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A note on exponentiated generalized inverse Pareto distribution

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Abstract

The Exponentiated Generalized inverse Pareto (EGIP) distribution is derived using Exponentiated-G generator which introduces additional two new parameters into the inverse Pareto model. Some basic statistical properties of the model are derived which includes: moment and incomplete moment, moment generating function, quantile function, Renyi and q −entropy, mean time to failure, mean residual life, probability weighted moment and order statistics. The model parameter estimated using maximum likelihood estimation method is also presented.

Keywords: stress-strength reliability; quantile function; Renyi and ρ −entropy; mean residual life

1 Introduction

In many real life situations, the standard probability distributions do not provide adequate fit to some lifetime data sets, most especially with those with non-monotone failure. Thus, researchers developed many generators by introducing one or more shape parameters to generate new distributions. The new generated distributions are more flexible and tractable when compare to the classical/ baseline distributions. Some well-known generators can be found in [2]-[13], [15], [17]-[19], [25]-[26], and many others.

In this article the Exponentiated family of distribution is considered to develop a new model with greater flexibility. It has been already used by [1], [20], [21], [23], among many others. The cumulative distribution function (CDF) and probability density function (pdf) of the Exponentiated family of

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distributions For a baseline continuous CDF $G(x;\ \xi)$, [11] defined the Exponentiated generalized (EG) class of distributions by

$$F(x;\ \xi) = [1 - (1 - G(x;\ \xi))^a]^b\ , \tag{1}$$

where a > 0 and b > 0 are two shape parameters whose role is to control skewness and kurtosis to enable the generation of distributions with heavier lighter tails. The PDF corresponding to (1) is given by

$$f(x;\ \xi) = abg(x;\ \xi)(1 - G(x;\ \xi))^{a-1}[1 - (1 - G(x;\ \xi))^a]^{b-1}\ , \tag{2}$$

where, $g(x;\ \xi)$ is the baseline pdf. Setting $a = 1$, the pdf (1) gives Lehmann Type I class. Also, for b = 1, the pdf (1) gives the Lehmann Type II family of distributions. So, the family (1) generalizes both Lehmann Types I and II family of distributions. It is essential to note that the baseline distribution $G(x;\ \xi)$ is a special case of (1) when $a = b = 1$.

2 Exponentiated generalized inverse Pareto distribution

A random variable X follows an inverse Pareto distribution if its CDF is given according to [23] is

$$F(z) = (\varphi x)^\lambda, \quad x > 0, \ \ \lambda, \varphi > 0, 0 < x < \frac{1}{\varphi}. \tag{3}$$

And the associated CDF to (3) is given by

$$f(x) = \lambda\varphi^\lambda x^{\lambda-1}, \quad x > 0, \ \ \lambda, \varphi > 0, 0 < x < \frac{1}{\varphi} \tag{4}$$

where, φ is scale and λ is shape parameter.

The CDF and PDF of the EGIP distribution is derived by putting eqn. (3) and eqn. (4) in eqn. (1) and eq. (2) and respectively, given as

$$F(x;\ a, b, \lambda, \varphi) = \left[1 - \left(1 - (\varphi x)^\lambda\right)^a\right]^b\ , \tag{5}$$

and

$$f(x;\ a, b, \lambda, \varphi) = ab\lambda\varphi^\lambda x^{\lambda-1}\left(1 - (\varphi x)^\lambda\right)^{a-1}\left[1 - \left(1 - (\varphi x)^\lambda\right)^a\right]^{b-1}\ , \tag{6}$$

where, φ is scale and a, b, λ is shape parameters.

2.1 Some special distributions of EGIP model

Here, we present special distributions to EGEP model as follows:

- For $b = 1$, the EGIP reduces to Generalized Exponentiated Inverse Pareto (GEIP) as a new model.
- For $a = 1$, the EGIP reduces to the Exponentiated Pareto (EP) distribution with
- For $\alpha = 1$ and $\beta = 1$, the EGEP reduces to IP distribution.

2.2 Some basic properties of the exponentiated generalized inverse Pareto distribution

In this section some standard reliability properties of the EGIP have been derived. An expression for the reliability function, hazard, cumulative hazard, and the odd function of the EGIP distribution is respectively, given by

$$R(x;\ a,b,\lambda,\varphi) = 1-\left[1-\left(1-(\varphi x)^{\lambda}\right)^{a}\right]^{b}, \tag{7}$$

$$h(x) = \frac{ab\lambda\varphi^{\lambda}x^{\lambda-1}\left(1-(\varphi x)^{\lambda}\right)^{a-1}\left[1-\left(1-(\varphi x)^{\lambda}\right)^{a}\right]^{b-1}}{1-[1-(1-(\varphi x)^{\lambda})^{a}]^{b}} \tag{8}$$

$$r(x) = \frac{ab\lambda\varphi^{\lambda}x^{\lambda-1}\left(1-(\varphi x)^{\lambda}\right)^{a-1}\left[1-\left(1-(\varphi x)^{\lambda}\right)^{a}\right]^{b-1}}{[1-(1-(\varphi x)^{\lambda})^{a}]^{b}} \tag{9}$$

and

$$O(x) = \frac{\left[1-\left(1-(\varphi x)^{\lambda}\right)^{a}\right]^{b}}{1-[1-(1-(\varphi x)^{\lambda})^{a}]^{b}} \tag{10}$$

The graphs of the distribution function and hazard function of the $EGIP$ are given in figure 1 and 2 with various values of the parameters of the distributions.

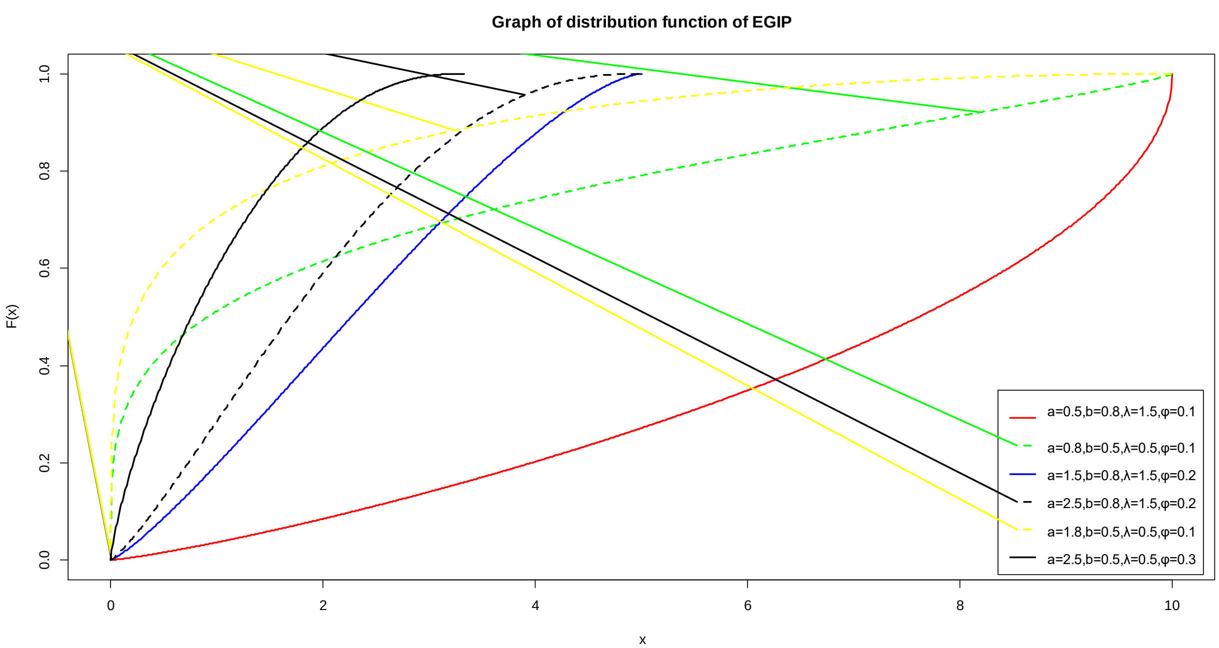

Figure 1. Graph of distribution function of EGIP distribution

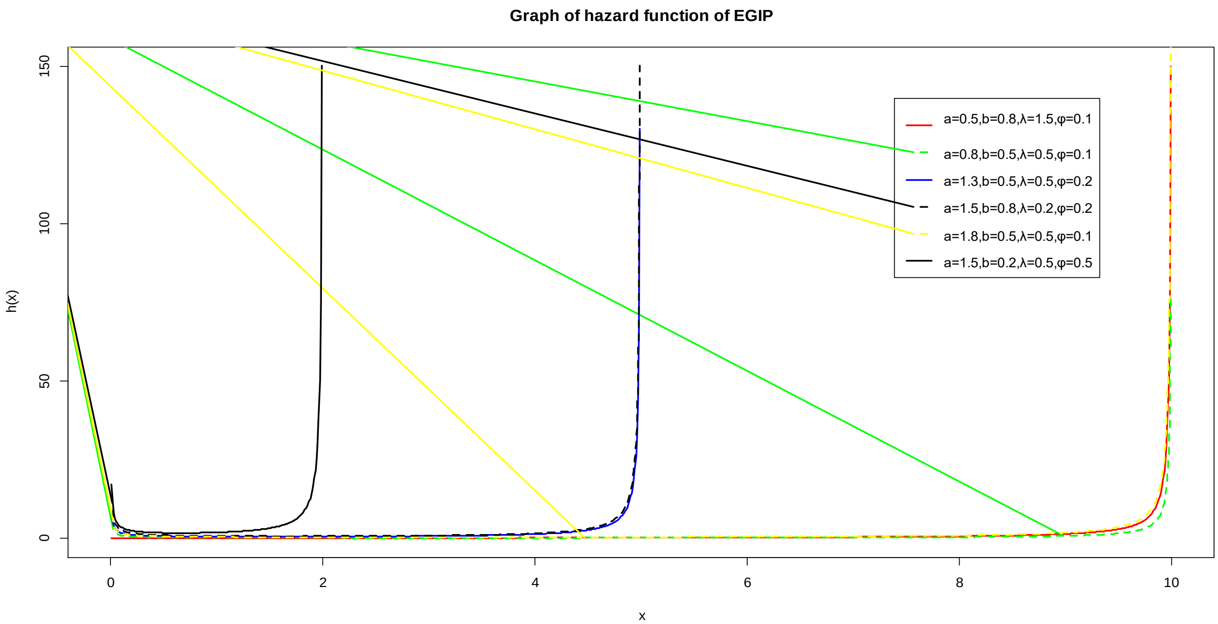

Figure 2. Graph of hazard function of EGIP distribution

2.3 Quantile function and median

In this section the quantile function, median (M) and mode are derived. Where u is Uniform (0,1), the quantile function of the EGIP distribution is obtain by inverting eqn.(5)

$$x_u = \frac{1}{\varphi}\left[1-\left(1-u^{1/b}\right)^{1/a}\right]^{1/\lambda} \tag{11}$$

median is obtain by taking $u = 0.5$ in eqn. (11) presented as

$$M = \frac{1}{\varphi}\left[1-\left(1-(0.5)^{1/b}\right)^{1/a}\right]^{1/\lambda} \tag{12}$$

The mode of $EGIP$ distribution can be obtained by solving the following non-linear equation $\frac{\partial}{\partial x}(log[f(x;a,b,\lambda,\varphi)]) = 0$, which is given as

$$\frac{\lambda(b-1)\varphi^\lambda x^\lambda\left(1-(\varphi x)^\lambda\right)^{a-1}}{[1-(1-(\varphi x)^\lambda)^a]} + \lambda = \frac{\lambda(a-1)\varphi^\lambda x^\lambda}{[1-(\varphi x)^\lambda]} + 1$$

The quantile function can be used to obtain the Bowley's skewness (B) by [14] and Moors's kurtosis (M) by [16]. These measures are given by

$$B = \frac{Q_{0.75}+Q_{0.25}-2Q_{0.5}}{Q_{0.75}-Q_{0.25}} \; and \; M = \frac{Q_{0.375}-Q_{0.125}+Q_{0.875}-Q_{0.625}}{Q_{0.75}-Q_{0.25}}$$

3 Moments

The r^{th} moment of $X \sim EGIP(a, b, \lambda, \varphi)$ is found by

$$E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx \tag{13}$$

Substitute from Equation (6) into Equation (13), we will obtain the r^{th} moment as

$$\mu_r' = ab\varphi^{-r} \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i} B\left(\frac{r}{\lambda} + 1, a(i+1)\right) \tag{14}$$

It should be noted the first moment represent the mean of $EGIP$ distribution and is obtained by taking $r = 1$, the second and the third moment is obtained by taking r=2 and 3 and are respectively given as

$$\mu_1' = ab\varphi^{-1} \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i} B\left(\frac{1}{\lambda} + 1, a(i+1)\right), \tag{15}$$

$$\mu_2' = ab\varphi^{-2} \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i} B\left(\frac{2}{\lambda} + 1, a(i+1)\right), \tag{16}$$

and

$$\mu_3' = ab\varphi^{-3} \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i} B\left(\frac{3}{\lambda} + 1, a(i+1)\right). \tag{17}$$

Thus the variance of EGIP distribution is given by

$$\sigma_X^2 = \frac{ab}{\varphi^2} Z^i B\left(\frac{2}{\lambda} + 1, a(i+1)\right) - \left(\frac{ab}{\varphi} Z^i B\left(\frac{1}{\lambda} + 1, a(i+1)\right)\right)^2 \tag{18}$$

where

$$Z^i = \sum_{i=0}^{b-1} (-1)^i \binom{b-1}{i}$$

3.2 The incomplete moment of EGIP distribution

The r^{th} incomplete moment of $X \sim EGIP(a,b,\lambda,\varphi)$ is obtained by

$$\zeta_r' = \int_0^{\infty} x^r f(x)dx \tag{19}$$

Substitute from Equation (6) into Equation (19), and we will obtain the incomplete moment as

$$\zeta_r' = ab\varphi^{-r}\sum_{i=0}^{b-1}(-1)^i\binom{b-1}{i}B\left(\frac{r}{\lambda}+1, a(i+1); \frac{t^{1/\lambda}}{\varphi}\right) \tag{20}$$

3.3 The moment generating function of EGIP distribution

Let $X \sim EGIP(a,b,\lambda,\varphi)$, then the moment generating function of the $EGIP$ can be can be obtain using

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x)dx = \sum_{j=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx = \sum_{j=0}^{\infty}\frac{t^r}{r!}\,\mu_r' \tag{21}$$

Putting eqn. (14) into Eqn. (21), we obtain an expression for the moment generating function for EGIP distribution as

$$M_X(t) = ab\varphi^{-r}\sum_{i=0}^{b-1}\sum_{j=0}^{\infty}\frac{t^r}{r!}(-1)^i\binom{b-1}{i}B\left(\frac{r}{\lambda}+1, a(i+1)\right) \tag{22}$$

3.4 Probability-weighted moment (PWM) of EGIP distribution

The PWM of X whose CDF $F(x)$, say $\psi^{v,r}$, is obtained by

$$\psi^{v,r} = E(X^v F^r(x)) = \int_0^{\infty} x^v F^r(x)\, f(x)dx\,. \tag{23}$$

If $X \sim EGIP(a,b,\lambda,\varphi)$, then the PWM $\psi^{v,r}$ is determined by

$$\psi^{v,r} = ab\varphi^{-r}\sum_{i=0}^{b-1}(-1)^i\binom{b(r+1)-1}{i}B\left(\frac{v}{\lambda}+1, a(i+1)\right) \tag{24}$$

3.5 Mean residual life of EGIP distribution

If $X \sim EGIP(a,b,\lambda,\varphi)$, then the mean residual life is determined by

$$MRL(t) = E[X - t/X > t] = \frac{\int_t^{\infty}(x-t)f(x)dx}{R(t)} = \frac{\mu_1' - \int_0^t xf(x)dx}{R(t)} - t \tag{25}$$

The integral $\int_0^t xf(x)dx$ is the incomplete moment of EGIP and μ_1' is the first moment obtained by taking $r = 1$ in eqn. (20) and $r(t)$ is given in eqn.(7)

$$MRL(t) = \frac{\mu_1' - ab\varphi^{-r}\sum_{i=0}^{b-1}(-1)^i\binom{b-1}{i}B\left(\frac{r}{\lambda}+1, a(i+1); \frac{t^{1/\lambda}}{\varphi}\right)}{1-[1-(1-(\varphi x)^{\lambda})^{a}]^{b}} - t \tag{25}$$

3.6 Entropy function and ρ −entropy

The entropy function is a measure of uncertainty related to X whose PDF f (x). It plays an important role in computer science, reliability, and others. The Renyi entropy [22] of X, say $I_\rho(X)$, is determined using

$$I_\rho(X) = \frac{1}{1-\rho} log \int_0^{\infty} f^{\rho}(x)dx, \quad \rho \neq 1 \tag{26}$$

If $X \sim EGIP(a, b, \lambda, \varphi)$, then $I_\rho(X)$is obtained by

$$I_\rho(X) = (ab)^{\rho}\varphi^{\rho-[\rho(\lambda-1)+1]}\lambda^{\rho-1}\sum_{i=0}^{b-1}(-1)^i\binom{\rho(b-1)}{i}B(P) \tag{27}$$

Where $P = \frac{(\lambda-1)(\rho-1)}{\lambda} + 1, a(i+\rho) - \rho + 1$

Consequently, the q −entropy of X, say $Z_q(X)$, can be obtained using

$$Z_q(X) = \frac{1}{1-\rho} log(1-[1-\rho])I_\rho(X)$$

3.7 Mean time to failure

Assume $X{\sim}EGIP(a, b, \lambda, \varphi)$, then the mean time to failure of X, say MTTF, is obtained using

$$MTTF = \int_0^{\infty} xf(x)dx = \mu_1' \tag{28}$$

Substitute from Equation (20), taking $r = 1$, to obtain

$$MTTF = b\varphi^{-1}\sum_{i=0}^{b-1}(-1)^i\binom{b-1}{i}B\left(\frac{1}{\lambda}+1, a(i+1)\right) \tag{29}$$

4 Maximum likelihood estimates of the parameters

The maximum likelihood approach is considered in estimating the unknown parameters of the EGIP distribution. Let $\underline{x} = x_1 \dots, x_n$ represent a random sample obtained from the EGIP distribution. The likelihood function $L(x;\ \zeta)$ and the log-likelihood function $logL(x;\ a,b,\lambda,\varphi) = l(x;\ a,b,\lambda,\varphi)$ corresponding to (30) are respectively given as

$$L(\underline{x}, a, b, \lambda, \varphi) = \prod_{i=1}^{n} ab\lambda\varphi^{\lambda}x^{\lambda-1}\left(1-(\varphi x)^{\lambda}\right)^{a-1}\left[1-\left(1-(\varphi x)^{\lambda}\right)^{a}\right]^{b-1} \tag{30}$$

and

$$l = nlog\ (a) + nlog(b) + nlog(\lambda) + \ n\lambda log\ (\varphi) + (\lambda-1)\sum_{i=1}^{n} log(x_i)$$
$$+(a-1)\sum_{i=1}^{n} log\left(1-(\varphi x_i)^{\lambda}\right) + (b-1)\sum_{i=1}^{n} log\left[1-\left(1-(\varphi x_i)^{\lambda}\right)^{a}\right] \tag{31}$$

The element of the score vector $\left(\frac{\partial l}{\partial a}, \frac{\partial l}{\partial b}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \varphi}\right)$ is given by

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\left(1-(\varphi x_i)^{\lambda}\right) - (b-1)\sum_{i=1}^{n}\left[\frac{\left(1-(\varphi x_i)^{\lambda}\right)^{a} log\left(1-(\varphi x_i)^{\lambda}\right)}{[1-(1-(\varphi x_i)^{\lambda})^{a}]}\right] \tag{32}$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log\left[1-\left(1-(\varphi x_i)^{\lambda}\right)^{a}\right] \tag{33}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + nlog(\varphi) + \sum_{i=1}^{n} log(x_i) + a\sum_{i=1}^{n}\left[\frac{(\varphi x_i)^{\lambda}\log(\varphi x_i)}{[1-(1-(\varphi x_i)^{\lambda})^{a}][1-(\varphi x_i)^{\lambda}]}\right] \tag{34}$$

$$\frac{\partial l}{\partial \varphi} = \frac{n\lambda}{\varphi} + (b-1)\lambda\varphi^{\lambda-1}\sum_{i=1}^{n}\left[\frac{{x_i}^{\lambda}}{(1-(\varphi x_i)^{\lambda})^{a}[1-(\varphi x_i)^{\lambda}]}\right] - (a-1)\lambda\varphi^{\lambda-1}\sum_{i=1}^{n}\frac{{x_i}^{\lambda}}{[1-(\varphi x_i)^{\lambda}]} \tag{35}$$

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