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Analytical Method for Estimation of Black-Scholes Equation of Option Pricing

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Abstract

The attainment of any investments hinge on the value of options which impels the entire financial authority of every trader and Black-Scholes equation are well known prevailing mathematical instrument used for the estimations of stock option prices. Therefore, this paper, considered the concept of option pricing by means of Black-Scholes equation which governs the growth of option price with esteem to the expiration and cost of the fundamental asset. These equations were modified to assume a probability which measures risk-free interest rate of the underlying asset for Call and Put options. The Black-Scholes exact values and Modified Black-Scholes values were obtained and compared to close form prices. Finally, graphical solutions and interpretations of relevant parameters were well discussed.

Keywords: stock prices; Black-Scholes equation; call option; put option; error

1 Introduction

An option is a tool whose worth is derived from the principal asset which is otherwise known as financial derivative. This type of derivative does have anything in common with mathematical meaning of derivative. In other words, an option on underlying asset is a business between parties who come together to agree on either buying or selling an underlying asset at a determined strike price in the future for a fixed price. The cost of the option lies on the underlying asset, which is usually a stock, commodity, currency or an index [1]. The holder has the right but cannot be compelled to buy, for call option where European put option involves the ability to sell an asset for a certain charge at a prescribed date in the future. Options are known as "*in- the money*", "*at- the money*", or "*out- of the money*". If S is a stock price and K is the strike price, a call option is in- the money as soon as S >K, at- the money when S=K, and out -of –the- money when S<K. A put option is in the money as soon as S<K, at- the -money once S=K, and out- of- the money once S>K. Obviously, an option is exercised only when it is in- the money. In the nonappearance of transactions costs, an in-the money option is always exercised on the expiration date if it has not been exercised earlier [2].

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The relevance of options valuation was first demonstrated by Black-Scholes [3] when option faced difficulties in valuation of option at expiration. They used no-arbitrage argument to explain a partial differential equation which governs the growth of the option price with esteem to the expiration and cost of the fundamental Asset. The Black-Scholes equation has been used widely in many financial applications.

For instance, [4] studied implied volatility and implied risk free rate of return in solving systems of Black- Scholes equations. In their research they established that options prices provides important information for market participants for future expectations and market policies. In the same vain [5] analyzed BS formula for the valuation of European options; Hermite polynomials were applied. They concluded that BS formula can easily be achieved devoid of the use of partial differential equation. In another study of BS [6] considered the BS terminal value problem and observed that their proposed method is better, simple than the previous methods. In the work of [7] time varying factor were incorporated in the explicit formula for different aspect of options with the aim of providing exact solution for dividend paying equity of option. In considering the stability of stock market price of stochastic model,[8] applied Crank-Nicolson numerical scheme to BS model. The results showed stock prices being stable and its increasing rate of stock shares was obtained. Not quite long,[9] investigated the variation of stock market price using BS PDE. The convergence to equilibrium of growth rate and sufficient conditions for stability was achieved. However,[10] studied Black-Scholes model because of its biasness in mispricing options. They established a new technique of assessing pricing effects on the premise to reduce pricing bias.

In financial markets, investors and financial analysts are generally too interested on how to maximize profit over particular trading days, that is, the changes in the price of goods and services. Therefore, modeling a behavior of a stock exchange market can be made through its relative change of the unstable market variables in time so as to predict stock price fluctuation, advice investors and corporative owners who are working out for convenient ways to do business by issuing of stocks in their corporations. The origin of this work lies in the study of [11]. This is so since the path of the stock price method can be allied to his description of the random collision of some tiny particles with the molecules of the liquid he introduced, hence the name Brownian motion. Now, the market price behavior shows the characteristics as a stochastic process called "Brownian motion" or Wiener process with drift. It is an important example of stochastic processes satisfying a Stochastic Differential Equation (SDE) displayed in Figure 1.

The analytical approach of solving BS equation for option pricing is not a difficulty one. It becomes multifarious when some levels of probability measures are considered to assess riskless interest rate of the underlying asset in the model. This difficulty depends when the analytical approach is being determined; which is the novelty of this paper. To propose an analytical approach of BS equation; this will be comparable or possibly better than the original BS equation. The great encounter in the modification is the ability of coming up with some realistic assumptions that will be incorporated into the model.

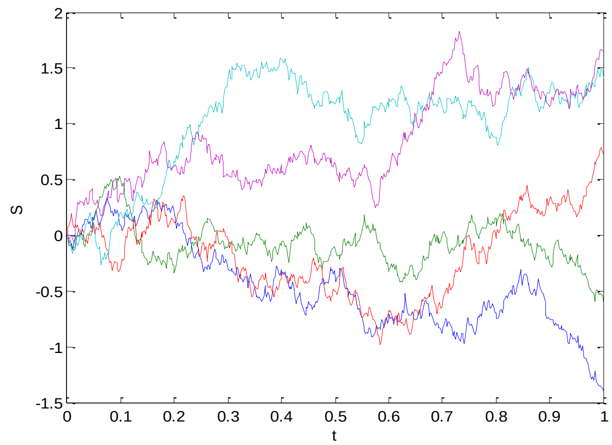

Figure 1: Sample trajectories of the stock price process following Black-Scholes Model

This study is aimed at modifying Black-Scholes equation such we obtain comparable results with BS. To this end, this type of work has not been seen elsewhere as these widens the area of application of problem of this nature.

For the purpose of this study the paper is arranged as follows: Section 2.1 presents the Mathematical preliminaries, problem formulation is seen in Subsection 2.1.1, method of solution is seen in Subsection 2.1.2, Results and discussion are seen in Section 3 and paper is concluded in Section 4.

2.1 Mathematical Preliminaries

Here we present some definitions as foundations of this mathematical finance models.

Definition 1. Probability space: This is a triple $(\Omega, \mathrm{F}, \tilde{A})$ where Ω represents a set of sample space, F represents a collection of subsets of Ω, while $\wp$ is the probability measure defined on each event $\mathrm{A} \in \mathrm{F}$. The collection F is a σ -algebra or σ -field such as $\Omega \in \mathrm{F}$ and F is closed under the arbitrary unions and finite intersections. Hence it is called probability measure when the following condition holds.

(i) $P(A) \geq 0 \;\; for \; all \;\; A \subset \Omega$ (1)

(ii) $P(\Omega) = 1$ (2)

(iii) $A, B \subset \Omega, \; A \cap B = \phi \; then \; P(A \cup B) = P(A) + P(B)$ (3)

Definition 2. Normal Distribution: A normal distribution function is a peculiar distribution in probability theory and is usually used for modeling asset returns. A normal distribution is used in the Black-Scholes Partial differential equation to value European options. A normal distribution depends on two parameters,[12].

Mean, $\mu \in \mathbb{R}$, is the expectation of a random variable normal distribution.

Variance , $\sigma^2 > 0$, deals with the magnitude of the spread from the mean.

In Black-Scholes formula, normal distributions are used. The cumulative distribution, usually denoted as $\phi(\mathrm{X})$, is the probability that X will be equal to or less than x ,expressed as $F_x(x) = P(X \le x)$. A standard normal cumulative distribution function is defined as.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \quad (4)$$

A normal distribution is a symmetric distribution, which means that it touches around a vertical axis of symmetry. Obviously, there is a connection between any given points with same distance to the vertical axis. This relationship is defined in equation 1.5.5

$$\phi(x) = 1 - \phi(-x). \quad (5)$$

Definition 3. A σ -algebra is a set F of subsets of Ω with the following axioms:

(i) $\phi, \Omega \in \mathrm{F}$ (6)

(ii) If $A \in \mathrm{F}, then\ A^c \in \mathrm{F}$ (7)

(iii) If $A_1, A_2, ..., \in \mathrm{F}, then \bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in \mathrm{F}$ (8)

Clearly $A^c := \Omega - A$ is the complement of A.

Definition 4. If F is a σ -algebra in Ω, then Ω is called a measurable space and the members of F are called the measurable sets in Ω.

Definition 5. Let (Ω, M) be a measurable space .A map $\mu : \mathrm{M} \to \mathbb{R} = [0, \infty) \cup \{\infty\}$ is called a measure provided that

(i) $\mu(\phi) = 0$ (9)

(ii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ (10)

Definition 5. Stochastic process: A stochastic process $X(t)$ is a relations of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each t in the index set $T, X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t . In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

It can also be seen as a statistical event that evolves time in accordance to probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete.

Definition 6. A stochastic process whose finite dimensional probability distributions are all Gaussian.(Normal distribution).

Definition 7. Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point $x = a$ to a point $x = b$. A random walk is a stochastic sequence $\{S_n\}$ with $S_0 = 0$, defined by

$$S_n = \sum_{k-1}^{n} X_k \tag{11}$$

where X_k are independent and identically distributed random variables

Definition 8: Stochastic Differential Equation (SDE)
Let $S(t)$ be the price of some risky asset at time t ,and μ ,an expected rate of returns on the stock and dt as a relative change during the trading days such that the stock follows a random walk which is govern by a stochastic differential equation.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW_t \tag{12}$$

Where, α is drift and σ the volatility of the stock, W_t is a Brownian motion or Wiener's process on a probability space $(\Omega, \xi, \wp), \xi$ is a $\sigma - \text{algebra}$ generated by $W_t, t \geq 0$.

Theorem 1.1: (Ito's formula) Let $(\Omega, \beta, \alpha, \mathrm{F}(\beta)$ be a filtered probability space $X = \{X, t \geq 0\}$ be an adaptive stochastic process on $(\Omega, \beta, \alpha, \mathrm{F}(\beta)$ possessing a quadratic variation(X) with SDE defined as:

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t) \tag{13}$$

$t \in \Re$ and for $u = u(t, X(t) \in C^{1\times 2}(\Pi \times \square))$

$$du(t, X(t)) = \left\{\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial x} + \frac{1}{2}f^2\frac{\partial^2 u}{\partial x^2}\right\}d\tau + f\frac{\partial u}{\partial x}dW(t) \tag{14}$$

Using theorem 1.1 and equation (12) comfortably solves the SDE with a solution given below:

$$S(t) = S_0 \exp\left\{\sigma dW(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}, \forall t \in [0,1].$$

2.1.1 Problem Formulation

This Section modifies BS model and presents the tools that can analyze empirically the dynamics of option pricing. In particular, we compare the results of original BS and that of modified one for the purpose of prediction and other capital investments. However, assuming there are N stocks in the capital market. Let $S_i,\ i = 1, 2, ..., N$ be the initial stock prices for T trading days. Since Black-Scholes equation is based on seven assumptions, we therefore, incorporate the probability of riskless asset. However, the seven assumptions are:

The asset price has characteristics of a Brownian motion with μ and σ as constants, the transaction costs or taxes are not allowed, the entire securities are absolutely divisible, dividend is not permitted during the period of the derivatives, unacceptable of riskless arbitrage opportunities, the security trading is constant, the option is exercised at the time of expiry for both call and put options.

Nevertheless, the dynamics of option pricing is given by the partial differential equation as follows:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0. \tag{15}$$

To eliminate the price process in (15) slightly gives the Black-Scholes analytic formula for Call and Put options:

The analytic formula for the prices of European call option is given as follows

$$\left.\begin{aligned} C &= SN(d_1)-Ke^{-rt}N(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1-\sigma\sqrt{T} \end{aligned}\right\} \tag{16}$$

where C is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate , T is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

Similarly, the formula for prices of European put option is given as

$$P = SN(d_1)-Ke^{-rt}N(d_2) \tag{17}$$

where P is the price of a put option and the meaning of other parameters remain the same as in (16) see [13] , [14] and [15] etc.

In option trading and capital investment, profit making is not known in advanced but is an uncertain stochastic variable. We therefore use ϕ_t to measure the probability levels of risk-free interest rate, r of the underlying asset which motivates to define as:

$$(1-\phi_t) \tag{18}$$

(18) Is the probability of risk free interest rate .Hence, it is a statistics concept that carries a major significance in financial mathematics. Therefore, using (18) modifies the BS equation as follows:

$$\left.\begin{aligned} &C = SN\left(d_1\right) - Ke^{-rt}\left(d_2\right) \\ &d_1 = \frac{\ln\left(\dfrac{S}{K}\right) + r\left(\left(1-\phi_t\right) + \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ &d_2 = d_1 - \sigma\sqrt{\tau} \end{aligned}\right\} \tag{19}$$

Similarly, the formula for prices of European Put option is given as

$$P = SN\left(d_1\right) - Ke^{-rt}N\left(d_2\right). \tag{20}$$

3 Results and discussion

This Section presents the simulated results for the problem stated in Subsection 2.1.1. The graphical results are implemented using MATLAB programming software.

Table 1: Comparing the performance of the Black-Scholes exact values and Modified Black-Scholes(MBS) for European Call Option when initial stock prices are 60 and 70 with K = 25, r = 0.2 and T = 1 , $\phi_t = 0.03$.

Sigma	So = 60, E = 25, ϕ =0.03			So =70, E =25,r=0.2		
	BS Exact Values	MBS Exact Values	Relative Error	BS Exact values	MBS Exact Values	Relative Error
0.25	39.5317	39.5317	0.0000	49.5317	49.5317	0.0000
0.3	39.5322	39.5322	0.0000	49.5318	49.5318	0.0000
0.35	39.5353	39.5353	0.0000	49.5325	49.5325	0.0000
0.4	39.5469	39.5469	0.0000	49.5361	49.5361	0.0000
0.45	39.5750	39.5750	0.0000	49.5476	49.5476	0.0000
0.5	39.6277	39.6276	0.000003	49.5736	49.5736	0.0000
0.55	39.7106	36.7106	0.0000	49.6208	49.6208	0.0000
0.6	39.8273	36.8273	0.0000	49.6946	49.6945	0.000002
0.65	39.9788	36.9756	0.00008	49.7985	49.7984	0.000002
0.7	40.1643	40.1642	0.000002	49.9342	49.9341	0.000002
0.75	40.3820	40.3819	0.000002	50.1020	50.1019	0.000002
0.8	40.6295	40.6294	0.000002	50.3009	50.3008	0.000002
0.85	40.9038	40.9037	0.000002	50.5291	50.5290	0.000002

0.9	41.2022	41.2021	0.000002	50.7846	50.7845	0.000002
0.95	41.5216	41.5215	0.000002	51.0650	51.0648	0.000004
1.0	41.8593	41.8592	0.000002	51.3675	51.3674	0.000002

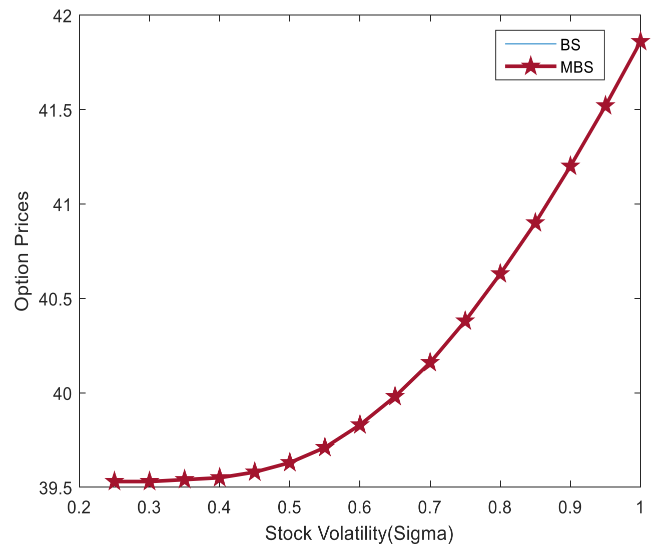

Figure 2: Profiles of BS and MBS for call option when initial stock price is 60.

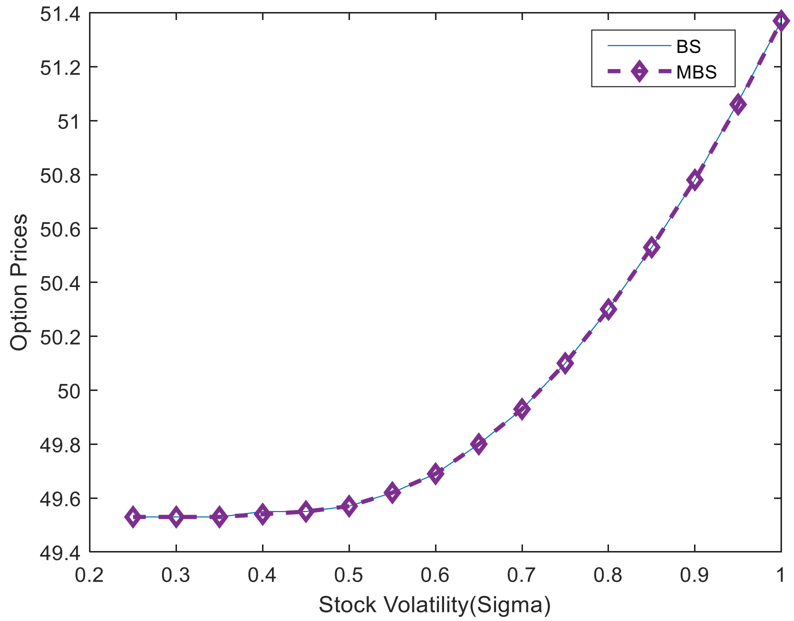

Figure 3: Profiles of BS and MBS for call option when initial stock price is 70.

Table 2: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option when initial stock prices are 60 and 70with K = 100, r = 0.2 and T = 1, $\phi_t = 0.03$.

Sigma	$S_o = 60$, $K = 100$, $\phi = 0.03$			$S_o = 70$, $K = 100$		
	BS Exact values	MBS Exact Values	Relative Error	BS Exact Values	MBS Exact Values	Relative Error
0.25	22.7672	22.7663	0.0004	14.9160	14.9143	0.0001
0.3	23.4960	23.4951	0.00004	16.1862	16.1847	0.0001
0.35	24.3677	24.3667	0.0053	17.5121	17.5107	0.00009
0.4	25.3408	25.3399	0.00004	18.8713	18.8701	0.00006
0.45	26.3860	26.3852	0.00003	20.2504	20.2492	0.00006
0.5	27.4826	27.4818	0.00003	21.6405	21.6395	0.00005
0.55	28.6158	28.6150	0.00003	23.0357	23.0348	0.00004
0.6	29.7745	29.7738	0.00003	24.4317	24.4309	0.00003
0.65	30.9508	30.9502	0.00003	25.8252	25.8244	0.00003
0.7	32.1384	32.1378	0.00002	27.2137	27.2130	0.00003
0.75	33.3325	33.3319	0.00021	28.5950	28.5944	0.00002
0.8	34.5291	34.5285	0.00002	29.9675	29.9669	0.00002
0.85	35.7250	35.7245	0.00001	31.3296	31.3290	0.00002
0.9	36.9177	36.9172	0.00001	32.6800	32.6794	0.00002
0.95	38.1049	38.1045	0.00001	34.0175	34.0170	0.00001
1.0	39.2848	39.2844	0.00001	35.3412	35.3407	0.00001

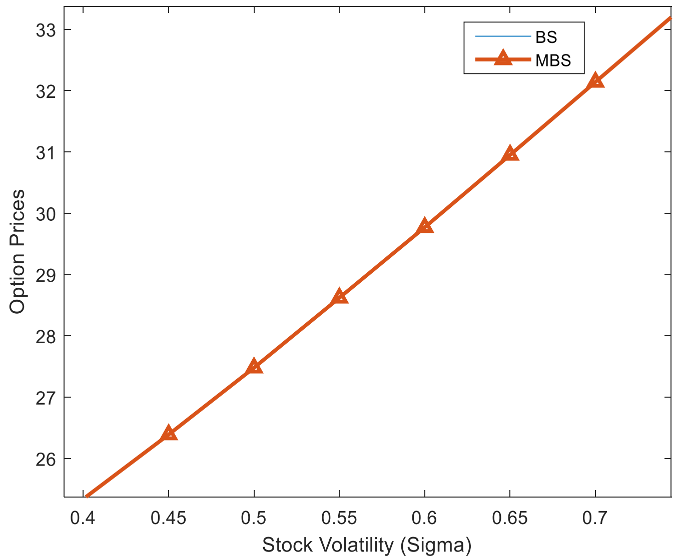

Figure 4: Profiles of BS and MBS for put option when initial stock price is 60.

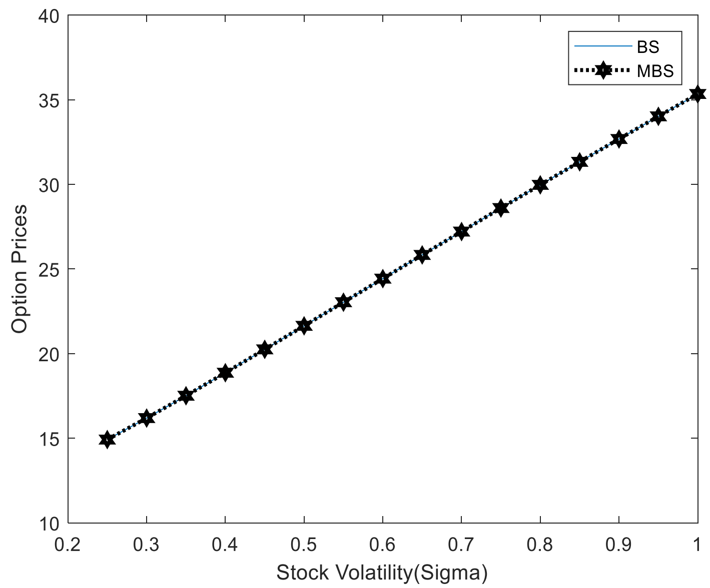

Figure 5: Profiles of BS and MBS for put option when initial stock price is 70.

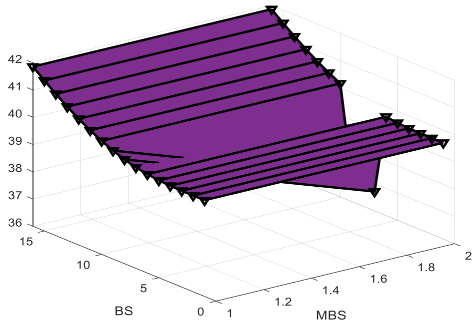

Figure 6: Surface view of BS call option when the initial stock price is 60 with variations of stock volatility (sigma).

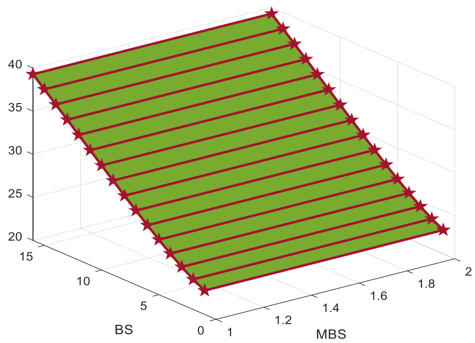

Figure 7: Surface view of BS put option when the initial stock price is 60 with variations of stock volatility (sigma).

Values in Tables 1 and 2 were generated by fixing r=0.2, k=25 $\phi_t = 0.03$ while allowing S_0 and σ vary in (16) - (20) for call and put options such that $S_0 = 60$ and 70 , $\sigma = (0.25, 0.30, ..., 1.00)$ presented in columns 1 are the difference values of σ ,columns 2 and 3 are the exact values of BS and MBS respectively. The 4th columns gives the relative error (RE) of the difference between the estimated prices using BS and MBS pricing schemes .RE is the ratio of absolute difference between BS and MBS to BS such that when this ratio is very small, the performances of both BS and MBS are equivalent.

However, it is clear that increasing stock volatility increases the value of option prices throughout the stipulated trading days. It implies that such trading business is profit maximizing in terms of long and short term investment plans as the future of investment is known against tomorrow. This situation informs investors on the activities of volatility in the capital investment hence it measures the price changes, see Tables 1 and 2 respectively.

Figures 2 and 3 describe the process of trading in respect to call option. The plots moves along sigma axis before it eventually make an upward trend. This price formation informs an investor the best decision to take in order to maximize profit; hence the process of this trading business is curve linear in nature.

As seen in Figures 4 and 5 respectively , the two plots defines changes over short and long term business plans which is linear; this means that there's a linear relationship during the time of trading activities of options. Carefully inspection of the plots shows that volatility increases linearly as option price increases throughout the trading days, with the trend of being 1; which informs an investor for proper decision making. The range of regression is within$1 \leq r \leq -1$; by this formation means that there is total sure of high rate of returns.

Figure 6 shows numerical illustration of different values of option prices. The plane at a point had jump discontinuity in the buying of underlying asset; which causes misrepresentation in stock market throughout the trading days.

In Figure 7, it is evident that there are positive option trajectories as a result of trading continuities. This continuity in selling of commodities is in relation to income price investments. This remark is profit maximizing which may be indexed in millions of naira in life of option trading.

4 Conclusion

The option pricing can be understood properly by the dynamics of Black-Scholes Equation. Consequently this paper, considered the concept of option pricing by means of Black-Scholes equation which governs the growth of option price with esteem to the expiration and cost of the fundamental asset. These equations were modified to assume a probability which measures risk-free interest rate of the underlying asset for Call and Put options. From the simulated results shows as follows:

MBS had comparable results even better than Black-Scholes (BS) exact values.

An increase in stock volatility increases option prices for call and put respectively.

The difference between BS values and MBS values are not significant.

However, we shall be looking at using the same assumption of modification of BS in harnessing the approximate solution of Black-Scholes partial differential equation in the next study.

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