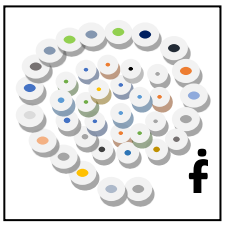

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Existence and uniqueness analysis of Atangana-Baleanu fractional order network model of Noro virus

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Abstract

This manuscript is concerned to the study of noro virus model and its complications with the Atangana-Baleanu fractional order derivative which is a new approach for such type of biological models. Further for the corresponding results, existence and uniqueness of the solution is examined by engaging the fixed point theory and Pichard Lidelof approach

Keywords: $SEII_tR$ Noro-virus model; Atangana-Baleanue fractional derivative; Picard-Linderlof approach; fixed point theory

1 Introduction

Gastroenteritis is still ranked second among the childhood killer diseases globally [19]. the greater burden of this disease can be observed in developing countries especially sub-Saharan Africa [26]. Decades ago, rotavirus was the predominant pathogen implicated in most cases of diarrhoea among children below five years of age but recent studies have shown that human norovirus (HuNoVs) has become the major aetiological agent responsible for viral gastroenteritis among children below five years of age, replacing rotavirus subsequent to the introduction of rotavirus vaccine in most countries [7].

Globally, an estimated 677 million episodes of HuNoV associated gastroenteritis and 200,000 mortalities have been recorded annually (Pires et al., 2015). Noroviruses belong to the caliciviridae family and are non-enveloped, single-stranded RNA viruses that are extremely infectious with just a few virions capable of causing gastroenteritis among infected individuals [2]. They are classified into ten genogroup (GI-GX) based on genetic similarity but most cases of acute gastroenteritis in humans were caused by two genogroups (GI and GII) while other genogroups can cause infections in the feline, canine, bovine etc. population [8,23,24]. Further genotypic classification based on the partial sequence of RNA-dependent RNA polymerase (RdRp) in ORF1 and complete sequence of the capsid in ORF2 led to the identification of 52 (49 accepted genotypes and 3 tentative genotypes) genotypes within the genogroups [8]. Noroviruses are formidable pathogens due to their ability to undergo genetic recombination and mutation leading to the emergence of a new strain every 2-3 years [16] .

HuNoVs affects all ages causing self limiting infections in healthy individuals with an incubation period of 12-48h [20]. But causes severe disease among children, the elderly and immunocompromised persons often accompanied with symptoms including: diarrhoea, stomach

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cramps, profuse vomiting, leading to loss of fluids that might result to severe dehydration [20]. Specifically, HuNoVs can result in chronic gastroenteritis among immunocompromised patients e.g. transplant patients with infection lasting from several weeks to a year (Brown et al., 2017). Despite causing gastroenteritis in some individuals, certain population of individuals $(5\% - 30\%)$ infected with HuNoVs become asymptomatic and several reasons have been provided for this observation including: lack of histo-blood group antigens (HBGA) absent in this population, patients recovering from a previous HuNoV infection and are still shedding the virions in their stool or passage of non-replicating virus through the human gastrointestinal tract [15].Norovirus is transmitted mainly through the fecal-oral route and close contact with infected persons. The virus has high stability in the environment and survives on formites for long periods [19].

Norovirus is responsible for sporadic outbreaks in enclosed spaces such cruise ships, military camps, and long term care facilities [21]. It is the primary pathogen implicated in most cases of food borne gastroenteritis due to several outbreaks linked to consumption of partially cooked, seafoods or ready to eat foods contaminated by infected food handlers [17,21]. Despite, the outbreaks and huge burden of disease linked to HuNoV, efforts to develop a vaccine for the prevention of the infection are underway although there is still limiting issues such as the lack of robust cell lines to culture and study these viruses. there is the challenge of lack of data especially from developing countries on the burden of the infections due to poor medical diagnostic facilities for viral pathogens [18].

The insight that the transmission dynamics of epidemic disease can be formulated using mathematical language dates back to 1766 when Daniel Bernouli published a paper where he described the effect of smallpox variolation on life expectancy [10]. Other early mathematical models in epidemiology where introduced in 1927 when Kermack and Mckendrick published a series of papers that described the dynamic of disease transmission in term of system of differential equation [12]. Epidemiological mathematical models provide several aspect for understanding the dynamic of the spread of an epidemiv and suggestion of effective control strategies. A large number of studies and publications have been made on modeling and analysis of epidemiology disease. However, the major of the studies were restricted to integer order differential equations.

Resent studies has revealed that fractional differential equations being a branch of mathematical analysis that studies calculus of derivatives and integral of arbitrary order can be used to model several phenomenon in different fields including epidemiology [10]. The non local property in the sense that the nth derivatives of a function g(x) at x is a local property of a function only when n is an integer, whereas a non integer fraction derivative of g(x) at x=b depends on all values of g(x) including those far away from b is the advantage of describing mathematical models using fractional derivatives. In other words, in the case of fractional order derivatives since it involves derivatives and integral of arbitrary real and complex orders,the future state depends on the present and previous.

Using fractional derivatives in modeling a dynamical system helps to describe the hereditary properties and efficacy of the memory of essential features in many biological mechanism [14].Many authors contributed to the development of fractional calculus starting from 1695 when L'Hospital asked Leibiz what if the order of a derivative is n=1/2? Other contributor include Euler 1730 and Langrange in 1812. other prominent name in fractional derivatives and integral include Reimann Liouvill,Hadamard, Caputo, Caputo-Fabrizio and recently Atangana- Baleanu. Atangana identified previous mistakes and advanced corrections to the use of fractional calculus related to fundamental theorem of calculus. Baleanu et al a fractional mathematical model for tumor-immune survillance mechanism and investigated the effect of chemotherapy on the model. Sene studied a fractional diffusion equation in the context of the fractional operator. Kolade and Owolabi performed analysis and numerical simulation of a system using a two-step family of Adams- Bashforth method to approximate the A-B fractional derivatives [28].

A-B fractional derivatives operator involving the Mittage -Leffler kernel is used to analyse SEIRA mathematical model [25]. The above surveyed works shows that fractional derivatives

have many applications in mathematical modeling and analysis of real life phenomena .The recent A-B fractional develope has earned popularity and respect due to its immense application in biological, physical,medical engerinnering and several other nonlinear analysis.

Motivated by the above argument , we study in the this paper $SEII_TR$ (susceptible-esposed-infected-infected on treatment-recovered cases).The authors have structured the rest of this paper as follows. In section 2 , we described and formulated the rota virus model with fractional order. In section 3 we established the existence and uniqueness of solution by exerting the fixed point theory and the Pichard Lindelof approch. In section 4 positivity and boundedness of solution in the sense of Atangana - Baleanu operator was shown.

2 Model description and formulation

In this section , we present the Atangana- Baleanu fractional derivative representation of HBV mathematical model. The basic definitions of Atangana- Baleanu is given by :

Definition 1 : Let $g \in C^1(a,b), a < b$, be a function, and let$\alpha \in [0,1]$. The Atangana- Baleanu (AB) fractional derivative in Caputo type of orderα given by [23,24]

$$D_t^{\alpha} g(t) = \frac{F(\alpha)}{1-\alpha} \int_0^t \frac{dg}{dk} E_{\alpha}\Big[-\frac{\alpha}{1-\alpha}(t-k)^{\alpha}\Big] dk, \tag{1}$$

where $F(\alpha)$ is the normalization function given by $F(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$, characterized by $F(0) = F(1) = 1$,and the Mittag-Leffler function $E_{\alpha}(z)$ with C the set of complex number given by

$$E_{\alpha} = \sum_{\beta=0}^{\infty} \frac{z^{\beta}}{\Gamma(1+\alpha\beta)}, \alpha, z \in C, R(\alpha) > 0.$$

Definition 2 : The AB fractional integral of the function $g \in C^1(a,b), a < b$, is given by [25,25,33]

$$I_t^{\alpha} g(t) = \frac{1-\alpha}{F(\alpha)} g(t) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_a^t g(k)(t-k)^{\alpha-1} dk. \tag{2}$$

Lemma 1 ([25]) The AB fractional derivative and AB fractional integral of the function $g \in C^1(a,b)$ satisfies the Newton-Leibniz equality

$$I_t^{\alpha}(D_t^{\alpha}) = g(t) - g(a).$$

Lemma 2 ([26,27]) For two functions f,g $\in \beta^1(a,b), b > a$, the AB fractional derivative satisfies the following inequality:

$$||D_t^{\alpha} f(t) - D_t^{\alpha} g(t)|| \leq \beta ||f(t) - g(t)||.$$

The mathematical model with integer order used in this study is expressed by the equation:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \beta - \mu S(t) - \alpha_1 C(N)(I(t) + Q(t)S(t),\\
\frac{dE(t)}{dt} &= \alpha_1 C(N)(I(t) + Q(t)S(t) - \lambda_2 E(t),\\
\frac{dI(t)}{dt} &= \lambda_2 I(t) - (\mu + \alpha_3 + \lambda_1)Q(t),\\
\frac{dI_T(t)}{dt} &= (1-\rho)\alpha_3 I(t) - (\mu + \lambda_2)I_T(t),\\
\frac{dQ(t)}{dt} &= \rho\alpha_3 I(t) - \Psi Q(t),\\
\frac{dR(t)}{dt} &= \alpha_4 I_T(t) - (\mu + \alpha_5)R(t),
\end{aligned} \tag{3}$$

where S(t) represent number of individuals susceptible to noro virus,E(t) represent exposed individuals,I(t) represent individuals infected with noro virus,I_t represent infected individual on treatment, Q(t) is contaminated surface , R(t) represent recovered individuals.The recruitment rate of the population is denoted by β,μ is death rate due to disease, λ_1, $\lambda_2, ..., \lambda_5$ is movement rate from one compartment to another, $\delta_1, ..., delta_4$ is rate at which individuals die due to natural cause. $C(N) = \dfrac{\alpha N}{1 + bN}$ derived in [28] is the saturating contact rate, α,b are positive contacts and the total population is given by: $N(t) = S(t) + E(t) + I(t) + T(t) + R(t)$. The natural death rate term is μN.

In the absence of disease the differential equation of total population is $\dfrac{dN}{dt} = \beta - \alpha_3 N$ and thus $\lim_{x\to\infty} N(t) = \dfrac{\beta}{mu}$ and it is the carrying capacity of the demographic structure under consideration. Observed that$C(N) \approx \alpha N$ for small N and $C(N) \approx \dfrac{\alpha}{b}$ for large N.We assume here also that a patient that recovered from noro virus is susceptible to the virus as he/she has no permanent immunity.

Thus the AB fractional order mathematical model taking into account the assumptions, the saturating contact rate is giving by the system of differential equations:

$$\begin{aligned}
{}_0^{ABC}D_t^{\alpha} S(t) &= G_1(t, S),\\
{}_0^{ABC}D_t^{\alpha} E(t) &= G_2(t, E),\\
{}_0^{ABC}D_t^{\alpha} I(t) &= G_3(t, I),\\
{}_0^{ABC}D_t^{\alpha} I_T(t) &= G_4(t, I_T),\\
{}_0^{ABC}D_t^{\alpha} Q(t) &= G_5(t, Q),\\
{}_0^{ABC}D_t^{\alpha} R(t) &= G_6(t, R),
\end{aligned} \tag{4}$$

where the kernels are given by

$$\begin{aligned}
G_1(t, S) &= \beta - \mu S(t) - b_0 \frac{(I(t) + Q(t))}{k(N)} S(t),\\
G_2(t, E) &= b_0 \frac{(I(t) + Q(t))}{k(N)} S(t) - \lambda_2 E(t),\\
G_3(t, I) &= \lambda_2 I(t) - (\mu + \alpha_3 + \lambda_1)Q(t),\\
G_4(t, I_T) &= (1-\rho)\alpha_3 I(t) - (\mu + \lambda_2)I_T(t),\\
G_5(t, Q) &= \rho\alpha_3 I(t) - \Psi Q(t),\\
G_6(t, R) &= \alpha_4 I_T(t) - (\mu + \alpha_5)R(t),
\end{aligned}$$

$S(0) = S_0, E(0) = E_0, I(0) = I_0, I_T(0) = I_T(0), Q(0) = Q_0, R(0) = R_0$ are the initial conditions and $b_0 = \alpha\alpha_1$, and α_1 is the probability per unit time of transmitting the infection between

individuals taking part in contact or a contaminated surface, $K(N) = 1 + BN$. In the presence of endemic, $\frac{dN}{dt} = \beta - \mu N - \alpha_3$ and this tells us that the population size is not constant.

3 Existence and uniqueness of solutions

We shall deliberate the existence and uniqueness of the solution of the fractional order model (4). To demonstrate the existence and uniqueness of the solution to (4) we engage the famous theorem referred to as the Banach fixed point theorem. For and extensive study of the fixed point and contraction, we refer the reader to [21] and the reference therein.

To show the existence and uniqueness of solution we proceed as follows. Applying the AB fractional integral defined in (2) to model (4), we have:

$$\begin{aligned}
S(t) - S(0) &= \frac{1-\alpha}{F(\alpha)} G_1(t,S) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_1(k,S)(t-k)^{\alpha-1} dk, \\
E(t) - E(0) &= \frac{1-\alpha}{F(\alpha)} G_2(t,E) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_2(k,E)(t-k)^{\alpha-1} dk, \\
I(t) - I(0) &= \frac{1-\alpha}{F(\alpha)} G_3(t,I) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_3(k,I)(t-k)^{\alpha-1} dk, \\
I_T(t) - I_T(0) &= \frac{1-\alpha}{F(\alpha)} G_4(t,I_T) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_4(k,I_T)(t-k)^{\alpha-1} dk, \\
Q(t) - Q(0) &= \frac{1-\alpha}{F(\alpha)} G_5(t,Q) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_5(k,Q)(t-k)^{\alpha-1} dk, \\
R(t) - R(0) &= \frac{1-\alpha}{F(\alpha)} G_6(t,R) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_6(k,R)(t-k)^{\alpha-1} dk.
\end{aligned} \tag{5}$$

The set $B = H(J)xH(J)x...andH(J) = C[0,T]$ is a Banach space of real value continuous functions defined on the interval J = [0,T] with the corresponding norm defined by:
$||(S,E,I,I_T,Q,R)|| = ||S|| + ||E|| + ||I|| + ||I_T|| + ||Q|| + ||R||$, where, $||S|| = \sup_{t\in J} |S(t)|$, $||E|| = \sup_{t\in J} |E(t)|$, $||I|| = \sup_{t\in J} |I(t)|$, $||I_T|| = \sup_{t\in J} |I_T(t)|$, $||Q|| = \sup_{t\in J} |Q(t)|$, $||R|| = \sup_{t\in J} |R(t)|$.

Theorem 1 (Lipschits Condition and Contraction)
For each of the kernels, $G_1(t,S), G_2(t,P), ..., G_6(t,Y)$ in (2) there exist $L_i > 0, i = 1,2,3,4,5,6$ such that

$$\begin{aligned}
||G_1(t,S) - G_1(t,S_1)|| &\leq L_1||S(t) - S_1(t)||, \\
||G_2(t,E) - G_1(t,P_2)|| &\leq L_2||E(t) - E_1(t)||, \\
||G_3(t,Q) - G_1(t,I_1)|| &\leq L_3||I(t) - I_1(t)||, \\
||G_4(t,I_T) - G_1(t,W_1)|| &\leq L_4||I_T(t) - I_{T1}(t)||, \\
||G_5(t,Q) - G_5(t,Q_1)|| &\leq L_5||Q(t) - Q_1(t)||, \\
||G_6(t,R) - G_6(t,R_1)|| &\leq L_6||R(t) - R_1(t)||,
\end{aligned}$$

are contraction for $0 \leq L_i \leq 1 \forall i = 1,2,3,4,5,6$.
Proof.

$$\begin{aligned}
||G_1(t,S) - G_1(t,S_1)|| &= ||\beta - \mu S(t) - b_0 \frac{(I(t)+Q(t))}{k(N)} S(t) - (\beta - \mu S_1(t) - b_0 \frac{(I(t)+Q(t))}{k(N)} S_1(t))|| \\
&\leq \left(b_0 \frac{m_2 + m_1}{K(N)} \right) ||(S_1(t) - S(t))|| \\
&\leq L_1 ||(S_1(t) - S(t))||,
\end{aligned}$$

where $L_1 = \left(b_0 \frac{m_3 + m_5}{K(N)}\right) + \mu$ $||S||$ is $\sup_{t\in J} = M_1$,$||E||$ is $\sup_{t\in J} = M_2$,$||I||$ is $\sup_{t\in J} = M_3$,$||I_T||$ is $\sup_{t\in J} = M_4$,$||Q||$ is $\sup_{t\in J} = M_5$,$||R||$ is $\sup_{t\in J} = M_6$.

In a similar way, we can show the existence of $L_1 = 2, 3, 4, 5, 6$, and a contraction principle for$G_2(t, E), G_3(t, I), G_4(t, I_T), G_5(t, Q), G_6(t, R), 0 \leq L_i < 1$.

We now define the recursive form of (5) for $t = t_n, n = 1, 2, 3, ...$

$$\begin{aligned}
S_n(t) &= \frac{1-\alpha}{F(\alpha)} G_1(t, S_{n-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_1(k, S_{n-1})(t-k)^{\alpha-1} dk, \\
E_n(t) &= \frac{1-\alpha}{F(\alpha)} G_2(t, E_{n-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_2(k, E_{n-1})(t-k)^{\alpha-1} dk, \\
I_n(t) &= \frac{1-\alpha}{F(\alpha)} G_3(t, I_{n-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_3(k, I_{n-1})(t-k)^{\alpha-1} dk, \\
I_{Tn}(t) &= \frac{1-\alpha}{F(\alpha)} G_4(t, I_{Tn-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_4(k, I_{Tn-1})(t-k)^{\alpha-1} dk, \\
Q_n(t) &= \frac{1-\alpha}{F(\alpha)} G_5(t, Q_{n-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_5(k, Q_{n-1})(t-k)^{\alpha-1} dk, \\
R_n(t) &= \frac{1-\alpha}{F(\alpha)} G_6(t, R_{n-1}) + \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t G_6(k, R_{n-1})(t-k)^{\alpha-1} dk.
\end{aligned} \tag{6}$$

The difference between successive terms in (6) are expressed as follows:

$$\begin{aligned}
A_{1n}(t) &= S_n - S_{n-1} = \frac{1-\alpha}{F(\alpha)}(G_1(t, S_{n-1}) - G_1(t, S_{n-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_1(t, S_{n-1}) - G_1(t, S_{n-1}))(t-k)^{\alpha-1} dk, \\
A_{2n}(t) &= E_n - E_{n-1} = \frac{1-\alpha}{F(\alpha)}(G_2(t, E_{n-1}) - G_2(t, E_{n-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_2(t, P_{n-1}) - G_2(t, P_{n-1}))(t-k)^{\alpha-1} dk, \\
A_{3n}(t) &= I_n - I_{n-1} = \frac{1-\alpha}{F(\alpha)}(G_3(t, I_{n-1}) - G_3(t, I_{n-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_3(t, I_{n-1}) - G_3(t, I_{n-1}))(t-k)^{\alpha-1} dk, \\
A_{4n}(t) &= I_{Tn} - I_{Tn-1} = \frac{1-\alpha}{F(\alpha)}(G_4(t, I_{Tn-1}) - G_4(t, I_{Tn-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_4(t, I_{Tn-1}) - G_4(t, I_{Tn-1}))(t-k)^{\alpha-1} dk, \\
A_{5n}(t) &= Q_n - Q_{n-1} = \frac{1-\alpha}{F(\alpha)}(G_5(t, Q_{n-1}) - G_5(t, Q_{n-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_5(t, Q_{n-1}) - G_5(t, Q_{n-1}))(t-k)^{\alpha-1} dk, \\
A_{6n}(t) &= R_n - R_{n-1} = \frac{1-\alpha}{F(\alpha)}(G_6(t, R_{n-1}) - G_6(t, R_{n-2})) + \\
&\quad \frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t (G_6(t, R_{n-1}) - G_6(t, R_{n-1}))(t-k)^{\alpha-1} dk.
\end{aligned} \tag{7}$$

Taking the norm of both sides of (5), we have

$$\begin{aligned}
||A_{1n}(t)|| &= ||S_n(t) - S_{n-1}(t)||\frac{1-\alpha}{F(\alpha)}||(G_1(t,S_{n-1}) - G_1(t,S_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_1(t,S_{n-1}) - G_1(t,S_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{2n}(t)|| &= ||PE_n - E_{n-1}|| = \frac{1-\alpha}{F(\alpha)}||(G_2(t,E_{n-1}) - G_2(t,E_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_2(t,E_{n-1}) - G_2(t,E_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{3n}(t)|| &= ||I_n - I_{n-1}|| = \frac{1-\alpha}{F(\alpha)}||(G_3(t,I_{n-1}) - G_3(t,I_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_3(t,I_{n-1}) - G_3(t,I_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{4n}(t)|| &= ||I_{Tn} - I_{Tn-1}|| = \frac{1-\alpha}{F(\alpha)}||(G_4(t,I_{Tn-1}) - G_4(t,I_{Tn-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_4(t,W_{n-1}) - G_4(t,W_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{5n}(t)|| &= ||Q_n - Q_{n-1}|| = \frac{1-\alpha}{F(\alpha)}||(G_5(t,Q_{n-1}) - G_5(t,Q_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_5(t,Q_{n-1}) - G_5(t,Q_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{6n}(t)|| &= ||R_n - R_{n-1}|| = \frac{1-\alpha}{F(\alpha)}||(G_6(t,R_{n-1}) - G_6(t,R_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_6(t,R_{n-1}) - G_6(t,R_{n-1}))||(t-k)^{\alpha-1}dk.
\end{aligned} \tag{8}$$

The first equation in (8) is reduced to the following expression

$$\begin{aligned}
||A_{1n}(t)|| &= ||S_n(t) - S_{n-1}(t)|| \leq \frac{1-\alpha}{F(\alpha)}||(G_1(t,S_{n-1}) - G_1(t,S_{n-2}))||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t ||(G_1(t,S_{n-1}) - G_1(t,S_{n-1}))||(t-k)^{\alpha-1}dk,\\
||A_{1n}(t)|| &\leq \frac{1-\alpha}{F(\alpha)}L_1||S_{n-1}(t) - S_{n-2}(t)||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}L_1\int_0^\tau ||S_{n-1}(t) - S_{n-2}(t)||(t-k)^{\alpha-1}dk,\\
||A_{1n}(t)|| &\leq \frac{1-\alpha}{F(\alpha)}L_1||S_{n-1}(t) - S_{n-2}(t)||+\\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)}L_1||S_{n-1}(t) - S_{n-2}(t)||\int_0^t (t-k)^{\alpha-1}dk,\\
||A_{1n}(t)|| &\leq L_1||S_{n-1}(t) - S_{n-2}(t)||\left|\frac{1-\alpha}{F(\alpha)} + \frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1}dk\right|\\
||A_{1n}(t)|| &\leq L_1||A_{1(n-1)}||\left|\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right|.
\end{aligned}$$

Therefore,

$$||A_{1n}(t)|| \leq L_1\left|\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}.\right| ||A_{1(n-1)}|| \tag{9}$$

In a similarly way , we reduced the remaining expression of (8) to the form:

$$\begin{aligned}
||A_{2n}(t)|| &\leq L_2 \left| \frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right| ||A_{2(n-1)}(t)|| \\
||A_{3n}(t)|| &\leq L_3 \left| \frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right| ||A_{3(n-1)}(t)|| \\
||A_{4n}(t)|| &\leq L_4 \left| \frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right| ||A_{4(n-1)}(t)|| \\
||A_{5n}(t)|| &\leq L_5 \left| \frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right| ||A_{5(n-1)}(t)|| \\
||A_{6n}(t)|| &\leq L_6 \left| \frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right| ||A_{6(n-1)}(t)||.
\end{aligned} \tag{10}$$

Theorem 2: The fractional model given in (2) has a solution if we can find M_0 satisfying

$$\left(\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_i \leq 1 i = 1, 2, ..., 6. \tag{11}$$

From (8) and (9) we have:

$$\begin{aligned}
||A_{1n}(t)|| &\leq ||S(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_1 \right|^n, \\
||A_{2n}(t)|| &\leq ||E(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_2 \right|^n, \\
||A_{3n}(t)|| &\leq ||I(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_3 \right|^n, \\
||A_{4n}(t)|| &\leq ||I_T(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_4 \right|^n, \\
||A_{5n}(t)|| &\leq ||Q(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_5 \right|^n, \\
||A_{6n}(t)|| &\leq ||R(0)|| \left| \left(\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)} \right) L_6 \right|^n.
\end{aligned} \tag{12}$$

The existence of the solution is confirmed by theorem 1, and we need now to show that the functions S(t), P(t),Q(t), W(t),X(t) and Y(t) are solutions of model (2). To show this we assumed that the following is satisfied:

$$\begin{aligned}
S(t) - S(0) &= S_n(t) - a_{1n}(t), \\
E(t) - E(0) &= E_n(t) - a_{2n}(t), \\
I(t) - I(0) &= Q_n(t) - a_{3n}(t), \\
I_T(0) - I_T(0) &= I_{Tn}(t) - a_{4n}(t), \\
Q(t) - Q(0) &= Q_n(t) - a_{5n}(t), \\
R(t) - R(0) &= R_n(t) - a_{6n}(t).
\end{aligned} \tag{13}$$

From (13),

$$\begin{aligned}
||a_{1n}(t)|| = \frac{1-\alpha}{F(\alpha)} &||(G_1(t, S_{n-1}) - G_1(t, S_{n-2}))|| + \\
&\frac{\alpha}{F(\alpha)\Gamma(\alpha)} \int_0^t ||(G_1(k, S_{n-1}) - G_1(k, S_{n-1}))|| (t-k)^{\alpha-1} dk,
\end{aligned}$$

$$||a_{1n}(t)|| \leq \frac{1-\alpha}{F(\alpha)}L_1||S_n(t) - S_{n-1}(t)|| \\ \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}L_1||S_n(t) - S_{n-1}(t)||.$$

If we were to repeat the process recursively, we have

$$||a_{1n}(t)|| \leq \left[\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right]^{n+1} L_1^n||S_n(t) - S_{n-1}(t)||^n,$$

which at $t = M_0^\alpha$ yields,

$$||a_{1n}(t)|| \leq \left[\frac{1-\alpha}{F(\alpha)} + \frac{M_0^\alpha}{F(\alpha)\Gamma(\alpha)}\right]^{n+1} L_1^n||S_n(t) - S_{n-1}(t)||^n. \tag{14}$$

Applying the limit to both sides of (13) as $n \to \infty$, we have see that $||a_{1n}(t)|| \to 0$ for

$$\left[\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right]L_1 \leq 1.$$

In a similar fashion, we show that

$$||a_{2n}(t)|| \to 0,\ ||a_{3n}(t)|| \to 0,\ ||a_{4n}(t)|| \to 0,\ ||a_{5n}(t)|| \to 0,\ ||a_{6n}(t)||, \to 0,$$

for

$$\left[\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right]L_i \leq 1 i = 2, 3, 4, 5, 6.$$

Theorem 1 and 2 guaranty the existence of solution of model (2) by the Banach fixed point theorem. we shall show the uniqueness of the solution in Theorem 3.

Theorem 3: (Uniqueness of Solution)

The fractional model (2) has a unique solution provided that

$$\left[\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right]L_i \leq 1 i = 2, 3, 4, 5, 6. \tag{15}$$

Proof.
We assume that $S_1(t), E_1(t), I_1(t), I_{T1}(t), Q_1(t), R_1(t)$, are solution to model (4). Then,

$$S(t) - S_1(t) = \frac{1-\alpha}{F(\alpha)}(G_1S(t) - G_1S_1(t)) + \\ \frac{\alpha}{F(\alpha)\Gamma(\alpha)}\int_0^t (G_1S(t) - G_1S_1(t))(t-k)^{\alpha-1}dk.$$

Taking the norm of both sides , we have:

$$||S(t) - S_1(t)|| = \frac{1-\alpha}{F(\alpha)}||L_1||||S(t) - S_1(t)|| + \\ \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}||L_1||||S(t) - S_1(t)||.$$

Since

$$\left(1 - ||L_1||\left(\frac{1-\alpha}{F(\alpha)} + \frac{t^\alpha}{F(\alpha)\Gamma(\alpha)}\right)\right) > 0,$$

we obtain $||S(t) - S_1(t)|| = 0$. Thus we have $S(t) = S_1(t)$. We can also show that $P(t) = P_1(t), Q(t) = Q_1(t), W(t) = W_1(t), X(t) = X_1(t), Y(t) = Y_1(t)$. We have completed the proof of Theorem 3.

4 Positivity and boundedness

The model (4) will be meaningful epidemiologically , if the solution of system (4) with no negative initial data will remain non-negative for all time $t \geq 0$.

Lemma (1) . For all time $t \geq 0$ and initial data $N(0) \geq 0$, where $N(t) = (S(t), E(t), I(t), I_T(t), Q(t), R(t))$. The solution of model () are non-negative for all $t \geq 0$r if there exist; furthermore $\limsup N(t) \leq \frac{\beta}{\mu}$. The epidemiologically feasible region of model (2) is given by

$$\begin{aligned} \Omega =: \{(S(t), E(t), I(t), I_T(t), Q(t), R(t)) \in \mathbb{R}^6_+ \leq \\ S(t) + P(t) + Q(t) + W(t) + X(t) + Y(t) \leq \mathbb{N} \leq \frac{\beta}{\mu}\}. \end{aligned} \tag{16}$$

Proof. Applying the AB fractional integral(2) of the first equation of (4) we have:

$$I_t^\alpha D_t^\alpha \left(S(t) e^{\int_0^t (\mu + b_0 \frac{(I(t)+Q(t))}{k(N)}) dS} \right) = \beta I_t^\alpha \left(e^{\int_0^t (\mu + b_0 \frac{(I(t)+Q(t))}{k(N)}) dS} \right).$$

It is obvious that

$$\begin{aligned} S(t) = \beta e^{-\int_0^t (\mu + b_0 \frac{(I(t)+Q(t))}{k(N)}) dS} \Big[\frac{1-\alpha}{F(\alpha)} e^{\int_0^t (\mu + b_0 \frac{(I(t)+Q(t))}{k(N)}) dS} + \\ \frac{\alpha}{F(\alpha)\Gamma(\alpha)} e^{\int_0^t (\mu + b_0 \alpha_1 \zeta) dS} (t-w) dw \Big] > 0. \end{aligned} \tag{17}$$

Similar argument goes for the remaining equations of (2)

$$\begin{aligned}
\mathbf{E(t)} &= b_0\frac{(I(t)+Q(t))}{k(N)}S(t)e^{-\int_0^t \lambda_2 dt}\left[\frac{1-\alpha}{F(\alpha)}e^{\int_0^t \lambda_2 dt}+\right.\\
&\left.\frac{\alpha}{F(\alpha)\Gamma(\alpha)}e^{\int_0^t \lambda_2 dt}(t-w)dw\right] > 0,\\
\mathbf{I(t)} &= \lambda_2 E(t)e^{-\int_0^t (\mu+\alpha_3+\lambda_1)dt}\left[\frac{1-\alpha}{F(\alpha)}e^{\int_0^t (\mu+\alpha_3+\lambda_1)dt}+\right.\\
&\left.\frac{\alpha}{F(\alpha)\Gamma(\alpha)}e^{\int_0^t (\mu+\alpha_3+\lambda_1)dt}(t-w)dw\right] > 0,\\
\mathbf{IT(t)} &= (1-\rho)\alpha_3 I(t)e^{-\int_0^t (\mu+\lambda_2)dt}\left[\frac{1-\alpha}{F(\alpha)}e^{\int_0^t (\mu+\lambda_2)dt}+\right.\\
&\left.\frac{\alpha}{F(\alpha)\Gamma(\alpha)}e^{\int_0^t (\mu+\lambda_2)dt}(t-w)dw\right] > 0,\\
\mathbf{Q(t)} &= \rho\alpha_3 I(t)e^{-\int_0^t \Psi dt}\left[\frac{1-\alpha}{F(\alpha)}e^{\int_0^t \Psi dt}+\right.\\
&\left.\frac{\alpha}{F(\alpha)\Gamma(\alpha)}e^{\int_0^t \Psi dt}(t-w)dw\right] > 0,\\
\mathbf{R(t)} &= \alpha_4 I_T(t)e^{-\int_0^t (\mu+\alpha_5)dt}\left[\frac{1-\alpha}{F(\alpha)}e^{\int_0^t (\mu+\alpha_5)dt}+\right.\\
&\left.\frac{\alpha}{F(\alpha)\Gamma(\alpha)}e^{\int_0^t (\mu+\alpha_5)dt}(t-w)dw\right] > 0.
\end{aligned} \tag{18}$$

It follows from (17)and (18) that each of the solution of (2) is non negative and remain in $\mathbb{R}^6_+$.

We now establish the boundedness of the solution of the fractional model (4) taking into account that all the parameters used in the model as said earlier are non negative. we shall continue by summing up all the equation of model (2) which gives

$$D_t^\alpha N = \beta - \mu N - \lambda_2 I_T - \Psi Q(t) - \alpha_5 R(t),$$

$$D_t^\alpha N(t) - \mu N(t) \leq \beta,$$

$$\left(D_t^\alpha N(t) - \mu N(t\right)e^{\int_0^t \mu dt} \leq \beta e^{\int_0^t \mu dt},$$

$$N(t) \leq \frac{\beta}{\mu}.$$

It is not difficult to observe that $N(t) \to \frac{\beta}{\mu}$ as $t \to \infty$. Hence (16) is the biologically feasible region of (4).

5 Conclusion

In this work ,we studied a $SEII_tR$ epidemic mathematical involving the Atangana-Baleanu fractional order which does not contain a singular kernel. further, the existence and uniqueness of solution for equilibrium solutions have been proved by engaging the the Banach fixed point theorem.

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