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Application of quadratic Lyapunov function for SIR model with demography

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Abstract

Lyapunov functions enable the stability analysis of dynamic systems described by ordinary differential equations without finding the solution of such equations. In this paper, we construct a quadratic Lyapunov function for SIR model with demography in analogy with a linear system, and show that the SIR model with demography is asymptotically stable using this quadratic Lyapunov function. This quadratic Lyapunov function satisfies the disease free and endemic equilibrium.

Keywords: Lyapunov function; asymptotic; stability; differential equation

1 Introduction

Lyapunov functions are scalar functions used to prove the stability of equilibrium of ordinary differential equations. It is very important to stability theory in dynamical systems and control theory. Lyapunov function is a necessary and sufficient condition for stability. There are many ways to construct Lyapunov functions and much remains to be done in this regard since there isn't a universal constructive method for finding simple and explicit Lyapunov functions which helps to determine the stability or instability of a system as well as acting as control.

Lyapunov himself [7] indicated the method of constructing the Lyapunov functions for an independent linear system. In [10], they constructed a Lyapunov function for a second and third order respectively with non-linear functions. Their method consists of finding a quadratic form function for a linear system, and analogy in the subsequent selection, a Lyapunov function for the non-linear system. The method also gives us the possibility of obtaining the Lyapunov function in a rather small proximity to the equilibrium position of the non-linear system [1]. Further results in the construction of Lyapunov functions were obtained in the connection with the so-called Aizerman problem [12]. The formulation of this problem prompts the idea of constructing Lyapunov function for non-linear systems by analogy with linear systems. [4] have studied the singular points of third order non-linear ordinary differential equations. According to them, only one singular point of each of the different equations is stable and the rest of the singular points are unstable. Also, they have not investigated the stability character of the system over some regions around a singular point.

For an autonomous polynomial system of differential equations, how to compute the Lyapunov function at equilibria is a basic problem. In [8], the author transformed the problem of computing the Lyapunov function into a quantifier elimination problem. The disadvantage of the method is that the computation complexity of quantifier elimination is doubly exponential in the

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number of total variables. In order to avoid this problem,[14] propose a symbolic method; they first construct a special semi algebraic system using the local properties of a Lyapunov function as well as its derivative and solving these inequations using cylindrical algebraic decomposition (CAD). Lyapunov functions are indispensable in verifying some quantitative properties of solutions in ordinary differential equations. The problems encountered in the applications are in the construction of appropriate Lyapunov functions for some non-linear ordinary differential equations [2]. Lyapunov function [9] is a good tool in the stability analysis of dynamical systems.

The SIR model is a classical model in mathematical epidemiology [13]. Particularly Kermack and McKendrick[5] use a SIR model to prove the existence of a threshold. The model of Kermack and McKendrick is without demography, i.e. without vital dynamics. As a conceptual model, the classical SIR models are very important [13] . However, the asymptotic behavior of the model changes when introducing vital dynamics. The global stability using Lyapunov functions of SIR model with a total constant population is proved in [3] and [13].

In this work, we established the application of quadratic lyapunov function [10] for the SIR model with demography in analogy with the linear system.

2 Methodology and Discussions

Consider the ordinary differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0, \tag{1}$$

resulting to the third scalar first order shown as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -az - by - cx, \end{aligned} \tag{2}$$

and the system of third- dimensional linear equation

$$\dot{X} = AX, \tag{3}$$

where A is positive definite metrics. Define

$$2V(x, y, z) = X^T AX = A_1x^2 + A_2y^2 + A_3z^2 + 2A_4xy + 2A_5xz + 2A_6yz \tag{4}$$

such that

$$\dot{V} = -cA_5x^2 - (bA_6 - A_4)y^2 - (aA_3 - A_6)z^2 - (cA_6 + bA_5 - A_1)xy - (cA_3 + aA_5 - A_4)xz - (aA_6 + bA_3 - A_2 - A_5)yz. \tag{5}$$

We have the following possibilities:

$$\begin{aligned} \dot{V} &\propto -x^2, \\ \dot{V} &\propto -y^2, \\ \dot{V} &\propto -z^2, \\ \dot{V} &\propto -(x^2 + y^2 + z^2). \end{aligned} \tag{6}$$

CASE I:

If $\dot{V} = -cA_5x^2$, then

$$\begin{aligned} cA_5 &> 0, \\ bA_6 - A_4 &= 0, \\ aA_3 - A_6 &= 0, \\ cA_6 + bA_5 - A_1 &= 0, \\ cA_3 + aA_5 - A_4 &= 0, \\ aA_6 + bA_3 - A_2 - A_5 &= 0, \end{aligned} \tag{7}$$

such that

$$\begin{aligned} A_1 &= \frac{(a^2c + ab^2 - cb)A_5}{k}, \\ A_2 &= \frac{(a^3 + c)A_5}{k}, \\ A_3 &= \frac{aA_5}{k}, \\ A_4 &= \frac{a^2bA_5}{k}, \\ A_6 &= \frac{a^2A_5}{k}, \end{aligned} \tag{8}$$

where $ab - c = k > 0$. Substituting (8) in (4) and set $A_5 = 1$ since $A_5 > 0$,

$$\begin{aligned} 2V(x,y,z) &= \frac{(a^2c + ab^2 - cb)}{k}x^2 + \frac{(a^3+c)}{k}y^2 + \frac{a}{k}z^2 + 2\frac{a^2b}{k}xy + 2xz + 2\frac{a^2}{k}yz \\ &= \frac{a^2c}{k}x^2 + bx^2 + \frac{a^3+c}{k}y^2 + \frac{a}{k}z^2 + \frac{2a^2b}{k}xy + 2xz + \frac{2a^2}{k}zy \\ &= bx^2 + \frac{2a^2b}{k}xy + \frac{a^3+c}{k}y^2 + \frac{a^2c}{k}x^2 + 2xz + \frac{a}{k}z^2 + \frac{2a^2}{k}zy \\ &= b\left(x + \frac{a^2y}{k}\right)^2 + \left(\frac{k(a^3+c) - ba^4}{k}\right)y^2 + \frac{a^2c}{k}\left(x + \frac{kz}{a^2c}\right)^2 + \left(\frac{a^3c - k^2}{ka^2c}\right)z^2 + \\ &\quad \frac{2a^2}{k}zy \qquad (9) \\ &= b\left(x + \frac{a^2y}{k}\right)^2 + \left(\frac{k(a^3+c) - ba^4}{k}\right)y^2 + \frac{a^2c}{k}\left(x + \frac{kz}{a^2c}\right)^2 + \\ &\quad \frac{a^3c - k^2}{ka^2c}\left(z + \frac{a^4cy}{a^3c - k^2}\right)^2 - \frac{a^6c}{k(a^3c - k^2)}y^2. \end{aligned}$$

This is positive definite and the corresponding time derivative

$$\dot{V}(x,y,z) = -cx^2. \tag{10}$$

Therefore

$$2V_1(x,y,z) = \frac{a^2c}{k}x^2 + bx^2 + \frac{a^3+c}{k}y^2 + \frac{a}{k}z^2 + \frac{2a^2b}{k}xy + 2xz + \frac{2a^2}{k}zy \tag{11}$$

is a Lypunov function for the third order differential equation (1) since $c > 0$ and $ab - c = k > 0$.

CASE II

If $\dot{V} = -(bA_6 - A_4)y^2$, then

$$\begin{aligned} cA_5 &= 0, \\ bA_6 - A_4 &> 0, \\ aA_3 - A_6 &= 0, \\ cA_6 + bA_5 - A_1 &= 0, \\ cA_3 + aA_5 - A_4 &= 0, \\ aA_6 + bA_3 - A_2 - A_5 &= 0, \end{aligned} \tag{12}$$

such that

$$\begin{aligned} A_5 &= 0, \\ A_1 &= caA_3, \\ a_2 &= (a^2 + b)A_3, \\ A_4 &= cA_3, \\ A_6 &= aA_3, \\ (ab - c)A_3 &> 0. \end{aligned} \tag{13}$$

Substituting (13) in (4) and set $A_3 = 1$,

$$\begin{aligned} 2V(x, y, z) &= cax^2 + (a^2 + b)y^2 + z^2 + 2cxy + 2ayz \\ &= cax^2 + 2cxy + a^2y^2 + by^2 + 2ayz + z^2 \\ &= ac\left(x + \frac{y}{a}\right)^2 + \left(a^2 - \frac{c}{a}\right)y^2 + b\left(y + \frac{az}{b}\right)^2 + \left(\frac{b - a^2}{b}\right)z^2, \end{aligned} \tag{14}$$

which is positive definite and the time derivative of V is

$$\dot{V}(x, y, z) = -(ab - c)y^2. \tag{15}$$

Since $ab - c > 0$,

$$2V_2(x, y, z) = cax^2 + (a^2 + b)y^2 + z^2 + 2cxy + 2ayz \tag{16}$$

is a Lyapunov function for (1).

CASE III

If $\dot{V} = -(aA_3 - A_6)z^2$, then

$$\begin{aligned} cA_5 &= 0, \\ bA_6 - A_4 &= 0, \\ aA_3 - A_6 &> 0, \\ cA_6 + bA_5 - A_1 &= 0, \\ cA_3 + aA_5 - A_4 &= 0, \\ aA_6 + bA_3 - A_2 - A_5 &= 0, \end{aligned} \tag{17}$$

which implies

$$\begin{aligned} A_5 &= 0, \\ A_1 &= cA_6, \\ a_2 &= (a + \frac{b^2}{c})A_6, \\ A_4 &= bA_6, \\ A_3 &= \frac{b}{c}A_6, \\ (\frac{ab}{c} - 1)A_6 &> 0. \end{aligned} \tag{18}$$

Setting $A_6 = 1$ and substitute (18) in (4),

$$\begin{aligned} 2V(x,y,z) &= cx^2 + (a + \frac{b^2}{c})y^2 + \frac{b}{c}z^2 + 2bxy + 2yz \\ &= cx^2 + 2bxy + ay^2 + \frac{b^2}{c}y^2 + 2yz + \frac{b}{c}z^2 \\ &= c\left(x + \frac{by}{c}\right)^2 + \left(\frac{ac - b^2}{c}\right)y^2 + \frac{b^2}{c}\left(y + \frac{cz}{b^2}\right)^2 + \left(\frac{b^3 - c^2}{cb^2}\right)z^2, \end{aligned} \tag{19}$$

which is positive definite and the corresponding time derivative

$$\dot{V}(x,y,z) = -(\frac{ab}{c} - 1)z^2 \tag{20}$$

and $(\frac{ab}{c} - 1) > 0$. Therefore

$$2V_3(x,y,z) = cx^2 + (a + \frac{b^2}{c})y^2 + \frac{b}{c}z^2 + 2bxy + 2yz \tag{21}$$

is suitable Lyapunov function for (7).

Remark 2.1 *The sum of two or more Lyapunov functions result to another Lyapunov function that satisfies the Lyapunov criteria.*

3 SIR Model with Demographic Term

We consider a population N divided into classes of susceptible, infectious and removed individuals, with numbers at time t denoted by $S(t)$, $I(t)$ and $R(t)$ respectively, that is $N = S(t) + I(t) + R(t)$. We assume that there is no vertical transmission, then all offspring are susceptible. We assume that the natality Λ compensates for the deaths and the state variables are proportions of the population. The parameter is the rate of recovery. Note that, in our model the disease confers a permanent immunity. The parameter β is the effective per capita contact rate of infectious individuals [13]. The dynamic of this model is given by the following system:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \mu S, \qquad (22) \\ \frac{dI}{dt} &= \beta SI - \gamma I - \mu I, \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned}$$

Since the variable R in SIR model (22) does not appear in the equations of S and I, for model (22) we only need to consider the subsystem [3]:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \mu S, \\ \frac{dI}{dt} &= \beta SI - \gamma I - \mu I. \end{aligned} \tag{23}$$

From model (22) we have,

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \mu N - \gamma I, \\ &\leq \Lambda - \mu N. \end{aligned} \tag{24}$$

Lemma 3.1 *The system (22) with initial conditions $S(0) > 0$, $I(0) \geq 0$, and $R(0) \geq 0$ have a positive invariant solution in the region* $\Omega = \left\{(S, I, R) \in \mathbb{R}^3_+ : N = S(t) + I(t) + R(t) \leq \frac{\Lambda}{\mu}\right\}$.

Proof. Without loss of generality,

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N(t).$$

Multiplying through by the integrating factor $I.F = e^{\mu t}$ and integrate both side, we have

$$N(t)e^{\mu t} \leq \frac{\Lambda e^{\mu t}}{\mu} + K. \tag{25}$$

Furthermore, applying the initial conditions $t = 0$ in (25), we have

$$K \geq N(0) - \frac{\Lambda}{\mu}. \tag{26}$$

Substituting (26) into (25), we have

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}, \tag{27}$$

where the $N(0) = N_0$ is the initial value of the total populations. As $t \longrightarrow \infty$ in (27), the population size $N(t)$ approaches $\frac{\Lambda}{\mu}$, ie

$$0 \leq N(t) \leq \frac{\Lambda}{\mu}.$$

However $N = S(t) + I(t) + R(t) \leq \frac{\Lambda}{\mu}$. This implies that $S(t) \leq \frac{\Lambda}{\mu}$, $I(t) \leq \frac{\Lambda}{\mu}$ and $R(t) \leq \frac{\Lambda}{\mu}$. Hence the solutions are bounded.

In particular, If $N(0) < \frac{\Lambda}{\mu}$, then $N(t) \leq \frac{\Lambda}{\mu}$. Therefore the region $\Omega \in \mathbb{R}^3_+$ is positively invariant. Therefore the model the model (22) makes biological sense

The basic reproduction ratio is given by

$$R_0 = \frac{\beta}{\gamma + \mu}.$$

3.1 Stability of Disease Free Equilibrium (DFE)

The system (22) has disease free equilibrium which is given as $(S, I) = (1, 0)$.

Theorem 3.1 *If $R_o \leq 1$ then the DFE is globally asymptotically stable on Ω.*

Proof. The quadratic Lyapunov function (4) for the given equation (23) is

$$2V(S,I) = A_1 S^2 + A_2 I^2 + 2A_4 SI,$$

such that

$$\dot{V}(S,I) = -(\beta A_1 I + A_1\mu - \beta A_4 I)S^2 - (\beta A_4 S - A_2\beta S + \gamma A_2 + \mu A_2)I^2 - (\mu A_4 + \gamma A_4 + \mu A_4)SI + A_1\Lambda S + \Lambda A_4 I. \tag{28}$$

The only possibility in (25) is that

$$\dot{V}(S,I) = -(\beta A_4 S - A_2\beta S + \gamma A_2 + \mu A_2)I^2,$$

such that

$$\begin{aligned} \beta A_4 S - A_2\beta S + \gamma A_2 + \mu A_2 &> 0, \\ \beta A_1 I + A_1\mu - \beta A_4 I &= 0, \\ \mu A_4 + \gamma A_4 + \mu A_4 &= 0, \\ A_1 &= 0, \\ A_4 = 0. \end{aligned} \tag{29}$$

Since $A_2 > 0$, set $A_2 = 1$ and we obtain

$$\dot{V}(S,I) = -(\gamma + \mu - \beta S)I^2,$$

such that

$$2V(S,I) = I^2.$$

Finally, the quadratic Lyapunov function for the disease free equilibrium is

$$2V(S,I) = I^2,$$

which is positive definite and the corresponding time derivative

$$\begin{aligned} \dot{V}(S,I) &= -(\gamma + \mu - \beta S)I^2 \\ &= -(\gamma + \mu)\left[1 - \frac{\beta}{\gamma + \mu}S\right]I^2 \\ &= -(\gamma + \mu)\left[1 - R_0 S\right]I^2 \\ &< 0, \end{aligned} \tag{30}$$

is negative definite since $\gamma + \mu - \beta S > 0, R_0 < 1$ and $S \approx 1$, hence asymptotically stable on Ω.

3.2 Global Stability of Endemic Equilibrium(EE)

The endemic equilibrium is give by $(S^*, I^*) = \left(\frac{1}{R_0}, \frac{\mu}{\beta}(R_0 - 1)\right)$.

Theorem 3.2 *If $R_0 > 1$, the DFE is unstable and there exists a unique endemic equilibrium $(S^*, I^*) = \left(\frac{1}{R_0}, \frac{\mu}{\beta}(R_0 - 1)\right)$ and this endemic equilibrium is stable on Ω and globally asymptotically stable if $S < \frac{1}{R_0}$.*

Proof. Consider the quadratic Lyapunov function defined by

$$2V(S,I) = I^2.$$

Clearly V is positive definite, that is $V(S,I) \geq 0$. The derivative along trajectories of (22) is given by:

$$\begin{aligned}\dot{V}(S,I) &= -(\gamma+\mu-\beta S)I^2 \\ &= -(\gamma+\mu)\left[1-R_0 S\right] I^2 \\ &= -(\gamma+\mu)\left[1-R_0\frac{1}{R_0}\right] I^2 \\ &\leq 0.\end{aligned} \tag{31}$$

Then we conclude $\dot{V}(S,I)$ is semi-definite positive. Then the endemic equilibrium is stable by Lyapunov theorems.

If $S < \dfrac{1}{R_0}$, then

$$\dot{V}(S,I) < 0.$$

Hence the Endemic Equilibrium is global asymptotically stable.

4 Conclusion

The construction of Lyapunov function is the fact that corresponding to any positive definite quadratic form $V(x)$, there exist another positive definite $U(x)$ such that $\dot{V} = -U$ is satisfied. We obtain the quadratic Lyapunov Function of SIR model near the equilibrium in analogy with the linear system without necessarily solving the differential equation. Although many methods for finding Lyapunov functions are available, quadratic Lyapunov stability analysis [10] provides suitable function that satisfied the disease free equilibrium and the endemic equilibrium of SIR Model instead of finding separately Lyapunov function for both disease free equilibrium and endemic equilibrium.

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