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On the transitivity and primitivity of permutation groups of degree 4p constructed via wreath products using numerical approach

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Abstract

Let p be an odd prime number and G a finite permutation group of degree 4p generated via wreath products of pairs of permutation groups. In this work, a group concept, namely isomorphism, is applied to discuss the transitivity and primitivity of G using numerical approach. The computational group theory (GAP) is applied to enhance and validate our work.

Keywords: permutation group; primitivity; wreath products; p-groups; isomorphism

1 Introduction

Let C and D be two groups of permutations on non empty sets Ω_1 and Ω_2, respectively, where $\Omega_1 \cap \Omega_2 = \emptyset$. Let $P = C^{\Omega 2} = \{f : \Omega_2 \to C\}$. The Wreath product of two permutation groups C and D denoted by $W = CwrD$ is the semi-direct product of P by D, so that, $W = \{(f,d) | f \in P, d \in D\}$, with multiplication in W defined as $(f_1,d_1)(f_2,d_2) = ((f_1,f_2 d_1^{-1})\ (d_1,d_2))$ $\forall$ $f_1, f_2 \in P$ and $d_1, d_2 \in D$ is a special form of permutation group as in [11].

When a group G acts on a set Ω, a typical point α is moved by elements of G to various other points. The set of these images is called the orbit of α under G and we denote it by $\alpha^G \coloneqq \{\alpha^g |\ g \in G\}$. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit which is equal to the entire set, that is $\alpha^G = \Omega$ for all $a \in \Omega$. Equivalently, G is transitive if for every pair of point $\alpha,\ \beta \in \Omega$ there exists $g \in \Omega$ such that $\alpha^g = \beta$. If a group is not transitive then it is called intransitive.

A permutation group G acting on a non empty set Ω is called primitive if G acts transitively on Ω and G preserves no non trivial partition of Ω where non-trivial partition means a partition that is not a partition into singleton set or partition into one set Ω. In other words, a group G is said to be primitive on a set Ω if the only sets of imprimitivity are the trivial ones otherwise G is imprimitive on Ω as in [1]. Transitive and Primitive finite permutation groups can be thought of

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as the building blocks of finite permutation groups, and questions about finite permutation groups can often be reduced to the primitive case according to [4].

Transitive and primitive permutation groups of special degrees have received much attention in the academic research space. [10] in a paper title "Wreath Product of Permutation Groups" provided an upgrade to the results of an earlier paper on some finite permutation groups which consist of the basic procedure of computing Wreath product of groups. They also discussed the condition under which the Wreath products of permutation groups are faithful, transitive and primitive. Also some properties of the centre of a group stabilizer and Wreath product of groups were investigated respectively in the work. Transitive and primitive p-subgroup of dihedral groups of degree pq, where p, q are any two distinct odd prime numbers were considered by [7], while more recently, Audu and Hamma in [1] discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most p^2 using the concept of p-groups. They used the standard program – The Groups, Algorithms and Programming (GAP) to validate their results while Hamma and Aliyu, in [8] worked "On transitive and primitive dihedral groups of degree $at\ most\ 2^r\ (r \geq 2)$". Also, in [9] Hamma and Mohammed discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most p^3. They proved theorems and validate them using the Groups, Algorithms and Programming (GAP). Cai and Zhang, in [2] presented "A Note on Primitive Permutation Groups of Prime Power Degree". Fengler, in his published work explored on "Transitive Permutation Groups of Prime Degree" as in [5].

In this paper, we obtained detailed description of the unique structure of Wreath product groups of degree 4p that are not p-groups and discussed their transitive and primitive nature using numerical approach. When the transitive and primitive nature of the Wreath products groups is well understood it facilitates comprehension of the classification of finite simple groups (CFSG) towards a productive review of same.

2 Materials and methods

2.1 p-group

A finite group G is said to be a p-group if its order is a power of p, where p is prime. A subgroup H of a group G is a p-subgroup if it (H) is a p-group.

2.2 Stabilizer

Any subset of G which fixes a specified element α is called the stabilizer of α in G and is denoted by $G_\alpha \coloneqq \{x \in G \mid \alpha^x = \alpha\}$.

2.3 Orbit

When a group G acts on a set Ω, a typical point α is moved by elements of G to various other points. The set of these images is called the orbit of α under G, and we denote it by $\alpha^G \coloneqq \{ \alpha^x \mid x \in G \}$.

2.4 Dihedral group

Dihedral groups are groups of symmetries of a regular polygon, which consist of rotations and reflections. The group is denoted D_n which symbolizes the symmetry of a regular polygon with n sides (that is, the dihedral group of degree n). Conventionally, we write $D_n = \langle r, f \mid r^n = f^2 = 1,\ fr = r^{n-1}f\ = r^{-1}f \rangle$ and we say that D_n is the group generated by the elements r and f subject to the conditions

$r^n = f^2 = 1$; $fr = r^{n-1}f = r^{-1}f$, and the 2n distinct elements of D_n are $1, r, r^2, \ldots, r^{n-1}, f, rf, r^2f, \ldots, r^{n-1}f$.

Here r is a rotation about the centre of the polygon through angle $2\pi^c/n$ and f is a reflection about an axis of symmetry of the polygon.

2.5 Isomorphism

Two abstract groups $(G, \cdot)$ and $(H, *)$ are said to be isomorphic if there is a bijection $\varphi : G \to H$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all x,y G. φ is called an Isomorphism.

Theorem 1

Any finite group G is isomorphic to a subgroup of the symmetric group S_n of degree n, where $n = |G|$.

Proof:

Let G act on itself by right multiplication $g^h = gh$ for all $g, h \in G$. If $g^h = g$ then $gh = g$ and so $h = 1$. That is, the kernel of the action is $\{1\}$. The mapping $f\colon G \to sym(G)$ define by $f\colon g \to f_g$ where $\alpha f_g = \alpha^g$ for any $\alpha \in G$ is a homomorphism. Then $G/_{\ker f} \cong im\, f$. But $ker\ f = \{1\}$ and $im\, f \leq sym(G) = S_n$. Accordingly $G \leq S_n$. In general we have that if G acts on Ω with k kernel of the action then $G/_k \leq sym(\Omega)$.

2.7 Transitivity

Let G be a permutation group on Ω, where Ω is a finite set.

We say that G is $\frac{1}{2}$ – transitive if all the orbits have the same size.

Suppose that G has just one orbit Ω. then for all $r \in \Omega$, $r^G = \Omega$ and as such for any $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$, and G is said to be transitive on Ω.

The group G is said to be k-fold transitive (or, simply k-transitive) on Ω if, for any sequences $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_i \neq \alpha_j$ when $i \neq j$; $\beta_1, \beta_2, \ldots, \beta_k$ such that $\beta_1 \neq \beta_j$ when $i \neq j$ of k elements of Ω, there exists $g \in$ G such that

$$\alpha_i^g = \beta_i \text{ for } 1 \leq i \leq k.$$

Thus for $k = 2$ we have that for $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Ω with $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$ there exists $g \in G$ such that;

$$\alpha_1^g = \beta_1,\ \alpha_2^g = \beta_2$$

and we say that G is doubly transitive.

If $k \geq 2$ then $k-$ transitivity implies $(k-1)-$transitively.

Let G act on itself by right multiplication. Then, $\Omega = G$. If $\alpha = x$, $\beta = y$ in Ω and we take $g = x^{-1}y$; then $\alpha^g = x(x^{-1}y) = y = \beta$ and so G is transitive.

Let $H \leq G$ and let G act on right cosets of H in G.
Then G is transitive on $\Omega := (G : H)$. For if $\alpha, \beta \epsilon \Omega$,
then $\alpha = H_x$, $\beta = H_y$ for some $x, y \in G$, and if we take $g := x^{-1}y$ then we have
$\alpha^g = (Hx)x^{-1}y = Hy = \beta$.

Lemma 1
Let G be a dihedral group of any order, then G is transitive.

Proof:

For given α_i, α_j as any two vertices of the regular polygon with i < j, we readily see that $(\alpha_1\alpha_2 \dots \alpha_i \dots \alpha_j \dots \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi/n$, (where n is the number of edges of the polygon) which takes α to αj . As such G is transitive.

2.9 Primitivity

A permutation group G acting on a non empty set Ω is said to be primitive on a set Ω if and only if it preserves the trivial block system otherwise G is imprimitive on Ω. For example, the group S_3 = {(1), (12), (13), (23), (123), (132)} is primitive as $\{1,2\}^{(1\ 2\ 3)} = \{2,3\}$ implying that $\Delta^g \neq \Delta$ and $\Delta^g \cap \Delta \neq \emptyset$ for $\Delta = \{1,2\}$.

On the other hand, a subset Δ of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g \in G$, either $\Delta^g = \Delta$ or Δ^g and Δ are disjoint. In particular, Ω itself, the 1-element subsets of Ω and the empty set are obviously sets of imprimitivity which are called trivial set of imprimitivity.

The group of symmetry D_4= {(1), (1234), (13)(24), (1432),(13), (24), (12)(34), (14)(23)} of the square with vertices 1,2,3,4 is imprimitive. For take $G_1 = \{(1), (24)\}$.

Let $H = \{(1), (13), (24), (13)(24)\}$ which is a normal subgroup of G. Then H is a group greater than G_1, but not equal to G.

Theorem 2 [12]
Let G be a transitive permutation group of prime degree on the set Ω. Then G is primitive
Proof
First and foremost, G been transitive, permutes the sets of imprimitivity where all the sets Ω_i have the same order. But $\Omega = \cup |\Omega_i|$, Ω_i being the sets of imprimitivity. As $|\Omega|$ is prime we have that either each $|\Omega_i| = 1$ or Ω or the only sets of imprimitivity. So, G is primitive.

Theorem 3 [12]
Let G be a non-trivial transitive permutation group on Ω. Then G is primitive if and only if $G_\alpha, \alpha \in \Omega$ is a maximal subgroup of G or equivalently G is imprimitive if and only if there is a subgroup H of G properly lying between $G_\alpha(\alpha \in \Omega)$ and G.
Proof
Let the group G be imprimitive and Ψ be a non-trivial subset of the imprimitivity of G.
Let $H = \langle g \in G | \Psi^g = \Psi \rangle$

It is obvious H is a proper subgroup of G because $\Psi \subset \Omega$ and G is transitive.
Now choose $\alpha \in \Psi$. If $g \in G$ then $\alpha \in \Psi \cap \Psi^g$ and so $\Psi = \Psi^g$.

Hence $G_\alpha \ \leq \ H \ \leq \ G$

Since $|\Psi| \neq 1$, choose $\beta \in \Psi$ such that $\beta \neq \alpha$. By transitivity of G, there exists some $h \in G$ with $\alpha^h = \beta$ so that $h \in G_\alpha$. Now $\beta \in \Psi \cap \Psi^h$, so $\Psi = \Psi^h$ and $h \in H - \ G_\alpha$. Thus $H \neq G_\alpha$
Conversely, suppose that $G_\alpha < H < G$ for some subgroup H.
Let $\Psi = \alpha^H$. Since $H > G_\alpha |\Psi| \neq 1$

Now if $\Psi = \Omega$, then H is transitive on Ω and hence $|\Omega| = |H : G_\alpha|$ showing that H = G, a contradiction.
Hence, $\Psi \neq \Omega$

Now we shall show that Ψ is a subset of imprimitivity of G.
Let $g \in G$ and $\beta \in \Psi \cap \Psi^g$ then $\beta = \alpha^h = \alpha^{h\,g}$ for some $h, h \in H$.

Hence $\alpha^{h\,gh^{-1}} = \alpha.$ so $h\,gh^{-1} \in G_\alpha < H$
This shows that $g \in H$

Thus $\Psi = \Psi^g$. Hence Ψ is a non-trivial subset of imprimitivity, so G is imprimitive.

Theorem 4 [11]
Let D act on P as $f^d(\delta) = f\ (\delta d^{-1})$ where $f \ \in P, d \ \in D \ and \ \delta \ \in \ \Delta$.
Let W be group of all juxtaposed symbols f d, with $f \ \in P, d \ \in D$ and multiplication given by $(f_1, d_1)(f_2, d_2) = (f_1\, f_2\, d_1^{-1})\ (d_1, d_2)$. Then W is a group called the semi-direct product of P by D with the defined action

Remark 1 [11]
If C and D are finite groups, then the Wreath product W determined by an action of D on a finite group is of order $|W| = |C|^{|\Delta|}|D|$
P is a normal subgroup of W and D is a subgroup of W
The action of W on $\Gamma \times \Delta$ is given by
$(\alpha, \beta)fd = (\alpha f(\beta), \beta d)$ Where $\alpha \in \Gamma \ and \ \beta \in \Delta$

3 Results

Proposition 1

Let G be the Wreath products of permutation groups of degree 4p. Then G is transitive and imprimitive.
Proof
From the remark, the order of G, that is $|G| = 4p^4$ and $\Omega = \{1, 2, 3, \ldots\ldots, 4p^4\}$.
That G is transitive follows easily from Lemma 1. Clearly, $|G| = 4p^4 = 2{\cdot}2p^4$ implies that the group G is isomorphic the dihedral group degree $2p^4$ (i.e. $G \cong D_{2p4}$).

Now, name the vertices of D_{2p4} as 1,2,3,…, $2p^4$ and let l be the line of symmetry joining the middle of the vertices 1 and $2p^4$ with the middle of the vertices $\frac{2p4}{2}$ and $\frac{2p4+4}{2}$ $so\ that$

$$\alpha = (1, 2p4)(2, 2p4 - 1)(3, 2p4 - 2) \dots \left(\frac{2p4}{2}, \frac{2p4\ +4}{2}\right)$$

is the reflection in l.

Then $G_{(1)} = D2p4_{(1)} = \left\{(1), (2, 2p4 - 1)(3, 4p2p4 - 2) \dots \left(\frac{2p4}{2}, \frac{2p4+4}{2}\right)\right\}$ is the stabilizer of the point 1. We readily see that $G_{(1)}$ is a non-identity proper subgroup of G which has

$$H = \left\{(1), (1, 4p), (2, 4p - 1), (3, 4p - 2), \dots, \left(\frac{2p4}{2}, \frac{2p4\ +4}{2}\right), \alpha\right\}$$

as a subgroup properly lying between $D2p4_{(1)}$ and D_{2p4}. That is, $D2p4_{(1)} \leq H \leq D_{2p4}$. By Theorem 4, D_{2p4} is imprimitive. Thus, G is transitive and imprimitive.

3.1 Illustrating Example1 (Primitivity of Wreath Product Group of Degree 4p where p = 3)

Let C be a group of degree 4 and D a group of degree 3 acting on the set $\Omega = \{1,2,3,4\}$ and $\Delta = \{5,6,7\}$ respectively. Let $P = C^{\Delta} = \{f:\Delta \to C\}$ with $|P| = |C|^{|\Delta|} = 4^3$. We show that the Wreath product $W = C_{wr}D$ of degree = 12 of order, $|W| = |C|^{|\Delta|} \text{ x } |D| = 4^3 3$ is transitive and imprimitive.

Proof:

We follow the procedure as described in Theorem 4 and Remark 1 to obtain the elements of the Wreath products group, W in cyclic form as:

W = {(1), (7,8,9,10,11,12), (7,9,11)(8,10,12), (7,10)(8,11)(9,12), (7,11,9)(8,12,10), (7,12,11,10,9,8), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8,9,10,11,12), (1,2,3,4,5,6)(7,9,11)(8,10,12), (1,2,3,4,5,6)(7,10) (8,11) (9,12), (1,2,3,4, 5,6)(7,11,9)(8,12,10), (1,2,3,4,5,6)(7,12,11,10,9,8), (1,3,5) (2,4,6), (1,3,5)(2,4,6) (7,8,9,10,11,12), (1,3,5) (2,4,6) (7,9,11)(8,10,12), (1,3,5)(2,4,6)(7,10) (8,11)(9,12), (1,3,5)(2,4,6) (7,11,9)(8,12,10), (1,3,5)(2,4,6)(7,12,11,10, 9,8), (1,4)(2,5)(3,6), (1,4)(2,5) (3,6) (7,8,9,10,11,12), (1,4) (2,5)(3,6)(7,9,11)(8,10,12), (1,4)(2,5)(3,6)(7,10) (8,11)(9,12), (1,4)(2,5)(3,6)(7,11,9)(8,12,10), (1,4) (2,5) (3,6)(7,12,11,10,9,8), (1,5,3)(2,6,4), (1,5,3)(2,6,4)(7,8,9,10,11,12), (1,5,3)(2,6,4)(7,9,11) (8,10, 12), (1,5,3)(2,6,4)(7,10)(8,11)(9,12), (1,5,3)(2,6,4)(7,11,9)(8,12,10), (1,5,3)(2,6,4) (7,12,11,10, 9,8), (1,6,5,4,3,2), (1,6,5,4,3,2)(7,8,9,10,11,12), (1,6,5,4,3,2)(7,9,11)(8,10,12), (1,6,5,4,3,2)(7,10)(8,11) (9, 12), (1,6,5,4,3,2)(7,11,9)(8,12,10), (1,6,5,4,3,2)(7,12,11,10,9,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12), (1,7,2,8,3,9,4,10,5, 11,6,12), (1,7,3,9,5,11)(2,8,4,10,6,12), (1,7,4,10)(2,8,5,11)(3,9,6,12), (1,7,5,11,3,9) (2,8,6,12,4,10), (1,7,6,12,5,11,4,10,3,9,2,8), (1,8,2,9,3,10,4,11,5,12,6,7), (1,8,3,10,5,12)(2,9,4,11,6,7), (1,8,4, 11)(2,9,5,12)(3,10,6,7), (1,8,5,12,3,10)(2,9,6,7,4,11), (1,8,6,7,5,12,4,11,3,10,2,9), (1,8)(2,9) (3,10)(4,11) (5,12) (6,7), (1,9,3,11,5,7)(2,10,4,12,6,8), (1,9,4,12)(2,10,5,7)(3,11,6,8), (1,9,5,7,3,11) (2,10,6,8,4,12), (1,9,6,8,5,7,4, 12,3,11,2,10), (1,9)(2,10) (3,11)(4,12)(5,7)(6,8), (1,9,2,10,3,11, 4,12, 5, 7,6,8), (1,10,4,7)(2,11,5,8)(3,12,6,9), (1,10,5,8,3,12)(2,11,6,9,4,7), (1,10,6,9,5,8,4,7,3,12,2,11), (1,10) (2,11)(3,12)(4,7)(5,8)(6,9), (1,10,2,11,3,12, 4,7,5,8,6,9), (1,10,3,12,5,8) (2,11,4,7,6,9), (1,11,5,9,3,7) (2,12,6,10,4,8), (1,11,6,10,5,9,4,8,3,7,2,12), (1,11) (2,12)(3,7)(4,8)(5,9)(6,10), (1,11,2,12,3,7,4,8,5, 9,6,10), (1,11,3,7,5,9)(2,12,4,8,6,10), (1,11,4,8)(2,12,5,9)(3,7,6,10), (1,12,6,11,5,10,4,9,3,8,2,7), (1, 12)(2,7)(3,8)(4,9)(5,10)(6,11), (1,12,2,7,3,8,4,9,5,10,6,11), (1,12,3,8,5,10) (2,7,4,9,6,11), (1,12,4,9)(2, 7,5,10)(3,8,6,11), (1,12,5,10,3,8) (2,7, 6,11,4,9)}.

Now $|W|$ = 196 and $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ is the set of the points of W. It follows from Lemma 1 that W is transitive as the orbit $\alpha^W = \Omega\ \forall\ \alpha \in \Omega$.

Also the stabilizer of the point 1 in W is given by $W_{\{1\}} = \{(1)$, (9,10,11,12), (9,11)(10,12), (9,12,1,10), (5,6,7,8), (5,6,7,8)(9,10,11,12), (5,6,7,8)(9,11)(10,12), (5,6,7,8)(9,12,11,10), (5,7) (6,8), (5,7)(6,8) (9, 10,11,12), (5,7)(6,8)(9,11)(10,12), (5,7)(6,8)(9,12,11,10), (5,8,7,6), (5,8,7,6)(9,10,11,12), (5,8,7,6) (9,11)(10,12), (5,8,7,6) (9,12,11,10)$\}$ which is obviously a non-identity proper subgroup of W. We readily see from the subgroups of W that the group (W) has a subgroup H = Syl$_2$(W) properly lying between $W_{\{1\}}$ and W that is, $W_{\{1\}} < H < W$ hence, W is imprimitive by Theorem 2. Thus, W is transitive and imprimitive.

3.2 Illustrating Example 2. Primitivity of Wreath Product Group of Degree 4p where p = 5)

Let C be a group of degree 5 and D a group of degree 4 acting on the set Ω = {1,2,3,4,5} and Δ = {6,7,8,9} respectively. Let $P = C^{\Delta} = \{f:\Delta \rightarrow C\}$ with $|P| = |C|^{|\Delta|} = 5^4$. Then the Wreath product $W = C_{wr}D$ of degree = 20 and order, $|W| = |C|^{|\Delta|}$ x $|D| = 2^2 5^4$ is transitive and imprimitive.
Proof:

We follow the procedure as described in Theorem 2 and Remark 1 to obtain the elements of the Wreath products group, W in cyclic form as:

W = {(1), (11,12,13,14,15,16,17,18,19,20), (11,13,15,17,19)(12,14,16,18,20), (11,14,17,20, 13,16, 19, 12,15,18), (11,15,19,13,17)(12,16,20,14,18), (11,16)(12,17)(13,18)(14,19)(15,20), (11,17,13,19,15) (12,18,14,20,16), (11,18,15,12,19,16,13,20,17,14), (11,19,17,15,13)(12,20,18,16,14), (11,20,19, 18,17,16,15,14,13,12), (1,2,3,4,5, 6,7, 8,9,10), (1,2,3,4,5,6,7,8,9,10)(11, 12,13,14,15,16,17,18,19,20), (1,2,3,4,5,6,7,8,9,10) (11,13,15,17,19)(12,14,16,18,20), (1,2,3,4,5,6,7,8,9,10)(11,14,17,20,13,16,19, 12,15,18), (1,2,3,4,5,6,7,8,9,10)(11,15,19,13,17)(12,16,20,14,18), (1,2,3,4,5,6,7,8,9,10)(11,16)(12,17) (13,18)(14,19)(15,20), (1,2,3,4,5,6,7,8,9,10)(11,17,13,19,15)(12,18,14,20,16), (1,2,3,4,5,6,7,8,9,10) (11,18,15,12,19,16,13,20,17,14), (1,2,3,4,5,6,7,8,9,10)(11,19,17,15,13) (12, 20,18,16,14), …….}.

Now $|W|$ = 2500 and $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ is the set of the points of W. It follows from Lemma 2.8 that W is transitive as the orbit $\alpha^W = \Omega\ \forall\ \alpha \in \Omega$.
Also the stabilizer of the point 1 in W is $W_{\{1\}}$ = {(1), (11,12,13,14,15,16,17,18,19,20), (11,13,15,17,19)(12,14,16,18,20), (11,14,17,20,13,16,19,12,15,18), (11,15,19,13,17)(12,16,20,14,18), (11,16)(12,17)(13,18)(14,19)(15,20), (11,17,13,19,15)(12,18,14,20,16), (11,18,15,12,19,16,13,20, 17, 14), (11,19,17, 15,13)(12,20,18,16,14), (11,20,19,18,17,16,15,14,13,12)$\}$,
which is obviously a non-identity proper subgroup of W. We readily see from the subgroups W that the group (W) has a subgroup $H = \{(1)$, (1,19,2,20,3,11,4,12,5,13,6,14,7,15,8,16,9,17,10,18), (1,2,3,4,5,6,7,8,9,10)(11, 12,13,14,15,16,17,18,19,20), (1,3,5,7,9)(2,4,6,8,10)(11,13,15,17,19) (12,14, 16,18, 20), (1,2,3,4,5,6) (7,8,9,10,11,12)}

properly lying between $W_{\{1\}}$ and W that is, $W_{\{1\}} < H < W$ clearly, W is imprimitive by theorem 2.11. Thus, W is transitive and imprimitive.

3.3 GAP Result for Wreath Product Group of Degree 4p, p = {3,5,7,11, …}

We validate our result in this Section by applying a standard programme - the groups, algorithms and programming version 4.11.1 as shown below.

```
GAP        GAP 4.11.1 of 2021-03-02
           https://www.gap-system.org
           Architecture: x86_64-pc-cygwin-default64-kv7

Configuration:  gmp 6.2.0, GASMAN, readline
Loading the library and packages ...
gap>
gap># Wreath Product (of permutation groups C and D) of Degree 4p for p = 3
gap> C := Group((1,2,3,4));
Group([ (1,2,3,4) ])
gap> D := Group((5,6,7));
Group([ (5,6,7) ])
gap> W := WreathProduct(C,D);
Group([ (1,2,3,4), (5,6,7,8), (9,10,11,12), (1,5,9)(2,6,10)(3,7,11)(4,8,12) ])
gap> Order(W);
192
gap> Elements(W);;
gap> IsTransitive(W);
true
gap> IsPrimitive(W);
false
gap>
gap>
gap># Wreath Product (of permutation groups C and D) of Degree 4p for p = 5
gap> C := Group((1,2,3,4,5));


Group([ (1,2,3,4,5) ])
gap> D := Group((6,7,8,9));
Group([ (6,7,8,9) ])
gap> W := WreathProduct(C,D);
<permutation group of size 2500 with 5 generators>
gap> Order(W);
2500
gap> Elements(W);;
gap> IsTransitive(W);
true
gap> IsPrimitive(W);
false
gap>
gap>
gap># Wreath Product (of permutation groups C and D) of Degree 4p for p = 7
gap> C := Group((1,2,3,4,5,6,7));
```

```
Group([ (1,2,3,4,5,6,7) ])
gap> D := Group((8,9,10,11));
Group([ (8,9,10,11) ])
gap> W := WreathProduct(C,D);
<permutation group of size 9604 with 5 generators>
gap> Order(W);
9604
gap> Elements(W);;
gap> IsTransitive(W);
true
gap> IsPrimitive(W);
false
gap>
gap>
gap># Wreath Product (of permutation groups C and D) of Degree 4p for p = 11
gap> C := Group((1,2,3,4,5,6,7,8,9,10,11));
Group([ (1,2,3,4,5,6,7,8,9,10,11) ])
gap> D := Group((12,13,14,15));
Group([ (12,13,14,15) ])
gap> W := WreathProduct(C,D);
<permutation group of size 58564 with 5 generators>
gap> Order(W);
58564
gap> Elements(W);;
gap> IsTransitive(W);
true
gap> IsPrimitive(W);
false
gap>
gap>
gap># Wreath Product (of permutation groups C and D) of Degree 4p for p = 13
gap> C := Group((1,2,3,4,5,6,7,8,9,10,11,12,13));
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13) ])
gap> D := Group((14,15,16,17));
Group([ (14,15,16,17) ])

gap> W := WreathProduct(C,D);
<permutation group of size 114244 with 5 generators>
gap> Order(W);
114244
gap> Elements(W);;
gap> IsTransitive(W);
true
gap> IsPrimitive(W);
false
gap>
```

4 Conclusion and Recommendation

4.1 Conclusion

This study showed that wreath products group of degree 4p where p is an odd prime number is imprimitive.

4.2 Recommendation

This study can be extended by considering for further research, one or a combination of two or more of other theoretic properties such as simplicity, nilpotency, regularity, etc of same algebraic structures.

References

[1] Audu M.S. and Hamma S. (2010). On transitive and primitive dihedral groups of degree at most p^2. Advances in Applied Science Research. Vol. 1(2):65-75.

[2] Cai Q. and Zhang H. (2015). A note on primitive permutation groups of prime power degree. Journal of Discrete Marthematics, 194741:4.

[3] Cayley A. (1854). On the theory of groups as depending on the symbolic equation $\theta^n = 1$. Philosophical Magazine, Vol. (42): 40–47.

[4] Fawcett J.M. (2009). The O'Nan-Scott theorem for fnite primitive permutation groups, and finite representability. Master's Thesis, University of Waterloo.

[5] Fengler S. (2018). Transitive permutation groups of prime degree. Masters (Published) Thesis, RWTH Aachen University, North Rhine-Westphalia, Germany.

[6] GAP 4.11.1 (2021). The GAP Group, GAP -- Groups, Algorithms, and Programming, Version 4.11.1; 2021. (https://www.gap-system.org).

[7] Hamma S. and Haruna Y. (2009). Transitive and primitive dihedral groups of degree pq. Icastor Journal of Mathematical Sciences, Vol. 3(1):101-111.

[8] Hamma S. and Aliyu S.O. (2010). On transitive and primitive dihedral groups of degree at most 2^r $(r \geq 2)$. Scholars research library, archive of applied science research, Vol. 2(5): 152-160.

[9] Hamma S. and Mohammed M.A. (2012). On transitive and primitive dihedral groups of degree p^3. The Journal Of The Mathematical Association Of Nigeria, Vol. 39 (2):65-74.

[10] Ibrahim A.A. and Audu M.S. (2007). On wreath product of permutation groups. Universidad Catolica del Note Antofagasta-Chile. Vol. 26 No 1.

[11] Joseph P.X. and Audu M.S. (1991). Wreath product of permutation groups. Nigeria Journal of Mathematics and Application, Vol. 4(5).

[12] Passman D. (1968). Permutation Groups. Yale University, U.S.A.