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Tracking oscillating edge-flames: a mathematical approach

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Abstract

Accurate simulation of oscillating edge flames is computationally very challenging because of the difficult calculations involved to incorporate most of the physics that is believed to be central to the phenomenon. This paper presents the design of a new simulation tool that can handle among other things the concept of flame speed. The conditions for the existence of unique solutions of energy and species equations are established by the actual solution method and the Lipschitz continuity approach. The analytical solution is obtained via Olayiwola's generalized polynomial approximation method (OGPAM), which shows the influence of the parameters involved on the system. Both the effects of an on-edge and off-edge convective flow, and the effects of a heat sink were examined. The effect of changes in values of parameters such as the Damkohler number and speed with which the flame advances are presented graphically and discussed.

Keywords: convective flow; edge-flame; flames; flame-speed; OGPAM; oscillating edge-flames

1 Introduction

The theory of oscillating edge flames has been investigated in numerous works (see [3], [4[, [5], [6], [18]). An edge flame is a two-dimensional structure in the neighborhood of the edge of a flame sheet (nominally one-dimensional) with an edge [4].

Flames with edges occur in many forms. They are: (i) Flame spreading over a fuel-bed, solid or liquid. (ii) A candle flame burning under microgravity conditions. (iii) A diffusion flame with edge-supported fluxes from a heterogeneous propellant. (iv) A wrinkled torn flame in a turbulent flow. (v) An axisymmetric diffusion flame supported on a tube burner, burning under microgravity conditions. (vi) A diffusion flame sitting on a plate through which fuel is injected. For the configuration of these flames with edges see [3].

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Oscillating edge flames in which the edge advances and retreats have long been observed in many experimental configurations. For example, when a flame spreads over a liquid fuel bed, Figure 1,

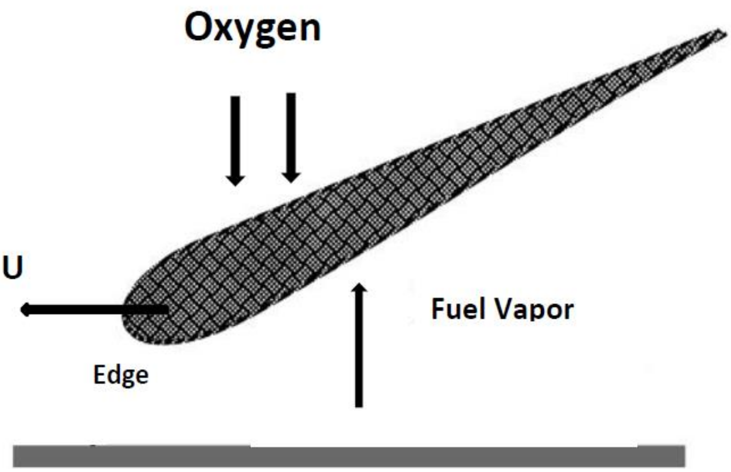

Figure 1: Flame spreading over a liquid pool of fuel

the speed U with which the flame advances is usually a constant, U_0 say. But there are circumstances that U fluctuate with time, viz.

$$U = U_0 + U'(t)$$

where U' is periodic [15].

In the context of flame spread over liquids, this phenomenon has been captured in numerical simulations, and various explanations have been proposed. One links the oscillations to a gas-phase eddy [10]; the other to Marangoni instabilities [8]. Oscillation instabilities were first predicted for one-dimensional deflagrations by Sivashinsky [16], for large Lewis numbers. Kirkby and Schmitz [11] examined one-dimensional diffusion flames in a theoretical study, and predict oscillating instabilities at near–extinction (blow-off) conditions. These occur only if the Lewis numbers are large, or if there are substantial volumetric heat losses. Joulin and Clavin [9] subsequently showed that volumetric heat losses exacerbate the instability, reducing the Lewis numbers for which they are achieved, and Margolis [12] and Buckmaster [7] showed that conductive heat losses to a burner have a similar effect. Buckmaster *et al.* ([5], [6]) proposed the simple hypothesis, that the edge oscillations are nothing more than a two-dimensional manifestation of the well-known one-dimensional instabilities described earlier. They examined a simple model which incorporates what we believe is the minimum necessary physics, and oscillations are predicted. In another development, Buckmaster *et al.* [4] added new physical ingredients to the model, to test the hypothesis further. They introduced $U\frac{\partial}{\partial x}$, an on-edge convective term and a "cold rectangle", a rectangle of interior mesh points at which the temperature is set equal to the boundary value T_0. They examined both the effects of an on-edge and off-edge convective flow, and the effect of a heat sink, and solutions are found using the method of lines.

This motivates the present analysis where we study oscillating edge flames with a view to providing analytical solutions capable of describing the influence of flame speed on the temperature field.

2 Model formulation

The model configuration consists of an oxygen-supply boundary at $y = -\frac{L}{2}$, $-\infty < x < \infty$, and a fuel-supply boundary at $y = \frac{L}{2}$, $0 < x < \infty$. The remainder of the upper wall ($y = \frac{L}{2}$, $-\infty < x < 0$) supplies neither oxygen nor fuel. The formulation of our model is guided by the following assumptions:

Initially (i.e., at a time $t = 0$), the medium is not clean (i.e., the domain is not fuel, oxygen, and temperature free). Let the initial medium temperature, fuel, and oxygen concentrations depend on space variables.

No convection in the $y-$direction.
How the reactants are supplied at the boundaries is of no great consequence, so we impose Dirichlet data at the lower boundary and Neumann data at the upper boundary.
Based on the above description and assumptions, the combustion field within the walls is governed by the two-dimensional equations:

$$\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(X, Y, c_p T\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\rho D_X X, \rho D_X X, kT\right) + \left(-\alpha_X, -\alpha_Y, Q\right)BXYe^{-\frac{E}{RT}} \tag{1}$$

with the initial and boundary conditions:

$$\left.\begin{array}{l}
X(x,y,0) = \alpha_X(x+y),\ X\left(-\frac{L}{2}, y, t\right) = X_0,\ X\left(x, -\frac{L}{2}, t\right) = X_0,\ \left.\frac{\partial X}{\partial x}\right|_{x=\frac{L}{2}} = 0,\ \left.\frac{\partial X}{\partial y}\right|_{y=\frac{L}{2}} = 0 \\
Y(x,y,0) = \alpha_Y(x+y),\ Y\left(-\frac{L}{2}, y, t\right) = 0,\ Y\left(x, -\frac{L}{2}, t\right) = 0,\ \left.\frac{\partial Y}{\partial x}\right|_{x=\frac{L}{2}} = 0,\ \left.\frac{\partial Y}{\partial y}\right|_{y=\frac{L}{2}} = 0 \\
T(x,y,0) = T_0(\in(x+y)+1),\ T\left(-\frac{L}{2}, y, t\right) = T_0,\ T\left(x, -\frac{L}{2}, t\right) = T_0,\ \left.\frac{\partial T}{\partial x}\right|_{x=\frac{L}{2}} = 0,\ \left.\frac{\partial T}{\partial y}\right|_{y=\frac{L}{2}} = 0
\end{array}\right\}, \tag{2}$$

where $U\frac{\partial}{\partial x}$ is an on-edge convective term and U is the speed of an applied convective flow in the x- direction, E is the activation energy, X is the mass fraction of oxidizer, Y is the mass fraction of fuel, T is temperature, x- coordinate parallel to edge-flame travel, y- coordinate perpendicular to edge-flame travel, T_0 is the initial temperature of the medium, X_0 is the initial mass fraction of oxidizer, Y_0 is the initial mass fraction of fuel.

3 Method of solution

3.1 Dimension analysis

Dimensionless variables are been introduced as:

$$\left.\begin{array}{l} x' = \dfrac{x}{L}, \quad y' = \dfrac{y}{L}, \quad t' = \dfrac{kt}{\rho c_p L^2}, \quad U' = \dfrac{\rho c_p LU}{k}, \quad v' = \dfrac{\rho c_p Lv}{k}, \\ \phi = \dfrac{X}{\alpha_X}, \quad \psi = \dfrac{Y}{\alpha_Y}, \quad \theta = \dfrac{E(T-T_0)}{RT_0^2}, \quad \in = \dfrac{RT_0}{E}, \end{array}\right\} \tag{3}$$

Using (3), and after dropping the prime, equations (1) and (2) become

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\phi, \psi, \theta) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{Le_X}\phi, \frac{1}{Le_Y}\psi, \theta\right) + (-\alpha, -\alpha, \sigma) D_a \phi\psi e^{\frac{\theta}{1+\in\theta}} \tag{4}$$

$$\left.\begin{array}{lllll} \phi(x,y,0) = (x+y), & \phi\left(-\dfrac{1}{2}, y, t\right) = \beta, & \phi\left(x, -\dfrac{1}{2}, t\right) = \beta, & \left.\dfrac{\partial\phi}{\partial x}\right|_{x=\frac{1}{2}} = 0, & \left.\dfrac{\partial\phi}{\partial y}\right|_{y=\frac{1}{2}} = 0 \\ \psi(x,y,0) = (x+y), & \psi\left(-\dfrac{1}{2}, y, t\right) = 0, & \psi\left(x, -\dfrac{1}{2}, t\right) = 0, & \left.\dfrac{\partial\psi}{\partial x}\right|_{x=\frac{1}{2}} = 0, & \left.\dfrac{\partial\psi}{\partial y}\right|_{y=\frac{1}{2}} = 0 \\ \theta(x,y,0) = (x+y), & \theta\left(-\dfrac{1}{2}, y, t\right) = 0, & \theta\left(x, -\dfrac{1}{2}, t\right) = 0, & \left.\dfrac{\partial\theta}{\partial x}\right|_{x=\frac{1}{2}} = 0, & \left.\dfrac{\partial\theta}{\partial y}\right|_{y=\frac{1}{2}} = 0 \end{array}\right\}, \tag{5}$$

where

$Le_X = \dfrac{k}{\rho c_p D_X}$ stands for the Lewis number for oxidizer, $Le_Y = \dfrac{k}{\rho c_p D_Y}$ stands for the Lewis number for fuel, $D_a = \dfrac{\alpha_X \alpha_Y BL}{\rho U}$ is the Damkohler number, $\alpha = \dfrac{\rho c_p LU}{k} e^{-\frac{E}{RT_0}}$, $\sigma = \dfrac{\rho LUQ}{k \in T_0} e^{-\frac{E}{RT_0}}$, $\beta = \dfrac{X_0}{\alpha_X}$.

Introducing the following new space variable (Olayiwola [14]):

$$\eta = x + y. \tag{6}$$

Then, equations (4) and (5) can be expressed as

$$\left.\begin{aligned}&\frac{\partial\phi}{\partial t}+U\frac{\partial\phi}{\partial\eta}=\frac{2}{Le_X}\frac{\partial^2\phi}{\partial\eta^2}-\alpha D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}}\\&\phi(\eta,0)=\eta,\quad\phi(-1,t)=\beta,\quad\left.\frac{\partial\phi}{\partial\eta}\right|_{\eta=1}=0\end{aligned}\right\}\tag{7}$$

$$\left.\begin{aligned}&\frac{\partial\psi}{\partial t}+U\frac{\partial\psi}{\partial\eta}=\frac{2}{Le_Y}\frac{\partial^2\psi}{\partial\eta^2}-\alpha D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}}\\&\psi(\eta,0)=\eta,\quad\psi(-1,t)=0,\quad\left.\frac{\partial\psi}{\partial\eta}\right|_{\eta=1}=0\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}&\frac{\partial\theta}{\partial t}+U\frac{\partial\theta}{\partial\eta}=2\frac{\partial^2\theta}{\partial\eta^2}+\sigma D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}}\\&\theta(\eta,0)=\eta,\quad\theta(-1,t)=0,\quad\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1}=0\end{aligned}\right\}\tag{9}$$

3.2 Properties of solution

Theorem 1: Let $Le_X=Le_Y=1$. Then there exists at most one bounded solution of equations (7) – (9).

Proof: Multiply (7) and (8) by σ, (9) by 2α and adding the resulting equations, we obtain

$$\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial\eta}=2\frac{\partial^2\varphi}{\partial\eta^2}\tag{10}$$

$$\varphi(\eta,0)=2(\sigma+\alpha)\eta,\quad\varphi(-1,t)=\sigma\beta,\quad\left.\frac{\partial\varphi}{\partial\eta}\right|_{\eta=1}=0,\tag{11}$$

where

$$\varphi(\eta,t)=\sigma(\psi(\eta,t)+\phi(\eta,t))+2\alpha\theta(\eta,t)$$

Using Olayiwola's generalized polynomial approximation method (OGPAM) (see, [13]), we obtain the solution to the problem (10) and (11) as:

$$\varphi(\eta,t)=\left(\begin{aligned}&\frac{1}{4}\left(3\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+\sigma\beta\right)+\\&\frac{1}{2}\left(\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)-\sigma\beta\right)\eta+\\&\frac{1}{4}\left(\sigma\beta-\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)\right)\eta^2\end{aligned}\right)\tag{12}$$

Then, we obtain

$$\phi(\eta,t)=\frac{1}{\sigma}\left(\begin{array}{l}\left(\begin{array}{l}\frac{1}{4}\left(3\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+\sigma\beta\right)+\\ \frac{1}{2}\left(\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)-\sigma\beta\right)\eta+\\ \frac{1}{4}\left(\sigma\beta-\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)\right)\eta^2\end{array}\right)-\\ (2\alpha\theta(\eta,t)+\sigma\psi(\eta,t))\end{array}\right) \tag{13}$$

$$\psi(\eta,t)=\frac{1}{\sigma}\left(\begin{array}{l}\left(\begin{array}{l}\frac{1}{4}\left(3\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+\sigma\beta\right)+\\ \frac{1}{2}\left(\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)-\sigma\beta\right)\eta+\\ \frac{1}{4}\left(\sigma\beta-\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)\right)\eta^2\end{array}\right)-\\ (2\alpha\theta(\eta,t)+\sigma\phi(\eta,t))\end{array}\right) \tag{14}$$

$$\theta(\eta,t)=\frac{1}{2\alpha}\left(\begin{array}{l}\left(\begin{array}{l}\frac{1}{4}\left(3\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+\sigma\beta\right)+\\ \frac{1}{2}\left(\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)-\sigma\beta\right)\eta+\\ \frac{1}{4}\left(\sigma\beta-\left(\frac{\sigma\beta(1+U)}{(2+U)}\left(1-e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)+2(\sigma+\alpha)\eta e^{-\frac{3}{2}\left(1+\frac{U}{2}\right)t}\right)\right)\eta^2\end{array}\right)-\\ \sigma(\psi(\eta,t)+\phi(\eta,t))\end{array}\right) \tag{15}$$

Hence, there exists a unique solution to the problem (7) – (9). This completes the proof.

Theorem 2: Suppose $f_0(\eta)$, $g_0(\eta)$, and $h_0(\eta)$ are bound for $\eta\in R^n$ and $f(\eta,t,\phi,\psi,\theta)$, $g(\eta,t,\phi,\psi,\theta)$, $h(\eta,t,\phi,\psi,\theta)$ satisfies the uniform Lipschitz condition. Then, there exists a unique solution $\phi(\eta,t)$, $\psi(\eta,t)$, $\theta(\eta,t)$ of equations (7) – (9).

Proof: Let proof as in [2]. We rewrite equations (7) – (9) in the form:

$$\frac{\partial \phi}{\partial t}+U\frac{\partial \phi}{\partial \eta}=\frac{2}{Le_X}\frac{\partial^2 \phi}{\partial \eta^2}+f(\eta,t,\phi,\psi,\theta) \tag{17}$$

$$\frac{\partial \psi}{\partial t}+U\frac{\partial \psi}{\partial \eta}=\frac{2}{Le_Y}\frac{\partial^2 \psi}{\partial \eta^2}+g(\eta,t,\phi,\psi,\theta) \tag{18}$$

$$\frac{\partial \theta}{\partial t}+U\frac{\partial \theta}{\partial \eta}=2\frac{\partial^2 \theta}{\partial \eta^2}+h(\eta,t,\phi,\psi,\theta) \tag{19}$$

$$\phi(\eta,0)=f_0(\eta),\quad \psi(\eta,0)=g_0(\eta),\quad \theta(\eta,0)=h_0(\eta),\quad \eta=(\eta_1,\eta_2,...\eta_n), \tag{20}$$

where

$$f(\eta,t,\phi,\psi,\theta)=-\alpha D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}},\ g(\eta,t,\phi,\psi,\theta)=-\alpha D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}},\ h(\eta,t,\phi,\psi,\theta)=\sigma D_a\phi\psi e^{\frac{\theta}{1+\epsilon\theta}}.$$

Ignoring the second term at the right-hand side, the fundamental solution of (17) – (19) are (see [17]):

$$F(\eta,t)=\frac{\eta}{2\sqrt{\frac{2\pi}{Le_X}t^{\frac{3}{2}}}}\exp\left(\frac{U}{2\left(\frac{2}{Le_X}\right)}\eta-\frac{U^2}{4\left(\frac{2}{Le_X}\right)}t-\frac{\eta}{4\left(\frac{2}{Le_X}\right)t}\right)$$

$$G(\eta,t)=\frac{\eta}{2\sqrt{\frac{2\pi}{Le_Y}t^{\frac{3}{2}}}}\exp\left(\frac{U}{2\left(\frac{2}{Le_Y}\right)}\eta-\frac{U^2}{4\left(\frac{2}{Le_Y}\right)}t-\frac{\eta}{4\left(\frac{2}{Le_Y}\right)t}\right)$$

$$H(\eta,t)=\frac{\eta}{2\sqrt{2\pi t^{\frac{3}{2}}}}\exp\left(\frac{U}{4}\eta-\frac{U^2}{8}t-\frac{\eta}{8t}\right).$$

Next, it suffices to show that the Lipschitz condition is satisfied. That is if we can show that:

$$|f(\eta,t,\phi_1,\psi_1,\theta_1)-f(\eta,t,\phi_2,\psi_2,\theta_2)|\le k_1(|\phi_1-\phi_2|+|\psi_1-\psi_2|+|\theta_1-\theta_2|) \tag{21}$$

$$|g(\eta,t,\phi_1,\psi_1,\theta_1)-g(\eta,t,\phi_2,\psi_2,\theta_2)|\le k_2(|\phi_1-\phi_2|+|\psi_1-\psi_2|+|\theta_1-\theta_2|) \tag{22}$$

$$|h(\eta,t,\phi_1,\psi_1,\theta_1)-h(\eta,t,\phi_2,\psi_2,\theta_2)|\le k_3(|\phi_1-\phi_2|+|\psi_1-\psi_2|+|\theta_1-\theta_2|) \tag{23}$$

where

$$k_1=\max\left\{\frac{\partial f}{\partial \phi},\frac{\partial f}{\partial \psi},\frac{\partial f}{\partial \theta}\right\},\quad k_2=\max\left\{\frac{\partial g}{\partial \phi},\frac{\partial g}{\partial \psi},\frac{\partial g}{\partial \theta}\right\},\quad k_3=\max\left\{\frac{\partial h}{\partial \phi},\frac{\partial h}{\partial \psi},\frac{\partial h}{\partial \theta}\right\}.$$

Since $\alpha,\ \sigma,\ Da,\ \epsilon>0$, $0\le\phi,\psi\le 1$ and $1\le e^{\frac{\theta}{1+\epsilon\theta}}\le e^{\frac{1}{\epsilon}}\quad\forall\quad 0\le\theta<\infty$.

Hence,

$$k_1=\max\left\{\frac{\partial f}{\partial \phi},\frac{\partial f}{\partial \psi},\frac{\partial f}{\partial \theta}\right\}=0,\ k_2=\max\left\{\frac{\partial g}{\partial \phi},\frac{\partial g}{\partial \psi},\frac{\partial g}{\partial \theta}\right\}=0,\ k_3=\max\left\{\frac{\partial h}{\partial \phi},\frac{\partial h}{\partial \psi},\frac{\partial h}{\partial \theta}\right\}=\sigma Da e^{\frac{1}{\epsilon}}$$

Clearly, $f(\eta,t,\phi,\psi,\theta)$, $g(\eta,t,\phi,\psi,\theta)$, $h(\eta,t,\phi,\psi,\theta)$ are Lipschitz continuous. Hence, the result follows. This completes the proof.

3.3 Analytical solution via Olayiwola's generalized polynomial approximation method

In the limit $\in\to 0$, equations (7) – (9) can be solved analytically using Olayiwola's generalized polynomial approximation method (OGPAM) [13]. Follow the idea in [1], that:

$$\exp(\theta)\approx 1+(e-2)\theta+\theta^2 \tag{24}$$

and let $0<\alpha<<1$, $\sigma=a\alpha$ such that

$$\left.\begin{aligned}\phi(\eta,t)&=\phi_0(\eta,t)+\alpha\phi_1(\eta,t)+\ldots\\ \psi(\eta,t)&=\psi_0(\eta,t)+\alpha\psi_1(\eta,t)+\ldots\\ \theta(\eta,t)&=\theta_0(\eta,t)+\alpha\theta_1(\eta,t)+\ldots\end{aligned}\right\} \tag{25}$$

and using OGPAM in [13], we obtain the solutions to (7) - (9) as:

$$\begin{aligned}\phi(\eta,t)=&\left(\frac{1}{4}\left(3\phi_0\big|_{\eta=1}+\beta\right)+\frac{1}{2}\left(\phi_0\big|_{\eta=1}-\beta\right)\eta+\frac{1}{4}\left(\beta-\phi_0\big|_{\eta=1}\right)\eta^2\right)+\\ &\alpha\left(\frac{3}{4}\phi_1\big|_{\eta=1}+\frac{1}{2}\phi_1\big|_{\eta=1}\eta-\frac{1}{4}\phi_1\big|_{\eta=1}\eta^2\right)\end{aligned} \tag{26}$$

$$\psi(\eta,t)=\left(\frac{3}{4}\psi_0\big|_{\eta=1}+\frac{1}{2}\psi_0\big|_{\eta=1}\eta-\frac{1}{4}\psi_0\big|_{\eta=1}\eta^2\right)+\alpha\left(\frac{3}{4}\psi_1\big|_{\eta=1}+\frac{1}{2}\psi_1\big|_{\eta=1}\eta-\frac{1}{4}\psi_1\big|_{\eta=1}\eta^2\right) \tag{27}$$

$$\theta(\eta,t)=\left(\frac{3}{4}\theta_0\big|_{\eta=1}+\frac{1}{2}\theta_0\big|_{\eta=1}\eta-\frac{1}{4}\theta_0\big|_{\eta=1}\eta^2\right)+\alpha\left(\frac{3}{4}\theta_1\big|_{\eta=1}+\frac{1}{2}\theta_1\big|_{\eta=1}\eta-\frac{1}{4}\theta_1\big|_{\eta=1}\eta^2\right), \tag{28}$$

where

$$\phi_0\big|_{\eta=1}=m_1+(\eta-m_1)e^{-m_0t},\qquad \psi_0\big|_{\eta=1}=\eta e^{-m_2t},\qquad \theta_0\big|_{\eta=1}=\eta e^{-m_3t},$$

$$\phi_1\big|_{\eta=1}=-\frac{3}{2}\begin{pmatrix}\frac{m_4}{m_0-m_2}\eta\left(e^{-m_2t}-e^{-m_0t}\right)-\frac{m_5}{m_2}\eta(\eta-m_1)\left(e^{-(m_0+m_2)t}-e^{-m_0t}\right)+\\ \frac{m_6}{m_0-m_2-m_3}\eta^2\left(e^{-(m_2+m_3)t}-e^{-m_0t}\right)-\frac{m_7}{m_2+2m_3}\eta^2(\eta-m_1)\left(e^{-(m_0+m_2+2m_3)t}-e^{-m_0t}\right)+\\ \frac{m_8}{m_0-m_2-2m_3}\eta^3\left(e^{-(m_2+2m_3)t}-e^{-m_0t}\right)-\frac{m_9}{m_2+2m_3}\eta^3(\eta-m_1)\left(e^{-(m_0+m_2+2m_3)t}-e^{-m_0t}\right)\end{pmatrix},$$

$$\psi_1\big|_{\eta=1}=-\frac{3}{2}\begin{pmatrix}m_4\eta te^{-m_2t}-\frac{m_5}{m_0}\eta(\eta-m_1)\left(e^{-(m_0+m_2)t}-e^{-m_2t}\right)-\frac{m_6}{m_3}\eta^2\left(e^{-(m_2+m_3)t}-e^{-m_2t}\right)-\\ \frac{m_7}{m_0+m_3}\eta^2(\eta-m_1)\left(e^{-(m_0+m_2+m_3)t}-e^{-m_2t}\right)-\frac{m_8}{2m_3}\eta^3\left(e^{-(m_2+2m_3)t}-e^{-m_2t}\right)-\\ \frac{m_9}{m_0+2m_3}\eta^3(\eta-m_1)\left(e^{-(m_0+m_2+2m_3)t}-e^{-m_2t}\right)\end{pmatrix},$$

$$\theta_1\big|_{\eta=1} = \frac{3}{2}a\left(\begin{array}{l} \frac{m_4}{m_3-m_2}\eta\left(e^{-m_2 t}-e^{-m_3 t}\right)+\frac{m_5}{m_3-m_0-m_2}\eta(\eta-m_1)\left(e^{-(m_0+m_2)t}-e^{-m_3 t}\right)- \\ \frac{m_6}{m_2}\eta^2\left(e^{-(m_2+m_3)t}-e^{-m_3 t}\right)-\frac{m_7}{m_0+m_2}\eta^2(\eta-m_1)\left(e^{-(m_0+m_2+m_3)t}-e^{-m_3 t}\right)- \\ \frac{m_8}{m_2+m_3}\eta^3\left(e^{-(m_2+2m_3)t}-e^{-m_3 t}\right)-\frac{m_9}{m_0+m_2+m_3}\eta^3(\eta-m_1)\left(e^{-(m_0+m_2+2m_3)t}-e^{-m_3 t}\right) \end{array}\right),$$

$$m_0 = \frac{3}{2}\left(\frac{2+Le_X U}{2Le_X}\right), \quad m_1 = \beta\left(\frac{1+Le_X U}{2+Le_X U}\right), \quad m_2 = \frac{3}{2}\left(\frac{2+Le_Y U}{2Le_Y}\right), \quad m_3 = \frac{3}{2}\left(\frac{2+U}{2}\right),$$

$$m_4 = \frac{32D_a}{240}(4m_1+\beta), \quad m_5 = \frac{128D_a}{240}, \quad m_6 = \frac{D_a(e-2)}{6720}(2823m_1+512\beta),$$

$$m_7 = \frac{2823D_a(e-2)}{6720}, \quad m_8 = \frac{D_a}{15886080}(22192444m_1+5870600\beta), \quad m_9 = \frac{22192444D_a}{15886080},$$

The computation was done on equations (26) – (28) using the computer symbolic algebraic package MAPLE 2021 version.

4 Results and discussion

The simulation was carried out to show the impact of the model parameters by employing Olayiwola's Generalized Polynomial Approximation Method (OGPAM) on the equations (7) – (9). We are concerned with the nature of the solutions obtained for different values of U and D_a. The computation of equations (26) - (28) was done using the MAPLE 2021 version.

Figure 2 shows the temperature history at $x = 0.75,\ y = 0.08$ for $Le_X = 1.0,\ Le_Y = 1.8,$ $U = 0,\ D_a = 12,\ 18,$ and 24. After the first oscillation, the graph displays a rapid transition to a steady solution.

Figure 3 displays the temperature topography for $Le_X = 1.0,\ Le_Y = 1.8,\ D_a = 12,\ 18,\ 24,\ U = 0$, no sink. The graph reveals oscillation solutions.

We turn now to the effects of U, the convective flow in the x direction.

Figures 4 and 5 show the effects of positive U for $Le_X = 1.0,\ Le_Y = 1.8,\ D_a = 12$, for which stable solutions exist and are achieved faster when $U = 1$ than when $U = 0$.

The effects of the negative U are shown in Figures 6 and 7. With the choice $Le_X = 1.0,\ Le_Y = 1.8,$ $D_a = 18$, the solutions are stable (the flame stabilizes) and this stability is achieved faster when $U = 0$ than when $U = -1$.

Figure 8 shows that when $U = 0$ and $U = 1$ steady burning solutions are generated faster than when $U = -1$.

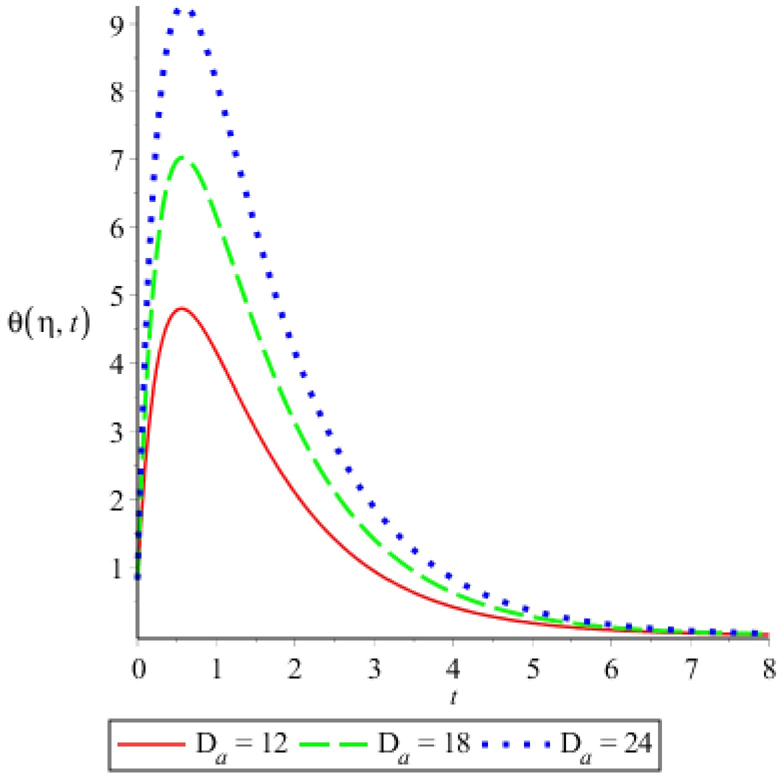

Figure 2: Temperature history at $x = 0.75,\ y = 0.08$ for $Le_X = 1.0,\ Le_Y = 1.8,$ $U = 0,\ D_a = 12, 18, 24$.

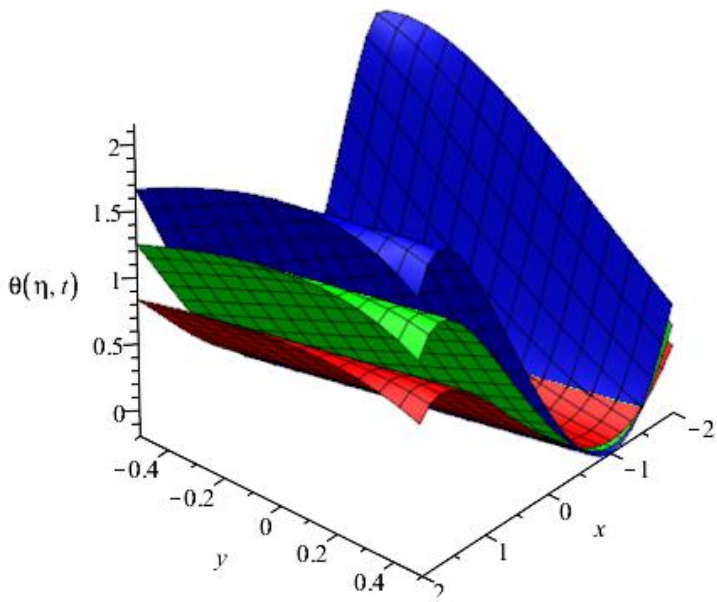

Figure 3: Temperature topography for $Le_X = 1.0,\ Le_Y = 1.8,\ D_a = 12, 18, 24,\ U = 0$, no sink.

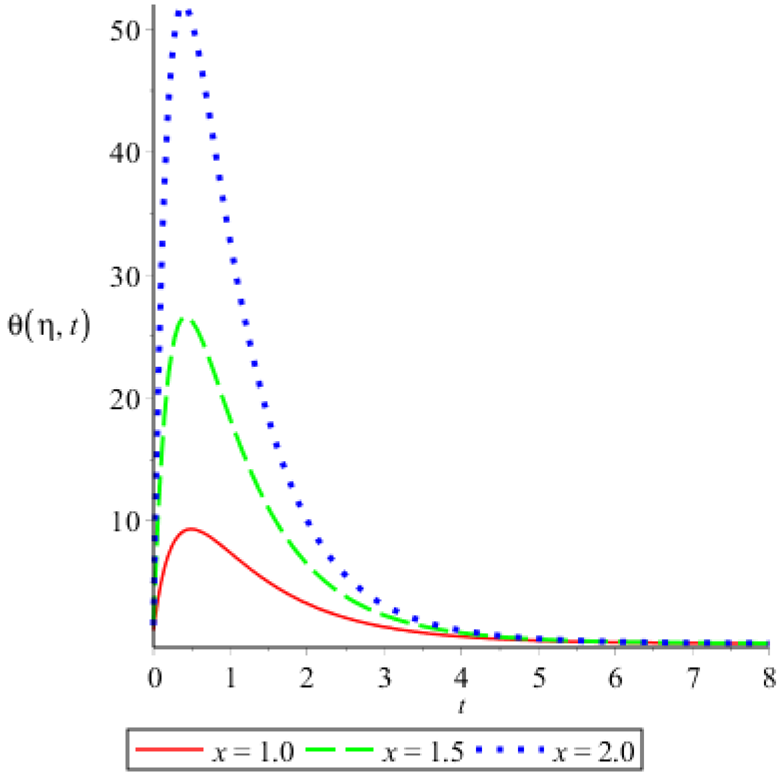

Figure 4: Effects of positive U: temperature history at $x = 1.0, 1.5, 2.0, y = 0.08$ for $Le_X = 1.0, Le_Y = 1.8, U = 0, D_a = 12$.

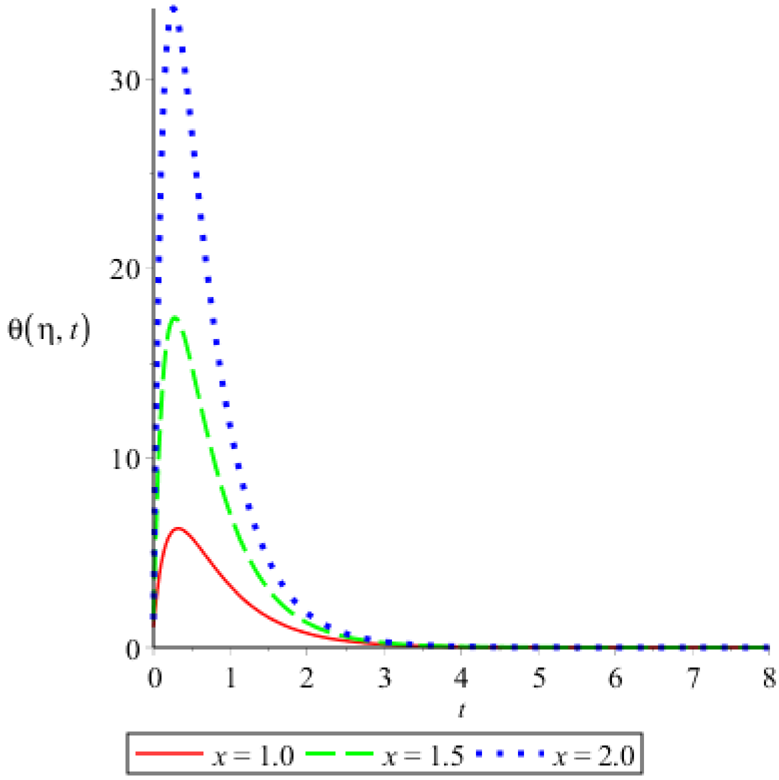

Figure 5: Effects of positive U: temperature history at $x = 1.0, 1.5, 2.0, y = 0.08$ for $Le_X = 1.0, Le_Y = 1.8, U = 1, D_a = 12$.

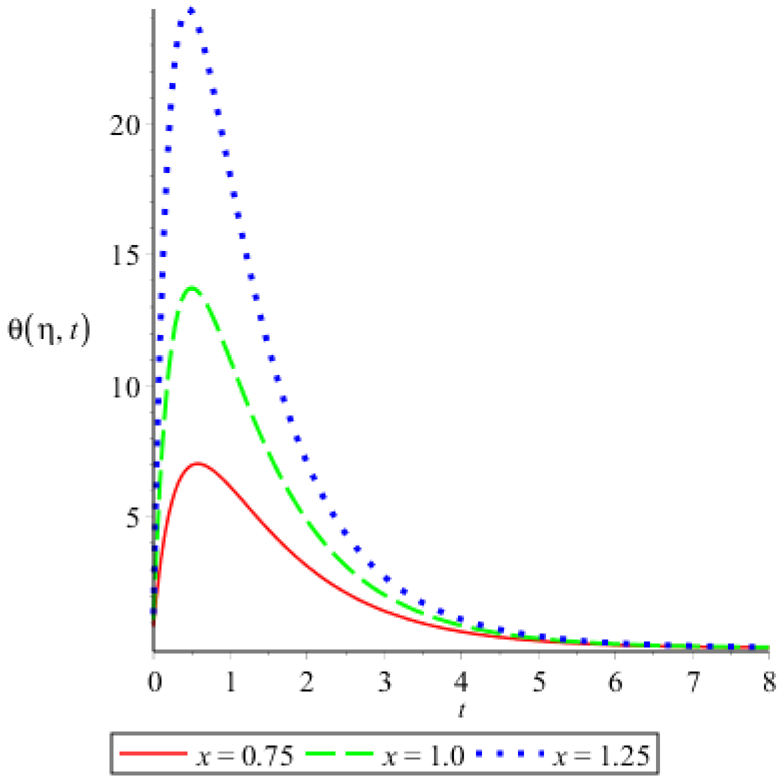

Figure 6: Effects of negative U : temperature history at $x = 0.75, 1.0, 1.25,\ y = 0.08$ for $Le_X = 1.0,\ Le_Y = 1.8,\ U = 0,\ D_a = 18$.

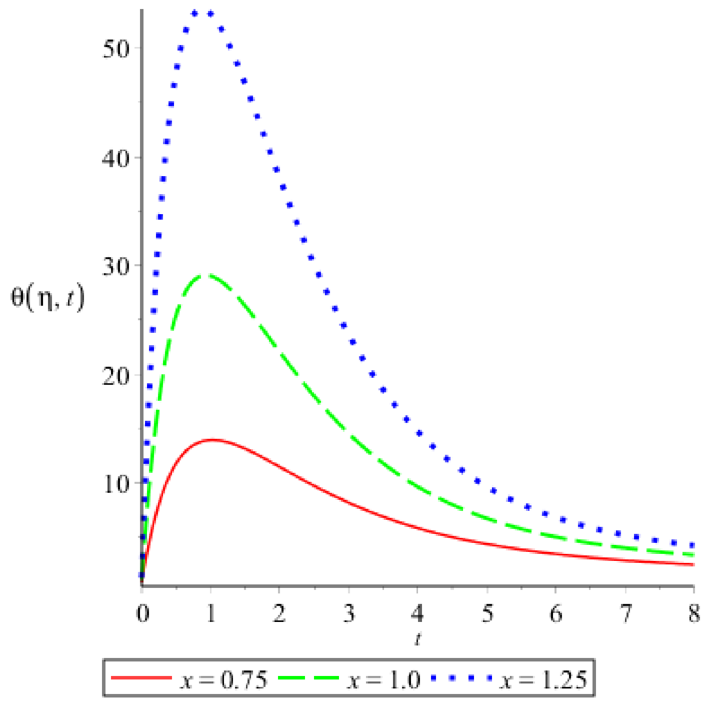

Figure 7: Effects of negative U : temperature history at $x = 0.75, 1.0, 1.25,\ y = 0.08$ for $Le_X = 1.0,\ Le_Y = 1.8,\ U = -1,\ D_a = 18$.

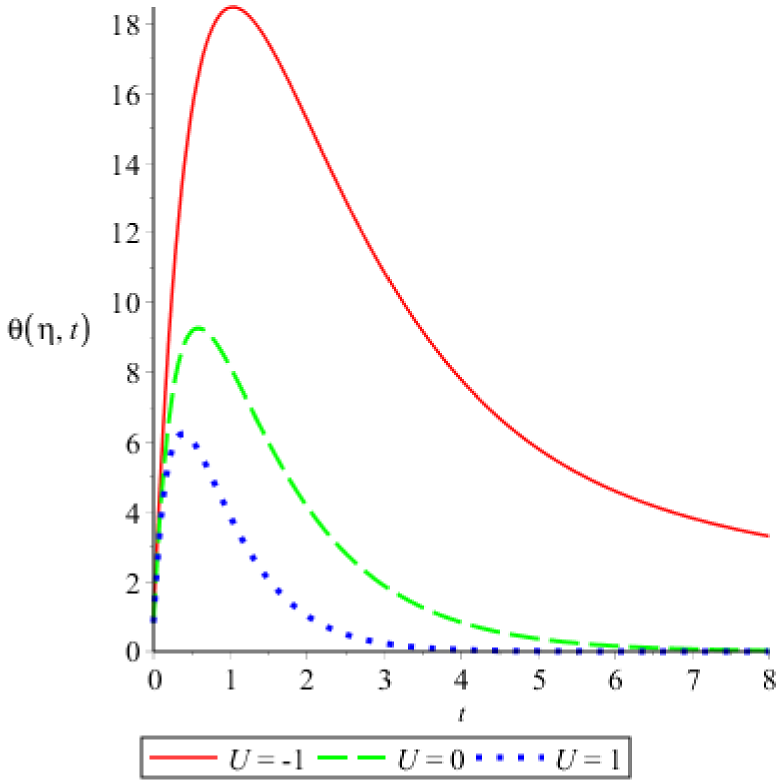

Figure 8: Temperature history at $x = 0.75,\ y = 0.08$ for $Le_X = 1.0,\ Le_Y = 1.8,$ $U = -1,\ 0,\ 1,\ \ D_a = 24$.

It is worth pointing out that the effects observed in Figures 2 - 8, are important to guide in controlling the flames.

5 Conclusion

In [6] a universal explanation of edge-flame oscillation is proposed while [4] considered additional physical ingredients (convective flow, heat sink). Here, we have established the conditions for the existence of a unique solution of the model in [4] and provided the approximate analytical solution. The existence of a unique solution to the problem implies that the problem represents a physical situation under specific conditions. The simulation results revealed the impact of the model parameters and the following conclusion can be drawn:

A reduction in the Damkohler number weakens the 2-D edge flame, eventually extinguishing it.
The positive speed $(U > 0)$ with which the flame advances weaken the 2-D edge flame.
The negative speed $(U < 0)$ enhanced the 2-D edge flame.

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