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Solving parabolic equations by Olayiwola's generalized polynomial approximation method

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Abstract

Polynomial approximation methods are in wide use today for approximately solving partial differential equations of mathematical physics. An evolution of the polynomial approximation methods has led to the development of Olayiwola's generalized polynomial approximation method (OGPAM) presented in this paper, which can be applied to solve parabolic equations with constant or variable coefficients in the cases of considering slab, cylindrical or spherical geometries. Different examples have been solved using the OGPAM. The result of the present analysis has been compared with earlier analytical results. A good agreement has been obtained between the present prediction and the available results.

Keywords: cylindrical geometry; OGPAM; parabolic equations; polynomial solution; slab; spherical geometry.

1 Introduction

Partial differential equations are the chief means of providing mathematical models in science, engineering, and other fields. Generally, these models must be solved numerically and in some special cases, analytically [2]. In many practical problems that involved Cartesian, Cylindrical, and Spherical coordinates the polynomial approximation method is used.

The polynomial approximation method is one of the simplest, and in some cases, accurate methods used to solve transient conduction problems. The method involves two steps: first, selection of the proper guess temperature profile, and second, converting a partial differential equation into an equation. This can then be converted into an ordinary differential equation, where the dependent variable is the average temperature and the independent variable is time. The steps are applied to the dimensionless governing equation [1].

The present work deals with the analytical solution of parabolic equations in different geometries using Olayiwola's generalized polynomial approximation method (OGPAM).

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2 Olayiwola's generalized polynomial approximation method

In this section, Olayiwola's generalized polynomial approximation method (OGPAM) for the solution of parabolic partial differential equations with constant or variable coefficients in different geometries is presented.

The parabolic equations are defined by

$$\frac{\partial \phi}{\partial t}=\frac{k}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial \phi}{\partial r}\right)+F(r,t,\phi), \quad t>0, \quad r\in\Omega \quad \left(\Omega\subset R^1, R^2 or\, R^3\right), \tag{1}$$

with the initial condition

$$\phi(r,0)=f(r), \tag{2}$$

and the boundary conditions

$$\alpha_1\frac{\partial \phi}{\partial r}\bigg|_{r=a}+\beta_1\phi\big|_{r=a}=g_1(t), \qquad \alpha_2\frac{\partial \phi}{\partial r}\bigg|_{r=b}+\beta_2\phi\big|_{r=b}=g_2(t), \tag{3}$$

where $F(r,t,\phi)$ are the other terms in equation (1), $n=0$ for a slab, $n=1$ for cylinder, $n=2$ for sphere, $f(r)$, $g_1(t)$, $g_2(t)$ are three known functions, $n,k,\alpha_1,\alpha_2,\beta_1,\beta_2$ are constants, and $a\le r\le b$ is the boundary of Ω.

To obtain an approximate solution of the partial differential equation (1) satisfying (2) and (3), the following steps are involved:

Case 1: When $\alpha_1\neq 0$ and $\alpha_2\neq 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\left(1+\frac{b^2\beta_1}{2(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\frac{b(2a-b)\beta_2}{2(b-a)\alpha_2}\right)+\\ \frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a\beta_2}{(b-a)\alpha_2}-\frac{b\beta_1}{(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)+\\ \frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{2(b-a)\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)-\frac{\beta_2}{2(b-a)\alpha_2}\right)\end{array}\right)$$

$$B=\left(\begin{array}{l}\left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{b^2}{2(b-a)\alpha_1}\right)+\\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{b}{(b-a)\alpha_1}-\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)\right)+\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{1}{2(b-a)\alpha_1}\right)\end{array}\right)$$

$$C=\left(\begin{array}{l}\left(\dfrac{b^2\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{b(2a-b)}{2(b-a)\alpha_2}\right)-\\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a}{(b-a)\alpha_2}+\dfrac{b\beta_1}{(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)+\\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{\beta_1}{2(b-a)\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{1}{2(b-a)\alpha_2}\right)\end{array}\right)$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$
\phi(r,t)=\left(\begin{array}{l}
\left(\begin{array}{l}
\left(1+\dfrac{b^{2}\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2(b-a)\alpha_{2}-\left(2ab-\left(b^{2}+a^{2}\right)\right)\beta_{2}\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)-\dfrac{b(2a-b)\beta_{2}}{2(b-a)\alpha_{2}}\right)\phi|_{r=b}+\\
\left(\dfrac{b^{2}\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)}\right)-\dfrac{b^{2}}{2(b-a)\alpha_{1}}\right)g_{1}+\\
\left(\dfrac{b^{2}\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)+\dfrac{b(2a-b)}{2(b-a)\alpha_{2}}\right)g_{2}
\end{array}\right)+\\
\left(\begin{array}{l}
\left(\dfrac{a\beta_{2}}{(b-a)\alpha_{2}}-\dfrac{b\beta_{1}}{(b-a)\alpha_{1}}\left(\dfrac{\left(2(b-a)\alpha_{2}-\left(2ab-\left(b^{2}+a^{2}\right)\right)\beta_{2}\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)\right)\phi|_{r=b}+\\
\left(\dfrac{b}{(b-a)\alpha_{1}}-\dfrac{b\beta_{1}}{(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)}\right)\right)g_{1}-\\
\left(\dfrac{a}{(b-a)\alpha_{2}}+\dfrac{b\beta_{1}}{(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)\right)g_{2}
\end{array}\right)r+\\
\left(\begin{array}{l}
\left(\dfrac{\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2(b-a)\alpha_{2}-\left(2ab-\left(b^{2}+a^{2}\right)\right)\beta_{2}\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)-\dfrac{\beta_{2}}{2(b-a)\alpha_{2}}\right)\phi|_{r=b}+\\
\left(\dfrac{\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)}\right)-\dfrac{1}{2(b-a)\alpha_{1}}\right)g_{1}+\\
\left(\dfrac{\beta_{1}}{2(b-a)\alpha_{1}}\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)+\dfrac{1}{2(b-a)\alpha_{2}}\right)g_{2}
\end{array}\right)r^{2}
\end{array}\right) \tag{4}
$$

where

$$
\phi|_{r=a}=\left(\begin{array}{l}
\left(\dfrac{\left(2(b-a)\alpha_{2}-\left(2ab-\left(b^{2}+a^{2}\right)\right)\beta_{2}\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)\phi|_{r=b}+\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)}\right)g_{1}\\
+\left(\dfrac{\left(2ab-\left(b^{2}+a^{2}\right)\right)\alpha_{1}}{\left(2(b-a)\alpha_{1}-\left(\left(b^{2}+a^{2}\right)-2ab\right)\beta_{1}\right)\alpha_{2}}\right)g_{2}
\end{array}\right)
$$

Step 3: By using (4), evaluate

$$\frac{(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\int_a^b r^n F(r,t,\phi)dr \tag{5}$$

and express the result in the form:

$$p_1\phi\big|_{r=b}+q_1(t) \tag{6}$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi\big|_{r=b}=e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx+f(r)e^{-\int p(t)dt} \tag{7}$$

where

$$p(t)=\frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{b^n\beta_2}{\alpha_2}-\frac{a^n\beta_1}{\alpha_1}\left(\frac{\left(2(b-a)\alpha_2-\left(2ab-\left(b^2+a^2\right)\right)\beta_2\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)\right)-p_1\right)$$

$$q(t)=\frac{1}{A}\left(\begin{array}{l}\dfrac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a^n\beta_1}{\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)}\right)-\dfrac{a^n}{\alpha_1}\right)g_1- \\ \dfrac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{a^n\beta_1}{\alpha_1}\left(\dfrac{\left(2ab-\left(b^2+a^2\right)\right)\alpha_1}{\left(2(b-a)\alpha_1-\left(\left(b^2+a^2\right)-2ab\right)\beta_1\right)\alpha_2}\right)+\dfrac{b^n}{\alpha_2}\right)g_2+ \\ q_1(t)-B\dfrac{\partial g_1}{\partial t}-C\dfrac{\partial g_2}{\partial t}\end{array}\right)$$

Case 2: When $\alpha_1=0$ and $\alpha_2\neq 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\left(\dfrac{a(a-2b)}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)+\dfrac{a^2\beta_2}{2\alpha_2 b}\right)+ \\ \dfrac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\dfrac{2b}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)- \\ \dfrac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\dfrac{1}{\left(a(a-2b)+b^2\right)}\left(1+\dfrac{\left(b^2-a^2\right)\beta_2}{2\alpha_2 b}\right)+\dfrac{\beta_2}{2\alpha_2 b}\right)\end{array}\right)$$

$$B=\begin{pmatrix}\frac{1}{\beta_1}-\left(\frac{a(a-2b)}{(a(a-2b)+b^2)\beta_1}\right)-\frac{(n+1)(b^{n+2}-a^{n+2})}{(n+2)(b^{n+1}-a^{n+1})}\left(\frac{2b}{(a(a-2b)+b^2)\beta_1}\right)+\\ \frac{(n+1)(b^{n+3}-a^{n+3})}{(n+3)(b^{n+1}-a^{n+1})}\left(\frac{1}{(a(a-2b)+b^2)\beta_1}\right)\end{pmatrix}$$

$$C=\begin{pmatrix}\left(\left(\frac{a(2b-a)}{(a(a-2b)+b^2)}\right)\left(\frac{(b^2-a^2)}{2\alpha_2 b}\right)-\frac{a^2}{2\alpha_2 b}\right)-\frac{(n+1)(b^{n+2}-a^{n+2})}{(n+2)(b^{n+1}-a^{n+1})}\frac{2b}{(a(a-2b)+b^2)}\left(\frac{(b^2-a^2)}{2\alpha_2 b}\right)\\ -\frac{(n+1)(b^{n+3}-a^{n+3})}{(n+3)(b^{n+1}-a^{n+1})}\left(\frac{1}{2\alpha_2 b}\left(1+\frac{(b^2-a^2)}{(a(a-2b)+b^2)}\right)\right)\end{pmatrix}$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$\phi(r,t)=\begin{pmatrix}\begin{pmatrix}\left(\frac{a^2\beta_2}{2\alpha_2 b}+\left(\frac{a(a-2b)}{a(a-2b)+b^2}\right)\left(1+\frac{(b^2-a^2)\beta_2}{2\alpha_2 b}\right)\right)\phi|_{r=b}+\\ \frac{1}{\beta_1}\left(1-\left(\frac{a(a-2b)}{a(a-2b)+b^2}\right)\right)g_1+\left(\left(\frac{a(a-2b)}{a(a-2b)+b^2}\right)\left(\frac{a^2-b^2}{2\alpha_2 b}\right)-\frac{a^2}{2\alpha_2 b}\right)g_2\end{pmatrix}+\\ \begin{pmatrix}\left(\left(\frac{2b}{a(a-2b)+b^2}\right)\left(1+\frac{(b^2-a^2)\beta_2}{2\alpha_2 b}\right)\right)\phi|_{r=b}-\frac{1}{\beta_1}\left(\frac{2b}{a(a-2b)+b^2}\right)g_1+\\ \left(\frac{2b}{a(a-2b)+b^2}\right)\left(\frac{a^2-b^2}{2\alpha_2 b}\right)g_2\end{pmatrix}r+\\ \begin{pmatrix}-\left(\left(\frac{1}{a(a-2b)+b^2}\right)\left(1+\frac{(b^2-a^2)\beta_2}{2\alpha_2 b}\right)+\frac{\beta_2}{2\alpha_2 b}\right)\phi|_{r=b}+\\ \frac{1}{\beta_1}\left(\frac{1}{a(a-2b)+b^2}\right)g_1+\left(\frac{1}{2\alpha_2 b}-\left(\frac{1}{a(a-2b)+b^2}\right)\left(\frac{a^2-b^2}{2\alpha_2 b}\right)\right)g_2\end{pmatrix}r^2\end{pmatrix} \tag{8}$$

Step 3: By using (8), evaluate

$$\frac{(n+1)}{(b^{n+1}-a^{n+1})}\int_a^b r^n F(r,t,\phi)dr\,, \tag{9}$$

and express the result in the form:

$$p_1\phi|_{r=b}+q_1(t) \tag{10}$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi|_{r=b}=e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx+f(r)e^{-\int p(t)dt} \tag{11}$$

where

$$p(t)=\frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^{n+1}-a^{n+1}\right)\beta_2}{\alpha_2 b}+\frac{a^n(b-a)\left(2\alpha_2 b+\left(b^2-a^2\right)\beta_2\right)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)-p_1\right)$$

$$q(t)=\frac{1}{A}\left(\begin{array}{l}\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^{n+1}-a^{n+1}\right)}{\alpha_2 b}+\frac{a^n(b-a)\left(b^2-a^2\right)\left(2\alpha_2 b+\left(b^2-a^2\right)\beta_2\right)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)g_2\\+\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a^n(b-a)}{\alpha_2 b\left(a(a-2b)+b^2\right)}\right)g_1+q_1(t)-B\frac{\partial g_1}{\partial t}-C\frac{\partial g_2}{\partial t}\end{array}\right)$$

Case 3: When $\alpha_1 \neq 0$ and $\alpha_2 = 0$

Step 1: Compare the equation to be solved with equations (1) – (3) and obtain A, B, C and $F(r,t,\phi)$, where

$$A=\left(\begin{array}{l}\frac{b(b-2a)}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)-\\\frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{\alpha_1}+\frac{2a}{(b-2a)}\frac{\beta_1}{\alpha_1}-\frac{2a}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)\right)+\\\frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\beta_1}{(b-2a)\alpha_1}-\frac{1}{b(b-2a)+a^2}\left(1+\frac{a}{\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\beta_1\right)\right)\end{array}\right)$$

$$B=\left(\begin{array}{l}-\frac{ab(b-2a)}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)+\\\frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{1}{\alpha_1}+\frac{2a}{(b-2a)\alpha_1}-\frac{2a^2}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)\right)+\\\frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\left(\frac{a}{\left(b(b-2a)+a^2\right)\alpha_1}\left(1+\frac{a}{(b-2a)}\right)-\frac{1}{(b-2a)\alpha_1}\right)\end{array}\right)$$

$$C=\left(\begin{array}{l}\frac{a^2}{\left(b(b-2a)+a^2\right)}-\frac{(n+1)\left(b^{n+2}-a^{n+2}\right)}{(n+2)\left(b^{n+1}-a^{n+1}\right)}\frac{2a}{b(b-2a)\beta_2}\left(1-\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)+\\\frac{(n+1)\left(b^{n+3}-a^{n+3}\right)}{(n+3)\left(b^{n+1}-a^{n+1}\right)}\frac{1}{b(b-2a)\beta_2}\left(1-\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)\end{array}\right)$$

Step 2: Assume Olayiwola's generalized polynomial solution:

$$\phi(r,t)=\left(\begin{aligned}
&\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\begin{aligned}&\left(\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right)\phi|_{r=a}-\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)g_1\right.\\ &\left.+\left(\frac{a^2}{b(b-2a)\beta_2}\right)g_2\right)\end{aligned}\right)+\\
&\left(\begin{aligned}&\left(\begin{aligned}&\frac{2a}{b(b-2a)}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right)-\\ &\frac{2a\beta_1}{(b-2a)\alpha_1}-\frac{\beta_1}{\alpha_1}\end{aligned}\right)\phi|_{r=a}+\\
&\left(\frac{1}{\alpha_1}+\frac{2a}{(b-2a)\alpha_1}-\frac{2a^2}{b(b-2a)\alpha_1}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{ab}{b(b-2a)}\right)\right)g_1\\
&+\left(\frac{2a}{b(b-2a)}\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\frac{a^2}{b(b-2a)\beta_2}\right)-\left(\frac{2a}{b(b-2a)\beta_2}\right)\right)g_2\end{aligned}\right)r+\\
&\frac{1}{b(b-2a)}\left(\begin{aligned}&\left(\frac{b\beta_1}{\alpha_1}-\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(1+\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)\beta_1\right)\right)\phi|_{r=a}+\\
&\left(\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\frac{a}{\alpha_1}\left(1+\frac{ab}{b(b-2a)}\right)-\frac{b}{\alpha_1}\right)g_1+\\
&\left(\frac{1}{\beta_2}-\left(\frac{b(b-2a)}{b(b-2a)+a^2}\right)\left(\frac{a^2}{b(b-2a)\beta_2}\right)\right)g_2\end{aligned}\right)r^2
\end{aligned}\right) \qquad (12)$$

Step 3: By using (12), evaluate

$$\frac{(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\int_a^b r^n F(r,t,\phi)dr\,, \qquad (13)$$

and express the result in the form:

$$p_1\phi|_{r=b}+q_1(t) \qquad (14)$$

Step 4: Obtain p_1 and $q_1(t)$

Step 5: Obtain

$$\phi\big|_{r=a} = e^{-\int p(t)dt}\int_0^t q(x)e^{\int p(x)dx}dx + f(r)e^{-\int p(t)dt} \tag{15}$$

where

$$p(t) = \frac{1}{A}\left(\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\begin{array}{l}\frac{\left(b^n-a^n\right)\beta_1}{\alpha_1}+\frac{2b^n(a-b)}{(b-2a)\alpha_1}+\frac{2b^n(b-a)}{\left(b(b-2a)+a^2\right)}+ \\ \frac{2ab^n(b-a)(b(b-2a)+a)\beta_1}{\left(b(b-2a)+a^2\right)b(b-2a)\alpha_1}\end{array}\right)-p_1\right)$$

$$q(t) = \frac{1}{A}\left(\begin{array}{l}\frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\left(\frac{\left(b^n-a^n\right)}{\alpha_1}+\frac{2b^n(a-b)}{(b-2a)\alpha_1}+\frac{2ab^n(b-a)(b(b-2a)+a)}{\left(b(b-2a)+a^2\right)b(b-2a)\alpha_1}\right)g_1+ \\ \frac{k(n+1)}{\left(b^{n+1}-a^{n+1}\right)}\frac{2b^n(b-a)}{b(b-2a)\beta_2}\left(1+\frac{a^2}{\left(b(b-2a)+a^2\right)}\right)g_2+q_1(t)-B\frac{\partial g_1}{\partial t}-C\frac{\partial g_2}{\partial t}\end{array}\right)$$

Case 4: When $\alpha_1 = 0$ and $\alpha_2 = 0$

In this case, no polynomial approximation solution to equation (1) exists. This assertion is proved by the following theorem:

Theorem 1: Let $\alpha_1 = 0$ and $\alpha_2 = 0$. Then no polynomial approximation solution of (1) that satisfies (2) and (3) exists.

Proof: Substitute $\alpha_1 = 0$ and $\alpha_2 = 0$ into (4), (8), and (12), gives $\phi(r,t) \to \infty$.
Hence, $\phi(r,t)$ does not exists. This completes the proof.

3 Application of OGPAM

This section provides the results obtained when solving partial derivatives equations in time-dependent (parabolic), using the OGPAM as proposed in this paper. Three sets of examples (the first one corresponding to slab, the second one corresponding to cylindrical geometry, and the third one corresponding to spherical geometry) are given.

3.1 Set 1: Slab

Here, two examples are considered.

3.1.1 Problem 1 (Kewat and Verma [3]):

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) \tag{16}$$

with the initial condition

$$\theta(x,0) = 1 \tag{17}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1} = -B^*\theta\big|_{x=1} \tag{18}$$

Compare (16) – (18) with (1) – (3), we have
$k=1,\ n=0,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=1,\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B^*,\ g_2(t)=0,$
$a=0,\ b=1,\ r=x,\ \ t=\tau$

Then
$$A=\frac{3+B^*}{3},\qquad B=-\frac{1}{6},\qquad C=-\frac{1}{3}.$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\theta\big|_{x=1}-\frac{B^*}{2}\theta\big|_{x=1}x^2. \tag{19}$$

Then $p_1=0$ and $q_1(t)=0$ since $F(r,t,\phi)=0$. So,
$$p(\tau)=\frac{B^*}{A}=\frac{3B^*}{3+B^*},\qquad q(\tau)=0.$$

Then
$$\theta\big|_{x=1}=e^{-\frac{3B^*}{3+B^*}\tau}, \tag{20}$$

as obtained by the authors in [3].

Thus,
$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)e^{-\frac{3B^*}{3+B^*}\tau}-\frac{B^*}{2}e^{-\frac{3B^*}{3+B^*}\tau}x^2. \tag{21}$$

3.1.2 Problem 2:

Let us consider (in dimensionless form) the equation
$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)-\lambda_1(1+x)e^{-\lambda_3 t} \tag{22}$$
with the initial condition
$$h(x,0)=1 \tag{23}$$
and the boundary conditions
$$h(0,t)=1,\qquad \frac{\partial h}{\partial x}\bigg|_{x=1}=0 \tag{24}$$

Compare (22) – (24) with (1) – (3), we have
$k=1,\ n=0,\ F(r,t,\phi)=-\lambda_1(1+x)e^{-\lambda_3 t},\ \phi=h,\ f(r)=1,\ \alpha_1=0,\ \beta_1=1,\ g_1(t)=1,\ \alpha_2=1,$
$\beta_2=0,\ g_2(t)=0,\ a=0,\ b=1,\ r=x$

Then
$$A=\frac{2}{3},\qquad B=\frac{1}{3},\qquad C=\frac{5}{6}\ .$$

Assume a polynomial solution of the form (8) as

$$h(x,t)=1+2\left(h|_{x=1}-1\right)x-\left(h|_{x=1}-1\right)x^2\ . \tag{25}$$

Then $p_1=0$ and $q_1(t)=-\frac{3}{2}\lambda_1 e^{-\lambda_3 t}$. So,
$$p(t)=0,\qquad q(t)=-\frac{9}{4}\lambda_1 e^{-\lambda_3 t}.$$

Then
$$h|_{x=1}=\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right). \tag{26}$$

Thus,
$$h(x,t)=1+2\left(\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right)-1\right)x-\left(\left(1+\frac{9\lambda_1}{4\lambda_3}\lambda_1\left(e^{-\lambda_3 t}-1\right)\right)-1\right)x^2. \tag{27}$$

3.2 Set 2: Cylindrical Shape

Here, two examples are considered.

***3.2.1 Problem 1*:**

Let us consider (in dimensionless form) the equation
$$\frac{\partial\theta}{\partial t}=\frac{Le}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\delta\psi^{\alpha}\phi^{\beta}\left(1+(e-2)\theta\right)-\delta_1\psi\left(1+(e-2)\theta\right)^m \tag{28}$$
with the initial condition
$$\theta(r,0)=0 \tag{29}$$
and the boundary conditions
$$\left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0,\qquad \left.\frac{\partial\theta}{\partial r}\right|_{r=1}=-Nu\theta|_{r=1} \tag{30}$$
Compare (28) – (30) with (1) – (3), we have
$k=Le,\ n=1,\ F(r,t,\phi)=\delta\psi^{\alpha}\phi^{\beta}\left(1+(e-2)\theta\right)-\delta_1\psi\left(1+(e-2)\theta\right)^m,\ \phi=\theta,\ f(r)=0,\ \alpha_1=1,$
$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Nu,\ g_2(t)=0,\ a=0,\ b=1$

Then

$$A=\left(1+\frac{Nu}{4}\right),\qquad B=-\frac{1}{12},\qquad C=-\frac{1}{4}.$$

Assume a polynomial solution of the form (4) as

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta\big|_{r=1}-\frac{Nu}{2}\theta\big|_{r=1}r^2. \tag{31}$$

Then $p_1=(e-2)\left(1+\frac{Nu}{4}\right)\left(\delta\psi^\alpha\phi^\beta-m\delta_1\psi\right)$ and $q_1(t)=\left(\delta\psi^\alpha\phi^\beta-\delta_1\psi\right)$. So,

$$p(t)=\left(\frac{2LeNu-(e-2)\left(1+\frac{Nu}{4}\right)\left(\delta\psi^\alpha\phi^\beta-m\delta_1\psi\right)}{1+\frac{Nu}{4}}\right),\qquad q(t)=\left(\frac{\delta\psi^\alpha\phi^\beta-\delta_1\psi}{1+\frac{Nu}{4}}\right).$$

Then

$$\theta\big|_{r=1}=\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right). \tag{32}$$

Thus,

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right)-\frac{Nu}{2}\frac{q(t)}{p(t)}\left(1-e^{-p(t)t}\right)r^2. \tag{33}$$

***3.2.2 Problem 2*:**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\rho}{\partial t}+\frac{u}{r}\frac{\partial}{\partial r}(r\rho)=\frac{1}{Pem\,r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right)+\frac{1}{Pem\,r}\frac{\partial\rho}{\partial r} \tag{34}$$

with the initial condition

$$\rho(r,0)=1 \tag{35}$$

and the boundary conditions

$$\left.\frac{\partial\rho}{\partial r}\right|_{r=0}=0,\qquad \left.\frac{\partial\rho}{\partial r}\right|_{r=1}=-Sh\,\rho(1,t) \tag{36}$$

Compare (34) – (36) with (1) – (3), we have

$k=\frac{1}{Pem}$, $n=1$, $F(r,t,\phi)=\frac{1}{Pem\,r}\frac{\partial\rho}{\partial r}-\frac{u}{r}\frac{\partial}{\partial r}(r\rho)$, $\phi=\rho$, $f(r)=1$, $\alpha_1=1$, $\beta_1=0$, $g_1(t)=0$, $\alpha_2=1$, $\beta_2=Sh$, $g_2(t)=0$, $a=0$, $b=1$

Then

$$A=\left(1+\frac{Sh}{4}\right),\qquad B=-\frac{1}{12},\qquad C=-\frac{1}{4}.$$

Assume a polynomial solution of the form (4) as

$$\rho(r,t)=\left(1+\frac{Sh}{2}\right)\rho\Big|_{r=1}-\frac{Sh}{2}\rho\Big|_{r=1}r^2. \tag{37}$$

Then $p_1=-\left(2u+\frac{Sh}{Pem}\right)$ and $q_1(t)=0$. So,

$$p(t)=\left(\frac{\left(2u+\frac{3Sh}{Pem}\right)}{1+\frac{Sh}{4}}\right),\qquad q(t)=0.$$

Then

$$\rho\Big|_{r=1}=e^{-p(t)t}. \tag{38}$$

Thus,

$$\rho(r,t)=\left(1+\frac{Sh}{2}\right)e^{-p(t)t}-\frac{Sh}{2}e^{-p(t)t}r^2. \tag{39}$$

3.3 Set 3: Spherical Shape

Here, four examples are considered.

3.3.1 Problem 1 **(Mohanty and Dalai [4]):**
Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) \tag{40}$$

with the initial condition

$$\theta(x,0)=f(x) \tag{41}$$

and the boundary conditions

$$\frac{\partial\theta}{\partial x}\Big|_{x=\varepsilon}=-Q,\qquad \frac{\partial\theta}{\partial x}\Big|_{x=1}=-B^*\theta\Big|_{x=1} \tag{42}$$

Compare (40) – (42) with (1) – (3), we have
$k=1,\ n=2,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=f(x),\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=-Q,\ \alpha_2=1,\ \beta_2=B^*,$
$g_2(t)=0,\ a=\varepsilon,\ b=1,\ r=x,\ t=\tau$

Then

$$A=\left(1+B^*\left(\frac{30\varepsilon(1-\varepsilon^4)-12(1-\varepsilon^5)+20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\right),$$

$$B=\left(\frac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right),\quad C=\left(\frac{12(1-\varepsilon^5)-30\varepsilon(1-\varepsilon^4)-20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)$$

Assume a polynomial solution of the form (4) as

$$\begin{aligned}\theta(x,\tau)=&\left(\left(1+\frac{(1-2\varepsilon)B^*}{2(1-\varepsilon)}\right)\theta|_{x=1}+\frac{Q}{2(1-\varepsilon)}\right)+\left(\frac{\varepsilon B^*}{(1-\varepsilon)}\theta|_{x=1}-\frac{Q}{(1-\varepsilon)}\right)x-\\&\left(\frac{B^*}{2(1-\varepsilon)}\theta|_{x=1}-\frac{Q}{2(1-\varepsilon)}\right)x^2\end{aligned}\tag{43}$$

Then $p_1=0$ and $q_1(t)=0$ since $F(r,t,\phi)=0$. So,

$$p(\tau)=\frac{3B^*}{(1-\varepsilon^3)A},\qquad q(\tau)=\frac{3\varepsilon^2 Q}{(1-\varepsilon^3)A}.$$

Then

$$\theta|_{x=1}=\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right).\tag{44}$$

Thus,

$$\begin{aligned}\theta(x,\tau)=&\left(\left(1+\frac{(1-2\varepsilon)B^*}{2(1-\varepsilon)}\right)\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)+\frac{Q}{2(1-\varepsilon)}\right)+\\&\left(\frac{\varepsilon B^*}{(1-\varepsilon)}\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{Q}{(1-\varepsilon)}\right)x-\\&\left(\frac{B^*}{2(1-\varepsilon)}\left(\frac{q(\tau)}{p(\tau)}+\left(f(x)-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{Q}{2(1-\varepsilon)}\right)x^2\end{aligned}\tag{45}$$

3.3.2 Problem 2 **(Prakash and Mahmood [7]):**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right)+Q \tag{46}$$

with the initial condition

$$\theta(x,0)=1 \tag{47}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1}=-B^*\theta\big|_{x=1}. \tag{48}$$

Compare (46) – (48) with (1) – (3), we have

$k=1,\ n=2,\ F(r,t,\phi)=Q,\ \phi=\theta,\ f(r)=1,\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B^*,\ g_2(t)=0,$
$a=0,\ b=1,\ r=x,\ \ t=\tau$

Then

$$A=\left(1+\frac{B^*}{5}\right)=\left(\frac{B^*+5}{5}\right),\qquad B=-\frac{1}{20},\qquad C=-\frac{1}{5}.$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\theta\big|_{x=1}-\frac{B^*}{2}\theta\big|_{x=1}x^2. \tag{49}$$

Then $p_1=0$ and $q_1(t)=Q$. So,

$$p(\tau)=\frac{3B^*}{A}=\left(\frac{15B^*}{B^*+5}\right),\qquad q(\tau)=\frac{Q}{A}=\left(\frac{5Q}{B^*+5}\right).$$

Then

$$\theta\big|_{x=1}=\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right) \tag{50}$$

Thus,

$$\theta(x,\tau)=\left(1+\frac{B^*}{2}\right)\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)-\frac{B^*}{2}\left(\frac{q(\tau)}{p(\tau)}+\left(1-\frac{q(\tau)}{p(\tau)}\right)e^{-p(\tau)\tau}\right)x^2 \tag{51}$$

3.3.3 Problem 3 **(Pradhan *et al.* [6]):**

Let us consider (in dimensionless form) the equation

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) \tag{52}$$

with the initial condition

$$\theta(x,0)=f(x) \tag{53}$$

and the boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=\varepsilon} = B_1\theta\big|_{x=\varepsilon}, \qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=1} = -B_2\theta\big|_{x=1}. \tag{54}$$

Compare (52) – (54) with (1) – (3), we have
$k=1,\ n=2,\ F(r,t,\phi)=0,\ \phi=\theta,\ f(r)=f(x),\ \alpha_1=1,\ \beta_1=-B_1,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=B_2,$
$g_2(t)=0,\ a=\varepsilon,\ b=1,\ r=x,\ t=\tau$

Then

$$A=\begin{pmatrix} 1+B_2\left(\dfrac{30\varepsilon(1-\varepsilon^4)-12(1-\varepsilon^5)+20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+ \\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{(2(1-\varepsilon)-(2\varepsilon-(1+\varepsilon^2))B_2)}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\end{pmatrix},$$

$$B=\begin{pmatrix} \left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+ \\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{(2\varepsilon-(1+\varepsilon^2))}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\end{pmatrix},$$

$$C=\begin{pmatrix} \left(\dfrac{12(1-\varepsilon^5)-30\varepsilon(1-\varepsilon^4)-20(1-2\varepsilon)(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)+ \\ B_1\left(\dfrac{30(1-\varepsilon^4)-12(1-\varepsilon^5)-20(1-\varepsilon^3)}{40(1-\varepsilon)(1-\varepsilon^3)}\right)\left(\dfrac{(2\varepsilon-(1+\varepsilon^2))}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\end{pmatrix}$$

Assume a polynomial solution of the form (4) as

$$\theta(x,\tau)=\begin{pmatrix} \left(1-\dfrac{B_1}{2(1-\varepsilon)}\left(\dfrac{(2(1-\varepsilon)-(2\varepsilon-(1+\varepsilon^2))B_2)}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)+\dfrac{(1-2\varepsilon)B_2}{2(1-\varepsilon)}\right)\theta\big|_{x=1}+ \\ \left(\dfrac{\varepsilon B_2}{(1-\varepsilon)}+\dfrac{B_1}{(1-\varepsilon)}\left(\dfrac{(2(1-\varepsilon)-(2\varepsilon-(1+\varepsilon^2))B_2)}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\right)\theta\big|_{x=1}x- \\ \left(\dfrac{B_2}{2(1-\varepsilon)}+\dfrac{B_1}{2(1-\varepsilon)}\left(\dfrac{(2(1-\varepsilon)-(2\varepsilon-(1+\varepsilon^2))B_2)}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\right)\theta\big|_{x=1}x^2 \end{pmatrix} \tag{55}$$

where

$$\theta\big|_{x=\varepsilon}=\left(\frac{(2(1-\varepsilon)-(2\varepsilon-(1+\varepsilon^2))B_2)}{(2(1-\varepsilon)+((1+\varepsilon^2)-2\varepsilon)B_1)}\right)\theta\big|_{x=1}$$

Then $p_1 = 0$ and $q_1(\tau) = 0$ since $F(r,t,\phi) = 0$. So,

$$p(\tau) = \frac{3}{(1-\varepsilon^3)A}\left(B_2 + \varepsilon^2 B_1\left(\frac{(2(1-\varepsilon) - (2\varepsilon - (1+\varepsilon^2))B_2)}{(2(1-\varepsilon) + ((1+\varepsilon^2) - 2\varepsilon)B_1)} \right)\right), \qquad q(\tau) = 0$$

Then

$$\theta\big|_{x=1} = f(x)e^{-p(\tau)\tau} \tag{56}$$

Thus,

$$\theta(x,\tau) = \begin{pmatrix} \left(1 - \frac{B_1}{2(1-\varepsilon)}\left(\frac{(2(1-\varepsilon) - (2\varepsilon - (1+\varepsilon^2))B_2)}{(2(1-\varepsilon) + ((1+\varepsilon^2) - 2\varepsilon)B_1)} \right) + \frac{(1-2\varepsilon)B_2}{2(1-\varepsilon)} \right) f(x)e^{-p(\tau)\tau} + \\ \left(\frac{\varepsilon B_2}{(1-\varepsilon)} + \frac{B_1}{(1-\varepsilon)}\left(\frac{(2(1-\varepsilon) - (2\varepsilon - (1+\varepsilon^2))B_2)}{(2(1-\varepsilon) + ((1+\varepsilon^2) - 2\varepsilon)B_1)} \right) \right) f(x)e^{-p(\tau)\tau} x - \\ \left(\frac{B_2}{2(1-\varepsilon)} + \frac{B_1}{2(1-\varepsilon)}\left(\frac{(2(1-\varepsilon) - (2\varepsilon - (1+\varepsilon^2))B_2)}{(2(1-\varepsilon) + ((1+\varepsilon^2) - 2\varepsilon)B_1)} \right) \right) f(x)e^{-p(\tau)\tau} x^2 \end{pmatrix} \tag{57}$$

***3.3.4 Problem 4* (Olayiwola *et al.* [5]):**

Let us consider (in dimensionless form) the equation

$$\left.\begin{array}{c} \frac{\partial u}{\partial t} = \frac{1}{Da\,\mathrm{Re}}u + \frac{4}{3\mathrm{Re}}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) - \frac{2}{3\mathrm{Re}}\frac{1}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \\ \frac{\partial \phi}{\partial t} = \frac{1}{Pem}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) \\ \frac{\partial \theta}{\partial t} = \frac{1}{Pe}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) + \frac{\beta}{Pem}\frac{\partial \phi}{\partial r}\frac{\partial \theta}{\partial r} + \frac{4Ec}{3\mathrm{Re}}g\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)^2 \end{array}\right\} \tag{58}$$

with the initial condition

$$u(r,0) = 1, \qquad \phi(r,0) = \frac{1}{2}, \qquad \theta(r,0) = \frac{1}{2} \tag{59}$$

and the boundary conditions

$$\left.\begin{array}{ll} \left.\frac{\partial u}{\partial r}\right|_{r=0} = 0, & \left.\frac{\partial u}{\partial r}\right|_{r=1} = \gamma \\ \left.\frac{\partial \phi}{\partial r}\right|_{r=0} = 0, & \left.\frac{\partial \phi}{\partial r}\right|_{r=1} = -Sh\,\phi\big|_{r=1} \\ \left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0, & \left.\frac{\partial \theta}{\partial r}\right|_{r=1} = -Nu\,\theta\big|_{r=1} \end{array}\right\} \tag{60}$$

Compare (58) – (60) with (1) – (3), we have
For the first equation in (58):

$$k=\frac{4}{3\text{Re}},\ n=0,\ F(r,t,\phi)=\frac{1}{Da\,\text{Re}}u-\frac{2}{3\text{Re}}\frac{1}{r}\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right),\ \phi=u,\ f(r)=1,\ \alpha_1=1,$$
$$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=0,\ g_2(t)=\gamma,\ a=0,\ b=1$$

For the second equation in (58):

$$k=\frac{1}{Pem},\ n=2,\ F(r,t,\phi)=0,\ \phi=\phi,\ f(r)=\frac{1}{2},\ \alpha_1=1,\ \beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Sh,$$
$$g_2(t)=0,\ a=0,\ b=1$$

For the third equation in (58):

$$k=\frac{1}{Pe},\ n=2,\ F(r,t,\phi)=\frac{\beta}{Pem}\frac{\partial\phi}{\partial r}\frac{\partial\theta}{\partial r}+\frac{4Ec}{3\text{Re}}g\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)^2,\ \phi=\theta,\ f(r)=\frac{1}{2},\ \alpha_1=1,$$
$$\beta_1=0,\ g_1(t)=0,\ \alpha_2=1,\ \beta_2=Nu,\ g_2(t)=0,\ a=0,\ b=1$$

Then, for

First equation: $A=1,\quad B=\frac{1}{6},\quad C=-\frac{1}{3}$

Second equation: $A=\left(1+\frac{Sh}{5}\right),\quad B=-\frac{1}{20},\quad C=-\frac{1}{5}$

Third equation: $A=\left(1+\frac{Nu}{5}\right),\quad B=-\frac{1}{20},\quad C=-\frac{1}{5}$

Assume polynomial solutions of the form (4) as

$$u(r,t)=\left(u|_{r=1}-\frac{\gamma}{2}\right)+\frac{\gamma}{2}r^2 \tag{61}$$

$$\phi(r,t)=\left(1+\frac{Sh}{2}\right)\phi|_{r=1}-\frac{Sh}{2}\phi|_{r=1}r^2 \tag{62}$$

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta|_{r=1}-\frac{Nu}{2}\theta|_{r=1}r^2. \tag{63}$$

Then, for

First equation: $p_1=\left(\frac{3-2Da}{3Da\,\text{Re}}\right)$ and $q_1(t)=-\frac{\gamma}{3Da\,\text{Re}}$

Second equation: $p_1=0$ and $q_1(t)=0$

Third equation: $p_1=\frac{\beta ShNu}{Pem}\phi|_{r=1}$ and $q_1(t)=\frac{2Ecg}{3\text{Re}}\gamma\left(1-\frac{\gamma}{3}\right)$.

So, for

First equation: $p(t)=\frac{1}{A}\left(\frac{2Da-3}{3Da\,\mathrm{Re}}\right),\qquad q(t)=-\frac{1}{A}\left(\frac{4Da+\gamma}{3Da\,\mathrm{Re}}\right)$

Second equation: $p(t)=\frac{1}{A}\left(\frac{3Sh}{Pem}\right),\qquad q(t)=0$

Third equation: $p(t)=\frac{1}{A}\left(\frac{3Nu}{Pe}-\frac{\beta ShNu}{Pem}\phi\big|_{r=1}\right),\qquad q(t)=\frac{2Ecg}{3\,\mathrm{Re}\,A}\gamma\left(1-\frac{\gamma}{3}\right)$.

Then

$$u\big|_{r=1}=\frac{q(t)}{p(t)}+\left(1-\frac{q(t)}{p(t)}\right)e^{-p(t)t} \tag{64}$$

$$\phi\big|_{r=1}=\frac{1}{2}e^{-p(t)t} \tag{65}$$

$$\theta\big|_{r=1}=e^{-\left(ct+\frac{c_1}{p(t)}e^{-p(t)t}\right)}\left(\frac{1}{2}+q(t)\left(\frac{1}{c}\left(e^{\left(ct+\frac{c_1}{p(t)}e^{-p(t)t}\right)}-1\right)-\frac{c_1}{c}\left(\left(1-e^{\left(t+\frac{c_1}{p(t)}e^{-p(t)t}\right)}\right)+\frac{c_1\sqrt{\pi}}{\sqrt{-c_1p(t)}}erf\left(\frac{\sqrt{-c_1p(t)}}{p(t)}\left(e^{-t}-1\right)\right)\right)\right)\right) \tag{66}$$

where

$$c=\frac{3Nu}{PeA},\qquad c_1=\frac{\beta ShNu}{2PemA}$$

Thus,

$$u(r,t)=\left(\frac{q(t)}{p(t)}+\left(1-\frac{q(t)}{p(t)}\right)e^{-p(t)t}-\frac{\gamma}{2}\right)+\frac{\gamma}{2}r^2 \tag{67}$$

$$\phi(r,t)=\left(1+\frac{Sh}{2}\right)\frac{1}{2}e^{-p(t)t}-\frac{Sh}{2}\frac{1}{2}e^{-p(t)t}r^2 \tag{68}$$

$$\theta(r,t)=\left(1+\frac{Nu}{2}\right)\theta\big|_{r=1}-\frac{Nu}{2}\theta\big|_{r=1}r^2. \tag{69}$$

4 Discussion

Readers should note that:

The p_1 as used in this paper could be a constant or a function. For cases 1 and 2, $b\neq 2$ and case 3, $a\neq 2$. When $b=2$ in cases 1 and 2 or $a=2$ in case 3, a new space variable needs to be introduced.

$\phi|_{r=a}$ as used in this paper, means $\phi(a,t)$. It meant $\phi(r,t)$ at $r=a$.

$\phi|_{r=b}$ as used in this paper, is the same as $\phi(b,t)$. It meant $\phi(r,t)$ at $r=b$.

$\phi(r,t)\neq\phi(a,t)$ and $\phi(r,t)\neq\phi(b,t)$ as being used by some authors in the literature. Most especially those that worked on the lumped model for transient heat conduction using the polynomial approximation method.

5 Conclusion

The use of Olayiwola's generalized polynomial approximation method (OGPAM) is important for solving partial differential equations. Also, the derivation of the generalized polynomial approximation method for solving hyperbolic equations by the author is ongoing and it will be made available to readers as soon as is ready.

Different examples have been solved using OGPAM in different geometries. The simplified relations obtained in this study can be used for science and engineering calculations in many conditions.

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