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An approximation of linear evolution equation with stochastic partial derivative in Sobolev space

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Abstract

In this paper, the existence and uniqueness of classical solution of stochastic elliptic Partial Differential Equation was considered. The analytical approach of weak solutions is presented in details which has enough financial implications in time-varying investments. Finally, the computations were basically on invocation of ellipticity; which gave analytical estimates from the structural algebraic ellipticity assumption in Sobolev space.

Keywords: Sobolev spaces; stochastic PDE; smooth function; weak solution; elliptic PDE

1 Introduction

Partial Differential Equations (PDEs) arises in every filed of science and technology. These equations have been used in mathematics to solve some societal problems. Consequently, the desire to understand the solutions of these equations and its roles in real-late situation is of great practical concern to mathematicians. PDEs has inspired so many fields of study such as Sciences, Engineering and Medical fields etc. in mathematical modeling PDEs of elliptic type has played crucial roles in making real life changes.

However, most mathematicians are more interested in solution which has topological and analytical characteristics namely: continuity, infinite differentiability, continuous derivatives etc.; these types of solutions exist and are not trivial to find, but require that which is loosen the conditions for one function to be the derivative of another. Therefore, the theory of Sobolev spaces is based on this type of redefinition of differentiability. In particular, a Sobolev space gives weak solutions to differential equations and built on the basis of weak derivative.

More so, so many scholars have used the formation of PDEs in different ways such as [5] studied the existences of unique solution of an elliptic PDE. Result showed that the key proofs lie in obtaining a

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priori estimates. [1] Considered the existence of solution for a PDE involving a singularity with a general non negative radon measure as its nonhomogeneous terms. In [3] have considered quasilinear elliptic equation involving the p-Laplacian and singular nonlinearities; they deduced a few comparison principles and have proved some unique results. In another dimension,[6] obtained some new existence, uniqueness and stability results for ODEs with coefficients Sobolev spaces. In their results linear transport equations were analyzed by the method of renormalization solutions.

The work in [4] looked at solution to nonlinear Black Scholes equations. They proved the existence of weak solutions in a bounded domain and extend the results to the whole domain using a diagonal process. [11] Considered a weak solution of nonlinear Black Scholes Equation with transaction cost and portfolio risk in Sobolev space. In their result they obtained a weak solution that is characterized by Fourier transform.

Previous studies have therefore considered a weak solution of non-linear Black-Scholes equation with transaction cost and portfolio risk in Sobolev space. In particular [11] but did not consider stochastic elliptic partial differential equation in Sobolev space which grantees the existence of weak solutions. In this study, we considered the analytical solution of stochastic elliptic PDE in Sobolev space. Apart from correctly obtaining weak formulation of the problem in Sobolev space, we also derive, in detail, closed form analytical solutions that are characterized by weak results. To the best of our knowledge this is the study that has solved elliptic PDE in Sobolev spaces.

This paper is arranged as follows: Section 2.1 Mathematical preliminaries, Section 2.1.1 presents Problem formulation, 2.1.2 Definitions, Assumptions and Main Results. The paper is concluded in Section 3.

2 Mathematical preliminaries

Let $u\left(s_{t,}V_{t,}t\right)$ represents the value of the underlying asset whose stock dynamics is given by:

$$\frac{\partial u}{\partial t}+VS^2\frac{\partial^2 u}{\partial S^2}+\rho\theta VS\frac{\partial^2 u}{\partial s\,\partial v}+\frac{1}{2}V\sigma^2\frac{\partial^2 u}{\partial v^2}+\gamma s\frac{\partial u}{\partial s}+\{k\,[\theta-V_t]-\lambda(St_1Vt_1t)\}$$

$$\frac{\partial u}{\partial V}-\gamma u=0, t>0, \qquad (1.1)$$

where $\lambda\,(St_1Vt_1t)$ demotes the price of volatility risk which is independent of a particular asset.

$$u\,(St_1Vt_1t)=f\,(S,V). \qquad (1.2)$$

Setting $[(S,V)\epsilon\,\mathbb{R}^2\colon\lambda(S,V,t)=0]$ (with different probability $\overline{\mathbb{p}}$ equivalent to the original$\overline{\mathbb{p}}$ [8].
In finance, for a contingent claim on one asset, the generic PDE can be written as follows:

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial^2 u}{\partial x^2}+b\left(x_,t\right)\frac{\partial u}{\partial x}+c\left(x_,t\right)u=0, \qquad (1.3)$$

where t represents time to maturity. x represents either the value of the underlying asset or some monotonic function of it, cd u is the value of the claim (as a function of $\mathcal{X}$ and t). The terms a (.), b (.) and c (.) are the diffusion, convention reaction Coefficients respectively and this type of PDE is called a convention diffusion PDE. Equation (1.3) can also be rewritten as

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}\right) + b(x,t)\ \frac{\partial}{\partial x}\ (\beta(x,t)u) + c(x,t)u = 0. \tag{1.4}$$

To eliminate the price process in (1.1) slightly modifies on stochastic elliptic PDE as:

$$\frac{1}{2}\ vs^2\ \frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\ v\sigma^2\ \frac{\partial^2 u}{\partial V^2} + K\ [\theta - V_t]\frac{\partial u}{\partial V} - \gamma u = 0. \tag{1.5}$$

In this paper we shall explor method of solving Stochastic Partial Differential Equation (SPDE) in Sobolev spaces. Writing (1.5) which is the boundary- value problem under study in the space format gives;

$$\begin{cases} L_u = f\ in\ \varphi \\ u = 0\ on\ \partial\varphi' \end{cases} \tag{1.6}$$

where u= 0 on $\partial\varphi$ is known as Dirichlet's boundary condition φ is open bounded subset of $\mathbb{R}^N$ and u: $\varphi \to \mathbb{R}$ is the unknown, u= $u(s,v)$. we have f: $u \to$. $\mathbb{R}$The linear second order PDE is denoted by L which is 2nd partial differential operator of the form:

$$L \equiv -\sum_{ij=1}^{N}\frac{\partial}{\partial s_i}\left(\frac{1}{2}vs^2\right)^{ij}\ (s,v)\ \frac{\partial}{\partial\ s_i}\Big)\ +\ \sum_{ij=1}^{N}\frac{\partial}{\partial v_i}\left(\frac{1}{2}\ v\sigma^2\ \right)^{ij}\ (s,v)\frac{\partial}{\partial v_i}\Big)$$
$$+\sum_{i=1}^{N}(\theta - V_t)^i\ (s,v)\frac{\partial}{\partial v_i}\ \ -\ \ r\ (s,v)u \tag{1.7}$$

or in non-divergence form of it

$$L = \ -\sum_{i=1}^{N}\left(\frac{1}{2}\ vs^2\right)^{ij}(s,v)\ \frac{\partial^2}{\partial s_i\partial s_j}\ \ +\ \ \sum_{ij=1}^{N}\left(\frac{1}{2}\ v\sigma^2\right)^{ij}(s,v)\ \frac{\partial^2}{\partial v_i\partial v_j} +$$
$$\sum_{i=1}^{N}(\theta - V_t)^i\ \left(s,v\right)^i\ \frac{\partial}{\partial v_i}\ \ -\ \mathrm{r}\left(s,v\right)u, \tag{1.8}$$

with the following deficient function $\left(\frac{1}{2}vs^2\right)^{ij}, \left(\frac{1}{2}\ v\sigma^2\right)^{ij}\ (\theta - V_t)^i,\ \ r(i,j = 1,2,\dots,N).$

We can comfortably say that the PDE L = f is the form of divergence assuming L is given by (3) also in non-divergence form on the premise that L is given by (1.8).

Observe that, if the coefficient $\left(\frac{1}{2}\ vs^2\right)^{ij}$ and $\left(\frac{1}{2}\ v\sigma^2\right)^{ij}$ respectively are of class C^1 functions, we can write an operator in divergence form (1.7) in the non-divergence form (1.8). We can indeed assume that the coefficients $\left(\frac{1}{2}\ vs^2\right)^{ij}$and $\left(\frac{1}{2}\ v\sigma^2\right)^{ij}$one symmetric by nature such that $\left(\frac{1}{2}\ vs^2\right)^{ij} = \left(\frac{1}{2}\ vs^2\right)^{ij}$and $\left(\frac{1}{2}\ Vs^2\right)^{ij} = \left(\frac{1}{2}\ vs^2\right)^{ji}$ which ranges for $1\leq i,j \leq N.$

So, because of the above assertion we shall assume that the partial differential operator L satisfies a condition of been uniform elliptic; if there exists u constant $\theta > o$ such that

$$\sum_{ij=1}^{N}\left(\frac{1}{2}\, vs^{2}\right)^{ij}(s,v)\mathcal{E}_i\mathcal{E}_j \geq \theta / \xi /^{2} + \sum_{ij=1}^{N}\left(\frac{1}{2}\, v\sigma^{2}\right)^{ij}(s,v)\mathcal{E}_i\mathcal{E}_j; \geq \theta / \xi /^{2} \tag{1.9}$$

$V_{\mathcal{E}} \in \mathbb{R}^{N}$, almost everywhere (a.e), s, v, $\in$ Ellipicity implies that in each point s, v, $\in \varphi$, then the symmetric matrix N×N is $A_1\,(s,v) = \left(\frac{1}{2}\, vs^{2}\right)^{ij}(s,v)$ and $A_2\,(s,v) = \left(\frac{1}{2}\, v\sigma^{2}\right)^{ij}(s,v)$ are positive definite with eigenvalues $\geq \theta$.

It is quite obvious $= \left(\frac{1}{2}\, vs^{2}\right)^{ij} \equiv \left(\frac{1}{2}\, v\sigma^{2}\right)^{ij} = \delta_{ij},\, (\theta - V_t)^{i} \equiv o, r \equiv o,$ in my case the partial differential operator L is $-\Delta$.

2.1 Problem formulation

Weak solution of the financial Dirichlet problem.

Firstly, we consider (1.6) when L has divergence form (1.7), then we explicitly define and construct an appropriate weak solution u of (1.6) and also examined the smoothness as well as other properties of u.

Let $\left(\frac{1}{2} vs^{2}\right)^{ij}, \left(\frac{1}{2}\, v\sigma^{2}\right)^{ij}, (\theta - V_t), r \in L^{\infty}(\varphi), ij = 1, \cdots,$N
and

$$f \in L^{2}(\varphi) \tag{1.11}$$

We shall motivate the definition of weak solution on the assumption that the moment u is a smooth enough, then we multiply the PDE Lu = f by a smooth test function $z\epsilon\, C_c^{\infty}(\varphi)$ and integrate over u to obtain (1.12)

$$\int_{\varphi}\left(\sum_{ij=1}^{N}\left(\frac{1}{2}\, V\sigma^{2}\right)^{ij} u_{(s,v)i}\, v_{(s,v)i} + \sum_{ij}^{N}\left(\frac{1}{2}\, v\sigma^{2}\right)^{ij} u_{(s,v)i}\, v_{(s,v)i} + \sum_{i=1}^{N}(\theta - V_t)^{i}\, u_{(s,v)i}\, v + ruvd(s,v)\right)\int_{\varphi} fvd = \int_{\varphi} fvdsv + \int_{\varphi} fvdsv \tag{1.12}$$

The first term on the left hand side when we have integrated by parts; there are no boundary term hence v=o on $\partial\varphi$. In term of approximation we observe the same identity satisfies following the smooth function v replaced by any $v\ \epsilon\ H_0^1\ (\varphi)$ and resulting identity makes enough sense when u ϵ $H_0^1\ (\varphi)$. We choose the space $H_0^1\ (\varphi)$ to incorporate the boundary condition from (1.6) that u=0 on $\partial\varphi$. The following definition is introduced.

2.1.1 Definitions, assumptions and main results

Definition (1.1): The bilinear from B[,] corresponding with divergence form elliptic operator L defined in (1.7) is as follow:

$$B\,[u,v] := \int_{\varphi} \left(\sum_{ij=1}^{N} \left(\frac{1}{2}\, v s^2\right)^{ij} u_{(s,v)i}\, v_{(s,v)j} + \sum_{ij}^{N} \left(\frac{1}{2}\, v\sigma^2\right)^{ij} u_{(s,v)i}\, v_{(s,v)j} + \sum_{i=1}^{N} (\theta - V_t)^i\, u_{(s,v)i}\, vruvd(s,v), u, v\, \epsilon H_0^1(\varphi) \qquad (1.13)\right)$$

Definition 1.2: A function $u, v\, \epsilon H_0^1(\varphi)$ is said to be weak solution of (1.6) if the condition below holds

$$B\,[u,v] = (f,v) \qquad (1.14)$$

For all $u, v\, \epsilon H_0^1(\varphi)$, where (,) represents the linear product in L^2 (φ).
Note that identify (1.14) is otherwise called the variational formulation of (1,7).

In general, we have the boundary-value problem

$$\begin{cases} L_{u\ = f^{\,o}} \\ \quad u = \ 0\ on\ \partial \end{cases} - \sum_{i=1}^{N} f^i\ (s_i v) + \sum_{i=1}^{N} f^i\ (s_i v) i \quad in \qquad \varphi, \qquad (1.15)$$

where L is linear defined in (3) and $f^i\ \epsilon\ L^2$ (φ)
i=v ⋯, N). In view of this we see the RHS term as f=f^0 - $\sum_i^N f'\ (siv)_i + \sum_{i=1}^N f'(s_i v)^i$
belongs to H$^-(\varphi)$, the dual space of $H_0^1(\varphi)$

Existence and uniqueness of weak solution
Here we assume H is a real Hilbert space with norm $\|,\|$ and inner product (,). We also let $<,>$ represent the pairing of it with its dual space [7].

So, we introduce Lax-Milgram. Assume that B: H x H → ℝ is bilinear mapping, for which here exist constants $\ltimes, \beta > o$ such that

$$B[u,v] \leq \propto \|\ u\ \|\ \|\ v\ \|,\ (u, v\, \epsilon\, H\,)$$
$$\beta\ \|\ u\ \|^2\ \leq B[u,u]. \qquad (u, \in\ H)$$

Lastly, let f : H⟶ ℝ be a bound linear functional on H. Then exists a unequal element, $u\ \in H$ such that

$$B[u,v] = < f, v > \text{ for all } v \in\ H \qquad (1.16)$$

Proof 1: for ach fixed element $u \in H$,the mapping V→ B$[u, v]$ is a bounded linear functional on H; when the $R_i esz$ representation theonem asserts the existence of a unique element W ϵ H satisfying
B$\left(v_, v\right) = \left(w_, v\right), (v \epsilon H)$. (1.17)

Let us write $A_u = w$ whenever (2) holds, so that

$$\mathrm{B}[u, v] = \left(A_{u,} v\right), \left(u_, v \epsilon H\right) \quad (1.18)$$

Proof 2: we claim A : H→ H is a bounded linear operation indeed if $\lambda_1 \lambda_2 \epsilon \mathbb{R}$ and u_1, u_2 we see for each $r. \epsilon H$ that

$$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = \mathrm{B}[\lambda_1 u_1 + \lambda_2 u_2 v] \text{ By} \quad (1.19)$$

$$(A (\lambda_1 u_1 + \lambda_2 u_2), v) = \mathrm{B}[\lambda_1 u_1 + \lambda_2 u_2 v]$$

$$= \lambda_1 B[u_1, v] + \lambda_2 B [u_2 v] = \lambda_1 (Au_1 v) + \lambda_2 (Au_2, v)$$

$$= (\lambda_1 Au_1 + \lambda_2 Au_2, v).$$

The equality obtains for each $v, \epsilon H$ and so A is linear.

More so, $\| Au \|^2 = (Au_1 Au) = B[u_1 Au] \leq \propto \| u \| \| Au \|$.
Consequently $\| Au \| \leq \alpha \| u \|$ for all $u, \epsilon H,$ and so A is bounded

Proof 3: Next we assert

$$\begin{cases} A \text{ is one} - \text{to} - \text{one and} \\ R(A), \text{the range of } A \text{ is closed in } H \end{cases}$$

To prove the above, we complete the following

$2\beta \| u \|^2 \leq \beta[u_1 u] = (Au_1 u) \leq \| Au \| \| u \|$
Hence $\beta \| u \| \leq \| Au \|$. (1.19)
This inequality easy implies
R (A) = H (1.20)

For if not, them, since R(A) is closed, there would exist a nonzero element, $\omega \in H$ with $\omega \in R(A)^2$. But this fact in term implies the contraction.

$$\beta \| \omega \|^2 \leq B[\omega_, \omega] = (A, \omega) = 0.$$

Proof 4: Next, we observe once none from the Rieze representation theorem that
$< f_1 v > = \left(w_, v\right) + \left(w_, v\right) for\ all\ v \in H.$

Finally, we show there in at most one element $u \in H$varying (1). For if both B[$u_,$v] = <$f_,$v> and B ($\bar{u}$,v)= <$f_,$v> + <$f_,$v> then B (u, $\bar{u}, u, v$) = 0, $v \in H$. We set v = u- $\bar{u}$
to find $\beta \| u - \bar{u} \|^2 \leq B(u - \bar{u}, u - \bar{u}) = 0.$

Energy Estimates

We take a look of a particular bilinear form B[,] defined

$$B[u,v] = \int_u \sum_{ij=1}^{N} \left(\frac{1}{2}VS^2\right)^{ij} u_{(s,v)i}\, V_{(s,v)j} + \sum_{ij=1}^{N} \left(\frac{1}{2}VS^2\right)^{ij} u_{(s,v)i}\, V_{(s,v)j}$$
$$+ \sum_{i=1}^{N} (\theta - V_t)^i\, u_{(s,v)i}\, v + ruvd\,(s,v).$$

For $u_, v \in H_0^1(\varphi)$ and try to verify the hypothesis of Lax-Milgram theorem [7].

Theorem 1 (Energy Estimates). There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$/\, B[u,v] \leq \propto \parallel U \parallel H_0^1(\varphi) \ \parallel V \ \parallel H_0^1(\varphi) + \alpha \ \parallel u \parallel H_0^1(\varphi) \parallel V \parallel_0^1 (\varphi)$$
$$\beta//u//^1{}_0(\varphi) \leq \ B[u,u] + \ r//u//^2 L^2(u) + \beta[u,u] + B[u,u] + r//u//^2 L^2(u) + \beta[u,u]$$
$$for\ u_, \, v_\in H^1{}_0\,(\varphi).$$

Proof: We prove the equalities by taking note of holder inequality, which follows that for each u, v∈ $H^1{}_0\,(\varphi)$

$$/B[u,v]\,/\leq \left(\sum_{ij=1}^{N} // \left(\frac{1}{2}vs^2\right)^{ij} //_{L^\infty} + \int_\varphi \sum_{ij=1}^{N} // \left(\frac{1}{2}v\sigma^2\right)^{ij} //_{L^\infty} \right) \int_\varphi /Du/Dv/d\,(s,v)$$
$$+ \sum_{i=1}^{N} //\,(\theta - V_t)//L^\infty \int_\varphi /Du/\ /v/d(s,v) + //r\,//L^\infty \int_\varphi /u//v/d(s,v)$$

$\leq \propto //\ u\ \ //\ H^1{}_0(\varphi) //\ v\ //\ H^1{}_0(\varphi)$ for some appropriate constants α.

In proving the 2nd inequality of ellipticity condition (4) therefore

$$\theta \int_\varphi /Du/^2 d(s,v) \leq \int_\varphi \ \sum_{i=1}^{N} \left(\frac{1}{2}vs^2\right)^{ij} u_{(s,v)i}\, u_{\ (s,v)j}\ d(s,v)$$
$$+ \int_\varphi \sum_{i=1}^{N} \left(\frac{1}{2}VO^{-2}\right)^{ij} u(s,v)i\ u(s,v)j\ d(s,v)$$
$$= B\,[u,u] - \sum_{i=1}^{N} (\theta - V_t)^i\ u(s,v)iu + ru^2\ d(s,v) \quad (6)$$

$\leq B\ [u,v] + \sum_{i=1}^{N} (\theta - V_t)^i\ //L^\infty \int_\varphi /Du//u/\ d(s,v) + //\gamma//L^\infty \int_\varphi{}^{u^2}\ \ d(s,v).$

Recall that Cauchy's inequality is given as ∈>0 for each a,b, >0 which states ab$\leq \in a^2 + \frac{b^2}{4\in}$
Using the inequality when a= /Du/ and b=/u/ then integrating over φ yields
$\int_\varphi /Du//u/ \leq \int_\varphi \left(\in/Du/^2 + \frac{1}{4\in} U^2\right)$.

Using the last inequality in (6) with ∈ such that
$0 < \in < \frac{\theta}{2\sum_{i=1}^{N}//\theta - V_t//_\infty}$.

We can conclude that appropriate for an constant C

$\frac{\theta}{2}\int_\varphi /Du/^2\, d(s,v) \leq B\,[u,u] + C\int_\varphi{}^{u^2} d(s,v)$.

According to point care's inequality, we know that

$$//u//L^2(u) \leq C^1\, //Du//L^2(\varphi)\ For\ constant C^1 > 0$$

Thus, we finally obtain that for some suitable constant

$$\beta > o, r \geq o.$$

Theorem 2. (Existence theorem for weak solution). There is a number $\gamma \geq 0$ such that for each

$$\mu \geq \gamma$$

And each function

$$f \in L^2(\varphi)$$

There exists a unique weak solution $u \in H_0^1(\varphi)\ of\ boundary\ value\ problem$

$$\oint \begin{cases} L_u + \mu_u = f & in\ \varphi \\ u = o & on\ \partial\varphi \end{cases}$$

Proof of 1: Take r from Theorem 1.1, let $\mu \geq \gamma$ and define then the bilinear form

$$B_\mu\,[u,v] := B[u,v] + \mu(u,v) \qquad \left(u, v \in H_0^1(\varphi)\right),$$

which corresponds to the operator L, $\mu u := \mu u$, (1.16) means the inner product in $L^2\ (\varphi)$. Then B_μ [1.16] Satisfies the hypothesis the Lax-Milgram theorem.

Proof of 2. Now fix f E $L^2(\varphi)$ and set <f,v> : = (f, v) L^2 (φ).
This is a bounded linear functional on L^2 (φ) and thus on H_0^1 (φ). we apply the Lax-Milgram theorem to find unique function u ∈ H_0' (φ) satisfying.

$$B_\mu[u,v] = < f, v >$$

For all v∈ H_0' (φ); u is consequently the unique weak solution of (8)
Note: we can also show that for all $f' \in L^2$ (φ) (I=0, …, N) there exists a unique weak solution is of the PDE

$$\{Lu + \mu u = f^0 - \sum_{i=1}^{N}$$

$$Indeed, it\ is\ enough\ to\ note\ < f, v > = \int_{\varphi} f, v + \sum_{i=1}^{N} f^i\ (\theta - vt)\ v_{\ (s,v)i}\ d_{\ (s,v)}$$

is bounded linear functional on $H_0^{'}(\varphi)$.

In particular, we deduce that the mapping

$L_{\mu}; = L + \mu \mathrm{I}: H_0^{'}(\varphi)\ n \to H^{-1}(\varphi)$ $\quad (\mu \geq r)$ is an isomorphism.

Examples: in the case Lu = -Δu so that B[u, v]= $\int_{\varphi} Du.Dv\ d\ (s,v)$
We easily check using point care's inequality that theorem 3.2 holds with γ= 0, A similar assertation holds for the general operator

$$L_u = -\sum_{i=1}^{N}\left(\frac{1}{2}\ vs^2\right)^{ij} u(s,v)i + \sum_{i=1}^{N}\left(\frac{1}{2}vo^{-2}\right)^{i} u(s,v)i$$
$$+ ru, provided\ r \geq o\ in\ \varphi.$$

Regularity
Here, we regularized to know whether a weak solution u is of the PDE.

$$\Longrightarrow L_u = f\ in\ \varphi$$

Is in fact smooth: it is the regularity issues for weak solutions. We shall ensure that a weak solution might be better 1 a typical function in $H_0^{'}(\varphi)$. Considering the problem.

$$-\Delta u = f\ \ in\ \mathbb{R}^N$$

In the sequel, we assume for heuristic purposes that is smooth and variables completely as /x/ → ∞ for justify the calculation as follows: Therefore, we calculate.

$$\int_{\mathbb{R}^N} f^2\ d\ (s,v) = \int_{\mathbb{R}^N} (\Delta u)^2\ d = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$+ \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ v\sigma^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v) +$$
$$\sum_{ij=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ vs^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)$$
$$= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\ VS^2\right) u_{(s,v)i(s,v)i}\ u_{(s,v)j\ (s,v)j}\ d(s,v)\ d(s,v) +$$

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{2}\, V0^2\right) u_{(s,v)i(s,v)i}\, u_{(s,v)j\,(s,v)j}\, d(s,v)$$
$$= \int_{\mathbb{R}^N} /D^2u\,/^2\, d(s,v) += \int_{\mathbb{R}^N} /D^2u/^2\, d(s,v)$$

It can be seen that L^2 – norm of the 2nd derivative of u can be estimated to give the L^2-norm of f. Also, the PDE (1.6) can be differentiated to obtain.

$$-\Delta\bar{u} = \bar{f},$$

Hence we have $\bar{u}$ = u$(s,v)_k$ and $\bar{f}$: f $(s,v)_k$, (k=1,…,N) using the same procedure, we find out L^2-norm of the third derivative of u is estimated via the first derivatives of f. Moving in the same manner, we notice the L^2-norm of the (m+2)nd derivatives of u is been controlled by the L^2-norm of mth derivative of f,m = 0, 1,⋯

The above computations is suggestive of Poisson's equation (1.6) therefore, we expect a weak solution u ∈ $H_0^{'}$ to belong to H^{m+2} as soon as the inhomogeneous term f belongs to H^m (m, = 1, ⋯). Informally we can conclude to say that u has "two more derivatives in L^2 than f". In particular, this will interestingly form = ∞ which is when u belong to H^m for all m = 1, . . . and it will be in the space C^{∞}. It can be seen that the computations did not justify a proof. u was assumed to be smooth or be in a special C^3, in order to carry out the computation (1.18); however, if we start these computations with a weak solution in the space $H_0^{'}$, the computation, cannot be justifies immediately. Otherwise, we rely upon an analysis of some certain difference quotients.

According to the computations they are technically difficult and obviously yielding powerful results and useful assertions in respect to the smoothness of weak solutions. The computations in geared toward the innovation of ellipticity; which of cause is to derive an analytical estimates from the structure algebraic ellipticity assumption.

3 Conclusion

The definitions, assumptions and analytical approach of stochastic of elliptic partial differential equation in Sobolev space has been illustrated; which gave powerful results and useful assertions in respect to the smoothness of weak solutions.

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