International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 3 (Oct.), 2022

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 - 5708

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 3, October 2022, pages 1 - 12

Iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mapping with application to monotone variational inclusion problems in Hilbert spaces

E. Ekuma-Okereke*, **F.M. Okoro**†‡, and **C.E. Abhulimen**†§

Abstract

Motivated by the recent result in [1], it is our aim in this paper to introduce the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex subset of real Hilbert spaces. We prove strong convergence theorem for it. Furthermore, we apply our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. In addition, we provide a numerical example to illustrate the effectiveness of our procedure. It is intended that our procedure complement our motivated result cited in the literature.

Keywords: fixed points; strictly pseudo-contractive mappings; modified Mann-type iteration; inertial term; strong convergence

1 Introduction

Throughout, we take C to represent a nonempty, closed and convex subset of real Hilbert space H with the inner product and induced norm denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. The metric projection mapping of H onto C denoted by $P_C(x)$ is defined by

$$\|x - P_C(x)\| = \inf\{\|x - z\| : z \in C\}, \tag{1}$$

and the metric projection holds true if and only if it satisfies the variational inequality

$$\langle x - P_C(x), z - P_C(x)\rangle \leq 0. \tag{2}$$

Remark 1.1 *Using this characterization, we get that a metric projection $P_C(x)$ is nonexpansive, that is $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2, \forall x, y \in X$.*

Definition 1.1 *A mapping T defined on a closed and convex subset $C \subset H$ is called*

(a) Lipschitzian if there exists $L \geq 0$ such that

$$\|Tx - Ty\|^2 \leq L\|x - y\|^2, \forall x, y \in C, \tag{3}$$

*Corresponding Author. Department of Mathematics, Federal University of Petroleum Resources, Effurun, P.M.B 1221, Nigeria; E-mail: ekuma.okereke@fupre.edu.ng

†Department of Mathematics, Ambrose Alli University, Ekpoma, Nigeria.

‡E-mail: moibi1@yahoo.com

§Email: cletusabhulimen@yahoo.co.uk

(b) *quasi nonexpansive if*

$$\|Tx - p\|^2 \leq \|x - p\|^2, \forall x \in C, p \in Fix(T). \tag{4}$$

(c) *strictly pseudocontractive [3] if there exists a constant* $k \in [0, 1)$ *such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in C. \tag{5}$$

or equivalently, if there exists a constant $\lambda > 0$ *such that*

$$\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in C. \tag{6}$$

(d) *demicontraction (or k-demicontractive) (see [1]) if there exists a constant* $k \in [0, 1)$ *and* $Fix(T) \neq \emptyset$ *such that*

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in C, p \in Fix(T). \tag{7}$$

Remark 1.2 *The mapping* T *is called a contraction if* $0 \leq L < 1$. *It is called nonexpansive if* $L = 1$. *It is demiclosed at a point* $p \in C$ *if* C *contains a sequence* $\{x_n\}_{n=1}^{\infty}$ *which converges weakly to* $\bar{x} \in C(x_n \rightharpoonup \bar{x})$ *and* Tx_n *converges strongly to* p $(Tx_n \to p)$ *and* $T\bar{x} = p$.

Numerous iterative procedures for the approximation of fixed points of nonexpansive mappings and its generalization have been constructed and studied by various authors (see [4], [10], [15] and the references therein) in the framework of Hilbert and Banach spaces. Notable among them are the Mann iteration procedure (see [13]), Ishikawa iteration procedure (see [9]), Halpern iteration procedure (see [8])and Krasnoselskij iteration (see [12]). One major setback in Mann iteration defined as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Tx_n, \tag{8}$$

is the weak convergence it only guarantees. However, in a bid to achieve strong convergence from Mann iteration, some authors on one hand, have imposed certain perceived stringent conditions on the $Fix(T)$ (see [19]) and as pointed out by [2], the unnecessary assumption like $\lim_{n\to\infty} \beta_n = 0$ used by some author to achieve convergence in their work, on the other hand. It is noteworthy to mention that it is only Halpern iteration procedure that satisfies this condition. Besides, in contributing to the volume of work on iterative procedures with inertial terms for the approximation of fixed points of various mappings and its applications, many authors have recently constructed and studied new procedures in this directions (see [16], [6], [7] [5] [14] and the references therein).

Very recently and in [1], Agbebaku, Nwokoro, Osilike, Chima and Onah, studied "The Iterative Algorithm with Inertial and Error Terms for Fixed Points of Strictly Pseudo-contractive Mappings and Zeros of Inverse Strongly Monotone Operators". They proved the following theorem:

Theorem 1.1 *Let* H *represent a real Hilbert space. Let* $T : H \to H$, *represent a* $k-$ *strictly pseudo-contractive mapping with* $Fix(T) \neq \emptyset$. *Choose* $x_0, x_1 \in H$ *arbitrarily and define the sequence* $\{x_n\}$ *generated from these as follows*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \beta_n y_n + \eta_n Ty_n + e_n, n \geq 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\eta_n\}_{n=1}^{\infty}$ *are sequences in* $(0, 1)$, $\{\theta_n\}_{n=1}^{\infty}$ *is a positive sequence and* e_n *is an error sequence satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\alpha_n + \beta_n + \eta_n = 1$, $\beta_n \geq k, \forall n \geq 1$, $\liminf_{n\to\infty} \eta_n[\beta_n - (1-\alpha_n)k] > 0$,

(iii) either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty}, \dfrac{\|e_n\|}{\alpha_n} = 0$,

(iv) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \text{whenever } x_n \neq x_{n-1} \\ \tau, & \text{otherwise} . \end{cases}$$

Then algorithm (9) converges strongly to metric projection of H onto $Fix(T)$.

It is important to note that their result extends the recent results of Shehu, Iyiola and Ogbuisi [16] from the class of nonexpansive mappings to the much more general class of strictly pseudocontractive mappings in Hilbert spaces.

Remark 1.3 *Observe that their map T as used in their theorem above is a self map on H. If on the contrary, the domain of their map T becomes a nonempty, closed and convex subset C of H (as it is noted to be the case in many applications) and T is a self map on C, then we can say that the iterative procedure of (9) may fail to be well-defined.*

Thus motivated by the result of Agbebaku et al. [1], it is our aim in this paper to introduce the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex set in real Hilbert spaces. We study strong convergence theorem for it. Furthermore, we apply our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. In addition, we provide a numerical example to illustrate the effectiveness of our procedure. It is intended that our procedure complement our motivated result cited in the literature.

2 Preliminaries

Lemma 2.1 *Let H be a real Hilbert space. Given $x, y \in H$, then the inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{10}$$

Lemma 2.2 *cf.([10]) Let H be a real Hilbert space. For all $x_i \in H, \alpha_i \in [0,1], i = 0,1,2,...,n$ satisfying $\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_n = 1$, then we have the equality condition as follows:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x\|^2 - \sum_{0\leq i,j\leq n} \|x_i - x_j\|^2. \tag{11}$$

Lemma 2.3 *([18]) Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$, represent a strictly pseudo-contractive mapping. Then*

(i) $(1-T)$ *is demiclosed at* 0,

(ii) $Fix(T)$ *is closed and convex subset of* C.

Lemma 2.4 *([14]) Let $\{a_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{e_n\}_{n=1}^{\infty}$ be sequences contain in $[0,\infty)$, $\{b_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{d_n\}_{n=1}^{\infty}$ be a sequence contain in $(-\infty_{n=1}, +\infty)$ such that*

$$a_{n+1} \leq [1 - b_n + c_n]a_n + d_n + e_n, n \geq 1. \tag{12}$$

If $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} e_n \infty$, then *(i) if* $d_n \leq Mb_n$ *for some* $M \geq 0$*, then* $\{a_n\}_{n=1}^{\infty}$ *is bounded.*

(ii) if $\lim_{n\to\infty} b_n = 0, \sum_{n=1}^{\infty} c_n = \infty$ *and* $\limsup_{n\to\infty} \dfrac{d_n}{b_n} \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 *([17]) Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0 \tag{13}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in* $(0,1)$*,* $\{\delta_n\}$ *is a sequence in* $(-\infty, +\infty)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,*

(ii) $\lim_{n\to\infty} sup\delta_n \leq 0$*. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 *([11]) Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of real numbers such that there exists a non-decreasing subsequence* $\{n_i\}$ *of* $\{n\}$ *that is* $a_{n_i} < a_{n_{i+1}}$ $\forall i \in N$*. Then there exists a non-decreasing subsequence* $\{m_k\} \subset N$ *such that* $m_k \to \infty$ *and the following properties are satisfied for all (sufficiently large number)* $k \in N$*:* $a_{m_k} \leq a_{m_{k+1}}$ *and* $a_k \leq a_{m_{k+1}}$*. In fact,* $m_k = max\{j \leq k : a_j \leq a_{j+1}\}$.

3 Main results

Theorem 3.1 *Let* C *represent a non-empty, closed and convex subset of a real Hilbert space* H*. Let* $T : C \to C$*, represent a strictly pseudo-contractive mapping with* $Fix(T) \neq \emptyset$*. Let the sequence* $\{x_n\} \in C$ *be generated by choosing* $x_1 < x_0 \in C$ *and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1-\alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1-\beta_n)Ty_n), n \geq 1, \end{cases} \tag{14}$$

where $\{\beta_n\}_{n=1}^{\infty} \in (k,1)$ *for any* $k \in [0,1)$*,* $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$*,* $\{\theta_n\}_{n=1}^{\infty}$ *is a positive sequence satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1-\beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise }. \end{cases}$$

Then $\{x_n\}$ *converges strongly to metric projection of* C *onto* $Fix(T)$.

Proof. We first show that the sequence $\{x_n\}$ generated by the procedure (14) is bounded. To do this, we let $p \in Fix(T)$. In view of Lemma 2.2 and the property of T, we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|P_C(\beta_n y_n + (1-\beta_n)Ty_n) - p\|^2 \\ &\leq \|\beta_n y_n + (1-\beta_n)Ty_n - p\|^2 \\ &= \|\beta_n(y_n - p) + (1-\beta_n)(Ty_n - p)\|^2 \\ &\leq \beta_n\|y_n - p\|^2 + (1-\beta_n)\|Ty_n - p\|^2 - \beta_n(1-\beta_n)\|Ty_n - y_n\|^2 \\ &\leq \beta_n\|y_n - p\|^2 + (1-\beta_n)[\|y_n - p\|^2 + k\|y_n - Ty_n\|^2] - \beta_n(1-\beta_n)\|Ty_n - y_n\|^2 \\ &\leq \|y_n - p\|^2 - (1-\beta_n)[\beta_n - k]\|y_n - Ty_n\|^2. \end{aligned} \tag{15}$$

Furthermore,

$$\begin{aligned}\|y_n-p\|^2 &= \|\alpha_n x_0+(1-\alpha_n)z_n-p\|^2\\ &\le \|\alpha_n x_0+(1-\alpha_n)z_n-p\|^2\\ &= \|\alpha_n(x_0-p)+(1-\alpha_n)(z_n-p)\|^2\\ &\le \alpha_n\|x_0-p\|^2+(1-\alpha_n)\|z_n-p\|-\alpha_n(1-\alpha_n)\|x_0-z_n\|^2. \end{aligned}\tag{16}$$

Substituting (16) into (15), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& \alpha_n\|x_0-p\|^2+(1-\alpha_n)\|z_n-p\|-\alpha_n(1-\alpha_n)\|x_0-z_n\|^2\\ &-(1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2. \end{aligned}\tag{17}$$

Besides

$$\begin{aligned}\|z_n-p\|^2 &= \|x_n+\tau_n(x_n-x_{n-1})-p\|^2\\ &= \|(x_n-p)+\tau_n(x_n-x_{n-1})\|^2\\ &= \|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2+2\langle\tau_n(x_n-x_{n-1},x_n-p)\rangle\\ &\le \|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2+2\tau_n\|x_n-x_{n-1}\|.\|x_n-p\|^2+2\tau_n\|x_n-x_{n-1}\|\\ &= [1+2\tau_n\|x_n-x_{n-1}\|].\|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2+2\tau_n\|x_n-x_{n-1}\|. \end{aligned}\tag{18}$$

Substituting (18) into (17), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& \alpha_n\|x_0-p\|^2\\ &+(1-\alpha_n)\{\alpha_n[1+2\tau_n\|x_n-x_{n-1}\|]\|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2+2\tau_n\|x_n-x_{n-1}\|\}\\ &-\alpha_n(1-\alpha_n)\|x_0-z_n\|^2-(1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2\\ &\le \alpha_n\|x_0-p\|^2+(1-\alpha_n)\{[1+2\tau_n\|x_n-x_{n-1}\|].\|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2\\ &+2\tau_n\|x_n-x_{n-1}\|\}\\ &\le \alpha_n\|x_0-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)2\tau_n\|x_n-x_{n-1}\|.\|x_n-p\|^2\\ &+(1-\alpha_n)\tau_n^2\|x_n-x_{n-1}\|^2+(1-\alpha_n)2\tau_n\|x_n-x_{n-1}\|\\ &\le \alpha_n\|x_0-p\|^2+(1-\alpha_n)\|x_n-p\|^2+2(1-\alpha_n)\alpha_n\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|.\|x_n-p\|^2\\ &+(1-\alpha_n)\alpha_n^2\frac{\tau_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2+2(1-\alpha_n)\alpha_n\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &= [(1-\alpha_n)+\alpha_n\{2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|\}]\|x_n-p\|^2\\ &+\alpha_n\{\|x_0-p\|^2+2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|+(1-\alpha_n)\alpha_n\frac{\tau_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2\}\\ &\le [(1-\alpha_n)+\alpha_n\{2(1-\alpha_n)\frac{\theta_n}{\alpha_n}\}]\|x_n-p\|^2\\ &+\alpha_n\{\|x_0-p\|^2+2(1-\alpha_n)\frac{\theta_n}{\alpha_n}+(1-\alpha_n)\alpha_n\frac{\theta_n^2}{\alpha_n^2}\}. \end{aligned}\tag{19}$$

By our hypothesis, we can find a number $M>0$ such that

$$2\{\|x_0-p\|^2+2(1-\alpha_n)\frac{\theta_n}{\alpha_n}+(1-\alpha_n)\alpha_n\frac{\theta_n^2}{\alpha_n^2}\}\le M,\ \text{for all } n\ge 1.$$

Also there exists $n(\epsilon)>0$ such that

$$2(1-\alpha_n)\frac{\theta_n}{\alpha_n}\le\frac{1}{2},\forall n\ge n(\epsilon).$$

Hence, (19) becomes

$$\|x_{n+1}-p\|^2 \le (1-\eta_n)\|x_n-p\|^2 + \eta_n M, \forall n \ge 1, \text{ where } \eta_n = \frac{\alpha_n}{2}. \tag{20}$$

This shows that $\{\|x_{n+1}-p\|\}_{n=1}^{\infty}$ is bounded. In view of Lemma 2.4, we get that the sequence $\{x_n\}$ is bounded.

We next show that the sequence $\{x_n\}$ generated by the procedure (14) converges to the metric projection mapping of C onto $Fix(T)$. To do this, we let $p = P_{Fix(T)}(x_0)$. In view of the construction of procedure (14), Lemma 2.1 and the approach used to get (19) we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2 = \|\alpha_n x_0 + (1-\alpha_n)z_n - p\|^2\\ &\le (1-\alpha_n)\|z_n-p\|^2 + 2\alpha_n\langle x_0-p, \alpha_n(x_0-p)+(1-\alpha_n)(z_n-p)\rangle. \end{aligned}\tag{21}$$

Observe from the definition of z_n, that

$$\|z_n-p\|^2 \le \|x_n-p\|^2 + 2\tau_n\|x_n-x_{n-1}\|^2 + \tau_n[\|x_n-p\|^2 - \|x_{n-1}-p\|^2]. \tag{22}$$

So that (21) becomes

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2 + 2(1-\alpha_n)\tau_n\|x_n-x_{n-1}\|^2 + (1-\alpha_n)\tau_n[\|x_n-p\|^2\\ &- \|x_{n-1}-p\|^2] + 2\alpha_n^2\|x_0-p\|^2 + \alpha_n(1-\alpha_n)\langle x_0-p, z_n-p\rangle\\ &\le (1-\alpha_n)\|x_n-p\|^2 + \alpha_n(2(1-\alpha_n)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|^2\\ &+ (1-\alpha_n)\frac{\tau_n}{\alpha_n}[\|x_n-p\|^2 - \|x_{n-1}-p\|^2] + 2\alpha_n\|x_0-p\|^2\\ &+ (1-\alpha_n)\langle x_0-p, z_n-p\rangle). \end{aligned}\tag{23}$$

Next, we consider two cases to complete the proof of our Theorem.

Case 1. If we suppose to find $n_0 \in N$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is decreasing, then we can get $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is convergent and hence

$$\|x_n-p\| - \|x_{n+1}-p\| \to 0, n \to \infty. \tag{24}$$

In view of (15), (24) and conditions (i) and (ii) of our hypothesis, we get

$$\begin{aligned}\|y_n-p\|^2 - \|Ty_n-p\|^2 &= \|y_n-p\|^2 - \|x_{n+1}-p\|^2 + \|x_{n+1}-p\|^2 - \|Ty_n-p\|^2\\ &\le \|y_n-p\|^2 - \|x_{n+1}-p\|^2 + \|y_n-p\|^2 - (1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2\\ &- \|Ty_n-p\|^2 \end{aligned}$$

$$(1-\beta_n)[\beta_n-k]\|y_n-Ty_n\|^2 \le \alpha_n(\|x_0-p\|^2 - \|x_n-p\|^2) + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \to 0, n \to \infty. \tag{25}$$

Thus,

$$\|y_n-Ty_n\| \to 0, n \to \infty. \tag{26}$$

Now from the definition of z_n and condition (i), we get

$$\begin{aligned}\|z_n-x_n\|^2 &= \|x_n+\tau_n(x_n-x_{n-1})-x_n\|^2\\ &= \|\tau_n(x_n-x_{n-1})\|^2\\ &\le \tau_n\|x_n-x_{n-1}\|^2\\ &\le \alpha_n\frac{\theta_n}{\alpha_n} \to 0, n \to \infty. \end{aligned}$$

Thus,

$$\|z_n-x_n\| \to 0, n \to \infty. \tag{27}$$

Moreover, in view of condition (i) and (27), we get

$$\|y_n - x_n\|^2 \leq \alpha_n \|x_0 - x_n\|^2 + (1-\alpha_n)\|z_n - x_n\| - \alpha_n(1-\alpha_n)\|x_0 - z_n\|^2 \to 0, n \to \infty.$$

Thus,

$$\|y_n - x_n\| \to 0, n \to \infty. \tag{28}$$

Since $\{z_n\}$ is bounded, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in C$ and

$$\begin{aligned} \limsup_{n\to\infty} \langle x_0 - p, z_n - p\rangle &= \limsup_{i\to\infty} \langle x_0 - p, z_{n_i} - p\rangle \\ &= \langle x_0 - p, z - p\rangle. \end{aligned} \tag{29}$$

Now, in view of (27) and (28), it follows that $y_{n_i} \rightharpoonup z \in C$. From (26) and Lemma 2.3 , we get $z \in Fix(T)$. By invoking (23), (29) and Lemma 2.5 we find that $\lim_{n\to\infty} \|x_n - p\| = 0$. Consequently, $x_n \to p = P_{Fix(T)}(x_0)$.

Case 2. Suppose we find a subsequence n_i of $n \neq n_0$ such that $\{\|x_n - p\|\}_{n=n_i}^{\infty}$ is decreasing, that is

$$\|x_{n_i} - p\| \leq \|x_{n_i+1} - p\|, \forall i \in N.$$

Now invoking Lemma 2.6, and following the same line of argument as in Case 1, we find that $x_n \to p = P_{Fix(T)}(x_0)$.

Corollary 3.1 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$, represent a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1-\beta_n) T y_n), n \geq 1, \end{cases} \tag{30}$$

where $\{\beta_n\}_{n=1}^{\infty} \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\theta_n\}_{n=1}^{\infty}$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty} [(1-\beta_n)\beta_n] > 0,$

(iii) $\tau \in [0,1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise} . \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(T)$.

4 Application

4.1 Monotone variational inclusion problem (MVIP)

By using our main result, we can approximate a solution of monotone variational inclusion problem (shortly, MVIP) for the sum of two mappings in real Hilbert spaces. Let $B : C \to 2^H$ be a set-valued mapping and let mapping $A : C \to H$ be a single-valued mapping. The MVIP is to find $x \in C$ such that

$$0 \in B(x) + A(x) \tag{31}$$

The solution set of (31) is denoted $MVIP(B, A)$.

Definition 4.1 *A mapping $A : dom(A) \subset C \to 2^H$ is*

(a) Monotone if for all $x, y \in dom(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{32}$$

(b) Maximal monotone if it is monotone and is not subset of another maximal monotone operator.

Now, given any set-valued mapping $A : dom(A) \subset C \to 2^H$, the resolvent and anti-resolvent of A is the mapping $A : C \to 2^H$ defined by $J_\lambda^A = (I + \lambda A)^{-1}$ and $H_\lambda^A = (I - \lambda A)$ respectively and I is identity.

Definition 4.2 *A mapping $A : C \to H$ is inverse strongly monotone (ism) or ($\alpha-$ism) if there exists $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C. \tag{33}$$

From this inequality (33) and that of (6), we can easily obtain that a mapping A is α-ism if and only if the mapping $T = I - A$ is $(1 - \alpha)$-strictly pseudo-contractive. Hence, the fixed point for this class of mapping is connected with the zero of α-ism mapping. That is $Fix(T) = \{x \in C : Tx = x\}$ is same as $A^{-1}0 = \{x \in C : Ax = 0\}$. Now let $B : dom(A) \subset C \to 2^H$ be a maximal monotone and $A : dom(A) \subset C \to 2^H$ be an α-inverse strongly monotone, then $J_\lambda^B = (I+\lambda B)^{-1}$ is known to be single-valued and nonexpansive and consequently $J_\lambda^B(I - \lambda A)$ is nonexpansive and hence $(1-\alpha)$-strictly pseudo-contractive such that $x \in (B+A)^{-1}0 \iff Fix(J_\lambda^B(I-\lambda A))$. Following from these facts, we have

Theorem 4.1 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$, represent an α-inverse strongly monotone mapping and let $B : dom(A) \in C \to 2^H$, represent a maximal monotone set-valued mapping with $Fix(J_\lambda^B(I - \lambda A)) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1 - \alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1 - \beta_n)J_\lambda^B(I - \lambda A)y_n), n \geq 1, \end{cases} \tag{34}$$

where $\{\beta_n\}_{n=1}^\infty \in (k, 1)$ for any $k \in [0, 1)$, $\{\alpha_n\}_{n=1}^\infty \in (0, 1)$, $\{\theta_n\}_{n=1}^\infty$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty, \sum_{n=1}^\infty \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1 - \beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0, 1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, \ \textit{whenever } x_n \neq x_{n-1} \\ \tau, \ \textit{otherwise} . \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(J_\lambda^B(I - \lambda A))$.

4.2 Variational inequality problem (VIP)

Similarly, using our main result, we can approximate a solution of variational inequality problem (shortly, VIP) for a single-valued mapping in real Hilbert spaces. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let C be a non-empty, closed and convex subset of $dom(A)$. Now the variational inequality problem corresponding to A is to find $x \in C$ such that

$$\langle Ax, z - x\rangle \geq 0, \forall z \in C. \tag{35}$$

The solution set of (35) is denoted by $VI(C, A)$. Let $H^A_\lambda = (I - \lambda A)$. If C is a non-empty, closed and convex subset of $dom(A)$, then following (2), $VI(C, A) = Fix(Proj_C \circ H^A_\lambda)$. Since, $Proj_C$ is nonexpansive, it is also a strictly pseudo-contractive. Thus, $Proj_C \circ H^A_\lambda$ is strictly pseudo-contractive. Following from these facts, we have

Theorem 4.2 *Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$, represent an α-inverse strongly monotone mapping with $Fix(Proj_C \circ (I - \lambda A)) \neq \emptyset$. Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1 - \alpha_n)z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1 - \beta_n)Proj_C \circ (I - \lambda A)y_n), n \geq 1, \end{cases} \tag{36}$$

where $\{\beta_n\}^\infty_{n=1} \in (k, 1)$ for any $k \in [0, 1)$, $\{\alpha_n\}^\infty_{n=1} \in (0, 1)$, $\{\theta_n\}^\infty_{n=1}$ is a positive sequence satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum^\infty_{n=1} \alpha_n = \infty, \sum^\infty_{n=1} \theta_n < \infty, \lim_{n\to\infty}, \dfrac{\theta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty}[(1 - \beta_n)[\beta_n - k]] > 0,$

(iii) $\tau \in [0, 1)$ *such that* $0 \leq \tau_n \leq \bar{\tau_n}$ *with*

$$\bar{\tau_n} := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \textit{whenever } x_n \neq x_{n-1} \\ \tau, & \textit{otherwise }. \end{cases}$$

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(Proj_C \circ (I - \lambda A))$.

5 Numerical illustration

In this section we would demonstrate the effectiveness, implementation and strong convergence results of our procedure (14) of Theorem 3.1 as discussed in section 3. All codes are written in Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz.

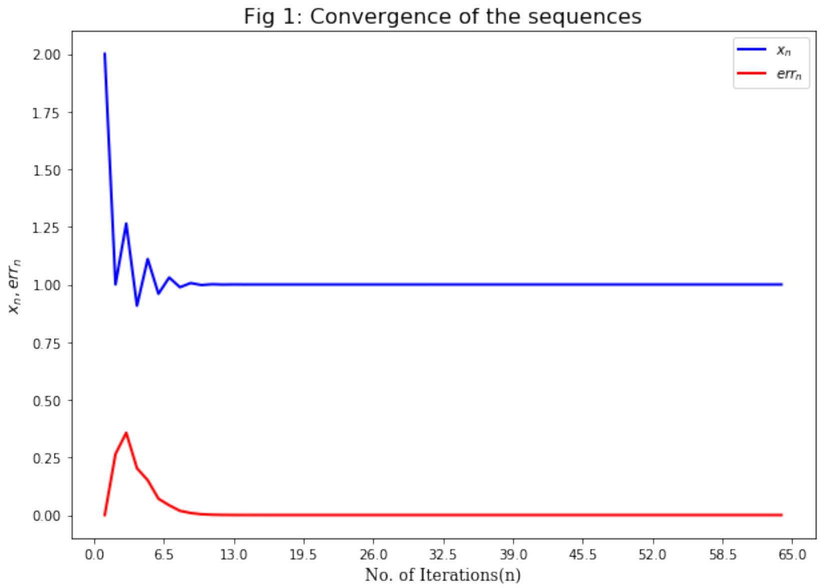
Example 5.1 *c.f [2]. Let $H = R$ denote the reals with the usual norm $|.|$ and the inner product $\langle x, y\rangle = xy$. Let $C = [\frac{1}{2}, 2]$ and define $T : C \to C$ by $T(x) = \frac{1}{x}$. Then T is strictly pseudo-contractive with $0 \leq k < 1$. Let $x_0 = 2, x_1 = 1, \alpha_n = \frac{1}{400(n+1)}, \beta_n = \frac{n+1}{2n+7}, \tau = \frac{2}{3}$, and $\theta_n = \alpha^2_n$ such that our hypothesis are satisfied. Also, the procedure of (14) is as follows:*

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \frac{1}{400(n+1)}x_0 + (1 - \frac{1}{400(n+1)})z_n, \\ x_{n+1} = \begin{cases} \frac{1}{2} \textit{ if } x < \frac{1}{2} \\ 2 \textit{ if } x > 2 \\ (\frac{n+1}{2n+7})y_n + (1 - \frac{n+1}{2n+7})\frac{1}{y_n}, & \textit{otherwise }. \end{cases} \end{cases} \tag{37}$$

n	10	15	20	30	40	50	55	61
x_n	1.0009862	0.9999745	0.9999876	0.9999941	0.9999965	0.9999977	0.9999999	1.00000000
$\|x_{n+1} - x_n\|$	0.00014017	0.0000086	0.0000010	0.0000003	0.0000002	0.0000001	0.0000000	0.0000000

Table 1: The numerical values of the sequences $\{x_n\}$ and $\|x_{n+1} - x_n\| = err_n$.

Then $\{x_n\}$ converges strongly to metric projection of C onto $Fix(T) = 1$. Moreover, the results are displayed below.



Remark 5.1 *We observe that the choice of the control sequences of $\{\alpha_n\}, \{\beta_n\}$ and $\{\theta_n\}$,satisfying the convergence hypothesis, influenced to a great deal, the strong convergence of the procedure (14) to the fixed point of the mapping which is $\{1\}$. If the stopping criteria is $\|x_{n+1} - x_n\| \leq 10^{-7}$, then the computation can be truncated after 50 iterations. If the stooping criteria is higher at $\|x_{n+1} - x_n\| \leq 10^{-5}$, then truncating the computation early may have a negligible computational error. Consequently, $\|x_n - p\| = \|x_n - 1\| \to 0$ as $n \to \infty$.*

6 Conclusion

We introduced the modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex subset of real Hilbert spaces. We proved strong convergence theorem for it. Furthermore, we applied our procedure to the approximation of monotone variational inclusion and variational inequality problems in real Hilbert spaces. We supported our main result with a numerical example to illustrate the effectiveness of our procedure. We observed that our procedure performed favourably with a "good" stopping criteria. Our procedure and the given example is found to complement the very recent result of Agbebaku et al [1] in the sense that our introduced procedure is different from theirs and which approximates the fixed point of strict pseudo-contractive self mapping on a closed and convex subset of a real Hilbert spaces. Therefore, our procedure or algorithm is recommended for approximation fixed point problem of this class of mapping in this direction.

References

[1] Agbebaku, D.F., Nwokoro, P.U., Osilike, M.O., Chima, E.E., and Onah, A.C. (2021).The iterative algorithm with inertial and error terms for fixed points of strictly pseudo-contractive mappings and zeros of inverse strongly monotone operators. Appl. Set-Valued Anal. Optim. 3: 95-107.

[2] Berinde, V. and Berinde, M. (2005). The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings. Carpathian J. Math. 21(1-2): 13-20.

[3] Browder, F.E. and Petryshyn, W.V. (1967). Construction of fixed points of nonlinear mappings in Hilbert spaces. J. Math. Anal. Appl. 20: 197-228.

[4] Chidume, C.E. and Shahzad, N. (2010). Weak convergence theorems for a finite family of strict pseudocontractions. Nonlinear Anal. 72: 1257-1265.

[5] Chidume, C.E., Ikechukwu, S.I., and Adamu, A. (2018). Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps. Fixed Point Theory and Applications, 9.

[6] Ekuma-Okereke, E. and Okoro, F.M. (2020). Common Solution for Nonlinear Operators in Banach Spaces, GUJ Sci 33(3):737-749.

[7] Ekuma-Okereke, E., Okoro, F., and Abhulimen, C (2021). Two modified hybrid inertial algorithms for Bregman weak relatively nonexpansive mapping in Banach spaces. Nigerian Journal of Mathematics and Applications, 31:149-174.

[8] Halpern, B. (1968). Fixed points of non-expanding maps. Bull. Amer. Math. Soc., 73:957-961.

[9] Ishikawa, S. (1974). Fixed point by a new iteration method. Proc. Amer. Math. Soc. 44:P147-150.

[10] Jeong, J.U. (2015). Convergence theorems of common elements for pseudo-contractive mappings and monotone mappings. Abstract and Applied Analysis, http://dx.doi.org/10.1155/2015/383579.

[11] Mainge, P.E. (2008). Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. 16:899-912.

[12] Mainge, P.E. and Maruster, S. (2011). Convergence in norm of modified Krasnoselkii-Mann iterations for fixed points of demicontractive mappings. Appl. Maths. Comput. 217: 9864-9874.

[13] Mann, W.R. (1953). Mean value methods in iteration. Proc. Amer. Math. Soc. 4:506-510.

[14] Nwokoro, P.U., Osilike, M.O., Agbebaku, D.F., Chima, E.E., and Onah, A.C. (2020). A new Halpern-type averaging algorithm with inertial and error terms for fixed points of asymptotically nonexpansive maps, J. Math. Comput. Sci. 10:1538-1558.

[15] Shehu, Y. (2016). Iterative procedure for left Bregman strongly relatively nonexpansive mappings with application to equilibrium problems. Fixed Point Theory, 17(1):173-188.

[16] Shehu, Y., Iyiola, O.S., and Ogbuisi, F.U. (2020). Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. Numer. Algor. 83:1321-1347.

[17] Xu, H.K. (2002). Another control condition in an iterative method for nonexpansive mappings. Bulletin of the Australian Mathematical Society, 65(1):109–113.

[18] Zhang, Q.B. and Cheng, C.Z. (2008). Strong convergence theorem for a family of Lipschitz pseudocontractive mappings in a Hilbert space. Mathematical and Computer Modelling, 48(3-4):480–485.

[19] Zegeye, H., Shahzad, N., and Alghamdi, M.A. (2011). Convergence of Ishikawa's iteration method for pseudocontractive mappings, Nonlinear Analysis, 74:7304-7311.