \subset C_{n+1}$, with $n \in N$.

Note $\Omega \subset C_1 = K$. Supposing $\Omega \subset C_n$, $n > 1$ and setting $w_{n,i} = b_n \nabla h(x_n) + (1 - b_n - c_n)\nabla h(T_i z_n) + c_n \nabla h(z_n)$, with $q \in \Omega$, we get using the convexity property of Bregman distance (see Remark4), together with (2) (20) and (21) that

$$\begin{aligned}
d_h(q, y_{n,i}) &= d_h(q, \nabla h^*(w_{n,i})) \\
&= v_h(q, w_{n,i}) \\
&= v_h(q, b_n \nabla h(x_n) + (1 - b_n - c_n)\nabla h(T_i z_n) + c_n \nabla h(z_n)) \\
&= h(q) - \langle q, b_n \nabla h(x_n) + (1 - b_n - c_n)\nabla h(T_i z_n) + c_n \nabla h(z_n)\rangle \\
&\quad + h^* b_n \nabla h(x_n) + (1 - b_n - c_n)\nabla h(T_i z_n) + c_n \nabla h(z_n) \\
&\leq b_n [h(q) - \langle q, \nabla h(x_n)\rangle + h^* \nabla h(x_n)] + (1 - b_n - c_n) \\
&\quad \times [h(q) - \langle q, \nabla h(T_i z_n)\rangle + h^* \nabla h(T_i z_n)] \\
&\quad + c_n [h(q) - \langle q, \nabla h(x_n)\rangle + h^* \nabla h(z_n)] \\
&= b_n v_h(q, \nabla h(x_n)) + (1 - b_n - c_n) v_h(q, \nabla h(T_i z_n)) \\
&\quad + c_n v_h(q, \nabla h(z_n)) \\
&= b_n d_h(q, x_n) + (1 - b_n - c_n) d_h(q, T_i z_n) + c_n d_h(q, z_n) \\
&\leq b_n d_h(q, x_n) + (1 - b_n - c_n)[d_h(q, z_n) + \rho_i d_h(z_n, T_i z_n)] \\
&\quad + c_n d_h(q, z_n) \\
&= b_n d_h(q, x_n) + (1 - b_n) d_h(q, z_n) + (1 - b_n - c_n)\rho_i d_h(z_n, T_i z_n) \\
&\leq b_n d_h(q, x_n) + (1 - b_n) d_h(q, z_n) + (1 - b_n - c_n) \times \\
&\quad \frac{\rho_i}{1-\rho_i}\langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - q\rangle \\
&\leq b_n d_h(q, x_n) + \frac{\rho_i}{1-\rho_i}\langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - q\rangle \\
&\quad + (1 - b_n)[d_h(q, x_n) + d_h(x_n, z_n) + \langle \nabla h(z_n) - \nabla h(x_n), x_n - q\rangle] \\
&\leq d_h(q, x_n) + d_h(x_n, z_n) + \langle \nabla h(z_n) - \nabla h(x_n), x_n - q\rangle \\
&\quad + \frac{\rho_i}{1-\rho_i}\langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - q\rangle .
\end{aligned} \tag{27}$$

Thus

$$\begin{aligned} d_h(q, y_n) &\leq d_h(q, x_n) + d_h(x_n, z_n) + \langle \nabla h(z_n) - \nabla h(x_n), x_n - q \rangle \\ &\quad + \frac{\rho_i}{1-\rho_i} \langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - q \rangle . \end{aligned} \tag{28}$$

So $q \in C_{n+1}$ and $C_{n+1} \subset C_n$. By implication, $\Omega \in C_n$ with $n \geq 1$.

Step 4: Demonstrate that all the sequences in (26) are bounded and convergent.

To justify the above claims, notice from our setting in (26), that $x_n = P^h_{C_n}(x_0)$, $x_{n+1} = P^h_{C_{n+1}}(x_0)$. Thus, using Lemma 2.6(2) we arrive at

$$\begin{aligned} d_h(x_n, x_0) &\leq d_h(x_{n+1}, x_0) - d_h(x_{n+1}, x_n) \\ d_h(x_{n+1}, x_0) &\geq d_h(x_n, x_0). \end{aligned} \tag{29}$$

This shows that $\{d_h(x_n, x_0)\}$ is monotone non-decreasing sequence. Again using Lemma 2.6(2), we obtain $\forall n \in N, q \in Fix(T)$ that

$$\begin{aligned} d_h(x_n, x_0) &= d_h(P^h_{C_n}(x_0), x_0) \\ &\leq d_h(q, x_0) - d_h(q, P^h_{C_n}(x_0)) \\ &\leq d_h(q, x_0). \end{aligned} \tag{30}$$

This shows for boundedness of $\{d_h(x_n, x_0)\}$ and through Lemma 2.4, the sequence $\{x_n\}$ becomes bounded. Combining (29) and (30) shows that $\lim_{n\to\infty} d_h(x_n, x_0)$ exist. Now, without loss of generality, let

$$\lim_{n\to\infty} d_h(x_n, x_0) = l. \tag{31}$$

In addition to (31) and Lemma 2.6(b), we get for any positive integer t and as $n \to \infty$, that

$$\begin{aligned} d_h(x_{n+t}, x_n) &= d_h(x_{n+t}, P^h_{C_n}(x_0)) \\ &\leq d_h(x_{n+t}, x_0) - d_h(x_n, x_0) \to 0. \end{aligned} \tag{32}$$

So that $\lim_{n\to\infty} d_h(x_{n+t}, x_n) = 0$. In particular,

$$\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0. \tag{33}$$

Thus we obtain using Lemma 2.1 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{34}$$

This shows the sequence $\{x_n\}$ to be Cauchy. Now since ∇h is bounded and going by Lemma 2.3,

$$\lim_{n\to\infty} ||\nabla h(x_{n+1}) - \nabla h(x_n)|| = 0. \tag{35}$$

Furthermore, we obtain from the definition of z_n and together with (34) as $n \to \infty$, that

$$\begin{aligned} ||x_n - z_n|| &= ||x_n - x_n - \alpha_n(x_n - x_{n-1})|| \\ &= ||\alpha_n(x_{n-1} - x_n)|| \\ &\leq ||x_{n-1} - x_n|| \to 0. \end{aligned}$$

We arrive at

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \tag{36}$$

Consequently,

$$\lim_{n\to\infty} ||\nabla h(x_n) - \nabla h(z_n)|| = 0. \tag{37}$$

By (36), z_n is bounded. Consequent upon the boundedness of ∇h and by (37) $\nabla h(z_n)$ is also bounded. Hence x_n, z_n and y_n are bounded. Moreover, since z_n is bounded we get by (Remark4) that $\lim_{n\to\infty} d_h(x_n, z_n) = 0$. Consequently, since ∇h is bounded we get together with (4) that

$$\begin{aligned} d_h(x_{n+1}, z_n) &= d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + \langle \nabla h(x_n) - \nabla h(z_n), x_{n+1} - x_n \rangle \\ &\leq d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + ||\nabla h(x_n) - \nabla h(z_n)||.||x_{n+1} - x_n||. \end{aligned}$$

Using (33), (34), (37) and the fact that $\lim_{n\to\infty} d_h(x_n, z_n) = 0$, we get

$$\lim_{n\to\infty} d_h(x_{n+1}, z_n) = 0. \tag{38}$$

Using Lemma 2.1,

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0. \tag{39}$$

Since from (26) $x_{n+1} \in C_{n+1} \subset C_n$, it follows from (33),(34),(35),(37),(39) and the boundedness of ∇h that

$$\begin{aligned} d_h(x_{n+1}, y_{n,i}) &\leq d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + \langle \nabla h(z_n) - \nabla h(x_n), x_n - x_{n+1} \rangle \\ &+ \frac{\rho_i}{1-\rho_i} \langle \nabla h(z_n) - \nabla h(T_i z_n), z_n - x_{n+1} \rangle . \\ &\leq d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + ||\nabla h(z_n) - \nabla h(x_n)||.||x_n - x_{n+1}|| \\ &+ \frac{\rho_i}{1-\rho_i} ||\nabla h(z_n) - \nabla h(T_i z_n)||.||z_n - x_{n+1}|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{40}$$

Hence

$$\lim_{n\to\infty} d_h(x_{n+1}, y_{n,i}) = 0, 1 \leq i \leq N. \tag{41}$$

Thus, by Lemma 2.1 we arrive at

$$\lim_{n\to\infty} ||x_{n+1} - y_{n,i}|| = 0, 1 \leq i \leq N. \tag{42}$$

Since ∇h is bounded and by X^* by Lemma 2.3, we arrive at

$$\lim_{n\to\infty} ||\nabla h(x_{n+1}) - \nabla h(y_{n,i})|| = 0, 1 \leq i \leq N. \tag{43}$$

Consequently,

$$\lim_{n\to\infty} ||x_n - y_{n,i}|| = 0, 1 \leq i \leq N, \tag{44}$$

and

$$\lim_{n\to\infty} ||\nabla h(x_n) - \nabla h(y_{n,i})|| = 0, 1 \leq i \leq N. \tag{45}$$

Using (38), (41), (43) and (39) we also obtain as $n \to \infty$ that

$$\begin{aligned} d_h(z_n, y_{n,i}) &= d_h(z_n, x_{n+1}) + d_h(x_{n+1}, y_{n,i}) + \langle \nabla h(x_{n+1}) - \nabla h(y_{n,i}), z_n - x_{n+1} \rangle \\ &\leq d_h(z_n, x_{n+1}) + d_h(x_{n+1}, y_{n,i}) + ||\nabla h(x_{n+1}) - \nabla h(y_{n,i})||.||z_n - x_{n+1}|| \\ &\to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} d_h(z_n, y_{n,i}) = 0, 1 \leq i \leq N. \tag{46}$$

Using Lemma 2.1, we obtain that

$$\lim_{n\to\infty} ||z_n - y_{n,i})|| = 0, 1 \leq i \leq N. \tag{47}$$

Since ∇h is bounded and by Lemma 2.3, we arrive at

$$\lim_{n\to\infty} ||\nabla h(z_n) - \nabla h(y_{n,i})|| = 0, 1 \leq i \leq N. \tag{48}$$

Using (48) and the two point identity of Bregman distance function (see Remark4) we have

$$\begin{aligned} d_h(z_n, y_{n,i}) + d_h(y_{n,i}, z_n) &= \langle \nabla h(z_n) - \nabla h(y_{n,i}), z_n - y_{n,i}\rangle \\ d_h(z_n, y_{n,i}) + d_h(y_{n,i}, z_n) &\leq ||\nabla h(z_n) - \nabla h(y_{n,i})||.||z_n - y_{n,i}|| \\ d_h(z_n, y_{n,i}) &\leq ||\nabla h(z_n) - \nabla h(y_{n,i})||.||z_n - y_{n,i}|| \to 0. \end{aligned} \tag{49}$$

Hence

$$\lim_{n\to\infty} d_h(z_n, y_{n,i}) = 0. \tag{50}$$

Consequently from the definition of $y_{n,i}$ in (26) we have that

$$\begin{aligned} ||\nabla h(z_n) - \nabla h(y_{n,i})|| &= ||\nabla h(z_n) - (\nabla h(\nabla h^*(b_n \nabla h(x_n) \\ &\quad + (1 - b_n - c_n)\nabla h(T_i z_n) + c_n \nabla h(z_n))))|| \\ &\geq (1 - b_n - c_n)\, ||\nabla h(z_n) - \nabla h(T_i z_n)|| \\ &\quad - b_n ||\nabla h(z_n) - \nabla h(x_n)|| + c_n ||z_n - z_n||, \end{aligned} \tag{51}$$

this becomes

$$\begin{aligned} (1 - b_n - c_n)\, ||\nabla h(z_n) - \nabla h(T_i z_n)|| &\leq ||\nabla h(z_n) - \nabla h(y_{n,i})|| \\ &\quad + b_n ||\nabla h(z_n) - \nabla h(x_n)||. \end{aligned} \tag{52}$$

Thus by (37) and (48) as $n \to \infty$ we obtain

$$(1 - b_n - c_n)\, ||\nabla h(z_n) - \nabla h(T_i z_n)|| \to 0. \tag{53}$$

Using our assumption that $\liminf_{n\to\infty}(1 - b_n - c_n) > 0$, we obtain

$$\lim_{n\to\infty} ||\nabla h(z_n) - \nabla h(T_i z_n)|| = 0, i = 1, 2, ..., N. \tag{54}$$

Using Lemma 2.3, we get that

$$\lim_{n\to\infty} ||z_n - T_i z_n|| = 0, i = 1, 2, ..., N. \tag{55}$$

Step 5: Demonstrate that $\overline{x} \in \Omega := \cap_{i=1}^{N} Fix(T_i)$

In view of the sequence x_n being Cauchy and from (39), we have that $z_n \to \overline{x}$ as $n \to \infty$. Following from (55), the fact that $z_n \to \overline{x}$, and by the closedness property of T_i, we arrive at $\overline{x} = T_i \overline{x}$, with $1 \leq i \leq N$.

Step 6: Demonstrate $\overline{x} = P^h_\Omega(x_0)$.

Set $u = P^h_\Omega(x_0)$. In **Step 3**, we demonstrated that $\Omega \subset C_n$. Since $P^h_\Omega(x_0) \in \Omega$, we get that $P^h_\Omega(x_0) \subset C_n$. It then follows from our setting with $x_n = P^h_{C_n}(x_0)$ that

$$d_h(x_n, x_0) \leq d_h(u, x_0). \tag{56}$$

Since $x_n \to \overline{x}$ as $n \to \infty$, we get from (56) that

$$d_h(\overline{x}, x_0) \leq d_h(u, x_0). \tag{57}$$

But by the Property of Bregman Projection mapping in Lemma 2.6, we have for $\forall w \in \Omega$ that

$$d_h(u, x_0) \leq d_h(w, x_0). \tag{58}$$

This implies

$$d_h(u, x_0) \leq d_h(\overline{x}, x_0). \tag{59}$$

Thus by combining (57) and (59) we then arrive at $u = \overline{x}$. Therefore, $\overline{x} = P^h_\Omega(x_0)$. This completes our proof of Theorem 3.1.

Now the following consequences of Theorem 3.1 are gotten.

If $N = 1$, we have an algorithm for a single BQSPC map.

$$\begin{cases} x_0, x_1 \in C_1 = K \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_n = \nabla h^* \left(b_n \nabla h\left(x_n\right) + \left(1 - b_n - c_n\right) \nabla h\left(T z_n\right) + c_n \nabla h\left(z_n\right)\right), \\ C_{n+1} = \{u \in C_n : d_h\left(u, y_n\right) \leq d_h\left(u, x_n\right) + d_h\left(x_n, z_n\right) \\ +\langle \nabla h\left(z_n\right) - \nabla h\left(x_n\right), x_n - u\rangle + \frac{\rho}{1-\rho}\langle \nabla h\left(z_n\right) - \nabla h\left(T z_n\right), z_n - u\rangle\}, \\ x_{n+1} = P^h_{C_{n+1}}(x_0), n \geq 1, \end{cases} \tag{60}$$

where ∇h and ∇h^* are called the gradient functions of h and h^* respectively. Our control sequences: $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are all in $(0, 1)$ such that

1. $c_n \in (0, (1 - \rho_i)(1 - b_n) \subset (0, 1)\ i = 1,$
2. $\liminf_{\to\infty} (1 - b_n - c_n) > 0.$

Corollary 3.1 *Let K be a nonempty, closed and convex subset of X with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X. Let $T : K \to K$ represent Bregman quasi strictly pseudo-contraction mapping which is also closed such that $\Omega = Fix(T) \neq \emptyset$. Then for each choice of $x_1, x_0 \in K$ with $x_1 < x_0$, we have sequences $\{x_n\}$ and $\{z_n\}$ satisfying (60) which converges strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0, as $n \to \infty$, with P^h_Ω as Bregman projection map of C_{n+1} onto Ω.*

If $T_i : K \to K, i = 1, 2, ..., N$ represent finite family of Bregman quasi nonexpansive mappings, we also have this algorithm for it.

$$\begin{cases} x_0, x_1 \in C_1 = K \\ C_{1,i} = \cap_{i=1}^N C_{1,i}, \\ z_n = x_n) + a_n(x_n) - x_{n-1}), \\ y_{n,i} = \nabla h^* \left(b_n \nabla h\left(x_n\right) + \left(1 - b_n - c_n\right) \nabla h\left(T_i z_n\right) + c_n \nabla h\left(z_n\right)\right), \\ C_{n+1,i} = \{u \in C_n : d_h\left(u, y_{n,i}\right) \leq d_h\left(u, x_n\right) + d_h\left(x_n, z_n\right) \\ +\langle \nabla h\left(z_n\right) - \nabla h\left(x_n\right), x_n - u\rangle \\ C_{n+1} = \cap_i^N C_{n+1,i}, \\ x_{n+1} = P^h_{C_{n+1}}(x_0), n \geq 1, \end{cases} \tag{61}$$

where ∇h and ∇h^* are called the gradient functions of h and h^* respectively. Our control sequences: $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are all in $(0, 1)$ such that

1. $c_n \in (0, (1 - b_n) \subset (0, 1),$
2. $\liminf_{\to\infty} (1 - b_n - c_n) > 0.$

Corollary 3.2 *Let K be a nonempty, closed and convex subset of X with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X. Let $T_i : K \to K$ represent* *finite family of closed Bregman quasi nonexpansive mappings such that* $\Omega = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. *Then for each choice of* $x_1, x_0 \in K$ *with* $x_1 < x_0$, *we have sequences* $\{x_n\}$ *and* $\{z_n\}$ *satisfying (61) which converges strongly to* $\overline{x} = P_{\Omega}^{h}(x_0)$ *nearest to* x_0, *as* $n \to \infty$, *with* P_{Ω}^{h} *as the Bregman projection mapping of* C_{n+1} *onto* Ω.

If $h(x) = \|x\|^2$, then $d_h(x, y) = \phi(x, y)$ (see Remark 1.2). In what follows, we have $T_i : K \to K$ as a finite family of closed quasi-ϕ -strictly pseudo-contraction mappings and the algorithm below for it .

$$\begin{cases} x_0, x_1 \in C_1 = K \\ C_{1,i} = \cap_{i=1}^{N} C_{1,i}, \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_{n,i} = J^* \left(b_n J\left(x_n\right) + \left(1 - b_n - c_n\right) J\left(T_i z_n\right) + c_n J\left(z_n\right) \right), \\ C_{n+1,i} = \{ u \in C_n : \phi\left(u, y_{n,i}\right) \leq \phi\left(u, x_n\right) + \phi\left(x_n, z_n\right) \\ + \langle J\left(z_n\right) - J\left(x_n\right), x_n - u \rangle + \frac{\rho_i}{1-\rho_i} \langle J\left(z_n\right) - J\left(T_i z_n\right), z_n - u \rangle \}, \\ C_{n+1} = \cap_{i}^{N} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \geq 1, \end{cases} \tag{62}$$

where J and J^* are called the normalized duality mappings on X and X^* respectively, $\{a_n\} \in (0, 1)$, $\{b_n\} \in (0, 1)$, $\{c_n\} \in (0, 1)$, such that

1. $c_n \in (0, (1 - \rho_i)(1 - b_n) \subset (0, 1)$, $i = 1, 2, ..., N$,
2. $\liminf_{\to \infty} (1 - b_n - c_n) > 0$.

Corollary 3.3 *Let* K *be a nonempty, closed and convex subset a uniformly convex and uniformly smooth real Banach space. Let* $T_i : K \to K$ *represent family of quasi-* ϕ *-strictly pseudo-contraction maps which is closed such that* $\Omega = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. *Then for each choice of* $x_1, x_0 \in K$ *with* $x_1 < x_0$, *we have sequences* $\{x_n\}$ *and* $\{z_n\}$ *satisfying (62) which converges strongly to* $\overline{x} = \Pi_{\Omega}(x_0)$ *nearest to* x_0, *as* $n \to \infty$, *with* Π_{Ω} *as the generalized projection mapping of* C_{n+1} *onto* Ω.

If $T : K \to K$ represent relatively nonexpansive mapping, we have algorithm for it as follows:

$$\begin{cases} x_0, x_1 \in C_1 = K \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_n = J^* \left(b_n J\left(x_n\right) + \left(1 - b_n - c_n\right) J\left(T z_n\right) + c_n J\left(z_n\right) \right), \\ C_{n+1} = \{ u \in C_n : \phi\left(u, y_n\right) \leq \phi\left(u, x_n\right) + \phi\left(x_n, z_n\right) \\ + \langle J\left(z_n\right) - J\left(x_n\right), x_n - u \rangle \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \geq 1, 8 \end{cases} \tag{63}$$

Corollary 3.4 *Let* K *be a nonempty, closed and convex subset a uniformly convex and uniformly smooth real Banach space. Let* $T : K \to K$ *represent relatively nonexpansive mapping such that* $\Omega = Fix(T) \neq \emptyset$. *Then for each choice of* $x_1, x_0 \in K$ *with* $x_1 < x_0$, *we have sequences* $\{x_n\}$ *and* $\{z_n\}$ *satisfying (63) which converges strongly to* $\overline{x} = \Pi_{\Omega}(x_0)$ *nearest to* x_0, *as* $n \to \infty$, *with* Π_{Ω} *as the generalized projection mapping of* C_{n+1} *onto* Ω.

4 Application

In this section we provide some applications of Theorem 3.1 as follows.

4.1. Convex feasibility problems

We let $K_1, K_2, ..., K_N$ represent a closed convex and nonempty sets such that the finite intersection of these sets is not empty. The convex feasibility problem (CFP) is to find a point say $z \in K$. The Bregman Projection onto the i^{th} constraint K_i with respect to a Legendre function h is denoted by $P^h_{K_i}$. It is easy to compute that $Fix(P^h_{K_i}) = K_i, \forall i = 1, 2, ..., N$. If in addition, the Legendre function is necessarily uniform Fréchet differentiable and totally convex on bounded subsets of a real reflexive Banach space X, we get that $P^h_{K_i}$ is in particular a Bregman relatively nonexpansive mappings (see [25]), thus a closed Bregman quasi strictly pseudo-contraction mappings such that $T_i = P^h_{K_i}, \forall i = 1, 2, ..., N$. With this fact, the fixed problem for finite family of closed Bregman quasi strictly pseudo-contraction nonexpansive-type mappings becomes solutions of finite family of convex feasibility problems. From Theorem 3.1, we have a strong convergence results for common solution of convex feasibility problems below:

Theorem 4.1 *Let K be a nonempty, closed and convex subset of X with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X. Let $K_i, \forall i = 1, 2, ..., N$ represent finite family of closed, nonempty subset of X such that $\Omega = \cap_{i=1}^N Fix(P^h_{K_i}) \neq \emptyset$. Choose $x_0, x_1 \in K$ such that $x_1 < x_0$ and define the sequence $\{x_n\}$ thus:*

$$\begin{cases} x_0, x_1 \in C_1 = K \\ C_{1,i} = \cap_{i=1}^N C_{1,i}, \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_{n,i} = \nabla h^* \left(b_n \nabla h\left(x_n\right) + (1 - b_n - c_n) \nabla h\left(P^h_{K_i} z_n\right) + c_n \nabla h\left(z_n\right)\right), \\ C_{n+1,i} = \{u \in C_n : d_h\left(u, y_{n,i}\right) \leq d_h\left(u, x_n\right) + d_h\left(x_n, z_n\right) \\ + \langle \nabla h\left(z_n\right) - \nabla h\left(x_n\right), x_n - u\rangle + \frac{\rho_i}{1-\rho_i} \langle \nabla h\left(z_n\right) - \nabla h\left(T_i z_n\right), z_n - u\rangle\}, \\ C_{n+1} = \cap_i^N C_{n+1,i}, \\ x_{n+1} = P^h_{C_{n+1}}(x_0), n \geq 1, \end{cases} \tag{64}$$

where $\{a_n\} \in (0, 1)$, $\{b_n\} \in (0, 1)$, and $\{c_n\} \in (0, 1)$ such that $c_n \in (0, (1-\rho_i)(1-b_n) \subset (0, 1)$, $i = 1, 2, ..., N$, with $\liminf_{\to\infty} (1 - b_n - c_n) > 0$. Then the sequences $\{x_n\}, \{z_n\}$ in (64) converges strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0, as $n \to \infty$ and where P^h_Ω is the Bregman projection mapping of C_{n+1} onto Ω.

4.2. Equilibrium problems

A mapping $\Theta : K \times K \to \mathbb{R}$ is called a bifunction so that the equilibrium problem with respect to $\Theta : K \times K \to \mathbb{R}$ is to find $z^0 \in K$ such that

$$\Theta(z^0, z) \geq 0, \forall z \in K. \tag{65}$$

The collection of solution of (65) is represented by

$$EP(K, \Theta) := \left\{z^0 \in K : \Theta(z^0, z) \geq 0, \forall z \in K\right\}.$$

To solve a problem of the form (65) (see [5]), certain conditions are imposed on the bifunction $\Theta : K \times K \to \mathbb{R}$ as follows:

Assumption 1

(A1) $\Theta(z, z) = 0, \forall z \in K$,

(A2) $\Theta : K \times K \to \mathbb{R}$ is monotone, that is $\Theta(z^0, z) + \Theta(z, z^0) \leq 0$,

(A3) $\limsup_{t\downarrow 0} \Theta(\mu x + (1 - \mu)z^0, z) \leq \Theta(z^0, z), \forall x, z, z^0 \in K$,

(A4) The function $z \mapsto \Theta(z^0, z)$ is convex and lsc.

The resolvent of a bifunctions $\Theta : K \times K \to \mathbb{R}$ is the mapping $Res^h_\Theta : X \to 2^K$ defined by

$$Res^h_\Theta(x) := \left\{z^0 \in K : \Theta(z^0, z) + \left\langle \nabla h(z^0) - \nabla h(x), z - z^0 \right\rangle \geq 0, \forall z \in K\right\}. \tag{66}$$

Lemma 4.1 *[25] Let K be a closed and convex subset of X and let $h : X \to R \cup \{\infty\}$ be a coercive and Gâteaux differentiable function. Let $\Theta : K \times K \to R$ be a bifunction satisfying **Assumption 1** conditions, then the domain of the resolvent of a bifunctions with respect to h is X (i.e) $dom(Res^h_\Theta) = X$.*

Lemma 4.2 *[29] Let K be a closed and convex subset of X and let $h : X \to R \cup \{\infty\}$ be a coercive Legendre function. Let $\Theta : K \times K \to R$ be a bifunction satisfying **Assumption 1** conditions, and let Res^h_Θ be a resolvent mapping $Res^h_\Theta : X \to 2^K$ defined by (66), then we these conditions are met:*
(a) Res^h_Θ is single-valued,
(b) $Fix(Res^h_\Theta) = EP(K, \Theta)$,
(c) $d_h(q, Res^h_\Theta x) + d_h(Res^h_\Theta x, x) \leq d_h(q, x), \forall q \in Fix(Res^h_\Theta)$,
(d) Res^h_Θ is Bregman quasi nonexpansive type mapping.

Theorem 4.2 *Let K be a nonempty, closed and convex subset of X with its dual space X^*. Let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X. Let $\Theta_i : X \times X \to \mathbb{R}, \forall i \in I$ represent finite family of bifunctions which satisfy **Assumption 1** conditions and let Res^h_Θ : $X \to 2^K$ be a resolvent mapping such that $\Omega = \cap_{i\in I} Fix(Res^h_{\Theta_i}) \neq \emptyset$. Choose $x_0, x_1 \in K$ such that $x_1 < x_0$ and define the sequence $\{x_n\}$ thus:*

$$\begin{cases} x_0, x_1 \in C_1 = K \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_n = \nabla h^* \left(b_n \nabla h\left(x_n\right) + (1 - b_n - c_n) \sum_{i\in I} \lambda^i_n \nabla h\left(v^i_n\right) + c_n \nabla h\left(z_n\right)\right), \\ C_{n+1} = \{u \in C_n : d_h\left(u, y_n\right) \leq d_h\left(u, x_n\right) + d_h\left(x_n, z_n\right) \\ +\langle \nabla h\left(z_n\right) - \nabla h\left(x_n\right), x_n - u\rangle\}, \\ x_{n+1} = P^h_{C_{n+1}}(x_0), n \geq 1, \end{cases} \tag{67}$$

where $v^i_n := Res^h_{\Theta_i}(z_n)$, $\{a_n\} \in (0,1)$, $\{b_n\} \in (0,1)$, $\{c_n\} \in (0,1)$, and $\{\lambda^i_n\} \in (0,1)$ such that (1) $\sum_{i\in I} \lambda^i_n = 1$, (2) $c_n \in (0, (1-\rho_i)(1-b_n) \subset (0,1)$ for each $i = 1, 2, ..., N$ and (3) $\liminf_{\to\infty}(1 - b_n - c_n) > 0$. Then the sequences $\{x_n\}, \{z_n\}, \{u_n\}$ in (67) converges strongly to $\overline{x} = P^h_\Omega(x_0)$ nearest to x_0, with P^h_Ω as Bregman projection of C_{n+1} onto Ω.

Proof. Following the same approach as demonstrated in the proof of Theorem 3.1, we have

$$\lim_{n\to\infty} ||v^i_n - z_n|| = 0. \tag{68}$$

In view of (68) and the fact that ∇h is uniformly continuous and for each $i \in I$, we obtain

$$\lim_{n\to\infty} ||\nabla h(v^i_n) - \nabla h(z_n)|| = 0. \tag{69}$$

Furthermore,

$$\Theta_i\left(v^i_n, z\right) + \frac{i}{r^i_n}\left\langle z - v^i_n, \nabla h\left(v^i_n\right) - \nabla h\left(z_n\right)\right\rangle \geq 0, z \in K, \forall i \in I. \tag{70}$$

So by the monotocity property $A1$ of the bifunction $\Theta_i : K \times K \to \mathbb{R}$, we get for $n \geq 1$ and for each $i \in I$ that

$$\frac{i}{r^i_n}\left\langle z - v^i_n, \nabla h\left(v^i_n\right) - \nabla h\left(z_n\right)\right\rangle \geq \Theta_i\left(z, v^i_n\right), z \in K. \tag{71}$$

By condition $A4$, (69) and the fact that $v_n^i \to \overline{x}$ for each $i \in I$ and as $n \to \infty$, we get

$$\begin{aligned} 0 &\geq -\Theta_i(\overline{x}, z) \geq \Theta_i(z, \overline{x}) \\ &\geq \Theta_i(z, \overline{x}) + \Theta_i(\overline{x}, z) \\ &\geq \Theta_i(z, \overline{x}), \forall z \in K. \end{aligned}$$

Now consider $z_\mu = \mu z + (1-\mu)\overline{x}$, $\forall \mu \in (0,1)$ and $z \in K$. This shows that $z_\mu \in K$ and so $\Theta_i(z_\mu \overline{x}) \leq 0$. Following from conditions $A1$ and $A4$, we get

$$\begin{aligned} \Theta_i(z_\mu, z_\mu) &= 0 = \Theta_i(z_\mu, \mu z + (1-\mu)\overline{x}) \\ &\leq \mu\Theta_i(z_\mu, z) + (1-\mu)\Theta_i(z_\mu, \overline{x}) \\ &\leq \mu\Theta_i(z_\mu, z). \end{aligned}$$

Since $\mu \in (0,1)$, we get

$$0 \leq \Theta_i(z_\mu, z)$$

for each $i \in I$. Using condition $A3$ we have that

$$0 \leq \Theta_i(z_\mu, z) \leq \Theta_i(\overline{x}, z),$$

$\forall \overline{x}, z \in K$ and for each $i \in I$. This has shown that $\overline{x} \in EP(\Theta_i)$ for each $i \in I$. Hence $\overline{x} \in \cap_{i \in I} EP(\Theta_i)$. Thus, we obtain that $\overline{x} \in \Omega := \cap_{i \in I} EP(\Theta_i))$.

Example 4.1 (as application) *Let $X = R$, $K = [-1,1]$, $h(x) := \frac{2}{3}x^2$. The equilibrium problem with respect to $\Theta : K \times K \to \mathbb{R}$ defined by*

$$\Theta_i(u, z) = z^2 - zu + iuz - iu^2, \forall z \in K, \tag{72}$$

*is to find $u \in K$, for each $i \in I$ (see [22]). It is very obvious that for each $i \in I$, $\Theta_i(u,z)$ satisfies **Assumption 1** conditions. Also using the fact that Res^h_{Θ,r_i} is single valued, we get using*

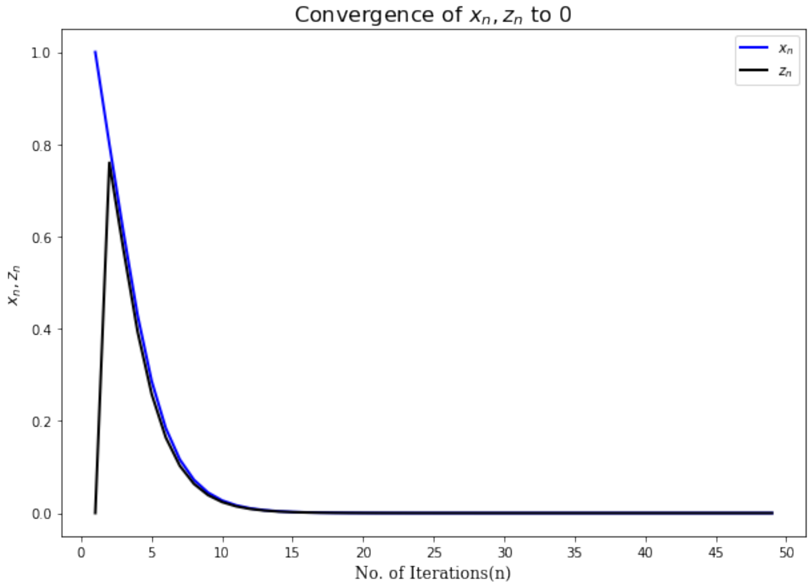
$$Res^h_{\Theta,r_i}(z_n) := \left\{ v_n^i \in K : \Theta(v_n^i, z) + \frac{1}{r_i} \left\langle \nabla h(v_n^i) - \nabla h(z_n), z - v_n^i \right\rangle \geq 0, \forall z \in K \right\},$$

that

$$v_n^i = Res^h_{\Theta,r_i}(z_n) = \frac{4z_n}{3ir_i + 3r_i + 4}. \tag{73}$$

We see also for each $i \in I$, that $Fix(Res^h_{\Theta_i}) = 0$. Now we set $i \in (1, \ldots 10)$, $x_0 = 1.0, x_1 = 0.5$ and $a_n = \frac{1}{(n+10)^2}, b_n = \frac{1}{n+1}, c_n = \frac{1}{4n}$, $r_n^i = \frac{2n}{in+n+i+1}, \lambda_n^i = \frac{1}{2}$, with $\sum_{i \in I} \lambda_n^i = 1$ (see Table and figure below). Thus, our algorithm of Theorem 67 simplifies to

$$\begin{cases} x_0, x_1 \in K, \\ C_1 = K = [-1,1] \\ z_n := \frac{x_n + (x_n - x_{n-1})}{(n+10)^2}, \\ v_n^1 = \cdots v_n^{10} = \frac{2z_n(n+1)}{5n+2}, \\ y_n := \frac{8n^2 z_n + 10nx_n^2 + 11nz_n + 4x_n^2 + 3z_n}{4(n+1)(5n+2)}, \\ C_{n+1} := \left\{ u \in C_n : z \leq \frac{28n^2 z_n + 10nx_n^2 + 39nz_n + 4x_n^2 + 11z_n}{8(n+1)(5n+2)} \right\}, \\ x_{n+1} := P^h_{K_{n+1}}(x_0), n \geq 1. \end{cases} \tag{74}$$



5 Numerical examples and implementation of (26)

In this section we demonstrate the effectiveness, implementation and convergence of our scheme (26) of Theorem 3.1. We do this by considering similar problem of [29]. We also use Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz to perform a side by side simulations of our scheme (26) with other similar results of some authors cited in the literature.

Example 5.1 *(cf.[29]) Let $K = [0, 1]$, $h(x) := x^2$. Let $T_1, T_2 : K \to K$ be defined by $T_1(x) = sinx$ and $T_2(x) = sin(\frac{1}{2}x)$ respectively. Then it is easy to see that both T_1 and T_2 are Bregman quasi strictly pseudo-contraction mappings with $Fix(T_i) = \{0\}$, and $\rho_i = 0.5$ for each $i = 1, 2$.*

*Now, with two different sets of scalar sequences as follows: **(a)** $a_n = 0.6$, $b_n = \frac{5}{10}$, $c_n = \frac{1}{4} \in (0,(1-\rho_i)(1-b_n)) \subset (0, 1)$ used in Table 1 and Figure 1, and **(b)** $a_n = 0.4$, $b_n = \frac{1}{2n}$, $c_n = \frac{2n-1}{4n} \in (0,(1-\rho_i)(1-b_n)) \subset (0, 1)$ used in Table 2 and Figure 2, then our assumptions in Theorem 3.1 are all satisfied. In addition, our scheme (26) is simplified thus:*

$$\begin{cases} x_0, x_1 \in K, \\ C_{1,i} = K = [0,1], i = 1, 2, \\ z_n := x_n + a_n(x_n - x_{n-1}), \\ y_{n,1} := b_n x_n + (1 - b_n - c_n) sin z_n + c_n z_n, \\ y_{n,2} := b_n x_n + (1 - b_n - c_n) sin\frac{1}{2} z_n + c_n z_n, \\ C_{n+1,1} := \{u \in C_n : u \leq \frac{(2\rho_1 sin(z_n) z_n - \rho_1 y_{n,1}^2 - \rho_1 z_n^2 + y_{n,1}^2 - z_n^2)}{2(\rho_1 sin(z_n) - \rho_1 y_{n,1} + y_{n,1} - z_n)}\}, \\ C_{n+1,2} := \{u \in C_n : u \leq \frac{(2\rho_2 sin(\frac{1}{2} z_n) z_n - \rho_2 y_{n,2}^2 - \rho_2 z_n^2 + y_{n,2}^2 - z_n^2)}{2(\rho_2 sin(\frac{1}{2} z_n) - \rho_2 y_{n,2} + y_{n,2} - z_n)}\}, \\ C_{n+1} := C_{n+1,1} \cap C_{n+1,2}, \\ x_{n+1} := P^h_{C_{n+1}}(x_0), n \geq 1. \end{cases} \tag{75}$$

Figure 1: Sequence values versus number of iterations showing comparison of our scheme (26) (indicated with Pink & Purple) and that of (24) (indicated with Black) respectively.

Table 1: Selected values of the sequences $\{x_n\}, \{z_n\}$ in our scheme (26) and $\{x2_n\}$ for that of (24) with the following cases.

Case 1 Take $x_0 = 1.0, x_1 = 0.8, z_0 = 0.0; x2_0 = 1.0, \rho_1 = \rho_2 = 0.5$

n	0	50	100	200	300	400	450	500
x_n	1.000000	0.054787	0.033168	0.012157	0.004456	0.001633	0.000989	0.000605
z_n	0.000000	0.054345	0.032900	0.012059	0.004420	0.001620	0.000981	0.000600
$x2_n$	1.000000	0.888736	0.790539	0.626695	0.497675	0.395650	0.352870	0.315479

Case 2 $x_0 = 1.0, x_1 = 1.0, z_0 = 0.0; x2_0 = 1.0, \rho_1 = \rho_2 = 0.5$

n	0	50	100	200	300	400	450	500
x_n	1.000000	0.871373	0.749156	0.555358	0.412674	0.307051	0.264935	0.229301
z_n	0.000000	0.869254	0.747349	0.554033	0.411695	0.306324	0.264309	0.228760
$x2_n$	1.000000	0.888736	0.790539	0.626695	0.497675	0.395650	0.352870	0.315479

Case 3 $x_0 = 0.5, x_1 = 0.3, z_0 = 0.0; x2_0 = 0.5, \rho_1 = \rho_2 = 0.5$

n	0	50	100	200	300	400	450	500
x_n	0.500000	0.097475	0.038147	0.005845	0.000896	0.00137	0.000054	0.000021
z_n	0.000000	0.095997	0.037569	0.005756	0.000882	0.000135	0.000053	0.000021
$x2_n$	0.500000	0.393700	0.310231	0.192864	0.119991	0.074674	0.058912	0.046698

Figure 2: Sequence values versus number of iterations showing comparison of our scheme (26) (indicated with Pink & Purple) and that of (24) (indicated with Black) respectively.

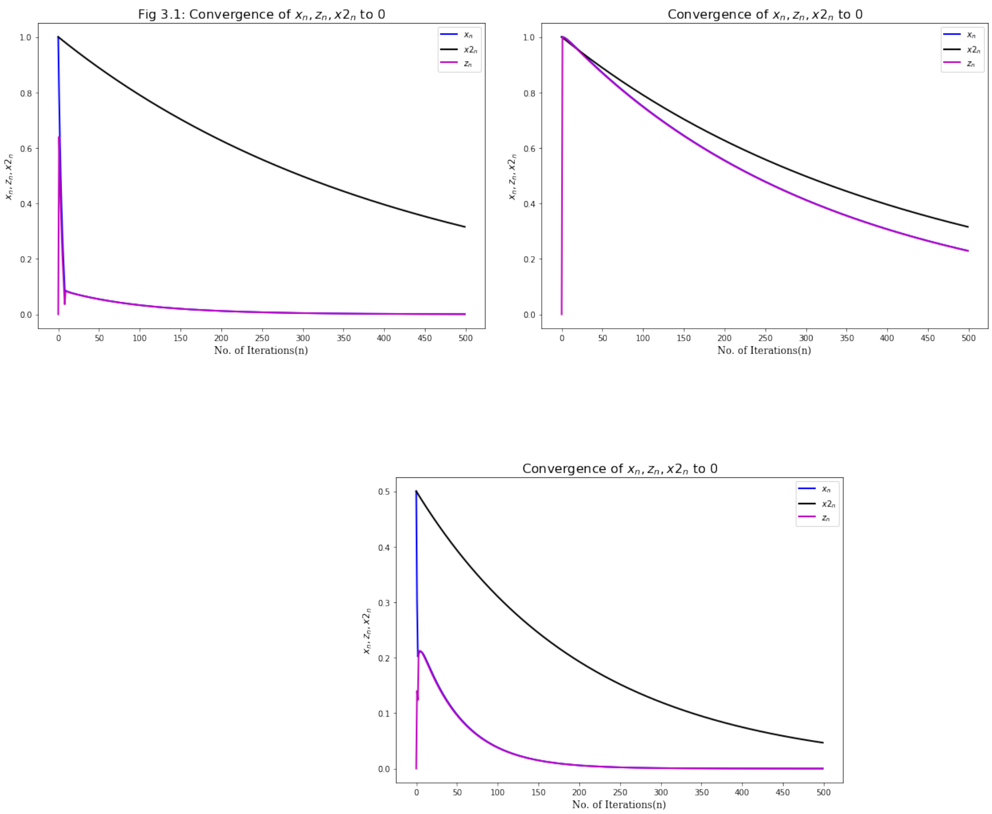


Table 2: Selected values of the sequences $\{x_n\}$ and $\{z_n\}$ in our scheme (26) and $\{x2_n\}$ with the following cases.

Case 1 Take $x_0 = 1.0, x_1 = 0.8, z_0 = 0.0; x2_0 = 1.0, \rho_1 = \rho_2 = 0.5$

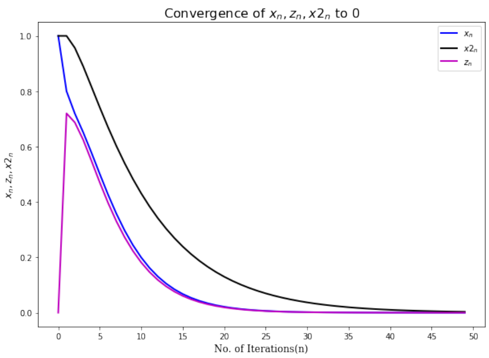
n	0	5	10	20	30	40	45	50
x_n	1.000000	0.501570	0.199426	0.021442	0.001968	0.000170	0.000049	0.000018
z_n	0.000000	0.470859	0.181492	0.019192	0.001753	0.000151	0.000041	0.000016
$x2_n$	1.000000	0.741965	0.431008	0.129957	0.037177	0.010388	0.005462	0.003260

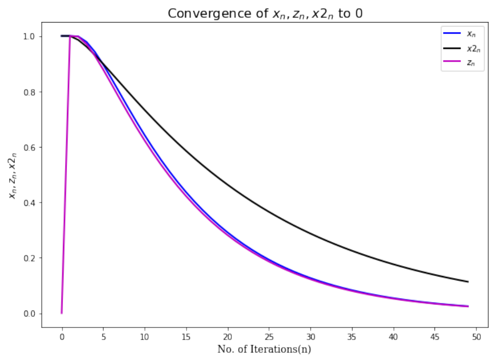
Case 2 $x_0 = 1.0, x_1 = 1.0, z_0 = 0.0; x2_0 = 1.0, \rho_1 = \rho_2 = 0.8$

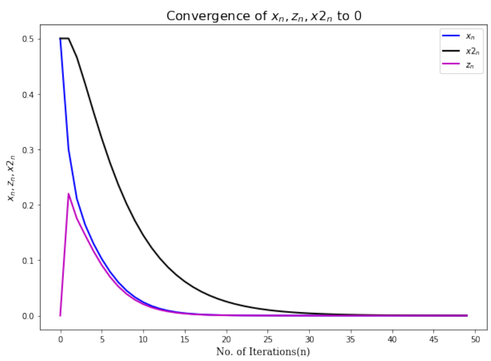
n	0	5	10	20	30	40	45	50
x_n	1.000000	0.897273	0.642592	0.291564	0.126819	0.054203	0.035294	0.025005
z_n	0.000000	0.879106	0.623060	0.281636	0.122361	0.052267	0.034027	0.024104
$x2_n$	1.000000	0.899435	0.733148	0.463565	0.287177	0.176368	0.137932	0.113223

Case 3 $x_0 = 0.5, x_1 = 0.3, z_0 = 0.0; x2_0 = 0.5, \rho_1 = \rho_2 = 0.3$

n	0	5	10	20	30	40	45	50
x_n	0.500000	0.102363	0.024258	0.000694	0.000014	0.00000	0.000000	0.000000
z_n	0.000000	0.091246	0.020658	0.000567	0.000011	0.000000	0.000000	0.000000
$x2_n$	0.500000	0.319823	0.145020	0.025390	0.004134	0.000653	0.000258	0.000122







We observe a better performance of our scheme when compared with an existing result for finite family of Bregman quasi-strict pseudocontarction mappings.

6 Conclusion

In this paper, we constructed a two-step iterative algorithm with inertial extrapolation term defined with respect to Bregman distance function which approximated common fixed points of the finite family of a closed Bregman quasi strictly pseudo-contraction mappings when the intersection of its set is assumed to be non empty. The inertial extrapolation term has a flexible relaxed condition on the initial iterates x_n and x_{n-1}. We proved a strong convergence theorem for it. Through the use of Python program, we obtained that the sequences generated by our algorithm showed faster convergence results as displayed in the tables and figures above. Thus, our result improves, generalizes and complements other previously and recently cited results of some authors in the literature.

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