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Shallow water flow over non-uniform varying and non-varying bottom topography

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Abstract

This study stemmed from the work of Okeke (1985) and Mbah(2008) where they studied the Shallow water wave equations for water depth below the undisturbed sea surface where they assumed the water depth (rough and horizontal) to be constant. We considered this as unrealistic and thus a case where the bottom topography is non-uniform (the depth of the water not constant) and varying is treated here. The depth of the water at all points of the bottom which we call h is defined as: $h = h' + x \text{ Cot } \beta$, where $h' =$ water depth at points where there is no non-uniform bottom, x is the points where there is non-uniform and varying bottom, and β is the angle of inclination of the non-uniform varying bottom to the leveled area. The solution to the wave equations for shallow water flow suggests the existence of a single wave elevation which still has a singularity in the profile. This consideration predicts more realistic results as we can see that the nature of the wave and also the velocity of flow is strictly a function of the angle, time and the size of the region that is non-uniform and varying. This is a very important result that must be taken note of in the study of shallow water flow in general and the management of shallow water flow problems, most importantly when the bottom topography is not horizontal or uniform at all or any regions of the flow.

1 Introduction

Shallow water problem in fluid dynamics for quite some time, have attracted the attention of researchers. In most of these works, assumptions were made of simpler bottom topography of such shallow waters. This is simply to avoid the complex nature of the physical features peculiar to such shallow water. However, it is these physical features that control the mechanism of wave propagation and probably its

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breaking along the line of flow, development of singularities along the line of flow and where possible, turbulence.

It is these important qualities of the bottom topography that led to our attempt to model the real nature of the bottom of shallow water as shown in our work, [7]. Here therefore, the analytical study of the development was based mainly on the traditional shallow water approximations with some little modifications.

Many researchers like [4, 5] had worked on shallow water waves where it was assumed that the shallow water has slopping bottom. He equally assumed the waves to be linear which in reality is not true. In our work, we assumed that the wave is generally non-linear and considered two cases; where it is dispersive and non-dispersive.

In his later work, [5], the author even assumed that such shallow water has constant Water depth. A comparison of the results we obtained fits very favourably with [4] if we assume the terms of the model the way he did.

To be able to obtain a closed solution for the water wave, we used the expression for the water depth as we obtained in the non linear dispersive wave train of the shallow water model here. [1, 3, 4, 6, 9 and 10,], were very helpful in the understanding of the assumptions made so as to obtain reasonable solution. Also, the solution we obtained in [7] which is redeveloped here showed a very good result which on our attempt of including the complex part of C in the graphing, gave us a totally different set of graphs. It is this behaviour that necessitated this current study.

2 A Model of the bottom topography of shallow water

In [7], we had considered the general bottom topography of Shallow water flow as:

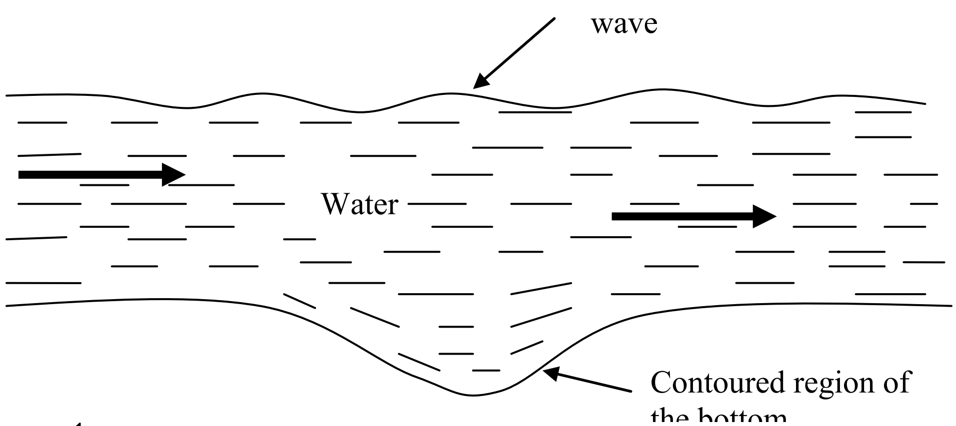

Figure 1

We had considered the non-uniform region as below:

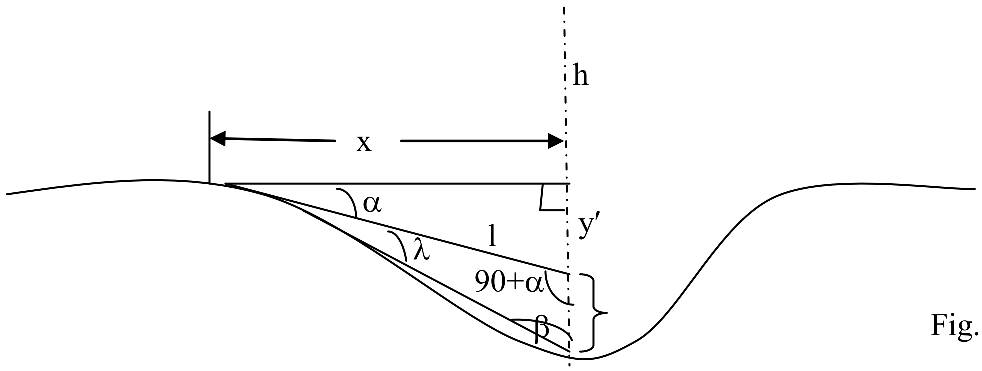

Fig. 2

From this diagram, we obtained the general expression for the height of the water at all points of the bottom of flow as:

$$H = y' + \delta y' + h$$

$$\Rightarrow H = x \tan\alpha + x\{\cot\beta - \tan\alpha\} + h = h + x\cot\beta \quad \text{... (1)}$$

where h = height at the level bottom (no contour), x = the width of the contoured region and β is the angle of inclination of the center of the contoured region with respect to the starting point of the width (x = 0).

In this case, we have assumed that β is not a function of x so that we consider it constant. Later, we shall study a case where the angle β is a function of x.

3 Governing equations for shallow water waves with rough bottom topography

Our development of the model here is based essentially on the shallow water equation governing the weakly non-linear waves on the surface of the shallow water with rough bottom topography. We shall incorporate the vertical component of water particles and the related energy transfer as the wave progresses. The effect of linear dispersion is included as against what mostly obtains in literatures.

Thus, we take the x – axis as the horizontal and normal to shore line, the z-axis is the vertically upward direction where z = $\eta(x, t)$ represents the wave profile occurrence on the water surface, z = h is the constant water depth of the shallow water as measured from the undisturbed water level . Thus, the equation, governing the evolution of the weakly non- linear train in shallow water with rough bottom topography can be written using H = h + x cot β as:

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[U(h+x\cot\beta)\right]=O \qquad (1).$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \eta}{\partial x}+\lambda\frac{\partial^3 u}{\partial x^3}=\frac{\left(h+x\cot\beta\right)^2}{3}\left(\frac{\partial^3 u}{\partial x^2\partial t}\right) \qquad (2)$$

Here u$\left(x,t\right)$ is the component of the particle velocity along the x – axis for t > 0 as the time. λ is the constant associated with the wave dispersion in the shallow water, g is the constant due to gravity with the term $\frac{\left(h+x\cot\beta^2\right)}{3}\frac{\partial^3 u}{\partial x^{2\partial t}}$ representing the vertical velocity effect on the horizontal velocity as well as the pressure.

We shall consider two cases of interest, namely, the case of non-varying and varying bottom topography respectively. Thus, we have the respective procedures to solution as:

Let us define another function w$\left(x,t\right)$ such that:

$$\text{U} = \frac{1}{htx\cot\beta}\frac{\partial w}{\partial t} \qquad (3)$$

$$\text{and } \eta=-\frac{\partial w}{\partial x} \qquad (4)$$

and they satisfy equations (1) and (2) when substituted into them.

Thus, substituting $\langle 3\rangle$ *and* $\langle 4\rangle$ into $\langle 1\rangle$ and $\langle 2\rangle$ have both satisfied provided that

$$\frac{1}{h+x\cot\beta}\frac{\partial^2 w}{\partial t^2}+\frac{1}{h+x\cot\beta}\left\{\frac{1}{h+x\cot\beta}\left[\frac{\partial^2 w}{\partial x\partial t}-u\cot\beta\right]\right\}-g\frac{\partial^2 w}{\partial x^2}$$

$$+\lambda\left\{\frac{1}{h+x\cot\beta}\frac{\partial^4 w}{\partial x^3\partial t}-\frac{3\cot\beta}{(h+x\cot\beta)^2}\frac{\partial^3 w}{\partial x^2\partial t}+\frac{2\cot^2\beta}{(h+x\cot\beta)^3}\frac{\partial^2 w}{\partial x\partial t}-\frac{6u\cot^2\beta}{\left(h+x\cot\beta\right)^3}\right\}$$

$$=\frac{\left(h+x\cot\beta\right)^2}{3}\left[\frac{1}{h+x\cot\beta}\frac{\partial^4 w}{\partial x^2\partial t^2}-\frac{2\cot\beta}{\left(h+x\cot\beta\right)^2}\frac{\partial^3 w}{\partial x\partial t^2}\right] \qquad (5)$$

Equation $\langle 5 \rangle$ simplifies to

$$\frac{\partial^2 w}{\partial t^2}+\lambda\frac{\partial^4 w}{\partial x^3 \partial t}-\frac{3\lambda\cot\beta}{h+x\cot\beta}\frac{\partial^3 w}{\partial x^2\partial t}+\frac{2\lambda\cot^2\beta}{\left(h+x\cot\beta\right)^2}\frac{\partial^2 w}{\partial x\partial t}+\frac{1}{h+x\cot\beta}\frac{\partial^2 w}{\partial x\partial t}$$

$$-g\left(h+x\cot\beta\right)\frac{\partial^2 w}{\partial x^2}-\frac{u\cot\beta}{h+x\cot\beta}-\frac{6u\lambda\cot^3\beta}{\left(h+x\cot\beta\right)^2}$$

$$=\frac{\left(h+x\cot\beta\right)^2}{3}\frac{\partial^4 w}{\partial x^2\partial t^2}-2\left(h+x\cot\beta\right)\cot\beta\frac{\partial^3 w}{\partial x\partial t^2} \qquad (6)$$

Since periodic wave trains are possible in nonlinear dispersive system, let us define a characteristic curve $\Gamma:\xi = x-ct$ which is the phase of the oscillation propagating with constant phase speed c and wave number k such that we can write

$$w=w\left(\xi\right) \qquad (7)$$

Now let us make the substitution $r=h+x\cot\beta.$ (8)

Then using equation $\langle 7\rangle$ and $\langle 8\rangle$ in equation $\langle 6\rangle$, we obtain

$$-\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^4 w(\xi)}{d\xi^4}+\left\{\frac{3\lambda c}{r}\cot\beta+\frac{2r^2c^2}{r}\cot\beta\right\}\frac{d^3 w(\xi)}{d\xi^3}+\left[c^2-gr-\frac{2\lambda c\, cot^{2\beta}}{r^2}-\frac{c}{r}\right]$$

$$+\left\{\frac{c}{r^2}\cot\beta+\frac{\partial\tau}{r2}\cot^3\beta\right\}\frac{\partial w\left[\xi\right]}{\partial\xi}=0 \qquad (9)$$

In equation $\langle 9\rangle$, the non-linear term is included so as to know its effect no matter how small on the phase speed. Thus, we shall obtain the phase speed c for the two cases:

1. where the dispersive effect λ is neglected that is, $\lambda = o$
2. where the dispersive effect λ is not neglected, that is, $\lambda \neq o.$

Case 1

If λ= 0 and we explicitly define $w\left(\xi\right)= Ae^{ik\xi}$*,then equation* $\langle 9\rangle$ *reduces* to

$$\frac{Ak^4r^2c^2}{3} - \frac{2iAk^3r^2c^2}{r}-Ak^2c^2+Ak^2gr+\frac{2\lambda cAk^2}{r2}\cot^2\beta+\frac{Ack^2}{r}+\frac{iAck\cot\beta}{r^2}=0$$

i.e. $$c^2\left[\frac{Ak^4r^2}{3}-2iAk^3r-Ak^2\right] + \text{c}\left[\frac{Ak^2}{r}+\frac{iAk}{r2}\cot\beta\right] + Ak^2gr=0$$

$$\Rightarrow \quad c^2\left[\frac{k^3r^2}{3}-2ik^2r-k\right] + \text{c}\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]+grk=0 \quad .$$

Solving for c, we obtained

$$c=\frac{-\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]\pm\left\{\left(\frac{k}{r}+\frac{i\cot\beta}{r^2}\right)^2-4grk\left[\frac{k^3r^2}{3}-2ik^2r-k\right]\right\}^{1/2}}{2\left[\frac{k^3r^2}{3}-2ik^2r-k\right]}$$

In shallow water wave, only low wave numbers are involved so that we can neglect k^4 and k^3 to obtain

$$c=\frac{-\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]\pm\left\{\left(\frac{k}{r}+\frac{i\cot\beta}{r^2}\right)^2+4grk^2\right\}^{1/2}}{-2\left[2ikr+1\right]k} \tag{10}$$

Case II:

We consider here a non-linear dispersive wave in which now $\lambda \neq 0$. Thus substituting for w(ξ) in equation (9), we obtain

$$-\left\{\lambda c+\frac{r^2c^2}{3}\right\}(Ak^4)+\left\{\frac{3\lambda c}{r}\cot\beta+\frac{2r^2c^2\cot\beta}{r}\right\}\left\{-iAk^3\right\} +$$
$$\left\{c^2-gr-\frac{2\lambda c}{r^2}\cot^2\beta-\frac{c}{r}\right\}(-Ak^2)+\left\{\frac{c}{r^2}\cot\beta+\frac{6\lambda}{r^2}\cot^3\beta\right\}(iAk)=0.$$

$$\Rightarrow \quad c^2\left[-\frac{Ak^4r^2}{3}-\frac{2iAr^2k^3}{r}\cot\beta-Ak^2\right] +$$
$$c\left\{-\lambda Ak^4-\frac{3\lambda iAk^3}{r}+\frac{2\lambda Ak^2}{r^2}\cot^2\beta+\frac{Ak^2}{r}+\frac{iAk}{r^2}\cot\beta\right\}+grAk^2+\frac{6\lambda iAk}{r^2}\cot^3\beta=0$$

Provided the roughness of the bottom of the shallow water is not a steep slope, we can neglect $Cot^3\beta$ since in this case β is large. Hence, the above equation reduces to

$$c^2\left\{\frac{k^3r^2}{3}+2irk^2\cot\beta+k\right\}+c\left[\lambda k^3+\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^2\beta-\frac{k}{r}-\frac{i\cot\beta}{r2}\right]-grk=0.$$

We can solve this equation to obtain a value for c which on neglecting $k^3 and, k^4$ gives:

$$c=\frac{-\left[\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^{2\beta}-\frac{k}{r}\frac{i\cot\beta}{r^2}\right]+\left\{\left(\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^{2\beta}-\frac{k}{r}-\frac{i\cot\beta}{r^2}\right)^2+4grk\left[2irk^2\cot\beta+k\right]\right\}^{1/2}}{2\left[2irk^2\cot\beta+k\right]}$$

Since k is small and β is large, we readily see that the value of c obtained in this case II is greater than that obtained in case 1. If we had linearised the wave, we would obtain c = $\left\{g\left[h+x\cot\beta\right]\right\}^{1/2}$. Therefore, for this non-linear dispersive wave, c as obtained is greater than the c for a linear wave speed given by c = $\left\{g\left(h+x\cot\beta\right)\right\}^{1/2}$ and this is as a result of additional terms that depended on the wave amplitude. Hence, we shall adopt that c in our case here is greater than $\left\{g\left(h+x\cot\beta\right)\right\}^{1/2}$ so that we will able to solve equation (9) to then obtain values for the velocity and the wave profile.

Hence, integrating equation (9) once and neglecting the constant of integration, we obtain:

$$\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^3w(\xi)}{d\xi^3}-\left\{\frac{3\lambda c}{r}\cot\beta+2rc^2\cot\beta\right\}\frac{d^2w(\xi)}{d\xi^2}-\left\{c^2-gr-\frac{2\lambda c}{r2}\cot^2\beta-\frac{c}{r}\right\}\frac{dw(\xi)}{d\xi}$$

$$-\left\{\frac{c}{r^2}\cot\beta+\frac{6\lambda}{r2}\cot^3\beta\right\}=0.$$

Suppose we let P = $\frac{dw(\xi)}{d\xi}$ and then substitute in the equation above. Then we have a second order non-homogenous differential equation with constant coefficients given as :

$$\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^2p}{d\xi^2}-\left\{\frac{3\lambda c}{r}\cot\beta+\frac{6\cot\beta}{r}\right\}\frac{dp}{d\xi}-\left\{c^2-gr-\frac{2\lambda c}{r2}\cot^{2\beta}-\frac{c}{r}\right\}p-\left\{\frac{c}{r^2}\cot\beta\,\&\,\frac{6\lambda}{r2}\cot^3\beta\right\}=0$$

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(12)

To simplify this equation further, let us make some substitutions like:

$$\sigma = \lambda c + \frac{r^2c^2}{3}, \quad \rho = \frac{3\lambda c}{\partial}\cot\beta + \frac{6\cot\beta}{r,}, \quad \tau = c^2 - gr - \frac{2\lambda c}{r^2}\cot^2\beta - \frac{c}{r}$$

$$and \quad \delta = \frac{c}{r^2}\cot\beta + \frac{6\lambda}{r^2}\cot^3\beta.$$

With these, we get equation $\langle 12 \rangle$ as:

$$\frac{\partial^2 P}{\partial \xi^2} - \rho\frac{\partial p}{\partial \xi} - \tau p = \delta \qquad (13)$$

and this is non-homogenous. When solved, the complementary solutions is obtained as:

$$m = \frac{\frac{\rho}{\sigma} \pm \left\{\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right\}^{1/2}}{2} = \frac{\rho}{2\sigma} \pm \frac{1}{2}\left\{\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right\}^{1/2}$$

$$\Rightarrow \quad P_{cf} = \ell^{\frac{\rho}{}\xi i}\left\{ \quad s\,{}^1\!/\!{}_2\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{1/2}\right\}\xi + B\sin\left\{{}^1\!/\!{}_2\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{1/2}\right\}\xi \qquad (14)$$

We can equally solve for the particular part solution of the equation $\langle 13 \rangle$ to get that

$$P_{pI} = -{}^{\delta}\!/\!{}_{\tau}. \qquad (15)$$

Thus, the general solution of equation $\langle 13 \rangle$ is:

$$P = P_{cf} + P_{pI}$$

$$\Rightarrow p = \ell^{\frac{\rho}{}\xi i}\left[\quad \text{)s}\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + B\sin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi\right] - \frac{\delta}{\tau} \qquad (16)$$

But $P = \dfrac{dw(\xi)}{d\xi}$ *and* $U = -\dfrac{c}{r}\dfrac{dw(\xi)}{d\xi}$, then we have that

$$U = -\frac{c}{r}\ell^{\frac{\rho}{\;}\xi i}\left[\quad ɔs\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2+4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + BSin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2+4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi - \frac{\delta}{\tau}\right] \quad (17)$$

$$\eta = \frac{dw(\xi)}{\partial\xi} = \rho \quad so \quad that$$

Also,

$$\eta = \ell^{\frac{\rho}{\;}\xi i}\left[\quad ɔs\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2+4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + BSin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2+4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi - \frac{\delta}{\tau}\right]$$

-------- (18)

Equation $\langle 17\rangle$ *and* $\langle 18\rangle$ *gives* the expression for the velocity and the wave profile of the shallow water wave over rough bottom topography as shown in the figures below.

4 Analysis of the Result

Here we investigate the effect of this non-uniform bottom topography by looking at the variations in the values of the wave number, the height of the undisturbed water surface and even the wave length and their subsequent effect on the velocity of flow as well as the wave profile.

In figures 1 to 4 below, we can see that the smaller the wave number, the less we can really see the effect of the rough bottom on both the velocity and the wave profile of the shallow water flow. This means that if the wave number is very small, it can not be used for any meaningful analysis of the flow mechanisms of shallow water.

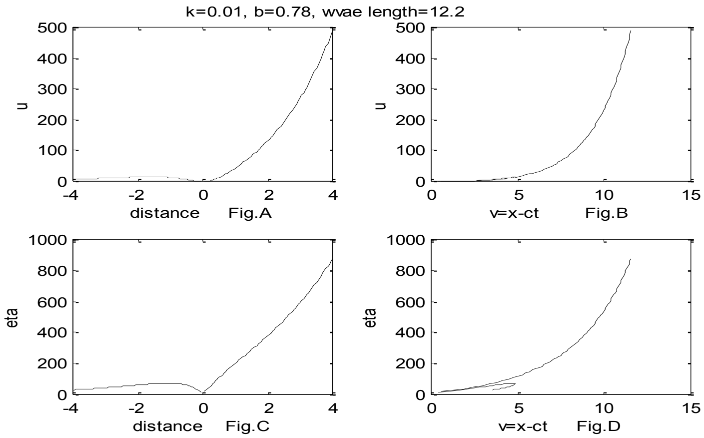

Figure 1

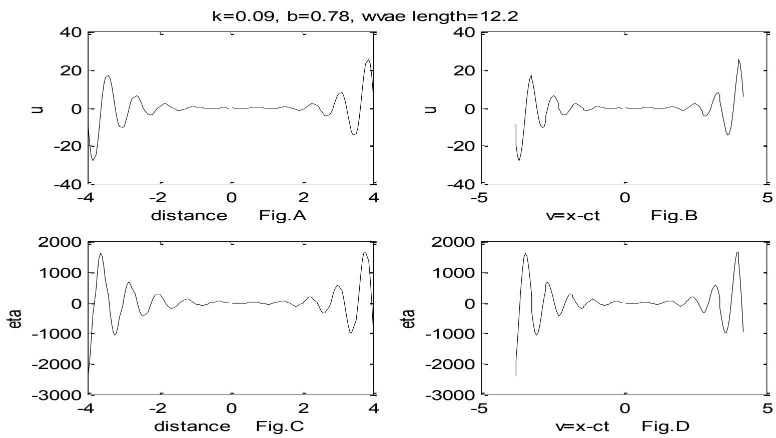

Figure 2

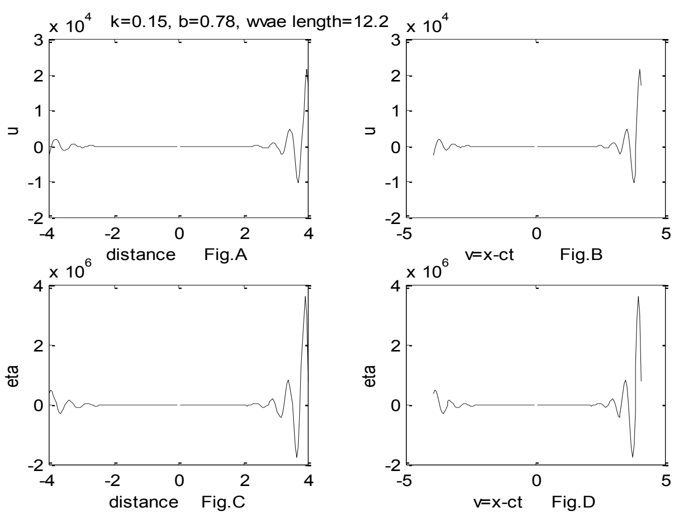

Figure 3

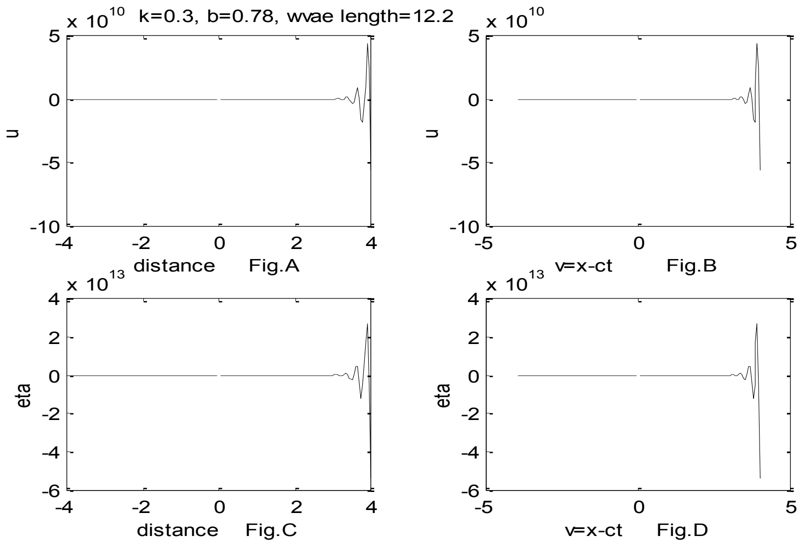

Figure 4

In Figures 5 and 6 below, we can see that while retaining the same values for the wave number, the wave length and time, a variation in the height of the water has great impact on the velocity of flow as well as the wave profile. The effect of the rough bottom on shallow water with larger h is smaller than when h is small. In these two examples, we can see that when h = 4, the impact of the rough bottom was highly felt as the velocity was highly irregular (Figure 6). Therefore to what extent the rough bottom shall affect the velocity and or the wave profile of shallow water flow shall be greatly determined by the height of the undisturbed water level .

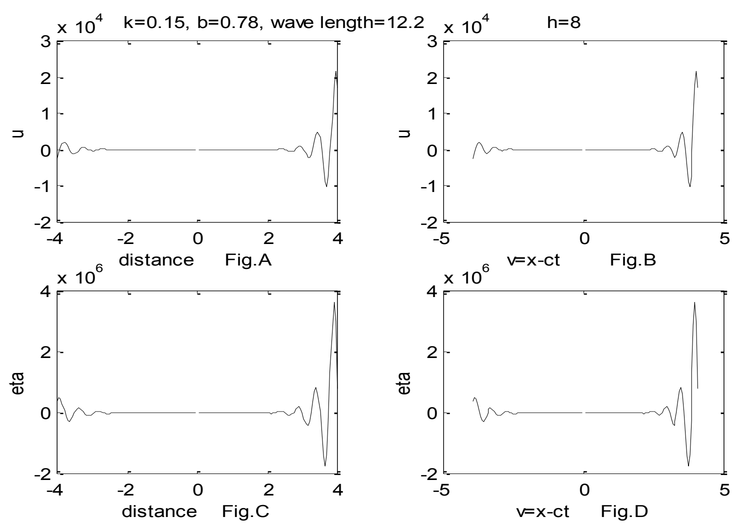

Figure 5

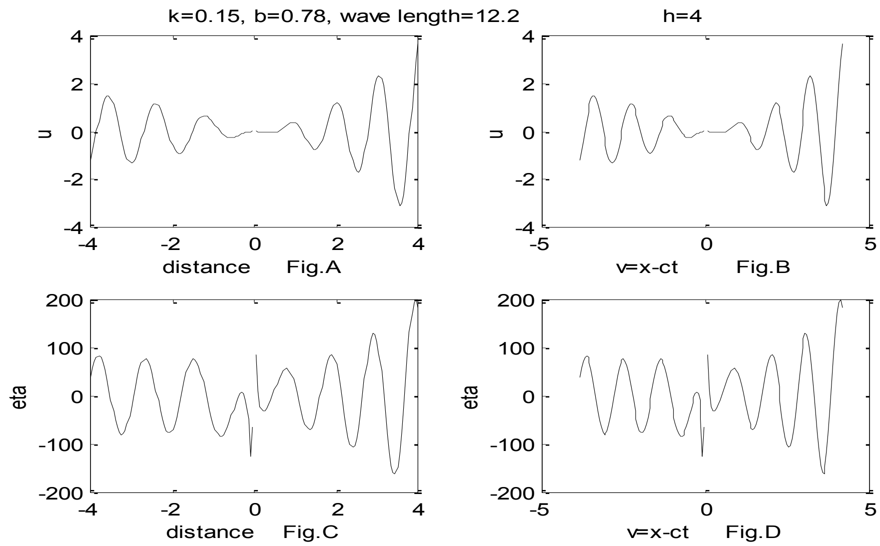

Figure 6

In Figure 7, we retained the same values for the wave number, the height of the water at the undisturbed surface and time and then varied the wave length. Comparing this with Figure 6 that has the same values except for the changed value of the wave length, we see that smaller wave length in the flow hides the impact of the rough bottom on the shallow water flow. Thus, depending on the source of such water, the use of wave length in the study of flow mechanism in shallow water must be taken seriously as we have seen that it can mask the effect of the rough bottom topography and probably other factors which we may not have shown here.

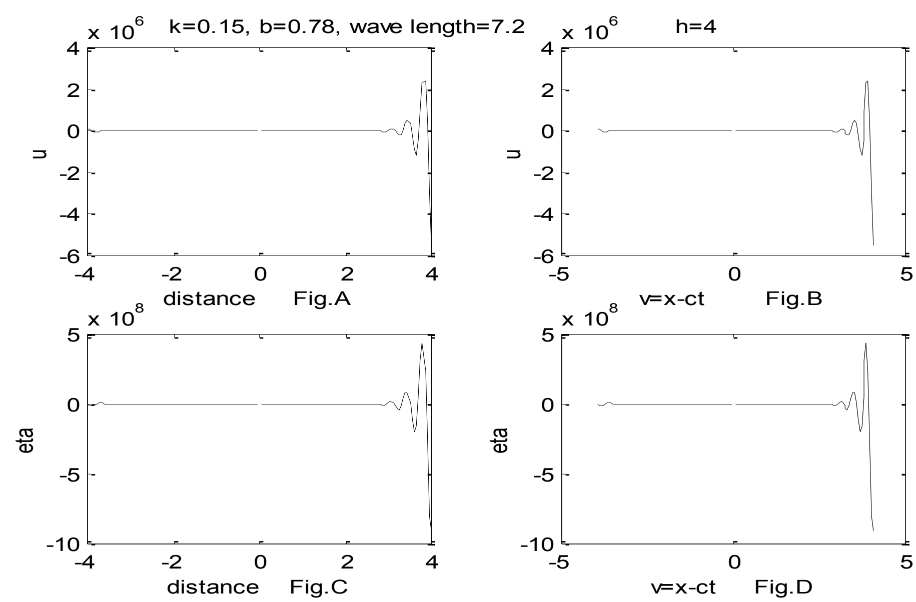

Figure 7

Section B

Let us consider the case where the flow is trying to fill the gully or rough bottom by dropping some sand or debris into the contoured regions. The model equation describing the flow is still the equations (1) and (2). The equations are simplified to obtain equation (9) as before. This equation (9) is a fourth order non-linear ordinary differential equation which we need to solve. Since this is a non-linear fourth order ordinary differential equation which is usually difficult to solve analytically, we adopt the asymptotic method to obtain solution to it.

We write equation (9) as

$$s\frac{d^4w}{d\in^4}-\rho\frac{d^3w}{d\in^3}+(\delta-g)\frac{d^2w}{d\in^2}+\sigma\frac{dw}{d\in}=\left(\tau\frac{d^2w}{d\in^2}-\mu\frac{dw}{d\in}\right)\frac{dw}{d\in}$$

Let w = λV and write $\frac{dV}{d\in}$ as V' and we shall have

$$\lambda\left[sV^{iv}-\rho V'''+(\delta-g)V''+\sigma V'\right]=\lambda^2\left[\tau V''-\mu V'\right]V'$$

$$\Rightarrow\ sV^{iv}-\rho V'''+(\delta-g)V''+\sigma V'=\lambda\left[\tau V''-\mu V'\right]V'$$

By perturbation method we shall write

$$V=V_0+\lambda V_1+\lambda^2V_2+\ldots$$

$$\lambda_0: LV_0\equiv sV_0^{iv}-\rho V_0^{III}+(\delta-g)V_0^{II}+\sigma V_0^{I}=0 \tag{10}$$

$$\lambda_1: LV_1\equiv sV_1^{iv}-\rho V_1^{III}+(\delta-g)V_1^{II}+\sigma V_1^{I}=\left[\tau V_0^{II}-\mu V^{I}_{\ 0}\right]V_0^{I} \tag{11}$$

We solve equation (10) as follows:

$$sV_0^{iv}-\rho V_0^{III}+(\delta-g)V_0^{II}+\sigma V_0^{I}=0$$

Integrate once to get

$$sV_0^{III}-\rho V_0^{II}+(\delta-g)V_0^{I}+\sigma V=\zeta$$

Assume ζ = 0 and we have

$$sV_0^{III}-\rho V_0^{II}+(\delta-g)V_0^{I}+\sigma V=0$$

Let $M=\frac{dV}{d\in}$, $M'=\frac{d^2V}{d\in^2}$ and $M''=\frac{d^3V}{d\in^3}$. Then we have

$$sM''[V]-\rho M'[V]+(\delta-g)M[V]+\sigma[V]=0$$

$$\left[sM''-\rho M'+(\delta-g)M+\sigma\right][V]=0$$

Since $V\neq 0$ then

$$sM''-\rho M'+(\delta-g)M+\sigma=0 \tag{12}$$

We solved (12) to obtain:

$$V_0\equiv M=\ell^{\left(ip/_{2s}\right)\in}\begin{Bmatrix}DCos\left(1/_{2s}\left[\rho^2-4s(\delta-g)\right]^{1/2}\right)\in+\\ BSin\left(1/_{2s}\left[\rho^2-4s(\delta-g)\right]^{1/2}\right)\in\end{Bmatrix}-\frac{\sigma}{(\delta-g)} \tag{13}$$

Let us denote (13) by K so that

$$K' = ih\ell^{ihe}\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + \ell^{ih\in}\left(-D\upsilon Sin\,\upsilon \in +B\upsilon Cos\,\upsilon \in\right)$$

$$K'' = -h^2\ell^{ih\in}\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + 2ih\ell^{ih\in}\left(-D\upsilon Sin\,\upsilon \in +B\upsilon Sin\,\upsilon \in\right) - \ell^{ih\in}\left(D\upsilon^2 Cos\,\upsilon \in +B\upsilon^2 Sin\,\upsilon \in\right)$$

$$K''K' = \ell^{ih\in}\left\{-h^2\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + 2ih\left(-D\upsilon Sin\,\upsilon \in +B\upsilon Cos\,\upsilon \in\right) - \left(D\upsilon^2 Cos\,\upsilon \in +B\upsilon^2 Sin\,\upsilon \in\right)\right\}\times$$
$$\ell^{ih\in}\left\{ih\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + \left(-D\upsilon Sin\,\upsilon \in +B\upsilon Cos\,\upsilon \in\right)\right\}$$

$$= \ell^{2ih\in}\left\{\begin{array}{l} -ih^3\left(D^2Cos^2\upsilon \in +B^2Sin^2\upsilon \in +2BDSin\,\upsilon \in Cos\,\upsilon \in\right) + h^2D^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in -h^2DB\upsilon Cos^2\upsilon \in \\ +h^2BD\upsilon Sin^2\upsilon \in -h^2B^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in +2h^2D^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in +2h^2BD\upsilon Sin^2\upsilon \in -2h^2BD\upsilon Cos^2\upsilon \in \\ -2h^2B^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in +2ihD^2\upsilon^2Sin^2\upsilon \in -2ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in -2ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in + \\ 2ihB^2\upsilon^2Cos^2\upsilon \in -ihD^2\upsilon^2Cos^2\upsilon \in -ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in -ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in -ihB^2\upsilon^2Sin^2\upsilon \in \\ +D^2\upsilon^3Sin\,\upsilon \in Cos\,\upsilon \in -BD\upsilon^3Cos^2\upsilon \in +DB\upsilon^3Sin^2\upsilon \in +B^2\upsilon^3Cos\,\upsilon \in Sin\,\upsilon \in \end{array}\right\}$$

Similarly,

$$K'K' = ih\ell^{ih\in}\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + \ell^{ih\in}\left(-D\upsilon Sin\,\upsilon \in +B\upsilon Cos\,\upsilon \in\right)\times$$
$$ih\ell^{ih\in}\left(DCos\upsilon \in +BSin\,\upsilon \in\right) + \ell^{ih\in}\left(-D\upsilon Sin\,\upsilon \in +B\upsilon Cos\,\upsilon \in\right)$$

$$= \ell^{2ih\in}\left\{\begin{array}{l} -h^2\left(D^2Cos^2\upsilon \in +2BDCos\,\upsilon \in Sin\,\upsilon \in +B^2Sin^2\upsilon \in\right) - ihD^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in + \\ ihBD\upsilon Cos^2\upsilon \in -ihBD\upsilon Sin^2\upsilon \in +ihB^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in -ihD^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in + \\ ihBD\upsilon Cos^2\upsilon \in -ihBD\upsilon Sin^2\upsilon \in +ihB^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in +D^2\upsilon^2Sin^2\upsilon \in - \\ 2BD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in +B^2\upsilon^2Cos^2\upsilon \in \end{array}\right\}$$

Now, $(\tau K'' - \mu K')K' = \tau K''K' - \mu K'K' =$

$$\tau\ell^{2ih\in}\left\{\begin{array}{l} -ih^3\left(D^2Cos^2\upsilon \in +B^2Sin^2\upsilon \in +2BDSin\,\upsilon \in Cos\,\upsilon \in\right) + h^2D^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in \\ -h^2DB\upsilon Cos^2\upsilon \in +h^2BD\upsilon Sin^2\upsilon \in -h^2B^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in \\ +2h^2D^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in +2h^2BD\upsilon Sin^2\upsilon \in -2h^2BD\upsilon Cos^2\upsilon \in \\ -2h^2B^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in +2ihD^2\upsilon^2Sin^2\upsilon \in -2ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in \\ -2ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in +2ihB^2\upsilon^2Cos^2\upsilon \in -ihD^2\upsilon^2Cos^2\upsilon \in -ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in \\ -ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in -ihB^2\upsilon^2Sin^2\upsilon \in +D^2\upsilon^3Sin\,\upsilon \in Cos\,\upsilon \in -BD\upsilon^3Cos^2\upsilon \in \\ +DB\upsilon^3Sin^2\upsilon \in +B^2\upsilon^3Cos\,\upsilon \in Sin\,\upsilon \in \end{array}\right\}$$

$$\mu\ell^{2ih\in}\left\{\begin{array}{l} -h^2\left(D^2Cos^2\upsilon \in +2BDCos\,\upsilon \in Sin\,\upsilon \in +B^2Sin^2\upsilon \in\right) - ihD^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in + \\ ihBD\upsilon Cos^2\upsilon \in -ihBD\upsilon Sin^2\upsilon \in +ihB^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in -ihD^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in + \\ ihBD\upsilon Cos^2\upsilon \in -ihBD\upsilon Sin^2\upsilon \in +ihB^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in +D^2\upsilon^2Sin^2\upsilon \in - \\ 2BD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in +B^2\upsilon^2Cos^2\upsilon \in \end{array}\right\}$$

With this, equation (11) becomes

$$sV_1^{iv} - \rho V_1^{iii} + (\delta - g)V_1^{ii} + \sigma V_1^{i} =$$

$$\tau\ell^{2ih\in}\left\{\begin{array}{l} -ih^3\left(D^2Cos^2\upsilon \in +B^2Sin^2\upsilon \in +2BDSin\,\upsilon \in Cos\,\upsilon \in\right) + h^2D^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in \\ -h^2DB\upsilon Cos^2\upsilon \in +h^2BD\upsilon Sin^2\upsilon \in -h^2B^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in \\ +2h^2D^2\upsilon Sin\,\upsilon \in Cos\,\upsilon \in +2h^2BD\upsilon Sin^2\upsilon \in -2h^2BD\upsilon Cos^2\upsilon \in \\ -2h^2B^2\upsilon Cos\,\upsilon \in Sin\,\upsilon \in +2ihD^2\upsilon^2Sin^2\upsilon \in -2ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in \\ -2ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in +2ihB^2\upsilon^2Cos^2\upsilon \in -ihD^2\upsilon^2Cos^2\upsilon \in -ihBD\upsilon^2Cos\,\upsilon \in Sin\,\upsilon \in \\ -ihBD\upsilon^2Sin\,\upsilon \in Cos\,\upsilon \in -ihB^2\upsilon^2Sin^2\upsilon \in +D^2\upsilon^3Sin\,\upsilon \in Cos\,\upsilon \in -BD\upsilon^3Cos^2\upsilon \in \\ +DB\upsilon^3Sin^2\upsilon \in +B^2\upsilon^3Cos\,\upsilon \in Sin\,\upsilon \in \end{array}\right\}$$

$$\mu\ell^{2ih\in}\begin{Bmatrix} -h^2\left(D^2Cos^2\upsilon\in+2BDCos\upsilon\in Sin\upsilon\in+B^2Sin^2\upsilon\in\right)-ihD^2\upsilon Cos\upsilon\in Sin\upsilon\in+ \\ ihBD\upsilon Cos^2\upsilon\in-ihBD\upsilon Sin^2\upsilon\in+ihB^2\upsilon Cos\upsilon\in Sin\upsilon\in-ihD^2\upsilon Cos\upsilon\in Sin\upsilon\in+ \\ ihBD\upsilon Cos^2\upsilon\in-ihBD\upsilon Sin^2\upsilon\in+ihB^2\upsilon Cos\upsilon\in Sin\upsilon\in+D^2\upsilon^2Sin^2\upsilon\in- \\ 2BD\upsilon^2Sin\upsilon\in Cos\upsilon\in+B^2\upsilon^2Cos^2\upsilon\in \end{Bmatrix}\text{Let}\,0\left(\upsilon^3\right),$$

$0\left(\upsilon^2\right)$ and $Cos\upsilon\in Sin\upsilon\in$ be very small so that

$$sV_1^{iv}-\rho V_1^{iii}+(\delta-g)V_1^{ii}+\sigma V_1^{i}=$$

$$\tau\ell^{2ih\in}\left\{ih^3\left(D^2Cos^2\upsilon\in+B^2Sin^2\upsilon\in\right)-3h^2BD\upsilon Cos2\upsilon\in\right\}-\mu\ell^{2ih\in}\begin{Bmatrix} -h^2\left(D^2Cos^2\upsilon\in+B^2Sin^2\upsilon\in\right) \\ +2ihBD\upsilon Cos2\upsilon\in \end{Bmatrix}$$

But $Cos^2\upsilon\in=\dfrac{(Cos2\upsilon\in+1)}{2}$ and $Sin^2\upsilon\in=\dfrac{1-Cos2\upsilon\in}{2}$.

Therefore we have that

$$sV_1^{iv}-\rho V_1^{iii}+(\delta-g)V_1^{ii}+\sigma V_1^{i}=$$

$$\tau\ell^{2ih\in}\left\{ih^3\left(D^2Cos^2\upsilon\in+B^2Sin^2\upsilon\in\right)-3h^2BD\upsilon Cos2\upsilon\in\right\}-\mu\ell^{2ih\in}\begin{Bmatrix} -h^2\left(D^2Cos^2\upsilon\in+B^2Sin^2\upsilon\in\right) \\ +2ihBD\upsilon Cos2\upsilon\in \end{Bmatrix}$$

$$=\frac{\tau\ell^{2ih\in}}{2}\left\{ih^3\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)-6h^2BD\upsilon Cos2\upsilon\in\right\}$$

$$-\frac{\mu\ell^{2ih\in}}{2}\left\{-h^2\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)-4ihBD\upsilon Cos2\upsilon\in\right\}$$

Since the right hand side is a product of exponential function with trigonometric functions, we let our choice of particular integral be

$$R_{PL}=\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$$

With this as a choice of particular solution, we go on and differentiate as below

$$R'_{PL}=2ih\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)+2\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$

$$R''_{PL}=-4h^2\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)+4ih\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$

$$+4ih\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)+4\ell^{ih\in}\left(-M\upsilon^2Cos2\upsilon\in-N\upsilon^2Sin2\upsilon\in\right)$$

$$=-4h^2\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)+8ih\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+NSin2\upsilon\in\right)$$

$$-4\ell^{2ih\in}\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)$$

$$R'''_{PL} = -8ih^3\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)-8h^2\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$
$$-16h^2\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$
$$+16ih\ell^{2ih\in}\left(-M\upsilon^2Cos2\upsilon\in-N\upsilon^2Sin2\upsilon\in\right)-8ih\ell^{2ih\in}\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)$$
$$-8\ell^{2ih\in}\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)$$

$$=-8ih^3\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)-24h^2\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$
$$-24ih\ell^{2ih\in}\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)-8\ell^{2ih\in}\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)$$

$$R^{iv}_{PL} = 16h^4\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)-16ih^3\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$
$$-48ih^3\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)-48h^2\ell^{2ih\in}\left(-M\upsilon^2Cos2\upsilon\in-N\upsilon^2Sin2\upsilon\in\right)$$
$$+48h^2\ell^{2ih\in}\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)-48ih\ell^{2ih\in}\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)$$
$$-16ih\ell^{2ih\in}\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)-64\ell^{2ih\in}\left(-M\upsilon^4Cos2\upsilon\in-N\upsilon^4Sin2\upsilon\in\right)$$

$$=16h^4\ell^{2ih\in}\left(MCos2\upsilon\in+NSin2\upsilon\in\right)-64ih^3\ell^{2ih\in}\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$$
$$+96h^2\ell^{2ih\in}\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)-64ih\ell^{2ih\in}\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)$$
$$+16\ell^{2ih\in}\left(M\upsilon^4Cos2\upsilon\in+N\upsilon^4Sin2\upsilon\in\right)$$

$$sR^{iv}_{PL}-\rho R'''_{PL}-(\delta-g)R''+\sigma R'_{PL}=\left[\tau K''-\mu K'\right]K' \tag{14}$$

On the R.H.S of (14) we have terms in powers of h as follows

$h^3 : \dfrac{i\tau}{2}\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)$

$h^2 : 3\tau BD\upsilon Cos2\upsilon\in+\dfrac{\mu}{2}\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Sin2\upsilon\in\right)$

$h : -2iBD\upsilon Cos2\upsilon\in$

On the L.H.S of (14), we also obtain the following as terms of powers of h

$h^4 : 16s\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$

$h^3 : -64is\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)+8i\rho\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$

$h^2 : 96s\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)+24\rho\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$
$+4(\delta-g)\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$

$h : -64is\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)+24i\rho\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)$
$-8i(\delta-g)\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)+2i\sigma\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$

$h^0 : 16s\left(M\upsilon^4Cos2\upsilon\in+N\upsilon^4Sin2\upsilon\in\right)+8\rho\left(-M\upsilon^3Sin2\upsilon\in+N\upsilon^3Cos2\upsilon\in\right)$
$+4(\delta-g)\left(M\upsilon^2Cos2\upsilon\in+N\upsilon^2Sin2\upsilon\in\right)+2\sigma\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)$

Equating terms of equal powers of h on both sides of (14), we obtain

$16s\left(MCos2\upsilon\in+NSin2\upsilon\in\right)=0$

$$\Rightarrow \quad M=-N\tan 2\upsilon\in \tag{15}$$

$\dfrac{i\tau}{2}\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)=-64is\left(-M\upsilon Sin2\upsilon\in+N\upsilon Cos2\upsilon\in\right)+8i\rho\left(MCos2\upsilon\in+NSin2\upsilon\in\right)$

$\Rightarrow$, using (15),

$\tau\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)=128\left(N\upsilon\tan 2\upsilon\in\sin 2\upsilon\in+N\upsilon\cos 2\upsilon\in\right)+16\rho\left(-NSin2\upsilon\in+NSin2\upsilon\in\right)$

$\tau\left(D^2+B^2+D^2Cos2\upsilon\in-B^2Cos2\upsilon\in\right)=128Ns\upsilon Sec2\upsilon\in$

$$\Rightarrow \qquad N = \frac{\tau\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right)}{128 s\upsilon Sec2\upsilon \in}$$

$$= \frac{\tau\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right) Cos2\upsilon \in}{128 s\upsilon} \qquad (16)$$

Again,

$$3\tau BD\upsilon Cos2\upsilon \in + \frac{\mu}{2}\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right) = 96s\left(M\upsilon^2 Cos2\upsilon \in + NSin2\upsilon \in\right) \quad +$$

$$24\rho\left(-M\upsilon Sin2\upsilon \in + NCos2\upsilon \in\right) + 4(\delta - g)\left(MCos2\upsilon \in + NSin2\upsilon \in\right)$$

$\Rightarrow$, using (16),

$$6\tau BD\upsilon Cos2\upsilon \in + \mu\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right) = 48\rho\left(N\upsilon \tan 2\upsilon \in \sin 2\upsilon \in + N\upsilon Cos2\upsilon \in\right)$$

$$\Rightarrow \qquad 6\tau BD\upsilon Cos2\upsilon \in + \mu\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right) = 48N\upsilon\rho \sec 2\upsilon \in$$

$$\Rightarrow \qquad N = \frac{6\tau BD\upsilon Cos2\upsilon \in + \mu\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right)}{48\upsilon\rho \sec 2\upsilon \in} \qquad (17)$$

But $M = -N \tan 2\upsilon \in$ from (15).

Therefore, using (17) in (15), we get that

$$M = -\frac{6\tau BD\upsilon Cos2\upsilon \in + \mu\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right)}{48\upsilon\rho \sec 2\upsilon \in} \tan 2\upsilon \in$$

$$= -\frac{6\tau BD\upsilon Cos2\upsilon \in + \mu\left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right)}{48\upsilon\rho} Sin2\upsilon \in$$

$$= \left(\frac{\tau H_1 + \mu H_2}{H_3}\right) Sin2\upsilon \in;$$

where,

$H_1 = \tau BD\upsilon Cos2\upsilon \in;$

$H_2 = \left(D^2 + B^2 + D^2 Cos2\upsilon \in - B^2 Cos2\upsilon \in\right)$ and

$H_3 = 48\upsilon\rho$.

Hence,

$$R_{PL} = \ell^{2ih\in}\left\{\left(\frac{\tau H_1 + \mu H_2}{H_3}\right) Sin2\upsilon \in Cos2\upsilon \in + \left(\frac{\tau H_1 + \mu H_2}{H_3}\right) Cos2\upsilon \in Sin2\upsilon \in\right\}$$

$$= 2\ell^{2ih\in}\left\{\left(\frac{\tau H_1 + \mu H_2}{H_3}\right) Sin2\upsilon \in Cos2\upsilon \in\right\}$$

$$= \ell^{2ih\in}\left\{\left(\frac{\tau H_1 + \mu H_2}{H_3}\right) Sin4\upsilon \in\right\}$$

Thus,

$$V_1 = \ell^{2ih\in}\left(C_1 \cos 2\upsilon \in + C_2 \sin \upsilon \in + \frac{\tau H_1 + \mu H_2}{H_3} \sin 4\upsilon \in\right)$$

It is assumed that our V_0 and V_1 will suffice.

Next we solve for the wave speed. To solve for c, the phase speed in the equation below, i.e. equation (9):

$$s\frac{d^4w}{d\in^4}-\rho\frac{d^3w}{d\in^3}+\delta\frac{d^2w}{d\in^2}+\sigma\frac{dw}{d\in} = \left(\tau\frac{d^2w}{d\in^2}-\mu\frac{dw}{d\in}\right)\frac{dw}{d\in}+g\frac{d^2w}{d\in^2}$$

Write (9) as:

$$s\frac{d^4w}{d\in^4}-\rho\frac{d^3w}{d\in^3}+(\delta-g)\frac{d^2w}{d\in^2}+\sigma\frac{dw}{d\in} = \left(\tau\frac{d^2w}{d\in^2}-\mu\frac{dw}{d\in}\right)\frac{dw}{d\in} \tag{18}$$

Recall that $w = A\ell^{ik\in}$ (19) so that into equation (18), we obtain

$$\left(\frac{\lambda c}{r}+\frac{rc^2}{3}\right)k^3 + \left(\frac{3c\lambda\cot\beta}{r^2}-\frac{x\zeta c\cos ec^2\beta}{3}+\frac{2c^2\cot\beta}{3}\right)ik^2$$
$$-\left\{\left(\frac{2c^2\cot^2\beta}{3r}-\frac{c^2}{r}+\frac{6\lambda c\cot^2\beta}{r^3}+2c\zeta(h-x\cot\beta)\cos ec^2\beta\right)-g\right\}k$$
$$+\left(\frac{cx\zeta Co\sec^2\beta}{r^2}+\frac{7c\lambda\cot^3\beta}{r^4}-4c\zeta\left[h-\tfrac{1}{2}x\cot\beta\right]\cot\beta\cos ec^2\beta\right)i$$
$$= \left(\frac{-c^2}{r^2}k^2-ik\frac{c^2\cot\beta}{r^3}\right)Ai\ell^{ik\in} \tag{20}$$

Arrange terms of (20) in powers of c, i.e. c^2, c and c^0 so as to obtain a quadratic equation in c as below:

$$\left(\frac{rk^3}{3}+\frac{2ik^2\cot\beta}{3}-\frac{2k\cot^2\beta}{3r}+\frac{k}{r}+\left[\frac{k^2}{r^2}+\frac{ik\cot\beta}{r^3}\right]Ai\ell^{ik\in}\right)c^2$$
$$+\left(\begin{array}{l}\frac{\lambda k^3}{r}+\frac{3i\lambda k^2\cot\beta}{r^2}-\frac{ix\zeta k^2\cos ec^2\beta}{3}+2\zeta k[h-x\cot\beta]\cos ec^2\beta\\ +\frac{ix\zeta\cos ec^2\beta}{r^2}+\frac{7i\lambda\cot^3\beta}{r^4}-4i\zeta\left[h-\tfrac{1}{2}x\cot\beta\right]\cot\beta\cos ec^2\beta\\ +\frac{6\lambda k\cot^2\beta}{r^3}\end{array}\right)c - gk = 0 \tag{21}$$

There are two cases here namely: when the wave is non-dispersive, the implication here is that $\lambda = 0$. And secondly when the wave is dispersive which implies that $\lambda \neq 0$

Case 1: $\lambda = 0$

If $\lambda = 0$, equation (21)

becomes $\left(\frac{rk^3}{3}+\frac{2ik^2\cot\beta}{3}-\frac{2k\cot^2\beta}{3r}+\frac{k}{r}+\left[\frac{k^2}{r^2}+\frac{ik\cot\beta}{r^3}\right]Ai\ell^{ik\in}\right)c^2 +$

$$\left(\begin{array}{l}-\frac{ix\zeta k^2\cos ec^2\beta}{3}+2\zeta k[h-x\cot\beta]\cos ec^2\beta+\frac{ix\zeta\cos ec^2\beta}{r^2}\\ -4i\zeta\left[h-\tfrac{1}{2}x\cot\beta\right]\cot\beta\cos ec^2\beta\end{array}\right)c - gk = 0 \tag{22}$$

Let $\frac{rk^3}{3}+\frac{2ik^2\cot\beta}{3}-\frac{2k\cot^2\beta}{3r}+\frac{k}{r}+\left[\frac{k^2}{r^2}+\frac{ik\cot\beta}{r^3}\right]Ai\ell^{ik\in}$ = m

$$-\frac{ix\zeta k^2\cos ec^2\beta}{3}+2\zeta k\left[h-x\cot\beta\right]\cos ec^2\beta+\frac{ix\zeta\cos ec^2\beta}{r^2} = l$$
$$-4i\zeta\left[h-\tfrac{1}{2}x\cot\beta\right]\cot\beta\cos ec^2\beta$$

and - gk = n.

with these, equation (23) becomes

$mc^2+lc+n=0$ so that

$$c=\frac{l\pm\sqrt{l^2-4mn}}{2m} \tag{23}$$

Case 2: $\lambda\neq 0$

For this case, we refer to equation (21) and represent the following argument as below:

$$\frac{rk^3}{3}+\frac{2ik^2\cot\beta}{3}-\frac{2k\cot^2\beta}{3r}+\frac{k}{r}+\left[\frac{k^2}{r^2}+\frac{ik\cot\beta}{r^3}\right]Ai\ell^{ik\in} = P$$

$$-\frac{ix\zeta k^2\cos ec^2\beta}{3}+2\zeta k\left[h-x\cot\beta\right]\cos ec^2\beta+\frac{ix\zeta\cos ec^2\beta}{r^2} = Q$$
$$-4i\zeta\left[h-\tfrac{1}{2}x\cot\beta\right]\cot\beta\cos ec^2\beta$$

gk = R

With these equation (21) becomes

$Pc^2+Qc+R=0$ and simplifies to

$$c=\frac{-Q\pm\sqrt{Q^2-4PR}}{P} \tag{24}$$

Analysis of results

Having used asymptotic method in obtaining our V_0 and V_1 which are enough to interpret the wave equation results, the following graphs will help in giving clearer pictures of the study made here. We need to be reminded that the given contour or sag to be filled is considered from certain point of view. For instance, the angle $\beta = \theta + \xi t, t\geq 0$, is measured in radians and are considered at different points. The distance from the left hand side is considered negative, that is, -x, while from the right hand side it is considered positive, that is +x. The rate at which the flow progresses is also considered. The velocity of the flow in terms of the associated wave pattern is also discussed in each case.

All these cases discussed helped in giving us the picture of what obtains assuming the case in each approach holds. And naturally looking at these facts, one will not hesitate to agree with us that we approached it from the realistic point of view unlike the other cases already referred to in our work.

In making this simple enough, we considered basically two cases thus:
at a fixed point say $x = 2$ at different times, dispersion rate and angles, and secondly; at different distances, dispersion rates and angles.

Each case is discussed in this work with the associated graph to enable us to comprehend the result obtained

To enable us use MATLAB 7.5 in plotting some of the graphs, appropriate and correct values for the equation parameters were chosen. As a case where the contoured region is to be filled over time, we choose some of these values for some of the parameters used in the equation as follow:

$g = 9.8$, $A = 19$, $h = 0$, $k = 0.02$, x the horizontal distance is in varied nature over time.

We considered $\lambda = 0$, $D = C_1 = C_2 = 1$, since they are constants.

Here we shall consider two cases, viz: fixed time over varying distance, and fixed distance over different times.

For case 1, our interest is to study the velocity of the wave at a particular point in time over different points and see what happens to the velocity of the flow at these points. For this case we fixed t at $t = 20$ seconds, i.e. 20 seconds of the flow over our point of interest, which here is the sag of interest. Different results for different values of θ are shown in the figures (8) to (10).

What is seen in fig (8) is an increase in the velocity of wave as water falls into the contoured region. It is also seen that the velocity decreases as the flow gets to the centre of the region. The physical meaning of this is that at the point of entry of water into the sag which is a fall, its velocity increases because of the fall. As the flows goes into the sag, it deposits sand and other debris, i.e. fills the contour which is seen in the decrease of the velocity as it approaches the centre where the filling occurs. At that point the velocity remains almost the same before slowing down further due to ascent in the other half of the region.

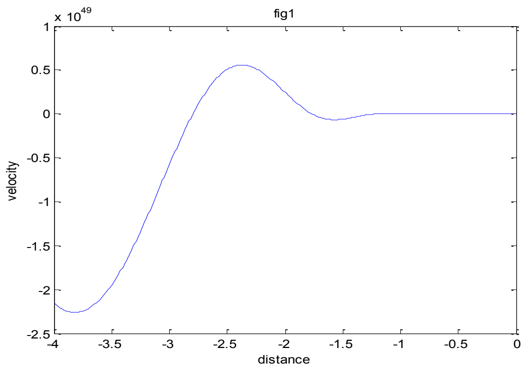

Fig. 8

In fig (9) the velocity increases rapidly for a short distance before becoming steady. This is because the angle θ has gone up as a result of the filling of the contoured region by deposits carried along by the flow, the steady velocity as seen from x = -3 shows that the v alley at such an angle is almost filled up.

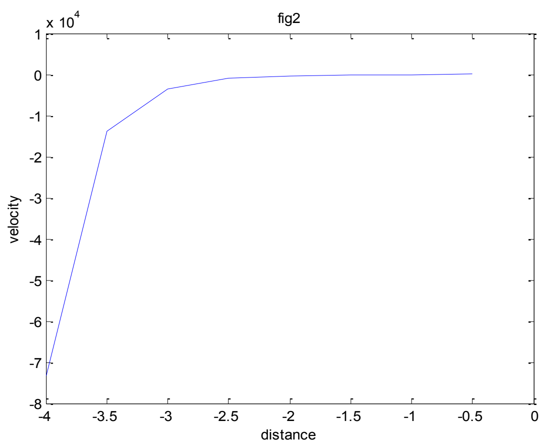

Fig. 9

Fig (10) shows a case of digging which in other words mean that the angle of flow θ, is small showing a steeper descent than the other cases shown above. This is why the velocity of the flow keeps increasing up to x = -1 before decreasing. This means that the area filled is small despite how long the flow has gone. The physical interpretation is that rather than filling, the flow digs causing the angle to increase leading to a case of steeper descent.

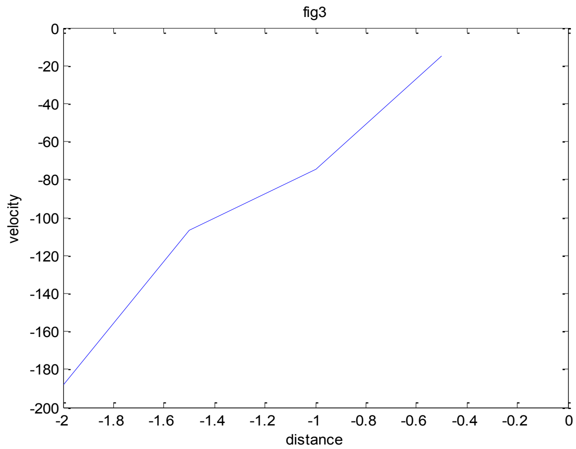

Fig. 10

Next we shall take the second case where we want to study the effect of the flow at a particular point of the contoured region over different period of time. Here the study is done at $x = 2$ and $x = -2$, with varying time respectively. This simulation is done considering changes in the angle, θ, of the flow.

In fig (12), we see that the velocity of the wave is noticed for a short period of time and that it is not turbulent but gentle. This is owing to the fact that the angle is large which means that the sag is almost getting filled up. This is why the wave is gentle. Unlike what is seen in fig (12) that of fig (13) is different in the sense that the wave is more noticed here. This is because we chose a smaller angle which means a steeper descent and ascent as well, for the flow. This is why the velocity wave is more noticed between $t = 2$ and $t = 7$. The time interval agrees with the physical observation. This is so because the wave forms immediately as the flow gets to the point of interests and subsequently dies out as the flow passes the point of interest. This is also guaranteed by steepness of the contour and subsequent rough region.

At $\theta = 0.40$, we noticed a different thing. This is shown in fig (14) where it is seen that at the time of departure of water from the valley a more turbulent wave is experienced. This is caused by the rough surface of the valley and ascent up the valley. The non-wave like motion seen from $t = 1.2$ to $t = 5$ shows that once the flow passes the point of exit its flow normalizes until it gets to a point where it will come to the level region which reflects as a wave form at $t = 5$ up to $t = 10$ on the graph.

What is seen in fig (15) to (18) are what happens as the flow enters the sag. The constant wave of the velocity seen here tells that there is no wave form before the sag, compared to what is seen as the flow goes into the sag. As the flow goes into the sag, the wave is produced resulting from fall into the sag and increase in velocity of the wave.

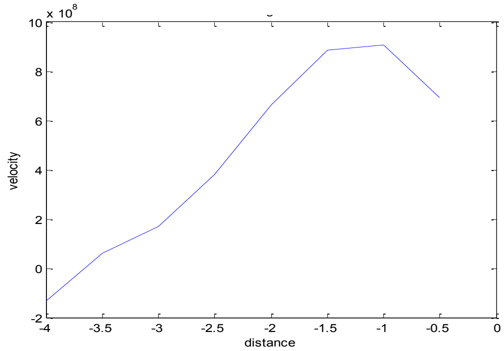

Fig.11

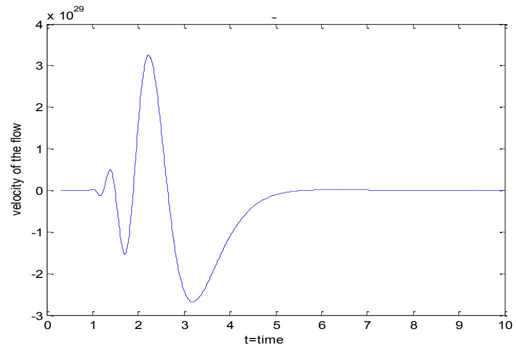

Fig.12

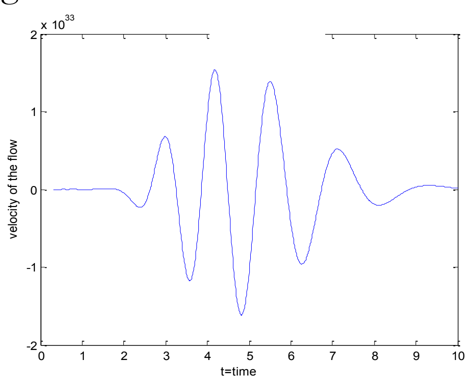

Fig. 13

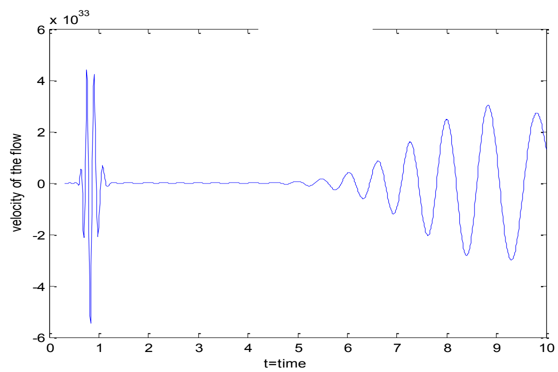

Fig. 14

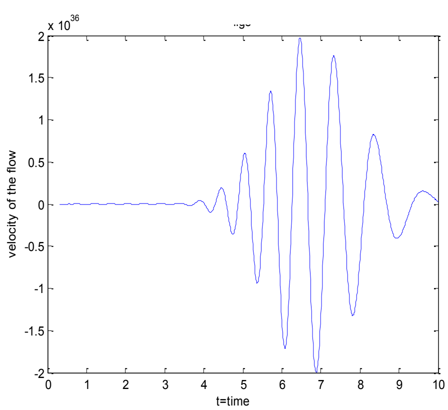

Fig. 15

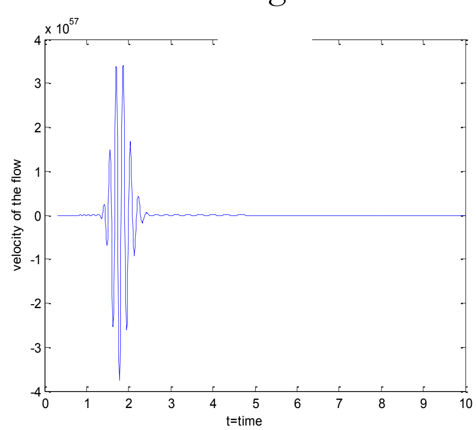

Fig. 16

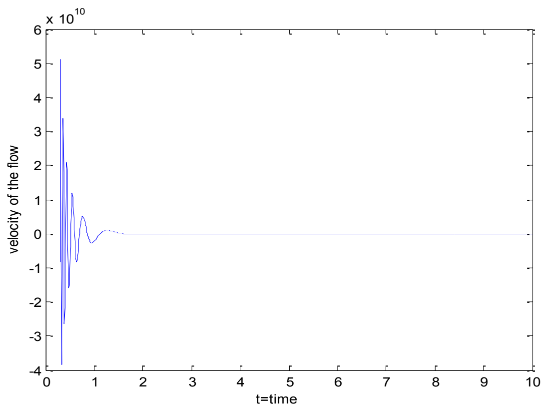

Fig. 17

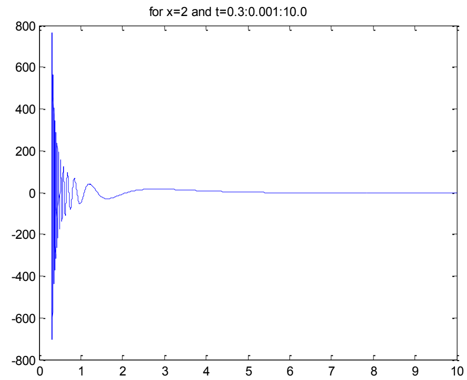

Fig. 18

Having seen all these, we may say in general that shallow water flow over rough bottom topography has such features that are not easily seen unless we take time to carry out this kind of studies. In [11], the case for sloping bottom was considered, which we said is not realistic, and this even though it showed the point of singularity in the wave profile, it did not for sure say the location of such singularity. We can see clearly in our case here that the point of singularity occurred at the point x = 0, that is , at the middle point in the width of the particular contoured region. In this sense therefore, we expect to have as many singular points as there are contoured points in the channel of flow.

Finally, we say that if the channel of flow of the shallow water is rough, any good study of the flow parameters like the velocity and the wave profile must consider the wave number, the height of water at the undisturbed surface and the wave length as the major determining factors of his results.

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