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On the constrained optimal control problems with mixed constraints

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Abstract

The analytical solutions of optimal control problems with mixed constraints are examined. The analytical solutions are obtained by applying the first order optimality conditions on the Lagrangian function. The method of fundamental matrix is then used to solve the resulting non-homogeneous system of first order ordinary differential equations. This leads to the analytical solutions of the state and control variables and the objective function value. Two examples of optimal control problems constrained by ordinary differential equations and inequalities are considered.

Keywords: analytical solution; Halmitonian; Lagrangian; fundamental matrix; fundamental solution

1 Introduction

Optimization involves finding the minimum or maximum value of a function called extremum. In many real life processes such as engineering and epidemiology, decision Makers have to take many technological and managerial decisions at several stages in order to minimize the effort required or maximize the benefit derived. Such decisions are crucial because it ensures optimum use of available resources.

Constrained optimization is the process of minimizing or maximizing an objective functional with respect to some decision variables in the presence of some constrained on the decision variables. These constraints create more problems while finding the minimum or maximum value of the objective function as compared to unconstrained ones. In most situation, the constraints have some effects on the objective functional. However, there are few situations under which the constraints do not have any influence on the minimum point. Here, the constrained minimum of the problem is the same as the unconstrained minimum ([20], [21]). Majority of the available methods are designed for unconstrained optimization,where no imitations are placed on the design variables.

Optimal Control theory deals with finding a control law for a dynamical system such that an objective function is optimized over a given period of time. It deals with systems that can be controlled. The theory of optimal control is well developed for many decades ([22]). It requires a performance index or cost functional $J(t, x(t), u(t))$, a set of state variables $(t, x(t) \in X)$, a set of control variables $(u(t) \in U(t))$ in a time t, with $t_0 \leq t \leq t_f$. To maximized a given objective functional, one must obtain the associated state variable $x(t)$ and a piecewise continuous control $u(t)$. Mathematical control theory deals with the basic principles underlying the analysis and design of control systems arising from some areas of applied Mathematics. To control any system means to influence its behaviour so as to achieve a desired goal. Mathematical techniques are

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built to implement this influence [4].

One of such techniques is Pontryagin's principle. Pontryagin's maximum (or minimum) principle was formulated in 1956 by the Russian mathematician Lev Pontryagin and his students as Pontryagin's maximum principle and proved historically based on maximizing the Hamiltonian [5]. For the validity of the statement to be established, the sufficient conditions must be satisfied in order to determine the nature of the turning point [7].

A brief historical survey of the development of the theory of optimal control and calculus of variations was given by ([18]). The calculus of variations uses variations, which are small changes in functionals in order to minimize or maximize the functionals. EulerLagrange equation of the calculus of variations is used to find functions that minimize or maximize functionals. Several other well-defined new areas in optimization theory were reviewed in [15]. A number of numerical techniques have also been developed over the years for optimal control problems that have analytical solutions and the ones that did not have analytical solutions by [2], [3], [6], [12], [13], [14], [16],[17]and [19].

In the early years of optimal control, the favoured approach for solving optimal control problems was that of indirect method. In an indirect method, the calculus of variations is employed to obtain the first-order optimality conditions [10]. [8] considered an optimal control problem involving equality and inequality state-control constraints with fixed initial and final endpoint state constraints. The control variables belong to the class of piecewise continuous functions while and state variables belong to the class of piecewise smooth functions respectively. The paper provided a direct derivation of second order necessary conditions, based on a variational approach where a certain quadratic form is non-negative. The inequality constraints are treated as equalities of the active constraints for this set, thus producing a restrictive set of optimality conditions.

The analytic and numeric solutions of discretized constrained optimal control problems were obtained ([11]). The analytic and numeric solutions of general continuous linear quadratic optimal control problem were presented. The associated general Riccati differential equation was solved by numerical-analytical approach using variational iteration method. Numerical solutions of the constrained optimal control problem were obtained via quadratic programming of the discretized continuous optimal control problems by shooting method and the conjugate gradient method (CGM). The results showed that both the analytical and numerical solutions agreed favourably. In the paper, the analytical solution and numerical solution by penalty function methods were presented and convergence analysis was conducted.

The analytical solutions of optimal control problems constrained by ordinary differential equations were obtained. The necessary condition of optimality was applied to the Hamiltonian function in order to obtain the analytical conditions of this class of problems. This leads to the optimal state, control and adjoint variables and hence the optimal objective function value of optimal control problems constrained by ordinary differential equations [1]. The theoretical analysis of [1] is extended to optimal control problems with mixed constrained in this paper. The analytical solutions of the state, control variables together with the objective function values were obtained. The effects of active and inactive inequality constraints on the state and control variables were also examined.

2 Methodology

Consider an optimal control problem with mixed constrained given as

$$\begin{array}{rl} \text{Minimize} & f(x,u,t), \\ \text{subject to} & h(x,u,t)=0, \\ & g(x,u,t)\leq 0, \end{array} \tag{1}$$

where $f : \Re^n \times \Re^m \times \Re \to \Re$, $h : \Re^n \times \Re^m \times \Re \to \Re^p$, $g : \Re^n \times \Re^m \times \Re \to \Re^r$ are given functions, $m \leq n$, $p \leq n$, $r \leq n$, $x \in \Re^n$ and $u \in \Re^m$. The components of h and g are denoted by $h_1, h_2, \ldots, h_p$ and $g_1, g_2, \ldots, g_r$, respectively. Equation (1) can be converted to an equality constrained problem by introducing a vector of additional variables $z = (z_1, z_2, \ldots, z_r)$. The problem then becomes

$$\begin{array}{rl} \text{Minimize} & f(x,u,t), \\ \text{subject to} & h_i(x,u,t)=0,\ i=1,2,\ldots p, \\ g_j(x,u,t)+ & z_j^2(x,u,t)=0,\ j=1,2,\ldots r. \end{array} \tag{2}$$

2.1 Analytical Solution of Optimal Control Problems with Mixed Constraints and Real Coefficients

$$\text{Minimize} \quad I(x,u)=\int_0^T (px^2(t)+qu^2(t))dt, \tag{3}$$

$$\text{subject to } \dot{x}(t)=ax(t)+bu(t), \tag{4}$$

$$cx(t)+du(t)\leq 0,\ x(0)=x_0, \tag{5}$$

where a, b , c, d are real constants and $p, q > 0$.

Theorem 1.1 *Given the optimal control variable $u^*(t)$ and the solution $x^*(t)$ of the state system (4) that minimizes $I(x,u,t)$ over U (where U is the permissible set of controls), then there exists an adjoint variable $\mu(t)$ satisfying*

$$\dot{\mu}(t)=-2px(t)-a\mu(t)-c\lambda, t\in[0,T], \tag{6}$$

and with the transversality condition

$$\mu(T)=0, \tag{7}$$

$$u^*(t)=\frac{-b\mu(t)-d\lambda}{2q}. \tag{8}$$

Proof. With appropriate conditions on the end points, adjoint variable $\mu(t)$ can be introduced by forming the required augumented functional from (3)-(5). The Hamiltonian function is given as

$$H(x,u,\mu)=px^2(t)+qu^2(t)+\mu(ax(t)+bu(t)), \tag{9}$$

and the Lagrangian function is defined as

$$L(x,u,\mu,\lambda)=px^2(t)+qu^2(t)+\mu(ax(t)+bu(t))+\lambda(cx(t)+du(t)). \tag{10}$$

From the knowledge of calculus of variation, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrange equations for L regarded as function of four variables (x,u,μ,λ). Thus, the E-L system can be written as

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{x}}]=\frac{\partial L}{\partial x}, \tag{11}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{u}}] = \frac{\partial L}{\partial u}, \tag{12}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\mu}}] = \frac{\partial L}{\partial \mu}, \tag{13}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\lambda}}] = \frac{\partial L}{\partial \lambda}. \tag{14}$$

Equations (11)-(14) now give the following equations

$$\dot{\mu}(t) = -2px(t) - a\mu(t) - c\lambda, t \in [0, T], \tag{15}$$

$$u^*(t) = \frac{-b\mu(t) - d\lambda}{2q}, \tag{16}$$

$$\dot{x}(t) = ax(t) - b(\frac{b\mu(t) + d\lambda(t)}{2q}), \tag{17}$$

$$cx + du = 0. \tag{18}$$

The optimality system of equations (15) and (17) represent a linear 2-point boundary value non-homogeneous ordinary differential equations, which are the necessary conditions for an optimal control $u^*(t)$. The two-point boundary value problem can be written in matrix form as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} -2p & -a \\ a & \frac{-b^2}{2q} \end{pmatrix} \begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} + \begin{pmatrix} -c \\ \frac{-bd}{2q} \end{pmatrix} \lambda(t). \tag{19}$$

Let

$$M = \begin{pmatrix} -2p & -a \\ a & \frac{-b^2}{2q} \end{pmatrix}. \tag{20}$$

The eigenvalues of M are

$$\lambda_1 = -\frac{4pq - A + b^2}{4q}, \tag{21}$$

$$\lambda_2 = -\frac{4pq + A + b^2}{4q}, \tag{22}$$

where $A = \sqrt{-(b^2 + 4aq - 4pq)(-b^2 + 4aq + 4pq)}$ and its eigenvectors are

$$U_1 = \begin{pmatrix} \frac{4b^2 - (4pq - A + b^2)}{4aq} \\ 1 \end{pmatrix}, \tag{23}$$

and

$$U_2 = \begin{pmatrix} \frac{4b^2 - (4pq + A + b^2)}{4aq} \\ 1 \end{pmatrix}. \tag{24}$$

Thus, the solution of the complementary part of (19) is given by

$$V(t) = c_1 \overrightarrow{U_1} e^{\lambda_1 t} + c_2 \overrightarrow{U_2} e^{\lambda_2 t}. \tag{25}$$

The general solution can be written in matrix form as

$$\begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} \frac{4b^2 - (4pq - A + b^2)}{4aq} e^{\lambda_1 t} & \frac{4b^2 - (4pq + A + b^2)}{4aq} e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{26}$$

Hence, applying the method of fundamental matrix [9], this implies that

$$X(t) = \psi(t)C \tag{27}$$

where

$$\psi(t) = \begin{pmatrix} \frac{4b^2-(4pq-A+b^2)}{4aq}e^{\lambda_1 t} & \frac{4b^2-(4pq+A+b^2)}{4aq}e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \tag{28}$$

is a 2×2 matrix with the fundamental solutions $x_1(t)$ and $x_2(t)$ along the columns.

Hence,

$$\psi^{-1}(t) = \frac{1}{-2A} \begin{pmatrix} e^{\lambda_2 t} & -\frac{4b^2-(4pq+A+b^2)}{4aq}e^{\lambda_2 t} \\ -e^{\lambda_1 t} & \frac{4b^2-(4pq-A+b^2)}{4aq}e^{\lambda_1 t} \end{pmatrix}. \tag{29}$$

This implies that

$$\psi^{-1}(t)g(s) = \begin{pmatrix} -\frac{e^{\lambda_2 t}}{2A} & \frac{4b^2-(4pq+A+b^2)}{8aqA}e^{\lambda_2 t} \\ \frac{e^{\lambda_1 t}}{2A} & -\frac{4b^2-(4pq-A+b^2)}{8aqA}e^{\lambda_1 t} \end{pmatrix} \begin{pmatrix} -c \\ \frac{-bd}{2q} \end{pmatrix} \lambda(s) \tag{30}$$

$$\psi^{-1}(s)g(s) = \begin{pmatrix} \frac{c\lambda(s)e^{\lambda_2 s}}{2A} - bd(\frac{4b^2-(4pq+A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \\ -\frac{c\lambda(s)e^{\lambda_1 s}}{2A} + bd(\frac{4b^2-(4pq-A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \end{pmatrix}, \tag{31}$$

where $g(t) = \begin{pmatrix} -c \\ \frac{-bd}{2q} \end{pmatrix} \lambda(t)$ is the non-homogeneous part of equation (19).

$$\int \psi^{-1}(s)g(s)d(s) = \int \begin{pmatrix} \frac{c\lambda(s)e^{\lambda_2 s}}{2A} - bd(\frac{4b^2-(4pq+A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \\ -\frac{c\lambda(s)e^{\lambda_1 s}}{2A} + bd(\frac{4b^2-(4pq-A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \end{pmatrix} d(s) \tag{32}$$

$$\psi(t)\int \psi^{-1}(s)g(s)d(s) = \begin{pmatrix} \frac{4b^2-(4pq-A+b^2)}{4aq}e^{\lambda_1 t} & \frac{4b^2-(4pq+A+b^2)}{4aq}e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \int \begin{pmatrix} \frac{c\lambda(s)e^{\lambda_2 s}}{2A} - bd(\frac{4b^2-(4pq+A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \\ -\frac{c\lambda(s)e^{\lambda_1 s}}{2A} + bd(\frac{4b^2-(4pq-A+b^2)}{16aq^2A})\lambda(s)e^{\lambda_2 s} \end{pmatrix} d(s) \tag{33}$$

This implies that the general solution is given by

$$X(t) = \psi(t)C + \psi(t)\int \psi^{-1}(s)g(s)d(s). \tag{34}$$

3 Solved examples

Example 3.1

$$Minimize\ I(t, x(t), u(t)) = \int_0^4 u^2(t)dt \tag{35}$$

$$\dot{x} = x + u,\ \ x(0) = 0 \tag{36}$$

$$u(t) \leq 5 \tag{37}$$

Solution:

The Hamiltonian function, H, is given by

$$H(x, u, \mu) = u^2(t) + \mu(x(t) + u(t)), \tag{38}$$

and the Lagrangian function L is given by

$$L(x, u, \mu, \lambda) = u^2(t) + \mu(x(t) + u(t)) + \lambda(u(t) - 5). \tag{39}$$

From the knowledge of calculus of variations, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrangian equations for L regarded as function of four variables $(x(t), u(t), \mu(t), \lambda(t))$. Thus, the E-L system can be written as

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{x}}] = \frac{\partial L}{\partial x}, \tag{40}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{u}}] = \frac{\partial L}{\partial u}, \tag{41}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\mu}}] = \frac{\partial L}{\partial \mu}, \tag{42}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\lambda}}] = \frac{\partial L}{\partial \lambda}. \tag{43}$$

Equation (40) implies that

$$\dot{\mu}(t) = -\mu, \tag{44}$$

whether the constraint is active or not. The transversality boundary condition implies that $\mu(4) = \frac{\partial \psi}{\partial x} = -1$ where $\psi = -x(4)$ is the terminal cost. Hence,

$$\mu(t) = -e^{4-t}. \tag{45}$$

Assuming that the control constraint is initially active for some period of time, then $\lambda \geq 0, u = 5$, then

$$L_u = 2u(t) + \mu + \lambda = 0, \tag{46}$$

$$\lambda(t) = -10 - \mu = e^{4-t} - 10. \tag{47}$$

For $\lambda \geq 0$, then

$$\lambda(t) = e^{4-t} - 10 \geq 0. \tag{48}$$

Hence,

$$t \leq 4 - ln(10) \text{ and } \lambda \geq 0. \tag{49}$$

On the other hand, if the constraint is inactive, then

$$L_u = 2u(t) + \mu = 0 \tag{50}$$

$$L_u = 2u(t) + \mu = 0 \tag{51}$$

$$u(t) = -\frac{1}{2}\mu(t). \tag{52}$$

The optimal control for this problem is given as

$$u^*(t) = \begin{cases} 5, & if\ t \leq 4 - ln(10) \\ \frac{1}{2}e^{4-t}, & if\ t \geq 4 - ln(10) \end{cases}$$

To obtain the state variable in the two arcs, we examine the two control inputs obtained above. When $u(t) = 5$, equation (3.2) becomes

$$\dot{x}(t) = x(t) + 5, \tag{53}$$

since $x(0) = 0$, then

$$x(t) = 5e^t - 5. \tag{54}$$

Similarly, in view of (52), equation (3.2) becomes

$$\dot{x}(t) = x(t) - \frac{1}{2}e^{4-t}, \tag{55}$$

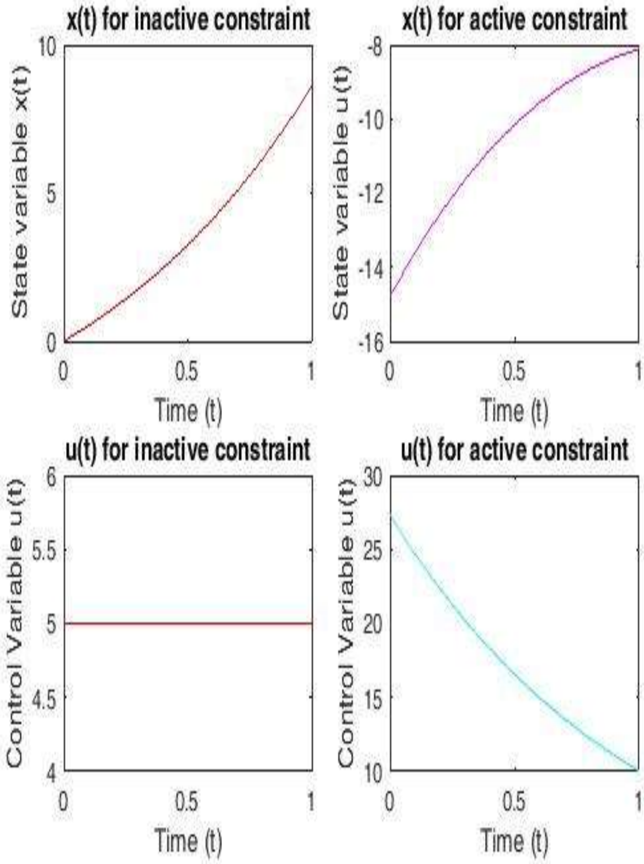

Figure 1: State Variable $x(t)$ and Control Variable $u(t)$ for Active and Inactive forms of the Inequality Constraint Against Time (t)

since $x(0) = 0$, then

$$x(t) = -\frac{1}{4}e^{4-t} + (5 - 25e^{-4})e^t. \tag{56}$$

Hence, the state variable is given by

$$x^*(t) = \begin{cases} 5e^t - 5, & if\ t \leq 4 - ln(10) \\ -\frac{1}{4}e^{4-t} + (5 - 25e^{-4})e^t, & if\ t \geq 4 - ln(10) \end{cases}$$

Hence, when $u^*(t) = 5$ and $x^*(t) = 5e^t - 5$ for $t \leq 4 - ln(10)$, then

$$\text{Minimize } I(t, x(t), u(t)) = (5e^4 - 5) + \int_0^4 5^2 dt = 372.9908. \tag{57}$$

Similarly, when $u^*(t) = \frac{1}{2}e^{4-t}$ and $x^*(t) = -\frac{1}{4}e^{4-t} + (5 - 25e^{-4})e^t$ for $t \geq 4 - ln(10)$, then

$$\text{Minimize } I(t, x(t), u(t)) = -\frac{1}{4}e^{4-4} + (5 - 25e^{-4})e^4 + \int_0^4 (\frac{1}{2}e^{4-t})^2 dt = 992.7302. \tag{58}$$

Example 3.2

$$Minimize\ I(t, x(t), u(t)) = \int_0^1 (x^2(t) + u^2(t))dt, \tag{59}$$

$$\dot{x} = 4x + 5u, \;\; x(0) = 1 \tag{60}$$

$$2x(t) + 4u(t) \leq 0. \tag{61}$$

Solution:
The Hamiltonian function function, H, is given by

$$H(x, u, \mu) = x^2(t) + u^2(t) + \mu(-4x - 5u), \tag{62}$$

and the Lagragian function, L, is given by

$$L(x, u, \mu, \lambda) = x^2(t) + u^2(t) + \mu(-4x - 5u) + \lambda(2x(t) + 4u(t)). \tag{63}$$

From the knowledge of calculus of variations, it seems plausible that the optimal solutions for our initial optimal control problem ought to be the Euler-Lagrangian equations for L regarded as function of four variables $(x(t), u(t), \mu(t), \lambda(t))$. Thus, the E-L system can be written as

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{x}}] = \frac{\partial L}{\partial x}, \tag{64}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{u}}] = \frac{\partial L}{\partial u}, \tag{65}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\mu}}] = \frac{\partial L}{\partial \mu}, \tag{66}$$

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{\lambda}}] = \frac{\partial L}{\partial \lambda}. \tag{67}$$

Equation (64) implies that

$$\dot{\mu}(t) = -2x(t) + 4\mu(t) - 2\lambda(t). \tag{68}$$

Equation (65) implies that

$$2u(t) - 5\mu(t) + 4\lambda(t) = 0. \tag{69}$$

This implies that

$$\lambda^*(t) = \frac{-2u(t) + 5\mu(t)}{4}. \tag{70}$$

Equation (66) leads to the state equation whereas equation (67) implies that

$$2x(t) + 4u(t) = 0. \tag{71}$$

Assuming that the control constraint is initially active for some period of time, then $\lambda \geq 0$, then

$$u^*(t) = \frac{x(t)}{2} \tag{72}$$

In view of equation (72), equation (60) becomes

$$\dot{x}(t) = \frac{13}{2}x(t). \tag{73}$$

Since $x(0) = 1$, then

$$x^*(t) = e^{\frac{13}{2}t}. \tag{74}$$

The optimal control given by equation (72) then becomes

$$u^*(t) = \frac{e^{\frac{13}{2}t}}{2}. \tag{75}$$

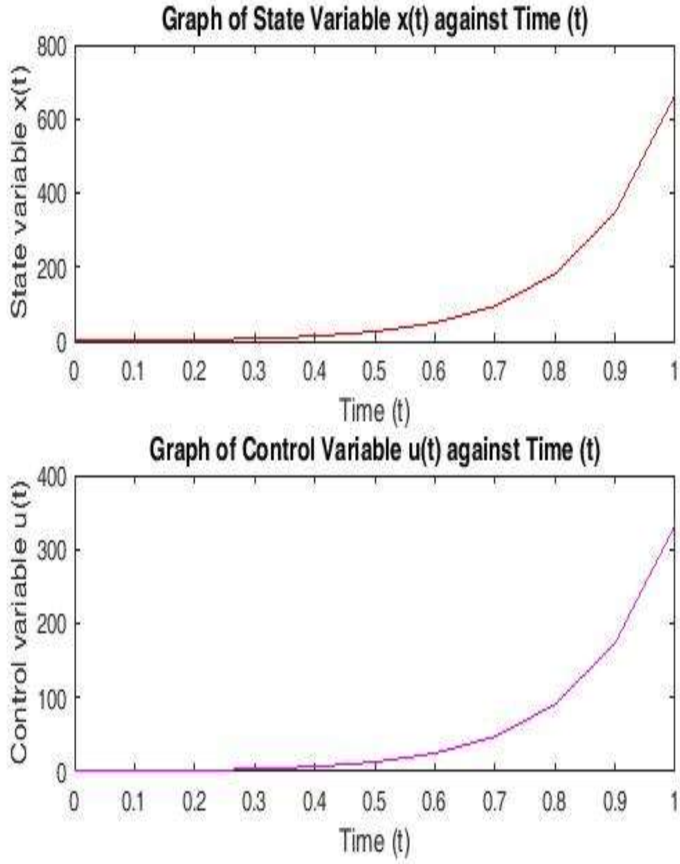

Figure 2: Graphical Representation of the Solution of Optimal Control Problem Constrained by Algebraic-differential Equations

In view of equations (70), (74) and (75), equation (68) then becomes

$$\dot{\mu}(t) = -\frac{3}{2}\mu + \frac{3}{2}e^{\frac{13}{2}t}, \tag{76}$$

whether the constraint is active or not. The transversality boundary condition implies that $\mu(1) = \frac{\partial \psi}{\partial x} = 0$ where $\psi = 0$ is the terminal cost. Hence,

$$\mu^*(t) = \frac{3}{16}e^{-\frac{3t}{2}}(e^{8t} - e^{8}). \tag{77}$$

It follows immediately that Equation (70) becomes

$$\lambda^*(t) = -\frac{1}{2}(\frac{e^{\frac{13}{2}t}}{2}) + \frac{5}{4}(\frac{3}{16}e^{-\frac{3t}{2}}(e^{8t} - e^{8})). \tag{78}$$

Hence,

$$\lambda^*(t) = \frac{15}{64}e^{-\frac{3t}{2}}(e^{8t} - e^{8}) - \frac{1}{4}e^{\frac{13}{2}t}. \tag{79}$$

The objective function value is given as 42540.

4 Concluding remarks

The analytical solutions of continuous quadratic optimal control problems constrained by ordinary differential equations and inequalities have been presented. The analytical solution was obtained by applying the first order optimality conditions on the Lagrangian function. The resulting system of non-homogeneous first order ordinary differential equations was then solved using the method of fundamental matrix to obtained the optimal values of the state, control and adjoint variables. This leads to the optimal value of the cost functional. Two examples are considered and the results show the effects of active and inactive constraints on the state and control variables and the objective function values.

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