International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 2 (Sept.), 2022

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 - 5708

Dynamics of an epidemiological model of drinking

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Abstract

In this paper, we review and carry out the bifurcation analysis of the drinking model proposed by Manthey *et al.* in 2008. We show that the model exhibits a drinking-free equilibrium point and a problem drinking-free equilibrium which exists when the reproductive number $\mathcal{R}_0^Y > 1$. Each of these equilibria is a stable node if the threshold parameters $\mathcal{R}_0^Y < 1, \mathcal{R}_0^Z < 1$ and $\mathcal{R}_1 < 1, \mathcal{R}_0^Y > 1$ respectively. Moreover, the existence of multiple endemic equilibria reveals the complex dynamics of the model by the occurrence of two saddle-node bifurcations leading to the phenomenon of backward bifurcation. Taking the transmission rate of non-drinkers to problem drinkers κ as a bifurcation parameter, we show the two saddle-nodes SN and SN_2 for the extinction and persistence of the drinking problem. Before the disappearance of E_1 and E_2 after SN in the interval $(0, \mathcal{R}_0^{Z_{c1}})$ the system is bistable with E_0 and E_2 being the stable equilibrium points. The system is rendered mono stable after the SN before the SN_2 emerged in $(\mathcal{R}_0^{Z_{c1}}, \mathcal{R}_0^{Z_{c2}})$ with E_0 as the only stable equilibrium point. Then after the emergence of SN_2 in the interval $(\mathcal{R}_0^{Z_{c2}}, \kappa^*)$, the system becomes bistable again with E_0 and E_4. But for $\kappa > \kappa^*$, there is endemic persistence of the drinking problem with E_4 as the only stable attractor. When such a scenario does not exist a forward bifurcation occurs for which the drinking-free equilibrium exchanges stability at $\kappa = \kappa^c$ with drinking-present equilibrium point. Numerical simulations of the model support these extinction and persistence regions of the drinking problem.

Keywords: campus drinking; bifurcation; stable node; saddle-node

1 Introduction

With recent improvement of our living standard, the life styles of people are becoming more and more diversified and drinking plays a significant role in daily life of some people. However the current situation of alcoholism around the world is really terrifying. In Europe for instance, alcohol is the third leading risk factor for disease and mortality after tobacco and high blood pressure [19]. The world health organization reported that more than 3 million people died because of alcohol and other substance abuse, which account to 5.3% of all death and 5.1% of the global disease burden attributable to alcohol consumption [20]. It was also presented in [14] that alcohol use rates are very high among college students in the US. In fact, approximately two of five such students were heavy drinkers, meaning that having had five or more drinks in a row in the past 2 weeks. Alcohol use is higher among male than female students as well. White students are highest in heavy drinking, black students are lowest and Hispanic students are intermediate. Use of alcohol alone is higher among college students than among non college age-mates. Therefore, alcoholism is not only a global public health issues, but also a societal problem that need to be addressed.

To address the problem of alcohol drinking among youth, several mathematical models have been proposed to analyze the spread and control of drinking behaviour, See for instance

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[1, 2, 3, 4, 11] and the references therein. Sánchez *et al.* [15], proposed a simple mathematical model to describe the dynamics of drinking behaviors of individuals in shared drinking environments. Despite its limitations, the model has generated some useful insights. More precisely, the basic reproductive number (as a function of treatment), is not always the key to controlling drinking within the population due to high rates of recovery and relapse. The high relapse rates will occur when treatment programs have only short-term positive effects. The drinking behaviours of college students motivated the study using mathematical model on how the 'disease' of drinking spread on campus [13]. The model suggests that the pattern of recruiting new members plays a significant role in the reduction of campus alcohol problems and that the reproductive numbers are not sufficient to predict whether drinking behaviour will persist on campus. More precisely, campus alcohol abuse may be reduced by minimizing the ability of problem drinkers to directly recruit non-drinkers. Bhunu developed a deterministic mathematical model for the spread of alcoholism to gain insights into its growing health and social problem. The associated reproduction number is calculated and Local asymptotic stabilities of equilibrium points were carried out. The analysis of the reproduction number reveals that encouraging and supporting moderate drinkers to quit alcohol consumption is more effective in the control of alcoholism than supporting and encouraging alcoholics to quit and vice-versa. Numerical simulations of the model support this results and also shown that encouraging and supporting all alcohol consumers to quit drinking will be the best strategy [4]. A mathematical model of alcohol abuse was presented in [17]. The basic reproduction number $\mathcal{R}_0$ is determined and they proved that the problem-free equilibrium is both locally and globally asymptotically stable when $\mathcal{R}_0 < 1$. Sensitivity analysis of the model parameters indicates that the transmission coefficient β_1 from moderate and occasional drinker to heavy drinker, is the most useful parameter to target for the reduction of $\mathcal{R}_0$.

In a recent development, the threshold dynamics of a multi-group SEAR alcoholism epidemic model with public health education was analysed [12]. It was proved that the basic reproduction number $\mathcal{R}_0$ plays the role of a threshold for the long-time behaviour of the model. They showed that the model exhibits global asymptotic stable alcohol-free and alcohol-present equilibrium points. Sensitivity analysis of the model parameters and numerical simulations were presented to illustrate the results. Zhang *et al.* proposed a delayed drinking model to carry out the bifurcation and optimal control analysis. A threshold value of the time delay for the local stability of drinking-present equilibrium and the existence of Hopf bifurcation were obtained. In addition, optimal strategies to lower down the number of drinkers were presented. Numerical simulations to support the obtained results and the effects of some parameters on dynamics of the model were provided [21]. An alcohol consumption model with health education and three time delays was formulated and analyzed [11]. The alcoholism generation number was determined and two equilibrium points of the model were obtained. The corresponding global dynamics of the model are analyzed respectively in four cases with different time delays. Furthermore, the effects of health education and three time delays in controlling the alcohol problem were discussed. Some numerical simulation results were also given to support the theoretical predictions.

We know that when a disease transmission model posses multiple endemic equilibria, the model exhibits complex dynamics such as backward bifurcation, saddle-node bifurcation and forward hysteresis [6, 7, 10, 18]. The existence of two endemic equilibrium points of the drinking model proposed by Manthey *et al.* [13], motivate us to review and investigate its complex dynamics. In fact, we show the occurrence of the phenomena of backward bifurcation and saddle-node bifurcation when the transmission rate of non-drinkers to problem drinkers κ is varied (increasing the threshold parameter $\mathcal{R}_0^Z$). The existence of these phenomena indicates the extinction and persistence regions of drinking problem.

The remaining paper is organized as follows: In Section 2, we describe the model. Section 3 presents the existence and stability results of the model equilibria. In order to complete the investigation, the special cases of the model are analyzed in Section 4. In Section 5, conditions for host population persistence and extinction for each of these cases are derived. Regions of host population persistence and extinction for certain parameter ranges are illustrated *via* numerical simulations also in Section 5.

2 Model description

For the drinking model proposed in [13], the student population is divided into three distinct classes of non-drinkers (X), social drinkers (Y) and problem drinkers (Z) and the population is assumed to be constant. It is well known that the definitions of non-drinkers, social drinkers, and problem drinkers are not consistently used in the literature [8]. Therefore, the following definitions that would be of interest to college administrators and parents who are concerned with the negative consequences of campus-drinking problem are presented in [13].

(a) Non-drinkers are susceptible which we define as students who have not taking or consume alcohol for an extended period of time, typically one year. Also student who stop drinking is regarded as non-drinkers

(b) Social drinkers are defined as students that consume alcohol occasionally and in moderation

(c) Problem drinkers are defined as student whose drinking habit and associated behavior have negative consequence for themselves and others

Since students are only in campus for a relatively short period of time typically five year or less then it's not easy to access whether or not a student who has stopped drinking has permanently recovered. Hence, the model does not include a recovered class and developed on the assumptions that

(i) Student joint campus in any of the three drinking state and mix randomly with the rest of the campus population. Once in campus, student may transit from any drinking state to another.

(ii) Transition to higher drinking state is as a result of social interaction and transition to lower drinking state results from recovery process [8].

(iii) There is a net positive flow towards a higher drinking state. This is an effect that has been termed the 'college effect' in the literature [1]. Such an effect is more pronounced in the first 2 years of college and declines in the last 2 years during which time students transition towards adult drinking patterns [4].

(iv) The student population is homogeneous with respect to variables such as age, academic class rank and socio-economic status. More precisely, the homogeneity here means that students within each compartment are similar with regard to their drinking behaviors.

(v) The model does not consider the effect of breaks and vacations.

(vi) The interaction terms $\alpha XY, \kappa XZ$, and γYZ, respectively modelled the conversion from non-drinkers to social drinkers, non-drinkers to problem drinkers, and social drinkers to problem drinkers.

(vii) The conversions in return of (vi) are to be the result of recovery process and are implemented via the terms $\beta Y, \epsilon Z$ and δZ, respectively.

Thus the dynamics of campus drinking is modelled using the following system of nonlinear differential equations (equation 1), where X, Y and Z have been resealed as proportion. The respective rates of transfer between the three classes are shown in the flow diagram in Figure 1.

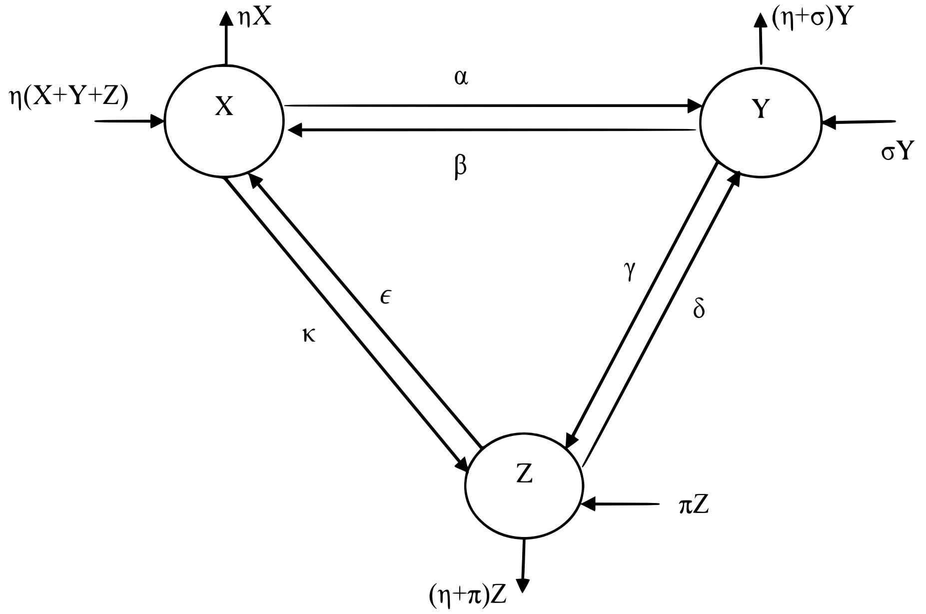

Figure 1: Flow diagram of model (1)

Parameter	Description	Typical value
α	Transmission rate of non- drinkers to social drinkers	1.0
β	Recovery rate of social drinkers	0.2
γ	Transmission rate of social drinkers to problem drinkers	1.0
δ	Recovery rate of problem drinkers to social drinkers	0.2
ϵ	Recovery rate of problem drinkers to non-drinkers	0.2
η	Departure rate from campus environment	0.25
κ	Transmission of non-drinkers to problem drinkers	1.5
σ	Entrance rate of social drinkers	0.25
π	Entrance rate of problem drinkers	0.1

Table 1: Parameter description

$$\begin{aligned}
\frac{dX}{dt} &= \eta - \eta X - \alpha XY - \kappa XZ + \beta Y + \epsilon Z,\\
\frac{dY}{dt} &= \sigma Y - (\eta+\sigma)Y + \alpha XY - \beta Y - \gamma YZ + \delta Z,\\
\frac{dZ}{dt} &= \pi Z - (\eta+\pi)Z + \gamma YZ + \kappa XZ - \delta Z - \epsilon Z,
\end{aligned} \tag{1}$$

with $X+Y+Z=1$. In system (1), the parameters α, γ, and κ are the mean transmission rates which measure the effectiveness of the interactions between non-drinkers and social drinkers, social drinkers and problem drinkers, and non-drinkers and problem drinkers, respectively. β is the recovery rate that represents the mean rate at which social drinkers revert to the non-drinking class. The recovery rates δ and ϵ are the mean rates at which problem drinkers move back to the social drinking and the non-drinking classes, respectively. Parameter η denotes the net departure rate from the campus environment. The other parameters of the model are described in Table 1.

For model (1) with initial condition $X > 0, Y > 0, Z > 0$ to be biologically meaningful, we need to show that the solutions of the model with positive initial conditions are non-negative and remain positive for all time $t > 0$. Moreover, the region these solutions exist is positively invariant and attractive.

We can use elementary techniques for solving ordinary differential equations to show that the state variables (X, Y and Z) are positive. In fact, using elementary integrating factor technique we have from the first equation of (1) the integrating factor

$$\rho(t) = e^{(\eta+\alpha Y+\kappa Z)},$$

so that

$$\frac{d(\rho X)}{dt} = \rho(\eta + \beta Y + \epsilon Z).$$

Integrating this, gives

$$\rho(t)X(t) = \rho(0)X(0) + (\eta + \beta Y + \epsilon Z)\int_0^t \rho ds \geq 0.$$

Similarly, from the second equation of (1)

$$\rho(t)Y(t) = \rho(0)Y(0) + (\eta + \beta Y + \delta Z)\int_0^t \rho ds \geq 0.$$

Then one can see by inspection from the third equation of (1) that

$$Z(t) = Z(0)\exp\left[\int_0^t \{\gamma Y(s) + \kappa X(s) - (\eta + \delta + \epsilon)\} ds\right] \geq 0.$$

Hence, $X(t), Y(t), Z(t) \geq 0$ for all $t \geq 0$.

Summing the three equations in (1), we have

$$\frac{d(X+Y+Z)}{dt} = \eta[1 - (X+Y+Z)].$$

It follows that $(X + Y + Z) \leq 1$, so that the feasible region

$$\mathcal{D} = \{(X, Y, Z) \in \mathbb{R}^3_+ : 0 < X + Y + Z \leq 1\}$$

is positively invariant for system (1).

3 Existence and stability of equilibria

3.1 Drinking problem free equilibria

This is an equilibrium which represents a free drinking environment (i.e. when the compartments of social and problem drinkers are empty). Thus, this equilibrium point is given by

$$E_0 = (X_0, Y_0, Z_0) = (1, 0, 0).$$

To identify condition under which a culture of drinking can be established a threshold parameter as in disease transmission models called the basic reproduction number is required. Considering the two level of infective, two reproductive numbers are obtained in the absence of drinking. The first reproductive number, denoted by $\mathcal{R}_0^Y$ is obtained from the second equation of system (1) as follows

$$\mathcal{R}_0^Y = \frac{\alpha}{\eta + \beta}.$$

This reproductive number is defined as the average number of secondary cases a typical social drinker generated in a nondrinking campus environment. Note that the secondary cases are cases of new social drinkers recruited from the nondrinking class. Similarly, from the third equation of system (1), we have the second reproductive number

$$\mathcal{R}_0^Z = \frac{\kappa}{\eta + \delta + \epsilon}.$$

The threshold parameter $\mathcal{R}_0^Z$ is defined similar to the $\mathcal{R}_0^Y$. The secondary cases here are the cases of new problem drinkers recruited from the nondrinking class.

Moreover, the reproduction numbers $\mathcal{R}_0^Y$ and $\mathcal{R}_0^Z$ are simply obtained from the product of the transmission rates α and κ and the times spent in the Y and Z classes $\left(\frac{1}{\eta+\delta+\epsilon} \; \& \; \frac{\alpha}{\eta+\beta}\right)$, respectively.

The Jacobian matrix of the system (1) evaluated at any equilibrium point is the following.

$$J(X,Y,Z) = \begin{pmatrix} -\eta - \alpha Y - \kappa Z & -\alpha X + \beta & -\kappa X + \epsilon \\ \alpha Y & -Q_1 + \alpha X - \delta Z & -\alpha Y + \delta \\ \kappa Z & \gamma Z & \alpha Y + \kappa X - Q_1 \end{pmatrix} \tag{2}$$

where $Q_1 = \eta + \beta, Q_2 = \eta + \delta + \epsilon$.

Theorem 3.1 The drinking-free equilibrium (DFE), E_0, of model (1) is a stable node on $\mathcal{D}$ if $\mathcal{R}_0^Y < 1, \mathcal{R}_0^Z < 1$ and is a saddle point otherwise.

Proof. Using (2), the Jacobian matrix evaluated at the drinking free equilibrium is

$$J(1,0,0) = \begin{pmatrix} -\eta & -\alpha + \beta & -\kappa + \epsilon \\ 0 & -Q_1 + \alpha & \delta \\ 0 & 0 & \kappa - Q_2 \end{pmatrix}. \tag{3}$$

It is easy to see that the eigenvalues of $J(1,0,0)$ are

$$-\eta, \alpha - Q_1 = Q_1(\mathcal{R}_0^Y - 1), \kappa - Q_2 = Q_2(\mathcal{R}_0^Z - 1).$$

These eigenvalues are real and negative whenever $\mathcal{R}_0^Y$ and $\mathcal{R}_0^Z$ are less than unity. Thus, E_0 is a stable node and a saddle point if $\mathcal{R}_0^Y$ and $\mathcal{R}_0^Z$ are greater than unity or one of the two is greater than one. Hence the proof. ■

3.2 Problem drinking-free equilibrium

At the problem drinking-free equilibrium, $X \neq 0, Y \neq 0$ and $Z = 0$. Therefore, from the second equation of (1), we obtain

$$X = \frac{\eta + \beta}{\alpha} = \frac{1}{\mathcal{R}_0^Y}. \tag{4}$$

Then using (4) in the first equation of (1) with Z=0, we have

$$Y = 1 - \frac{\eta + \beta}{\alpha} = 1 - \frac{1}{\mathcal{R}_0^Y}.$$

Thus, in the absence of problem drinkers on campus the second equilibrium point is given by

$$E_1 = (X_1, Y_1, Z_1) = \left(\frac{1}{\mathcal{R}_0^Y}, 1 - \frac{1}{\mathcal{R}_0^Y}, 0\right). \tag{5}$$

Thus, this equilibrium point exists when $\mathcal{R}_0^Y > 1$ and different from E_0. It is considered as second 'drinking-free' equilibrium if social drinking is not regarded as a problem. In this case, Z is the only infective compartment of system (1) and so, the third reproductive number, $\mathcal{R}_1$, which is the average number of secondary cases a typical problem drinker can produce in a campus environment with only non-drinkers and social drinkers. Here, the secondary cases are cases of new problem drinkers recruited from the non-drinking and social drinking classes. The threshold parameter $\mathcal{R}_1$ is then computed from the third equation of (1) as follows

$$\mathcal{R}_1 = \frac{\gamma(\alpha - \eta - \beta) + \kappa(\alpha + \beta)}{\alpha(\eta + \delta + \epsilon)} = \frac{\gamma\left(1 - \frac{1}{\mathcal{R}_0^Y}\right) + \frac{\kappa}{\mathcal{R}_0^Y}}{\eta + \delta + \epsilon}.$$

Theorem 3.2 The problem drinking-free equilibrium E_1 is a stable node on $\mathcal{D}$ if $\mathcal{R}_1 < 1, \mathcal{R}_0^Y > 1$ and a saddle point otherwise.

Proof. The Jacobian matrix evaluated at E_1 is obtained from (2) as follows

$$J(X_1, Y_1, 0) = \begin{pmatrix} -\eta - \alpha Y_1 & -\alpha X_1 + \beta & -\kappa X_1 + \epsilon \\ \alpha Y_1 & -Q_1 + \alpha X_1 & -\alpha Y_1 + \delta \\ 0 & 0 & \alpha Y_1 + \kappa X_1 - Q_2 \end{pmatrix}. \tag{6}$$

It can be observed that one of the eigenvalues of $J(X_1, Y_1, 0)$ is

$$\alpha Y_1 + \kappa X_1 - Q_2 = Q_2(\mathcal{R}_1 - 1)$$

and the other two are of the 2×2 sub-matrix $J^*(X_1, Y_1, 0)$ of $J(X_1, Y_1, 0)$ obtained by deleting the third row and the third column of $J(X_1, Y_1, 0)$.

$$J^*(X_1, Y_1, 0) = \begin{pmatrix} -\alpha X_1 + \beta & -\kappa X_1 + \epsilon \\ -Q_1 + \alpha X_1 & -\alpha Y_1 + \delta \end{pmatrix}.$$

Now, using (5) the characteristics equation of $J^*(X_1, Y_1, 0)$ is

$$\lambda^2 + \left[\eta + \alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right)\right]\lambda + \eta\alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right) = 0$$

so that

$$\lambda = \frac{-\left[\eta + \alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right)\right] \pm \left[\eta - \alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right)\right]}{2}$$

implying that $\lambda_1 = -\eta$ and $\lambda_2 = -\alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right)$. Then the three eigenvalues of $J(X_1, Y_1, 0)$ are $Q_2(\mathcal{R}_1 - 1), -\eta, -\alpha\left(1 - \frac{1}{\mathcal{R}_0^Y}\right)$, which are real and negative when $\mathcal{R}_1 < 1$ and $\mathcal{R}_0^Y > 1$. Moreover, the first eigenvalue is positive if $\mathcal{R}_1 > 1$ and the third one is positive when $\mathcal{R}_0^Y < 1$. It follows that the problem drinking-free equilibrium E_1 is a stable node if $\mathcal{R}_1 < 1$ and $\mathcal{R}_0^Y > 1$ and a saddle point otherwise . ■

The result of this theorem shows that, it is possible for non-drinkers and social drinkers to coexists at equilibrium provided $\mathcal{R}_1 < 1$ and $\mathcal{R}_0^Y > 1$.

3.3 Endemic equilibria

With the persistence of the problem of drinking abuse in college campuses, all the three drinking states can be present at a time in a campus. From the third equation of (1), we obtain at endemic state

$$X = \frac{\eta - \gamma Y + \delta + \epsilon}{\kappa}. \tag{7}$$

Then using (7) in the second equation of system (1), we have

$$\sigma Y - (\eta + \sigma)Y + \alpha\frac{\eta - \gamma Y + \delta + \epsilon}{\kappa}Y - \beta Y - \gamma Y Z + \delta Z = 0$$

and, so

$$Z = \frac{\alpha\gamma Y^2 + (\eta\kappa + \beta\kappa - \alpha\eta - \alpha\delta - \alpha\epsilon)Y}{\kappa(\delta - \gamma Y)}. \tag{8}$$

Substituting the expressions of X and Z in (7) and (8), respectively, into equation $X+Y+Z=1$, we obtain after algebraic manipulation the following quadratic equation in Y.

$$aY^2 + bY + c = 0, \tag{9}$$

where

$$\begin{aligned} a &= \gamma(\kappa - \alpha - \gamma), \\ b &= -\kappa\gamma + \alpha\delta + \epsilon\gamma + \eta\alpha - \delta\kappa + \alpha\epsilon - \beta\kappa + \eta\gamma + 2\gamma\delta - \eta\kappa \\ &= -\gamma(\eta + \delta + \epsilon)(\mathcal{R}_0^Z - 1) - \kappa(\delta + \beta + \eta) + \delta(\alpha + \gamma) + \alpha(\eta + \epsilon), \\ c &= \delta(\eta + \delta + \epsilon)(\mathcal{R}_0^Z - 1). \end{aligned} \tag{10}$$

Then the following theorem provides condition under which an equilibrium proportion of social drinkers can exist.

Theorem 3.3 The system of equations (1) has

(i) a unique endemic equilibrium if $a > 0, c < 0$.

(ii) a unique endemic equilibrium if $a > 0, b < 0$ and $c = 0$ or $b^2 - 4ac = 0$.

(iii) two endemic equilibria if $a > 0, b < 0, c > 0$ and $b^2 - 4ac > 0$.

(iv) two endemic equilibria if $a < 0, b > 0, c < 0$ and $b^2 - 4ac > 0$.

(v) no endemic equilibrium otherwise.

Proof. The proof follows from the properties of the roots of quadratic equation if $\mathcal{R}_0^Z$ is less than or greater than unity. For $\mathcal{R}_0^Z = 1$, the statements can be verified by inspection.

The possible existence of multiple endemic equilibrium points is given in (iii) and (iv) of the Theorem above which suggest the possibility of backward bifurcation, where the stable DFE coexist with a stable endemic equilibrium, when the associated reproductive number is less than unity.

4 Bifurcation analysis and numerical results

To investigate the dynamics of model (1) using the Center manifold theory as presented in [5], we make the following transformation on the model. Let $X = p_1, Y = p_2$ and $Z = p_3$ so that the total population is $1 = p_1 + p_2 + p_3$. In vector notation, we have $P = (p_1, p_2, p_3)^T$ with the system (1) written in the form $\frac{dP}{dt} = (f_1, f_2, f_3)^T$. It follows that

$$\begin{aligned} \frac{dP}{dt} &= f_1 = \eta - \eta p_1 - \alpha p_1 p_2 - \kappa p_1 p_3 + \beta p_2 + \epsilon p_3, \\ \frac{dY}{dt} &= f_2 = \sigma p_2 - (\eta + \sigma)p_2 + \alpha p_1 p_2 - \beta p_2 - \gamma p_2 p_3 + \delta p_3, \\ \frac{dZ}{dt} &= f_3 = \pi p_3 - (\eta + \pi)p_3 + \gamma p_2 p_3 + \kappa p_1 p_3 - \delta p_3 - \eta p_3. \end{aligned} \tag{11}$$

The trace and determinant of $J(E_0)$ as presented in (3) are respectively

$$tr(J(E_0)) = -\eta - Q_1 + \alpha + \kappa - Q_2 \neq 0,$$

and

$$\det(J(E_0)) = -\eta(\alpha - Q_1)(\kappa - Q_2) = -\eta Q_1 Q_2(\mathcal{R}_0^Y - 1)(\mathcal{R}_0^Z - 1).$$

This determinant is zero if either $\mathcal{R}_0^Y = 1$ or $\mathcal{R}_0^Z = 1$ or $\mathcal{R}_0^Y = \mathcal{R}_0^Z = 1$. The first two relations are equivalent to $\alpha = \eta+\beta$ or $\kappa = \eta+\delta+\epsilon$. Thus, $J(E_0)$ have a simple zero eigenvalue and so, the Center Manifold Theory can be applied. Let the left eigenvectors for $J(E_0)$ be $V = (v_1, v_2, v_3)^T$ then using standard computation of eigenvectors, we have

$$v = \left(1, \frac{\alpha - \beta}{\eta + \beta + \alpha}, \frac{\delta(\alpha - \beta) + (\kappa - \epsilon)(\eta + \beta + \alpha)}{Q_2(\eta + \beta + \alpha)(\mathcal{R}_0^Z - 1)}\right). \tag{12}$$

Similarly, the right eigenvectors corresponding to the simple zero eigenvalue $w = (w_1, w_2, w_3)^T$ are the following.

$$w = \left(\frac{-\delta(\beta - \alpha) + Q_1(\epsilon - \kappa)(\mathcal{R}_0^Y - 1)}{\eta Q_1(\mathcal{R}_0^Y - 1)}, \frac{-\delta}{Q_1(\mathcal{R}_0^Y - 1)}, 1\right). \tag{13}$$

The nonzero partial derivatives of the vector field f of (11) evaluated at E_0 are as follows

$$\begin{aligned}
&\frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\partial^2 f_1}{\partial x_2 \partial x_1} = -\alpha, \ \ \frac{\partial^2 f_1}{\partial x_1 \partial x_3} = \frac{\partial^2 f_1}{\partial x_3 \partial x_1} = -\kappa,\\
&\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \alpha, \ \ \frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_2} = -\gamma,\\
&\frac{\partial^2 f_3}{\partial x_1 \partial x_3} = \frac{\partial^2 f_3}{\partial x_3 \partial x_1} = \kappa, \ \ \frac{\partial^2 f_3}{\partial x_2 \partial x_3} = \frac{\partial^2 f_3}{\partial x_3 \partial x_2} = \gamma.
\end{aligned} \tag{14}$$

Computation of a

Using equations (12), (13) and (14), we have

$$\begin{aligned}
a &= \sum_{k,i,j=1}^{3} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}\\
&= 2\frac{-\delta(\beta - \alpha) + Q_1(\epsilon - \kappa)(\mathcal{R}_0^Y - 1)}{\eta Q_1(\mathcal{R}_0^Y - 1)}\left(\frac{\alpha\delta}{Q_1(\epsilon - \kappa)(\mathcal{R}_0^Y - 1)} - \kappa\right)\\
&+ 2\frac{-\delta(\beta - \alpha) + Q_1(\epsilon - \kappa)(\mathcal{R}_0^Y - 1)}{\eta Q_1(\mathcal{R}_0^Y - 1)}\frac{\delta\phi(\beta - \alpha) + Q_1\varphi(\epsilon - \kappa)(\mathcal{R}_0^Y - 1)}{Q_1 Q_2 \varphi(\mathcal{R}_0^Y - 1)(\mathcal{R}_0^Z - 1)}\\
&- 2\frac{\delta\gamma[\varphi(\kappa - \epsilon) - (\beta - \alpha)(Q_2\varphi + \delta)]}{Q_1 Q_2 \varphi(\mathcal{R}_0^Y - 1)(\mathcal{R}_0^Z - 1)}.
\end{aligned} \tag{15}$$

with $\phi = Q_1(\mathcal{R}_0^Y - 1) + Q_2(\mathcal{R}_0^Z - 1), \varphi = \eta + \beta + \alpha$.

Computation of b

Now, using the vectors v and w as presented in (12) and (13), respectively with the following associated partial derivatives

$$\frac{\partial^2 f_1}{\partial x_3 \partial \kappa^*}(E_0) = -1, \ \ \frac{\partial^2 f_3}{\partial x_3 \partial \kappa^*}(E_0) = 1$$

we have

$$b = \sum_{k,i}^{3} v_k x_i \frac{\partial^2 f_k}{\partial x_i \partial \kappa^*} = \frac{(\delta + \eta)(\eta + \beta) + \eta\alpha + \delta\beta}{Q_2(\eta + \beta + \alpha)(\mathcal{R}_0^Z - 1)}$$

where $\kappa = \kappa^*$ is a bifurcation parameter. Since b is positive if $\mathcal{R}_0^Z > 1$, it follows that model (1) will undergo backward bifurcation if $a > 0$. On the other hand, the model will exhibit a forward bifurcation when $a < 0$.

We find the bifurcation curve by solving for the bifurcation parameter κ^* in equation (9) as follows.

$$\kappa(Y) = \frac{\gamma(\alpha+\gamma)Y^2 - [\gamma(\eta+\delta+\epsilon) + \delta(\alpha+\delta) + \alpha(\eta+\epsilon)]Y + \delta(\eta+\delta+\epsilon)}{\gamma Y^2 - (\gamma+\delta+\beta+\eta)Y + \delta}.$$

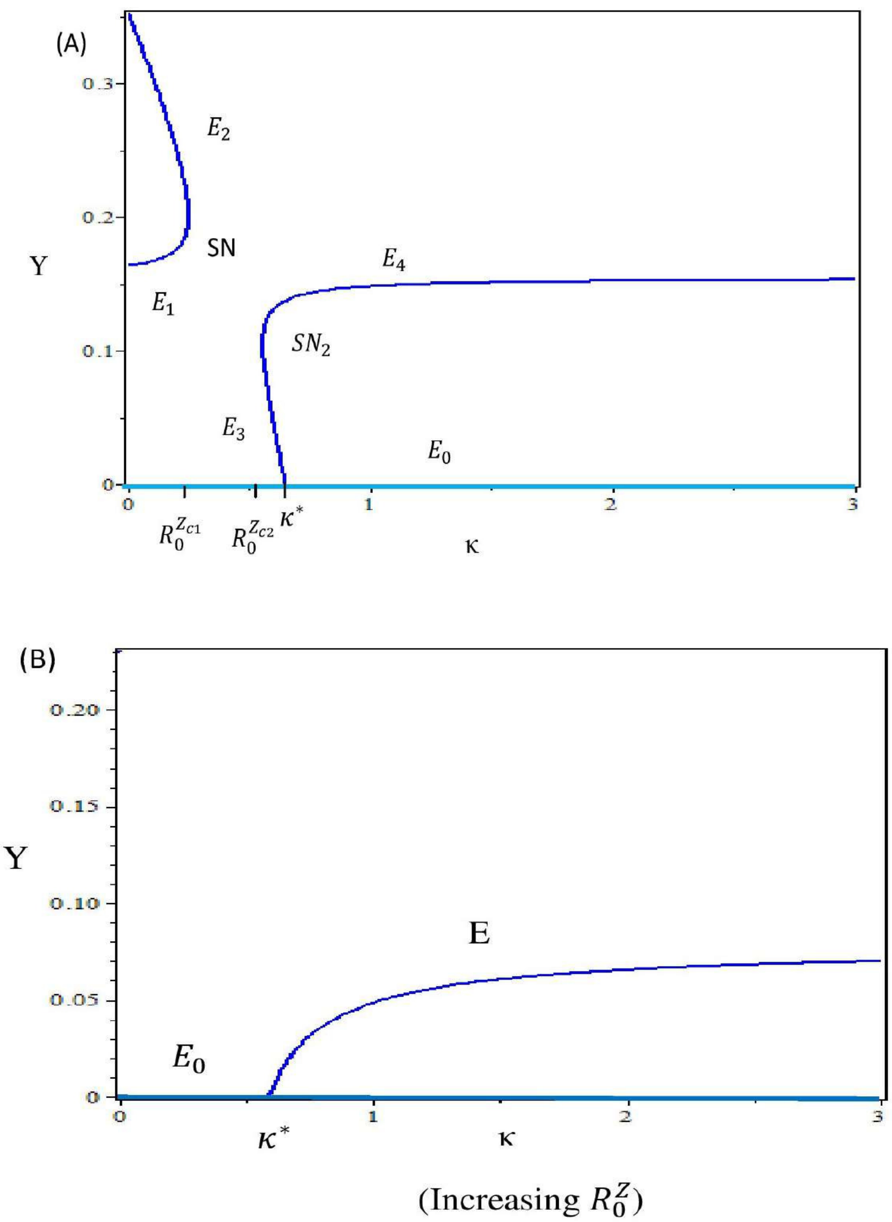

Figure 2: Bifurcation diagrams for varying the transmission rate of non-drinkers to problem drinkers κ (A) showing backward bifurcation, SN indicate the saddle-node for which the proportion of social drinkers collapses, SN_2 indicates another saddle-node for which the problem will persist in the campus and (B) forward bifurcation.

We use the parameter values $\alpha = 1.25, \beta = 0.0000002, \gamma = 1, \delta = 0.2, \epsilon = 0.2$, and $\eta = 0.25$ to plot the bifurcation diagram in figure 2(A) which shows how social drinkers proportion changes with varying the bifurcation parameter $\kappa = \kappa^*$ or equivalently the basic reproductive number $\mathcal{R}_0^Z$. The threshold parameter $\mathcal{R}_0^Z$ is varied by altering the transmission rate of non-drinkers to problem drinkers, κ. If $\mathcal{R}_0^Z < 1 < \mathcal{R}_0^{Z_{c1}}$, the drinking problem invades the population as the stable drinking-free equilibrium E_0 coexists with a stable endemic equilibrium point E_2. These equilibrium points coalesce and disappear by a saddle-node (SN) bifurcation at $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c1}}$. Thus, the proportion of social drinkers collapses. However, if $\mathcal{R}_0^Z$ is between $\mathcal{R}_0^{Z_{c1}}$ and $\mathcal{R}_0^{Z_{c2}}$ the only stable equilibrium point is the drinking-free equilibrium E_0. After the extinction regime another critical threshold parameter $\mathcal{R}_0^{Z_{c2}}$ exists for which a second saddle-node bifurcation occurs for the persistence of the drinking problem. This scenario resulting to the emergence of two endemic equilibrium points (E_3 and E_4) at $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c2}}$ after the extinction regime $(\mathcal{R}_0^{Z_{c1}}, \mathcal{R}_0^{Z_{c2}})$. In this case, the drinking-free equilibrium (E_0) and endemic equilibrium E_4 are locally stable in the interval $(\kappa^*, \mathcal{R}_0^{Z_{c2}})$ due to the occurrence of backward bifurcation at the bifurcation point $\kappa = \kappa^*$. But for $\kappa > \kappa^*$ the equilibrium E_0 loses stability and a locally stable endemic equilibrium is only E_4.

It follows that, the critical threshold parameter $\mathcal{R}_0^{Z_{c1}}$ is a tipping point for an unexpected population collapse. This leads to a mono stable system whenever $\mathcal{R}_0^Z > \mathcal{R}_0^{Z_{c1}}$ such that the only global attractor is the drinking-free equilibrium point E_0. More precisely, there is a saddle-node bifurcation for $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c1}}$, after which the proportion of social drinkers goes extinct. Moreover, there is another critical threshold parameter $\mathcal{R}_0^{z_{c2}}$ which is greater than $\mathcal{R}_0^{z_{c1}}$, for the occurrence of a second saddle-node bifurcation. This scenario gives rise to two endemic equilibrium points E_3 and E_4, as rapidly as the endemic attractor disappears at $\mathcal{R}_0^Z = \mathcal{R}_0^{z_{c1}}$. They re-emerge after the extinction regime $(\mathcal{R}_0^{z_{c1}}, \mathcal{R}_0^{z_{c2}})$.

One can observe from Figure 2(A) that these two saddle-node bifurcations, if they exist are separate from each other and not related by sharing the same saddle-node as in the scenarios with three endemic equilibrium points [6, 7, 9]. This discontinuity has a profound implication for control measure aimed at impacting the threshold parameter $\mathcal{R}_0^Z$. In fact, increasing $\mathcal{R}_0^Z$ before and after the extinction regime (i.e., below $\mathcal{R}_0^{z_{c1}}$ and above $\mathcal{R}_0^{z_{c2}}$) can be beneficial for the social drinkers sub-population since it facilitates endemic persistence not extinction. On the other hand, reducing $\mathcal{R}_0^Z$ below $\mathcal{R}_0^{z_{c2}}$ but above $\mathcal{R}_0^{z_{c1}}$ can be detrimental to social drinkers, as it causes eradication of this sub-population. Moreover, decreasing $\mathcal{R}_0^Z$ below $\mathcal{R}_0^{z_{c2}}$ and above $\mathcal{R}_0^{z_{c1}}$ can be beneficial to the host population since it leads to the stability of the system at drinking-free equilibrium point E_0.

In Figure 2(B), we show using the parameter values $\alpha = 1.207, \beta = 1.2, \gamma = 1, \delta = 0.2, \epsilon = 0.141$, and $\eta = 0.25$ a forward (transcritical) bifurcation for which the stable drinking-free equilibrium point E_0 loses stability at $\kappa > \kappa^*$ or equivalently $\mathcal{R}_0^Z > 1$ due to the emergence of a unique stable endemic equilibrium point (E). In this case, the classical requirement for drinking problem control (reducing κ below κ^* or equivalently $\mathcal{R}_0^Z$ below 1) could be sufficient to eradicate the drinking problem in the campus.

4.1 Numerical simulations

We plot the numerical solutions of model (1) using Runge-Kutta RK5 with parameter values as presented in Table 1 to see the impact of the transmission rate of non-drinkers to problem drinkers on the dynamics of the model. Figure 3(A) depicts the persistence of drinking problem when $\mathcal{R}_0^Z = 0.3846153846 < 1$. That is, the social and problem drinkers recruit the non drinkers. This scenario correspond to the persistence before $\mathcal{R}_0^{Z_{c1}}$ in Figure 2(A). In Figure 3(B) an extinction scenario corresponding to Figure 2(A) which occur between $\mathcal{R}_0^{Z_{c1}}$ and $\mathcal{R}_0^{Z_{c2}}$ with only a stable drinking-free equilibrium is shown. Such a scenario occurs when $\mathcal{R}_0^Z = 0.5 < 1$. Another persistence scenario is depicted in Figure 3(C) showing that the drinking problem persist in the campus when $\mathcal{R}_0^Z = 2.307692308 > 1$.

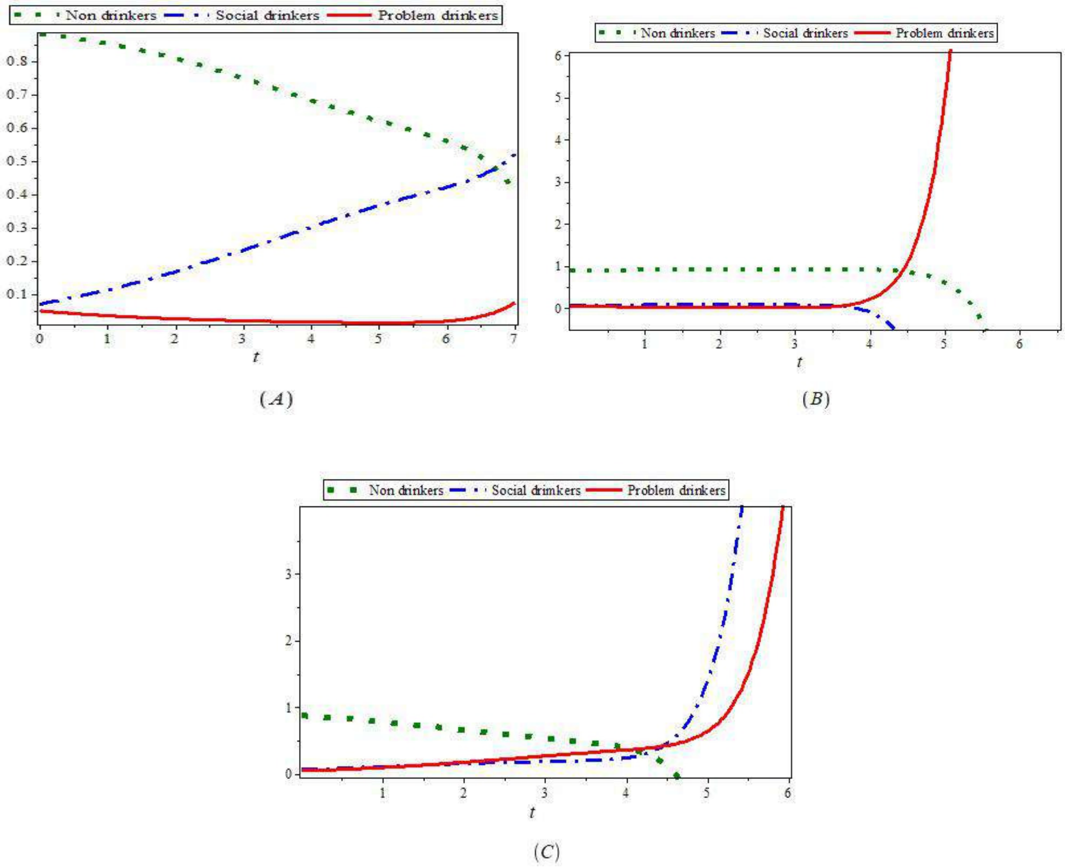

Figure 3: Time diagrams of model (1) with: (A) using the parameter values in Table 1 showing that when $R_0^Z = 0.3846153846 < 1$ drinking problem persist on campus; (B) using $\kappa = 0.75, \epsilon = 0.25, \eta = 0.5, \delta = 0.5$ and other parameter values are as in Table 1 showing that when $R_0^Z = 0.5 < 1$ there is a drinking problem free campus; (C) parameter values as in Table 1 showing that when $R_0^Z = 2.307692308 > 1$ drinking Problem persist on the campus. The graphs of non-drinkers (dot), social drinkers (dash dot) and problem drinkers(solid).

5 Discussion and conclusion

We review the drinking model presented in [13] and investigate its complex dynamics noting that the model exhibits multiple endemic equilibrium points. In particular, the model have a drinking-free equilibrium and problem drinking-free equilibrium point which exists if $\mathcal{R}_0^Y > 1$. These equilibrium points are stable nodes if $\mathcal{R}_0^Y < 1, \mathcal{R}_0^Z < 1$ and $\mathcal{R}_1 < 1, \mathcal{R}_0^Y > 1$, respectively and are saddle points otherwise. Then when the drinking problem persist on campus two equilibrium points emerged called endemic or drinking present equilibrium points. The existence of multiple drinking-present equilibrium points enable us to use Center manifold theory as presented in [5] to investigate the existence of the phenomenon of a backward bifurcation in section 4. In fact, we show in Figure 2 the bifurcation diagram of such phenomenon by varying the bifurcation parameter $\kappa = \kappa^*$ or equivalently the basic reproductive number $\mathcal{R}_0^Z$. That is, the threshold parameter $\mathcal{R}_0^Z$ is varied by altering the transmission rate of non-drinkers to problem drinkers, κ. More precisely, figure 2(A) shows that if $\mathcal{R}_0^Z < 1 < \mathcal{R}_0^{Z_{c1}}$, the drinking problem invades the population as the stable drinking-free equilibrium E_0 coexists with a stable endemic equilibrium point E_2. These equilibrium points coalesce and disappear by a saddle-node (SN) bifurcation at $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c1}}$. Thus, the proportion of social drinkers collapses. However, if $\mathcal{R}_0^Z$ is between $\mathcal{R}_0^{Z_{c1}}$ and $\mathcal{R}_0^{Z_{c2}}$ the only stable equilibrium point is the drinking-free equilibrium E_0. After the extinction regime another critical threshold parameter $\mathcal{R}_0^{Z_{c2}}$ exists for which a second saddle-node bifurcation occurs resulting in the persistence of the drinking problem. This scenario results to the emergence of two endemic equilibrium points (E_3 and E_4) at $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c2}}$ after the extinction regime $(\mathcal{R}_0^{Z_{c1}}, \mathcal{R}_0^{Z_{c2}})$ due to the occurrence of the phenomenon of backward bifurcation. In this case, the stable drinking-free equilibrium (E_0) coexist with a stable endemic equilibrium E_4 in the interval $(\mathcal{R}_0^{Z_{c2}}, \kappa^*)$. Then for $\kappa > \kappa^*$ the equilibrium E_0 loses stability and a locally stable endemic equilibrium is the only E_4. This result is similar to the findings in [10, 18].

Thus, the critical threshold parameter $\mathcal{R}_0^{Z_{c1}}$ is a tipping point for an unexpected population collapse. This leads to a mono stable system whenever $\mathcal{R}_0^Z > \mathcal{R}_0^{Z_{c1}}$ such that the only global attractor is the drinking-free equilibrium point E_0. In particular, there is a saddle-node (SN) bifurcation at $\mathcal{R}_0^Z = \mathcal{R}_0^{Z_{c1}}$, after which the proportion of social drinkers goes extinct. Moreover, there is another critical threshold parameter $\mathcal{R}_0^{z_{c2}}$ which is greater than $\mathcal{R}_0^{z_{c1}}$, for the occurrence of a second saddle-node bifurcation (SN_2). This scenario gives rise to two endemic equilibrium points E_3 and E_4, as rapidly as the endemic attractor disappears at $\mathcal{R}_0^Z = \mathcal{R}_0^{z_{c1}}$. These equilibrium points re-emerge after the extinction regime $(\mathcal{R}_0^{z_{c1}}, \mathcal{R}_0^{z_{c2}})$.

We observe from Figure 2(A) that these two saddle-node bifurcations, if they exist are separate from each other and not related by sharing the same saddle-node as in the scenarios with three endemic equilibrium points [9, 6, 7]. This discontinuity has a profound implication for control measure aimed at impacting the threshold parameter $\mathcal{R}_0^Z$. In fact, increasing $\mathcal{R}_0^Z$ before and after the extinction regime (i.e., below $\mathcal{R}_0^{z_{c1}}$ and above $\mathcal{R}_0^{z_{c2}}$) can be beneficial for the social drinkers sub-population since it facilitates endemic persistence not extinction. On the other hand, reducing $\mathcal{R}_0^Z$ below $\mathcal{R}_0^{z_{c2}}$ but above $\mathcal{R}_0^{z_{c1}}$ can be detrimental to social drinkers, as it causes eradication of this sub-population. Moreover, decreasing $\mathcal{R}_0^Z$ below $\mathcal{R}_0^{z_{c2}}$ and above $\mathcal{R}_0^{z_{c1}}$ can be beneficial to the host population since it leads to the stability of the system at drinking-free equilibrium point E_0.

We also show in Figure 2(B), a forward (transcritical) bifurcation for which the stable drinking-free equilibrium point E_0 loses stability at $\kappa > \kappa^*$ or equivalently $\mathcal{R}_0^Z > 1$ due to the emergence of a unique stable endemic equilibrium point (E). In this case, the classical requirement for drinking problem control (reducing κ below κ^* or equivalently $\mathcal{R}_0^Z$ below 1) could be sufficient to eradicate the problem in the campus.

In order to complete our investigations, we simulate the numerical solution of model (1) using Runge-Kutta RK5 with Maple 18 software and the parameter values presented in Table 1. Indeed, we show in Figure 3 the persistence and extinction scenarios of the drinking problem. To be more precise, when $\mathcal{R}_0^Z = 0 : 3846153846 < 1$ ($\kappa \ll \kappa^*$) and $\mathcal{R}_0^Z = 2 : 3076692308 > 1$ ($\kappa > \kappa^*$) the drinking problem persist in the campus environment (see Figures, 3(A), and 3(C), respectively for the persistence). This scenario is similar to that of Figure 2(A) below $\mathcal{R}_0^{Z_{c1}}$ and above $\mathcal{R}_0^{Z_{c2}}$. On the other hand, Figure 3(B) shows that the drinking problem goes to extinct if $\mathcal{R}_0^Z = 0.5 < 1$ since the population collapse as in the extinction regime $(\mathcal{R}_0^{Z_{c1}}, \mathcal{R}_0^{Z_{c2}})$ shown in Figure 3(A).

In conclusion, although the campus alcohol abuse may be reduced by minimizing the ability of problem drinkers to directly recruit non-drinkers as reported in [13] but our analysis reveals that such strategy is to be applied with caution. This is because the occurrence of two separate saddle-node bifurcations make the control more difficult. Indeed, reducing the direct recruitment of non-drinkers to problem drinkers below the first saddle-node SN and above the second one SN_2 can leads to the persistence of drinking problem in the campus.

Conflict of interest

All the authors state that there is no conflict of interest.

References

[1] Adu I.K., EL-Nor Osman M., and Yang C. (2017). Mathematical model of drinking epidemic. British Journal of Mathematics and Computer Science, Vol. 22(5): 1-10.

[2] Agrawal A., Tenguria A., and Modi G. (2018). Role of epidemic model to control drinking problem. International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol. 5(4): 341-354.

[3] Benedict F. (2007). Modeling alcoholism as a contagious disease: how "infected" drinking buddies spread problem drinking, SIAM News, Vol. 4(3).

[4] Bhunu C.P. (2012). A mathematical analysis of alcoholism. World Journal of Modelling and Simulation, Vol. 8(2): 124-134.

[5] Castillo-Chavez C. and Song B. (2004), Dynamical models of tuberculosis and their applications. Mathematical Biosciences and Engineering, Vol. 1(2): 361-404.

[6] Greenhalgh D. and Griffiths M. (2009). Dynamic phenomena arising from an extended core group model. Math. Biosci., Vol. 221(2): 136–149.

[7] Gulbudak H. and Martcheva M. (2013., Forward hysteresis and backward bifurcation caused by culling in an avian influenza mode. Math. Biosci., Vol. 246(1): 202–212.

[8] Ham L.S. and Hope D.A. (2003. College students and problematic drinking: a review of literature. Clin. Psychol. Rev., Vol. 23: 719–759.

[9] Hilker F., Langlias M., and Malchow H. (2009). The Allee effect and infectious diseases: Extinction, multiplicity, and the (dis-)appearance of oscillations. Am. Nat., Vol. 173: 72–88.

[10] Hilker F.M. (2010., Population collapse to extinction: the catastrophic combination of parasitism and Allee effect. J. Biol. Dyn., Vol. 4(1): 86–10.

[11] Ma S. *et al.* (2021). Global dynamics of a delayed alcoholism model with the effect of health education. Mathematical Biosciences and Engineering, Vol. 18(1): 904–932.

[12] Ma S., Huo H., and Xiang H. (2019). Threshold dynamics of a multi-group SEAR alcoholism model with public health education. International Journal of Biomathematics, Vol. 12(2): 1-24.

[13] Manthey J.L., Aidoo A.Y., and Ward K.Y. (2008). Campus drinking: an epidemiological model. Journal of Biological Dynamics, Vol. 2(3): 346-356.

[14] O'Malley P.M., and Johnston L.D. (2002). Epidemiology of alcohol and other drug use among American college student. Journal of Studies on Alcohol, Supplement No. 14: 23-39.

[15] Sánchez B., Wang X., *et al.* (2007). Drinking as an Epidemic-A Simple Mathematical Model with Recovery and Relapse. Therapist's Guide to Evidence-Based Relapse Prevention, 353-368.

[16] Sharma S. and Samanta G.P. (2015). Analysis of a drinking epidemic model. Int. J. Dynam. Control, doi: 10.1007/s40435-015-0151-8.

[17] Sharma S. and Samanta G.P. (2013). Drinking as an epidemic: A mathematical model with dynamic behaviour. J. Appl. Math. and Informatics, Vol. 31(1-2): 1-25.

[18] Usaini S., Lloyd A.L., Anguelov R., and Garba S.M. (2017). Dynamical behavior of an epidemiological model with a demographic Allee effect. Mathematics and Computers in Simulation, Vol. 133: 311-32

[19] World Health Organization (2012). Alcohol in the European Union: Consumption, harm and policy approaches, www.who.org.

[20] World Health Organization (2018). Global status report on alcohol and health. http://www.who.int/substance-abuse/publication/global-alcohol-report/en/.

[21] Zhang Z., Zou J. and Kundu S. (2020). Bifurcation and optimal control analysis of a delayed drinking model. Advances in Difference Equations, Vol. 522: 1-20.