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Simulation of the optimal control strategies to mitigate smartphone virus infection transmission dynamics

T. I. Chinebu[*†], G. C. E. Mbah[‡], E. A. Jooda[*], E. O. Ezennorom[§], and I. V. Udegbe[‡]

Abstract

In this study, a mathematical model of the optimal control strategies of Smartphone virus infection transmission dynamics incorporating vaccination, quarantine and treatment as the control parameters is presented. Pontryagin's maximum principle is used to characterize the optimal controls and the optimality system is solved by an iteration method. We performed some numerical simulations to verify the theoretical analysis with Runge-Kutta method of order four in MATLAB. The numerical simulation of the model shows that effective vaccination, quarantine and treatment as control strategies can eradicate Smartphone virus infection form the population. Finally, our findings strongly suggest that high coverage of vaccination, quarantine and treatment are critical to the success of Smartphone virus infection control.

Keywords: smartphone virus; optimal control; vaccination; quarantine; treatment; pontryagin's maximum principle

1. Introduction

The Smartphone enable users to have a device with a good processing power, internet access and functions of traditional mobile phones in one device, small enough to fit in a pocket. However, the wealth of private information which is stored on those devices made them an interesting target for malicious users and cyber criminals [18], who then invade these Smartphone with virus. Smartphone virus is a mischief program written to proliferate from one phone to another and it takes control of Smartphone devices by exploiting its vulnerability. Continuous improvement in Smartphone device hardware and Smartphone communication technologies has led to a highly interconnected world,

[*]Department of Applied Sciences, Federal College of Dental Technology and Therapy, P.M.B. 01473, Trans Ekulu Enugu, Nigeria
[†] Corresponding Author
[‡] Department of Mathematics, University of Nigeria, Nsukka, Nigeria
[§] Department of Computer Science, Madonna University Nigeria, Elele, Nigeria

but also a world grown highly vulnerable. Cyber criminals have proven significantly efficient in uncovering new vulnerabilities in popular applications (apps)[14]. As a result, more and more

Smartphone malware families are introduced every year [19]. Just in 2017, more than 20 million malware samples have been detected [17], while other studies show that the chances for monetization is a key factor responsible for the rise of Smartphone malware [12][23].

The cyber attacker who was able to install malware on some system can install or delete programs, modify files, download sensitive information and use them to impersonate the user's action and keystrokes, capture users screen, use camera and retrieve photos or videos, use the infected system as a source of DDoS attack [18][22]. Smartphone can become infected either through downloading infected files using the phone internet browser, transferring files between phones using Bluetootth interface, synchronizing with an infected computer, assessing an infected physical memory card, opening infected files attached to multimedia message service (MMS) or email [4][5]. Zhang [28] discovered that Smartphone virus can also propagate through multi-layered network. He concluded that Smartphone viruses can use any network (3G, 4G, Wi-Fi or Bluetooth). When an unsuspected user accepts the infected attachment file, using a phone susceptible to the virus, then the virus is installed and infects the target phone which then begins to perform as an attacked phone [6][25]. Smartphone virus can seize the victim's mobile device by running malicious exploit and this infected device will in turn, scan and infect other Smartphones in the mobile network [25].

Some response mechanisms completely stop further virus propagation, but others simply slow the propagation rate of the virus. Both types of response can be used. Ideally, the response mechanism would always quickly and completely stop Smartphone virus propagation. However, some viruses spread so quickly that a first response mechanism that slows the spread could buy time to enable activation of a second response mechanism that completely halts the propagation process. Actually, in phone user education response, the consent of the phone user to accept the message and infect the phone is required, therefore, a user not accepting an infected message has a direct effect on virus propagation. Also, other response mechanism such as immunization can prevent infection even if the user accepts the MMS message attachment [9]. A virus scans of all MMS attachments as they pass through an MMS gateway is completely effective against viruses with known virus signature. For that reason, after the new virus signature is added to the list of known viruses, the gateway virus scan is able to completely halt virus propagation [24].

The immunization response mechanism operates using software placed directly on each Smartphone [21]. After a Smartphone service provider detects virus that exploits a vulnerable Smartphone, the service provider begins developing a patch to fix that vulnerability [26]. Once the patch is developed, the immunization software resident in each Smartphone automatically installs any immunization patch available [28]. Bandwidth constraint prevents most Smartphones from receiving the patch simultaneously, so the patch is rolled out to the entire Smartphone population uniformly over a period of time [16]. The more servers that are dedicated to distributing these patches, the faster the deployment to susceptible Smartphones in the network [2]. When the patch arrives at a particular Smartphone, it becomes immunized from virus if not infected, or the patch stops further propagation attempts from Smartphone if it is already infected [24].

Because Smartphone viruses have potential to attack in many different ways, an optimal response strategy must incorporate mechanisms to counteract a wide variety of virus behaviors. The hope of stopping Smartphone virus before it degrades the utility and value of Smartphones is quick and concerted action on the part of all universal mobile telecommunications service (UMTS) data networksthat their mobile devices use; Wi-Fi networks have no such protection. And while some carriers already filter their MMS streams to remove messages bearing malicious attachments, all should do so [12].

Some of the manufacturers have joined the trusted computing group, which has been hammering on industry standards micro circuiting inside phones that will make it harder for malware to get a sensitive data in the devices memory or to hijack its payment mechanism. Symbian released a version of its operating system that does an improved job of protecting key files which requires software to obtain digital certificate from the company. This Symbian system refuses to install programs not accompanied by a certificate. Unless disabled by a user the system effectively excludes all mobile malware discovered [12].

2. Related literature

Virus attack is a serious threat in Smartphone network environment and is still multiplying as time continues to pass. Controlling these virus attacks in Smartphone is very necessary. Several mathematical models have been developed to simulate, analyze and understand Smartphone virus propagation and its control strategies, in related works, Guillen et al [11] used the theory of optimal control that is associated to systems of ordinary differential equations to find a new function to control malware epidemic through time, while Liu and Zhong [15] employed the optimal control theory to study malware immunization strategies, aiming to keep the total economic loss of security investment and infection loss as low as possible.

Van Ruitenbeek and Stevens [24] proposed response mechanisms for mobile phone viruses and used their model to obtain insight on the effectiveness of the virus mitigation technique. To study the fundamental spreading patterns that characterize a mobile virus outbreak, Wang et al [25] modeled the mobile benign worm based on two stages of repairing mechanisms. Chinebu et al [6] proposed an SVEIQRS model that control virus attack through vaccination and quarantine. Their model has a globally asymptotically stable virus free equilibrium when $R_0 < 1$ and a unique endemic equilibrium when $R_0 > 1$. Yang et al [27] proposed two different propagation models of mobile phone viruses under the complex network.

Our aim in this paper is to highlight the seriousness of Smartphone virus propagation, particularly exposed $(E(t))$ Smartphone that is infected without symptom which can transmit the infection to susceptible$(S(t))$ Smartphone. Likewise, there can be transmission from infected Smartphone $(I(t))$ to susceptible. In addition, we included complication with virus infected Smartphone $(C(t))$. Furthermore, we have the quarantined Smartphone $(Q(t))$ and finally the recovered Smartphone $(R(t))$.

A continuous mathematical model was proposed in this paper and it describes the dynamics of a population infected by Smartphone virus. Moreover, we gave the optimal control model in

section 3. We presented the optimal control problem for the model where various results regarding the existence and characterization of the optimal control using Potryagin's Maximum Principle was given in section 4. Matlab was used for the Numerical Simulation and is presented in section 5. Lastly, the conclusion was presented in section 6.

3. Mathematical model formulation

We consider a mathematical model that describes the Sensitivity analysis of Smartphone virus infection transmission dynamics as proposed in Chinebu et al [8]. The entire Smartphone population denoted by $N(t)$ was divided into six compartments.

3.1. Description of the smartphone model compartments

The schematic diagram of the proposed Smartphone virus transmission dynamics is shown in Figure 1

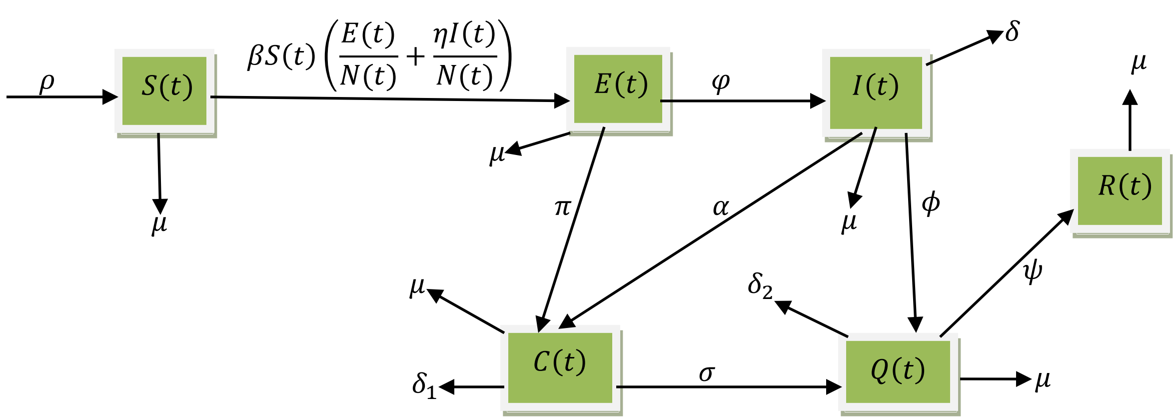

Figure 1: Schematic diagram of the model.

Let us denote the force of infection by

$$\lambda = \beta\left(\frac{\eta E(t) + I(t)}{N}\right), \tag{1}$$

where β is the Smartphone virus infection transmission rate and η is the modification factor for the exposed Smartphones.

The susceptible compartment is increased by ρ, which denotes the incidence of susceptible Smartphone. All the compartments is decreased by the amount μ (natural Death), $\frac{\beta\eta S(t)E(t)}{N}$ (the number of Smartphone that are infected with the virus by contact with the exposed Smartphone) and $\frac{\beta S(t)I(t)}{N}$ (the number of Smartphone that were infected with the virus by communication with infected Smartphone). φ, is the number of Smartphones that become infectious, andπ is the number of Smartphones that have upgraded to the development of rapid and dangerous virus, thereby, having complications due to antivirus deficiency. Smartphones that have severe complications such as Apps crashing over and over again is denoted by α. δ is death rate due to virus infection and ϕ, progression rate from infected to quarantine. Death rate due to virus infection is denoted by δ_1 and progression rate from infected with complications to quarantine by σ. ψ represent the movement from quarantined to recovered Smartphones and quarantined Smartphones that cannot be repaired again is represented by δ_2. The natural death rate is denoted by μ.

Based on the above assumptions and the schematic diagram, we thus present the Smartphone virus infection transmission dynamics model by the following systems of differential equation as proposed in Chinebu et al [8]:

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= \rho - \mu S(t) - \lambda S(t),\\
\frac{dE(t)}{dt} &= \lambda S(t) - (\mu + \varphi + \pi)E(t),\\
\frac{dI(t)}{dt} &= \varphi E(t) - (\mu + \alpha + \phi + \delta)I(t),\\
\frac{dC(t)}{dt} &= \pi E(t) + \alpha I(t) - (\mu + \delta_1 + \sigma)C(t),\\
\frac{dQ(t)}{dt} &= \phi I(t) + \sigma C(t) - (\mu + \delta_2 + \psi)Q(t),\\
\frac{dR(t)}{dt} &= \psi Q(t) - \mu R(t),
\end{aligned}\right\} (2)$$

where the initial states are $S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, C(0) \geq 0, Q(0) \geq 0, R(0) \geq 0$.

4. Formulation as an optimal control problem

The goal of this study is to use three control strategies $\tau_1(t), \tau_2(t)$ and $\tau_3(t)$ in terms of combined effort of vaccination, quarantine and treatment (repair) for $t \in [0, t_f]$ that would minimize the number infected Smartphones which will subsequently be destroyed (When there is less/no possibility of the Smartphone being repaired) due to virus infection and its complications. And at the same time, also minimize the cost of vaccination, quarantine and treatment of the population. First control is the proportion of Smartphones that are subjected to vaccination. Therefore, τ_1 denotes vaccination strategy to susceptible Smartphones at time t. Second control is the proportion of Smartphones that were quarantined and is denoted by $\tau_2\big(I(t) + C(t)\big)$ which are infected and complicated Smartphones that are quarantined at time t. The third control is denoted by $\tau_3 Q(t)$ which represents quarantined Smartphones that underwent treatment (repair) at time t. Thus, we obtain

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= \rho - \mu S(t) - \beta\left[\big(1-\tau_1(t)\big)\left(\frac{\eta E(t) + I(t)}{N}\right)\right]S(t),\\
\frac{dE(t)}{dt} &= \beta\left[\big(1-\tau_1(t)\big)\left(\frac{\eta E(t) + I(t)}{N}\right)\right]S(t) - (\mu + \varphi + \pi)E(t),\\
\frac{dI(t)}{dt} &= \varphi E(t) - (\mu + \alpha + \phi + \delta)I(t) - \tau_2(t)I(t),\\
\frac{dC(t)}{dt} &= \pi E(t) + \alpha I(t) - (\mu + \delta_1 + \sigma)C(t) - \tau_2(t)C(t),\\
\frac{dQ(t)}{dt} &= \phi I(t) + \sigma C(t) - (\mu + \delta_2 + \psi)Q(t) + \tau_2(t)I(t) + \tau_2(t)C(t) - \tau_3(t)Q(t),\\
\frac{dR(t)}{dt} &= \psi Q(t) - \mu R(t) + \tau_3(t)Q(t).
\end{aligned}\right\} \quad (3)$$

The problem is to minimize the objective functional

$$\begin{aligned}
J(\tau_1, \tau_2, \tau_3) = {} & E(t_f) + I(t_f) + C(t_f) + Q(t_f)\\
& + \int_0^{t_f}\left(E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t)\right)dt, \quad (4)
\end{aligned}$$

where $B_1 \geq 0, B_2 \geq 0$ and $B_3 \geq 0$ are the cost coefficients of $\tau_1(t), \tau_2(t)$ and $\tau_3(t)$ at time t, t_f being the final time. Hence, we seek the optimal controls τ_1^*, τ_2^* and τ_3^* such that

$$J(\tau_1^*, \tau_2^*, \tau_3^*) = \min_{\tau_1,\tau_2,\tau_3 \in W} J(\tau_1, \tau_2, \tau_3), \tag{5}$$

where W is the set of admissible control defined by

$$\begin{aligned} W = \{\tau_1, \tau_2, \tau_3 : 0 \leq \tau_{1\min} \leq \tau_1(t) \leq \tau_{1\max} \leq 1,\\ 0 \leq \tau_{2\min} \leq \tau_2(t) \leq \tau_{2\max} \leq 1,\\ 0 \leq \tau_{3min} \leq \tau_3(t) \leq \tau_{3max} \leq 1,\\ t \in [0, t_f]\} \end{aligned} \tag{6}$$

4.1. The optimal control: existence and characterization

We shall first show the existence of solutions of system (3), after which we shall prove the existence of optimal control [10].

4.2. Existence of an optimal control

Theorem 4: Consider the control problem with system (3). There exists an optimal control $(\tau_1^*, \tau_2^*, \tau_3^*) \in W$ such that

$$J(\tau_1^*, \tau_2^*, \tau_3^*) = \min_{\tau_1,\tau_2,\tau_3 \in W} J(\tau_1, \tau_2, \tau_3).$$

Proof: We can prove the existence of the optimal control through the following four conditions as was discussed in [7]:

Proof of Condition 1.
The theorems 3 and 4 indicate that the set of controls and corresponding state variables is not empty. We shall use Boyce and DiPrima [3] simplified version of an existence result to prove that the set of controls and the corresponding state variables are non empty. Let $y_i' = Fy_i(t, y_1, y_2, \dots, y_6)$ with $i = 1, 2, \dots, 6$ where $\left(y_1, y_2, y_3, y_4, y_{5,} y_6\right) = \left(S(t), E(t), I(t), C(t), Q(t), R(t)\right)$ where $y_1, y_2, y_3, y_4, y_5, y_6$ form the right hand side of the system of equation (3). Let τ_1, τ_2 and τ_3 for some constants and since all parameters are constant with y_1, y_2, y_3, y_4, y_5 $and y_6$ being continuous, then, it implies that F_S, F_E, F_I, F_C, F_Q and F_R are also continuous. More so, the partial derivatives $\frac{\partial Fy_i}{\partial y_i}$ with $i = 1, 2, \dots, 6$ are all continuous. Therefore, there exists a unique solution $\left(S(t), E(t), I(t), C(t), Q(t), R(t)\right)$ that satisfies the initial conditions. Consequently, the set of control and the control variables is non-empty, hence, condition 1 is satisfied.

Proof of Condition 2.
The control space of equation (4)

$$J(\tau_1,\tau_2,\tau_3) = E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \int_0^{t_f}\left(E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t)\right)dt$$

is convex in W and closed by definition. Take any controls $\tau_1, \tau_2 \in W\ and\ \Omega \in [0,1]$. Then, $0 \leq \Omega\tau_1 + (1-\Omega)\tau_2$. Furthermore, we observe that $\Omega\tau_1 \leq \Omega\ and (1-\Omega)\tau_2 \leq (1-\Omega)$, then, $\Omega\tau_1 + (1-\Omega)\tau_2 \leq \Omega + (1-\Omega) = 1$. Therefore, $0 \leq \Omega\tau_1 + (1-\Omega)\tau_2 \leq 1$, for all $\tau_1, \tau_2 \in W$ and $\Omega \in [0,1]$. So, W is convex and therefore, condition 2 is satisfied.

Proof of Condition 3.
All the right hand sides of the equations in system (4) are continuous, bounded above by a sum of bounded control and state, and can be written as a linear function of τ_1, τ_2, τ_3 with coefficients depending on time and state. The control space $W = \{\tau_1, \tau_2, \tau_3 : 0 \leq \tau_{1min} \leq \tau_1(t) \leq \tau_{1max} \leq 1, 0 \leq \tau_{2min} \leq \tau_2(t) \leq \tau_{2max} \leq 1, 0 \leq \tau_{3min} \leq \tau_3(t) \leq \tau_{3max} \leq 1, t \in [0, t_f]\}$ is convex and closed by definition.

From system (3), we have

$$\frac{dN(t)}{dt} = \rho - \mu N,$$
$$\lim_{t\to\infty} Sup\ N(t) \leq \frac{\rho}{\mu}.$$

Therefore, all solutions of system (3) are bounded. So, there exists $y_1, y_2, y_3, y_4, y_5\ and\ y_6$ such that$\forall\ t \in [0, t_f]$: $S(t) \leq y_1, E(t) \leq y_2, I(t) \leq y_3, C(t) \leq y_4, Q(t) \leq y_5\ and\ R(t) \leq y_6$. If we consider

$$\begin{cases} F_S = S(t) \leq \rho \\ F_E = E(t) \leq D_1 E \\ F_I = I(t) \leq \varphi E - D_2 I \\ F_C = C(t) \leq \pi E(t) + \alpha I(t) - \tau_2(t) y_4 \\ F_Q = Q(t) \leq \phi I(t) + \sigma C(t) + \tau_2(t)(y_3 + y_4) - \tau_3(t) y_5 \\ F_R = R(t) \leq \psi Q(t) + \tau_3(t) y_5 \end{cases}$$

So, we can rewrite system (3) in matrix form as

$$F\big(t, S(t), E(t), I(t), C(t), Q(t), R(t)\big) \leq \rho + VY(t) - FW(t)$$

where $F\big(t, S(t), E(t), I(t), C(t), Q(t), R(t)\big) = \big[F_S, F_E, F_I, F_C, F_Q, F_R\big]^T$

$$Y(t) = [S(t), E(t), I(t), C(t), Q(t), R(t)]^T$$
$$W(t) = [\tau_1, \tau_2, \tau_3]^T$$

and

$$V := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi & \alpha & 0 & 0 & 0 \\ 0 & 0 & \phi & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi & 0 \end{pmatrix}$$

$$F := \begin{pmatrix} -\beta\left(\dfrac{\eta ES + IS}{N}\right) & 0 & 0 \\ \beta\left(\dfrac{\eta ES + IS}{N}\right) & 0 & 0 \\ 0 & -I & 0 \\ 0 & -C & 0 \\ 0 & I + C & -Q \\ 0 & 0 & Q \end{pmatrix}$$

which gives the linear function of the control vector defined by time and state variables. Then we find out the bound of the right hand side. It is noted that all parameters are constant and greater than or equal to zero. Therefore, we can write

$$F\big(t, S(t), E(t), I(t), C(t), Q(t), R(t)\big) \leq \|\rho\| + \|V\|\|Y(t)\| + \|F\|\|W(t)\|$$
$$\leq \mathfrak{D} + \gamma(\|Y(t)\| + \|W(t)\|)$$
$$where\ \ \mathfrak{D} \leq \rho\ and\ \gamma = max(\|V\|, \|F\|).$$

.

Hence, we say that the right hand side is bounded by a sum of the state and control. Therefore, condition 3 is satisfied.

Proof of Condition 4.

The integrand in the objective functional $E\big(t_f\big) + I\big(t_f\big) + C\big(t_f\big) + Q\big(t_f\big) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t)$ is clearly convex in W.

We then show that there exists constants $\xi_1, \xi_2, \xi_3, \xi_4 > 0$, and ξ such that $E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t)$ satisfies $E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t) \geq \xi_1 + \xi_2|\tau_1|^{\xi} + \xi_3|\tau_2|^{\xi} + \xi_4|\tau_3|^{\xi}$.

The state variable being bounded, let $\xi_1 = \inf_{t\in[0,t_f]} E(t_f) + I(t_f) + C(t_f) + Q(t_f), \xi_2 = \frac{B_1}{2}, \xi_3 = \frac{B_2}{2}, \xi_4 = \frac{B_3}{2}$ $and\ \xi = 2$, then it follows that: $E(t_f) + I(t_f) + C(t_f) + Q(t_f) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t) \geq \xi_1 + \xi_2|\tau_1|^{\xi} + \xi_3|\tau_2|^{\xi} + \xi_4|\tau_3|^{\xi}$.

Therefore, condition 4 is also satisfied. From the above discussion the existence of the objective functional has been established. Then from Fleming and Rishel [10], we conclude that there exists an optimal control.

4.3. Characterization of the optimal control

In order to derive the necessary conditions for the optimal control, we apply Pontryagin's Maximum Principle (PMP)[26] to the Hamiltonian H.

Theorem 5: Given optimal controls $(\tau_1^*, \tau_2^*, \tau_3^*)$ and solutions $S^*, E^*, I^*, C^*, Q^*\ and\ R^*$ of the corresponding system (3), there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\ and\ \lambda_6$ satisfying

$$\left.\begin{aligned}
\dot{\lambda}_1 &= \lambda_1\left(\mu + \beta(1-\tau_1(t))\left(\frac{\eta E(t) + I(t)}{N}\right)\right) - \lambda_2\left(\beta(1-\tau_1(t))\left(\frac{\eta E(t) + I(t)}{N}\right)\right)\\
\dot{\lambda}_2 &= -1 + \lambda_1\left(\beta(1-\tau_1(t))\left(\frac{\eta S(t)}{N}\right)\right) - \lambda_2\left(\beta(1-\tau_1(t))\frac{\eta S(t)}{N} - A\right) - \lambda_3\varphi - \lambda_4\pi\\
\dot{\lambda}_3 &= -1 + \lambda_1\left(\beta(1-\tau_1(t))\left(\frac{S(t)}{N}\right)\right) - \lambda_2\left(\beta(1-\tau_1(t))\frac{S(t)}{N}\right) + \lambda_3(B + \tau_2(t)) - \lambda_4\alpha - \lambda_5(\phi + \tau_2(t))\\
\dot{\lambda}_4 &= -1 - \lambda_4(G + \tau_2(t)) - \lambda_5(\sigma + \tau_2(t))\\
\dot{\lambda}_5 &= -1 + \lambda_5(K + \tau_3(t)) - \lambda_6(\psi + \tau_3(t))\\
\dot{\lambda}_6 &= \lambda_6\mu
\end{aligned}\right\} \quad (7$$

With transversality conditions at time $t_f: \lambda_1(t_f) = 0, \lambda_2(t_f) = -1, \lambda_3(t_f) = -1, \lambda_4(t_f) = -1, \lambda_5(t_f) = -1, \lambda_6(t_f) = 0.$ Moreover, the optimal controls $\tau_1^*, \tau_2^*\ and\ \tau_3^*$ are given by

$$\left.\begin{aligned}
\tau_1^* &= min\left(1, max\left(0, \frac{\lambda_2 - \lambda_1}{B_1}\left(\beta\left(\frac{\eta E(t) + I(t)}{N}\right)S(t)\right)\right)\right)\\
\tau_2^* &= min\left(1, max\left(0, \frac{\lambda_3 I(t) + \lambda_4 C(t) - \lambda_5(I(t) + C(t))}{B_2}\right)\right)\\
\tau_3^* &= min\left(1, max\left(0, \frac{(\lambda_5 - \lambda_6)Q(t)}{B_3}\right)\right)
\end{aligned}\right\} (8)$$

Proof: The Hamiltonian H is defined as follows:

$H(t) = E\left(t_f\right) + I\left(t_f\right) + C\left(t_f\right) + Q\left(t_f\right) + \frac{B_1}{2}\tau_1^2(t) + \frac{B_2}{2}\tau_2^2(t) + \frac{B_3}{2}\tau_3^2(t) + \dot{\lambda}_1 S(t) + \dot{\lambda}_2 E(t) + \dot{\lambda}_3 I(t) + \dot{\lambda}_4 C(t) + \dot{\lambda}_5 Q(t) + \dot{\lambda}_6 R(t)$.

The adjoint variables $\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3, \dot{\lambda}_4, \dot{\lambda}_5, \dot{\lambda}_6$ are obtained by the following system:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial S}, \dot{\lambda}_2 = -\frac{\partial H}{\partial E}, \dot{\lambda}_3 = -\frac{\partial H}{\partial I}, \dot{\lambda}_4 = -\frac{\partial H}{\partial C}, \dot{\lambda}_5 = -\frac{\partial H}{\partial Q}, \dot{\lambda}_6 = -\frac{\partial H}{\partial R} (9)$$

For $t \in \left[0, t_f\right]$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle [20][13][1], such that

$$
\left.\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial S} = \lambda_1\left(\mu + \beta\big(1-\tau_1(t)\big)\left(\frac{\eta E(t)+I(t)}{N}\right)\right) - \lambda_2\left(\beta\big(1-\tau_1(t)\big)\left(\frac{\eta E(t)+I(t)}{N}\right)\right)\\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial E} = -1 + \lambda_1\left(\beta\big(1-\tau_1(t)\big)\left(\frac{\eta S(t)}{N}\right)\right) - \lambda_2\left(\beta\big(1-\tau_1(t)\big)\frac{\eta S(t)}{N} - A\right) - \lambda_3\varphi - \lambda_4\pi\\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial I} = -1 + \lambda_1\left(\beta\big(1-\tau_1(t)\big)\left(\frac{S(t)}{N}\right)\right) - \lambda_2\left(\beta\big(1-\tau_1(t)\big)\frac{S(t)}{N}\right) + \lambda_3\big(B+\tau_2(t)\big)\\
&\qquad -\lambda_4\alpha - \lambda_5\big(\phi+\tau_2(t)\big)\\
\dot{\lambda}_4 &= -\frac{\partial H}{\partial C} = -1 - \lambda_4\big(G+\tau_2(t)\big) - \lambda_5\big(\sigma+\tau_2(t)\big)\\
\dot{\lambda}_5 &= -\frac{\partial H}{\partial Q} = -1 + \lambda_5\big(K+\tau_3(t)\big) - \lambda_6\big(\psi+\tau_3(t)\big)\\
\dot{\lambda}_6 &= -\frac{\partial H}{\partial R} = \lambda_6\mu
\end{aligned}\right\} \quad (10)
$$

The optimal control τ_1^*, τ_2^* *and* τ_3^* derived from the optimality condition

$$
\frac{\partial H}{\partial \tau_1} = 0, \frac{\partial H}{\partial \tau_2} = 0, \frac{\partial H}{\partial \tau_3} = 0
$$

$$
\left.\begin{aligned}
-\frac{\partial H}{\partial \tau_1} &= -B_1\tau_1(t) + (\lambda_2-\lambda_1)\left(\beta\left(\frac{\eta E(t)+I(t)}{N}\right)S(t)\right) = 0\\
-\frac{\partial H}{\partial \tau_2} &= -B_2\tau_2(t) + \lambda_3 I(t) + \lambda_4 C(t) - \lambda_5(I(t)+C(t)) = 0\\
-\frac{\partial H}{\partial \tau_3} &= -B_3\tau_3(t) + (\lambda_5-\lambda_6)Q(t) = 0
\end{aligned}\right\}
$$

Then, we have

$$\left.\begin{aligned}\tau_1(t) &= \frac{\lambda_2-\lambda_1}{B_1}\left(\beta\left(\frac{\eta E(t)+I(t)}{N}\right)S(t)\right)\\ \tau_2(t) &= \frac{\lambda_3 I(t)+\lambda_4 C(t)-\lambda_5(I(t)+C(t))}{B_2}\\ \tau_3(t) &= \frac{(\lambda_5-\lambda_6)Q(t)}{B_3}\end{aligned}\right\}$$

By substituting the optimal control $\tau_1^*, \tau_2^*\ and\ \tau_3^*$ ino the state system (3) and the adjoint system (9), we obtain the corresponding $S^*, E^*, I^*, C^*, Q^*, R^*$and $\lambda_i^*, i = 1, 2, \dots, 6$, with the help of the transversality condition $\lambda_i^*(t_f), i = 1, 2, \dots, 6$

5. Numerical simulations

In this section, we present the result obtained by solving the optimal control system numerically having the initial conditions for the state variables. Thus, the optimal control system is a two-point boundary value problem with separate boundary conditions at time steps $i = 0\ and\ i = t_f$. We solve the optimality system by using Runge Kutta method of order four (RK4). The numerical simulations were carried out with the parameters in Table 1.

Paramete	*Description*	*Value*
ρ	Incidence of Smartphone	100
μ	Natural death rate	0.057
β	Contact rate of infected Smartphones with susceptible	0.85
η	Modification factor for the exposed Smartphones	0.0002
φ	The rate at which exposed Smartphones become infectious	0.045
π	The rate at which exposed Smartphones develop complication due to virus infection	0.04
α	The rate at which infected Smartphones develop complication due to virus infection	0.21
ϕ	Quarantine rate of infected Smartphones before treatment	0.083
δ	Disease induced death rate of infected Smartphones	0.01
δ_1	Disease induced death rate of infected with complication Smartphones	0.01
δ_2	Disease induced death rate of quarantined Smartphones	0.01
σ	Quarantine rate of infected with complication Smartphones before treatment	0.02
ψ	The rate at which quarantined Smartphones recover from virus infection after treatment.	0.017

Table 1: Parameter value for the simulation

For the optimality of the model, we have the initial value for the susceptible, exposed, infected, infected with complication, quarantined and recovered as $S(0) = 100, E(0) = 10, I(0) = 4, C(0) = 2, Q(0) = 3\ and\ R(0) = 5$.

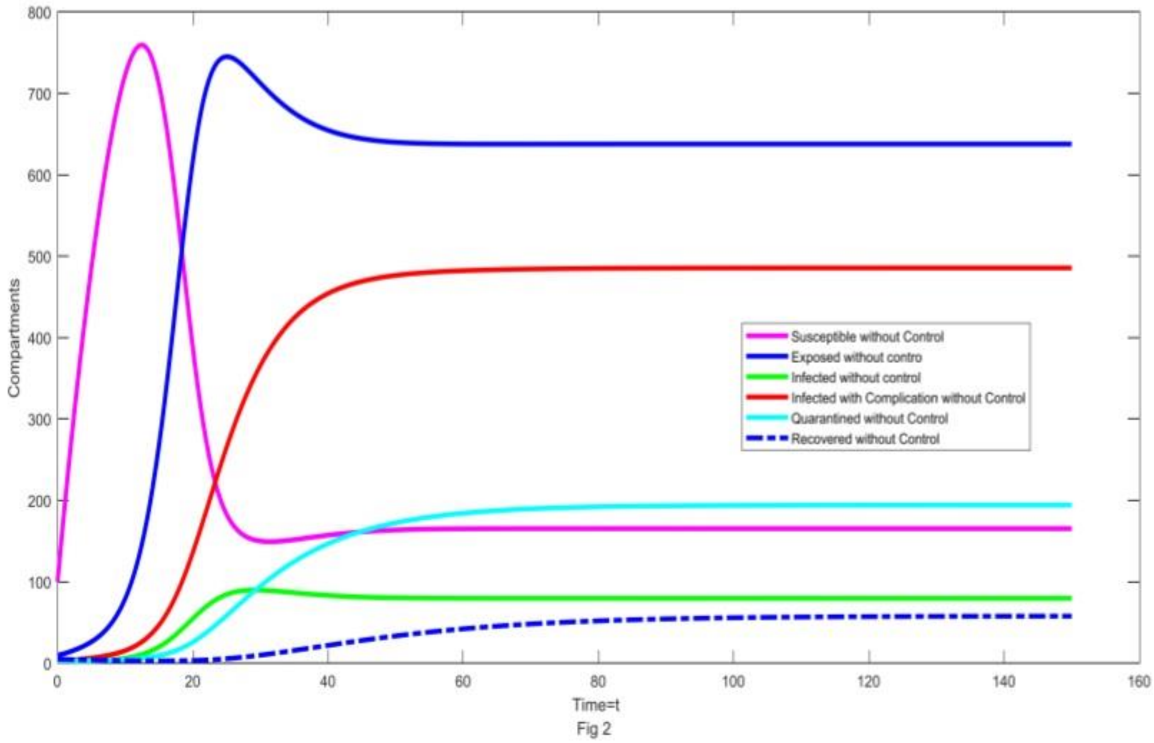

Figure 2: The Dynamics of Susceptible, Exposed, Infected, Infected with Complication, Quarantined and Recovered Smartphones without Vaccination, quarantine and treatment Control strategies $(\tau_1(t), = \tau_2(t) = \tau_3(t) = 0)$.

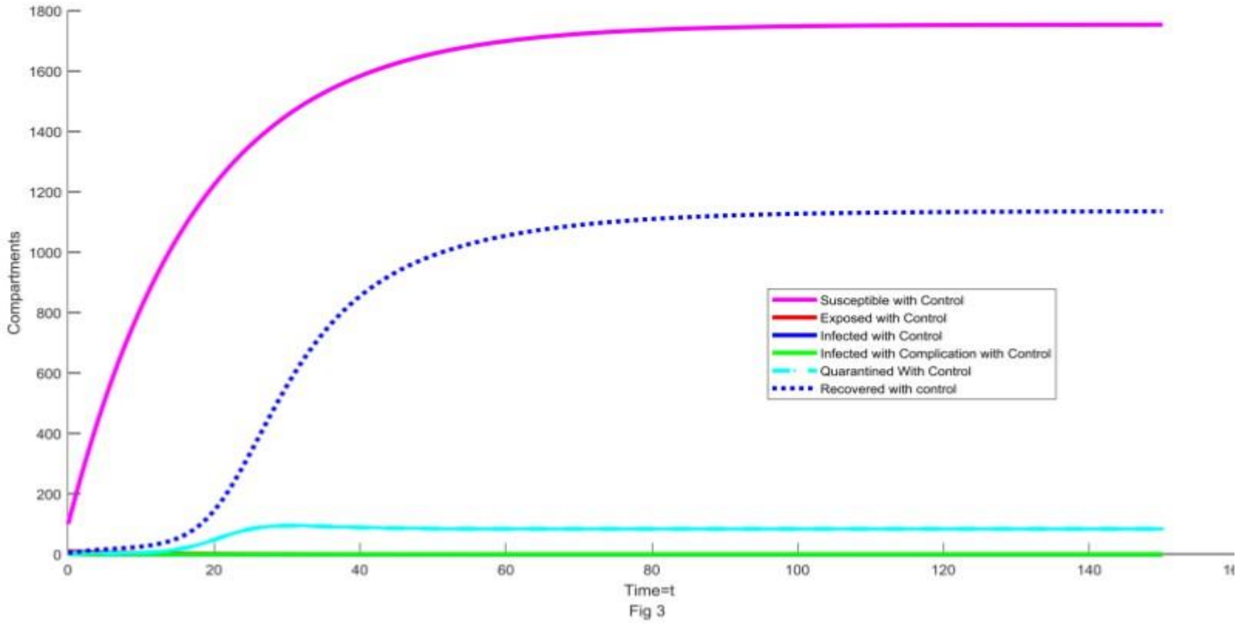

Figure 3: The Dynamics of Susceptible, Exposed, Infected, Infected with Complication, Quarantined and Recovered Smartphones with Vaccination, quarantine and treatment Control strategies $(0 < \tau_3(t), \tau_3(t), \tau_3(t) < 1)$.

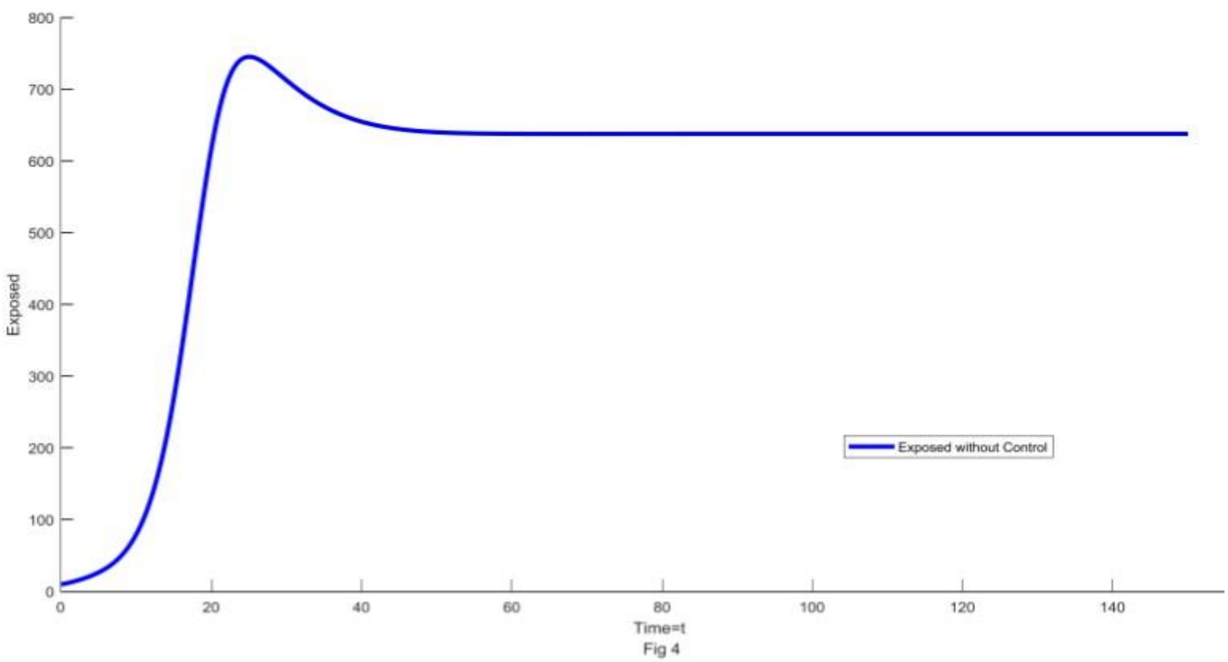

Figure 4: The Dynamics of Exposed Smartphones without Vaccination, quarantine and treatment Control strategies $(\tau_1(t), = \tau_2(t) = \tau_3(t) = 0)$.

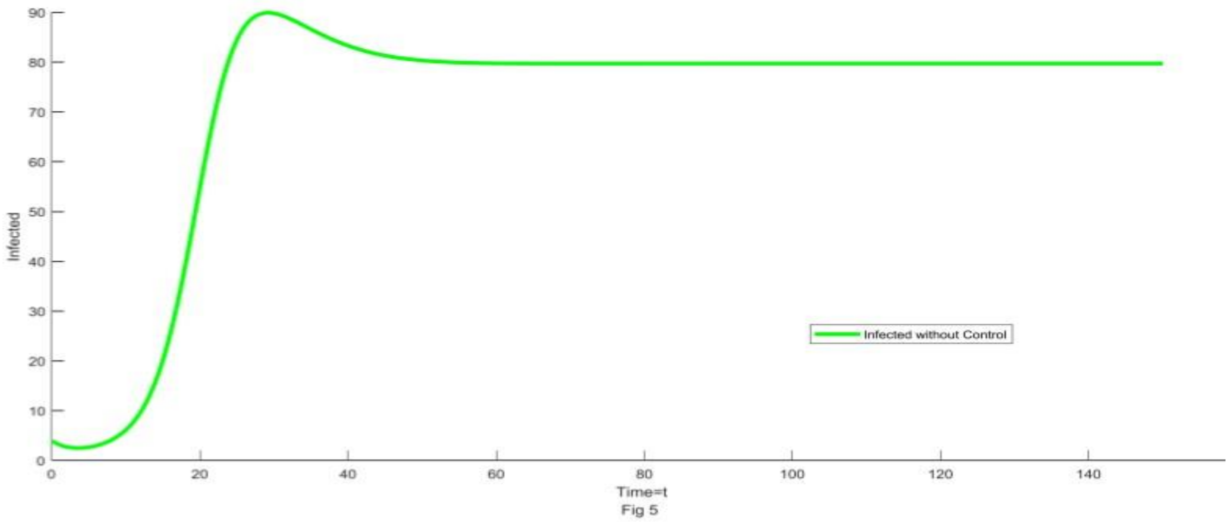

Figure 5: The Dynamics of Infected Smartphones without Vaccination, quarantine and treatment Control strategies $(\tau_1(t), = \tau_2(t) = \tau_3(t) = 0)$.

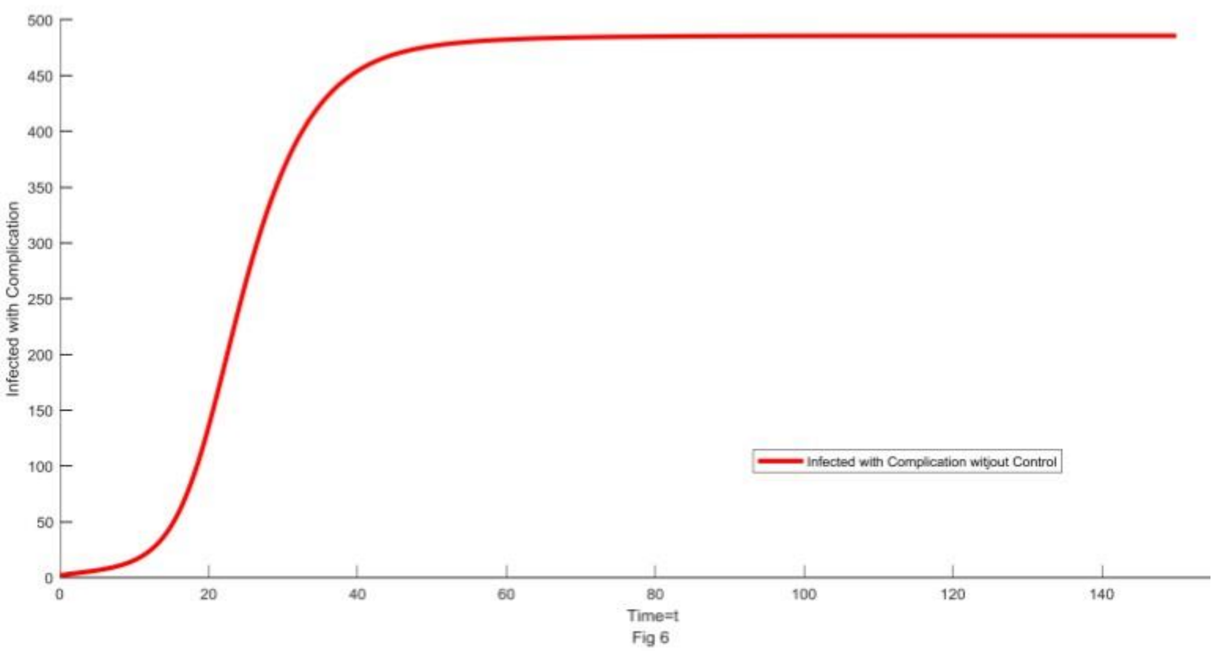

Figure 6: The Dynamics Infected with Complication Smartphones without Vaccination, quarantine and treatment Control strategies $(\tau_1(t), = \tau_2(t) = \tau_3(t) = 0)$.

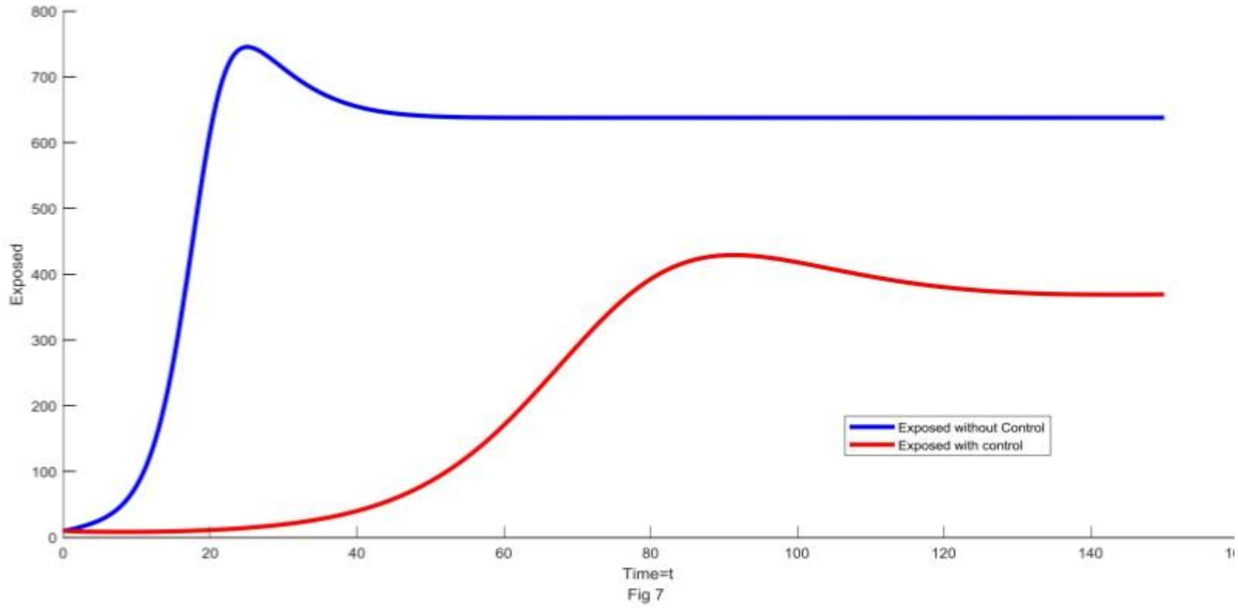

Figure 7: The Dynamics of Exposed Smartphones with vaccination Control strategy only $(0 < \tau_1(t) < 1, and\ \tau_2(t) = \tau_3(t) = 0)$.

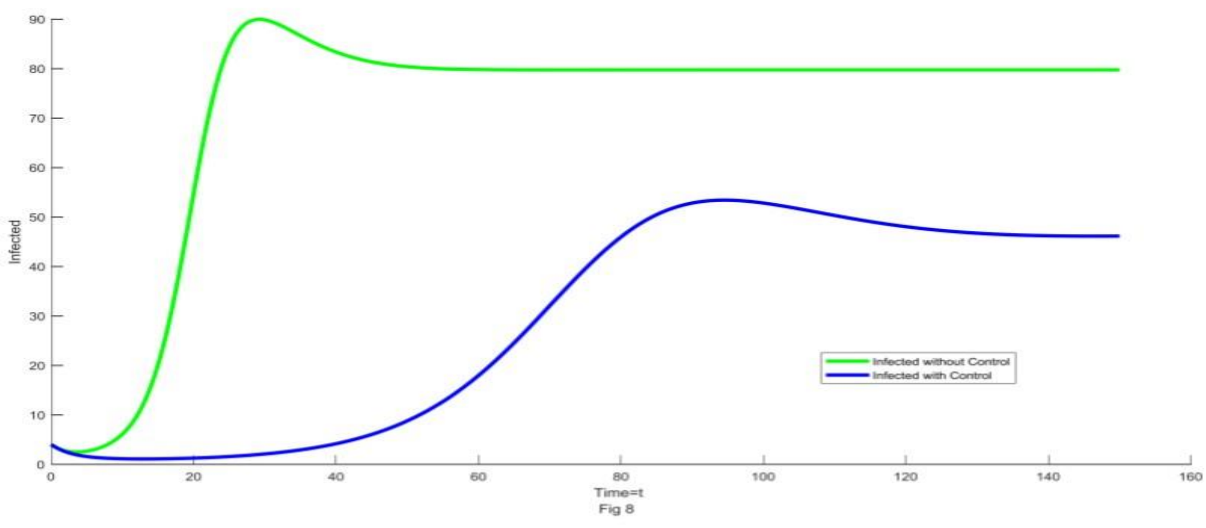

Figure 8: The Dynamics of Infected Smartphones with vaccination Control strategy only $(0 < \tau_1(t) < 1, and\ \tau_2(t) = \tau_3(t) = 0)$.

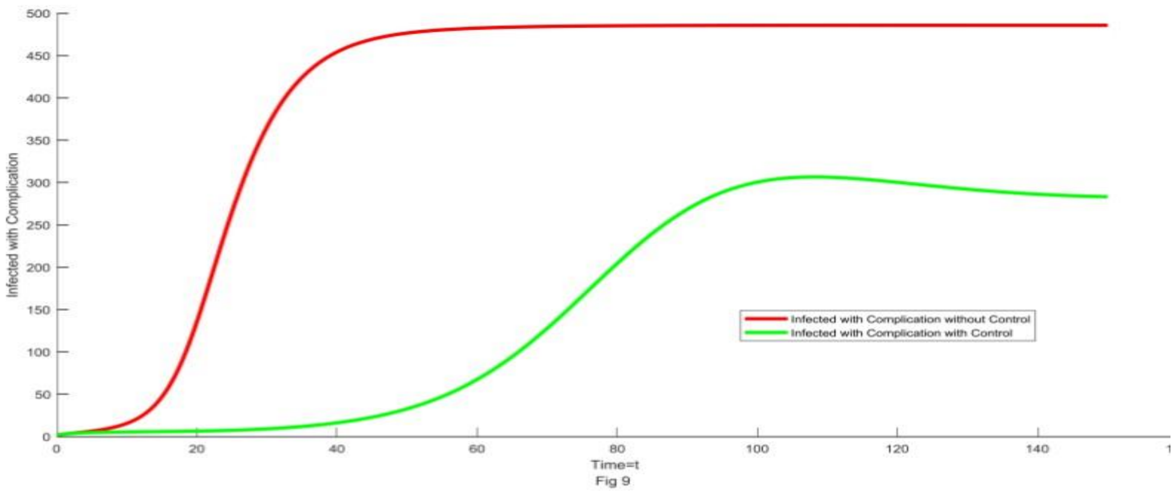

Figure 9: The Dynamics of Infected with Complication Smartphones with vaccination Control strategy only $(0 < \tau_1(t) < 1, and\ \tau_2(t) = \tau_3(t) = 0)$.

Without and with control $\tau_1(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	369.1904	42.12
Infected	79.7262	46.1296	42.14
Infected with Complication	637.8093	369.1904	41.70

Table 2: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_1(t)$

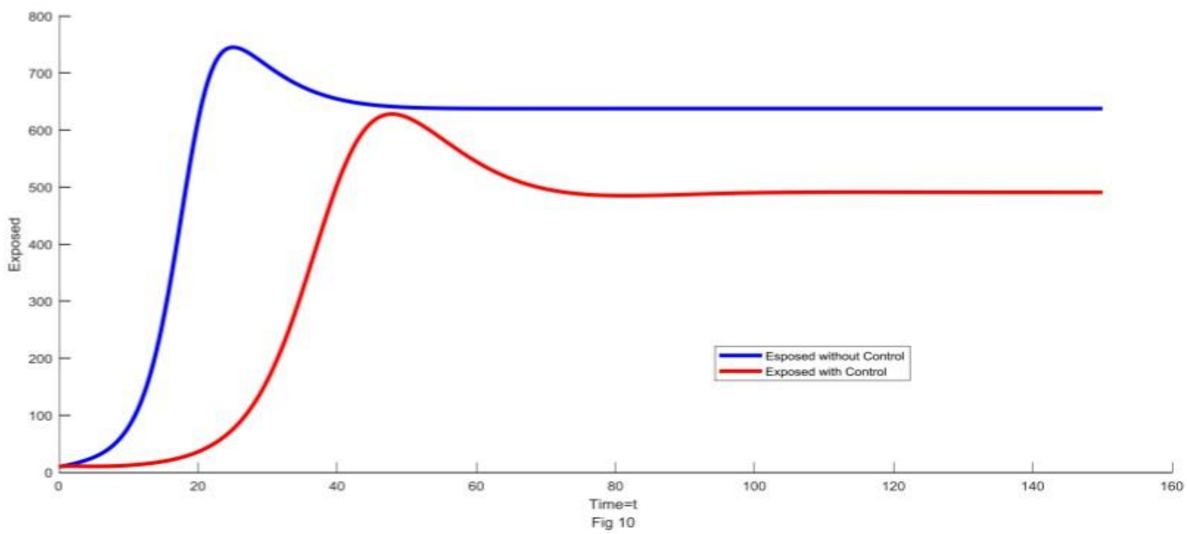

Figure 10: The Dynamics of Exposed Smartphones with quarantine control strategy only $(0 < \tau_2(t) < 1, and\ \tau_1(t) = \tau_3(t) = 0)$.

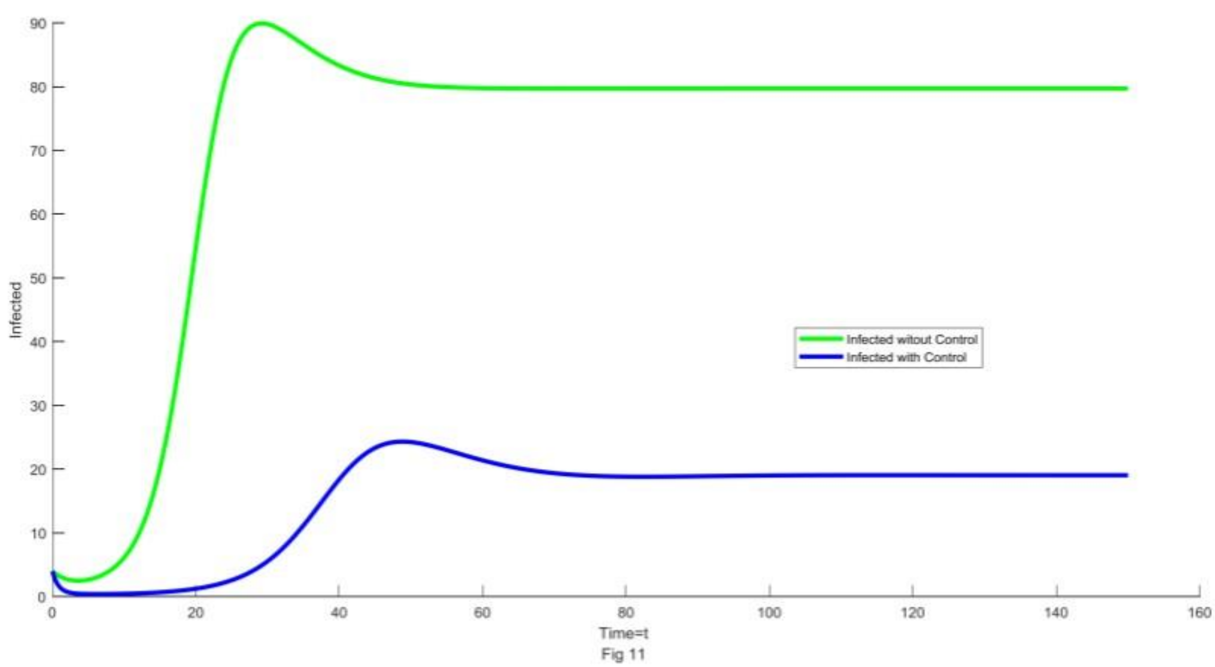

Figure 11: The Dynamics of Infected Smartphones with quarantine control strategy only $(0 < \tau_2(t) < 1, and\ \tau_1(t) = \tau_3(t) = 0)$.

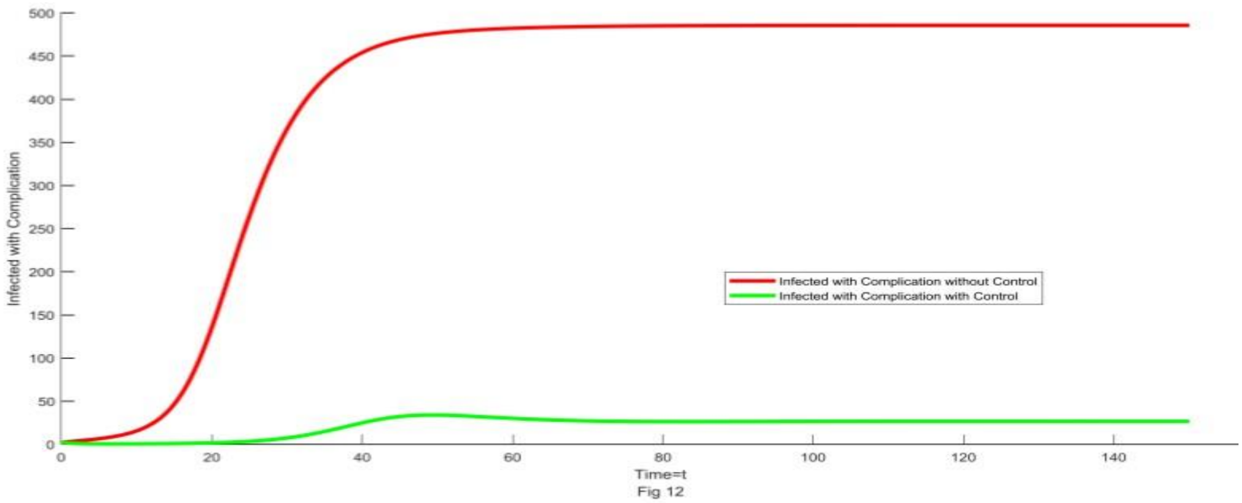

Figure 12: The Dynamics of Infected with Complication Smartphones with quarantine control strategy only $(0 < \tau_2(t) < 1, and\ \tau_1(t) = \tau_3(t) = 0)$.

Without and with control $\tau_2(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	490.9569	23.04
Infected	79.7262	19.0457	76.11
Infected with Complication	637.8093	26.6492	94.51

Table 3: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_2(t)$

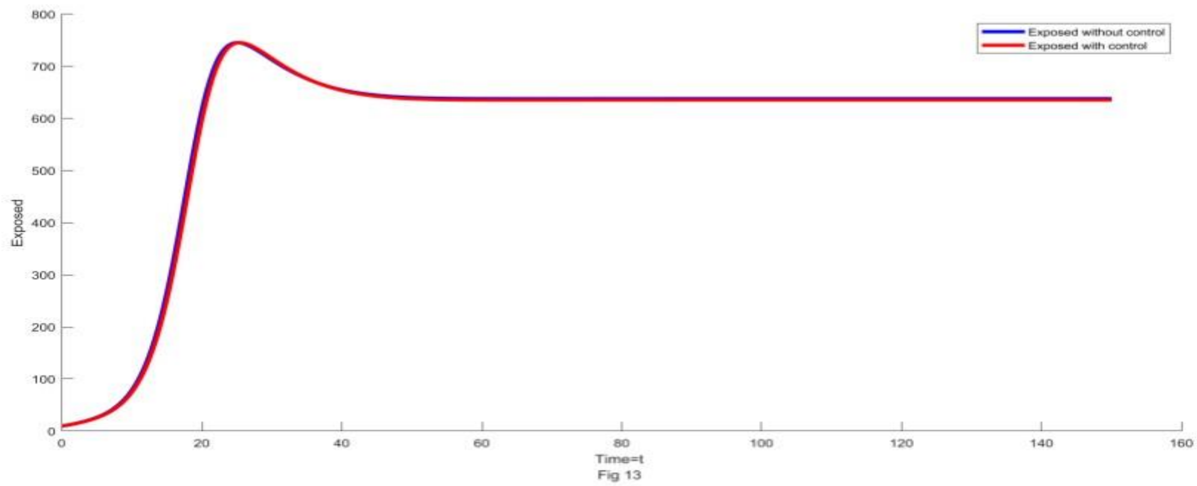

Figure 13: The Dynamics of Exposed Smartphones with treatment control strategy only $(0 < \tau_3(t) < 1, and\ \tau_1(t) = \tau_2(t) = 0)$.

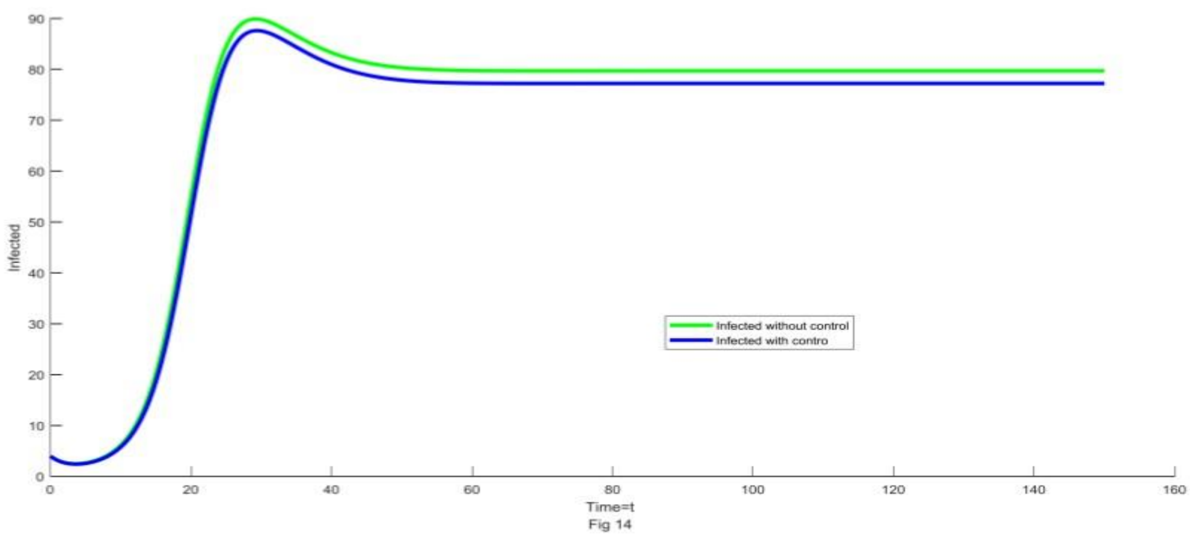

Figure 14: The Dynamics of Infected Smartphones with treatment control strategy only $(0 < \tau_3(t) < 1, and\ \tau_1(t) = \tau_2(t) = 0)$.

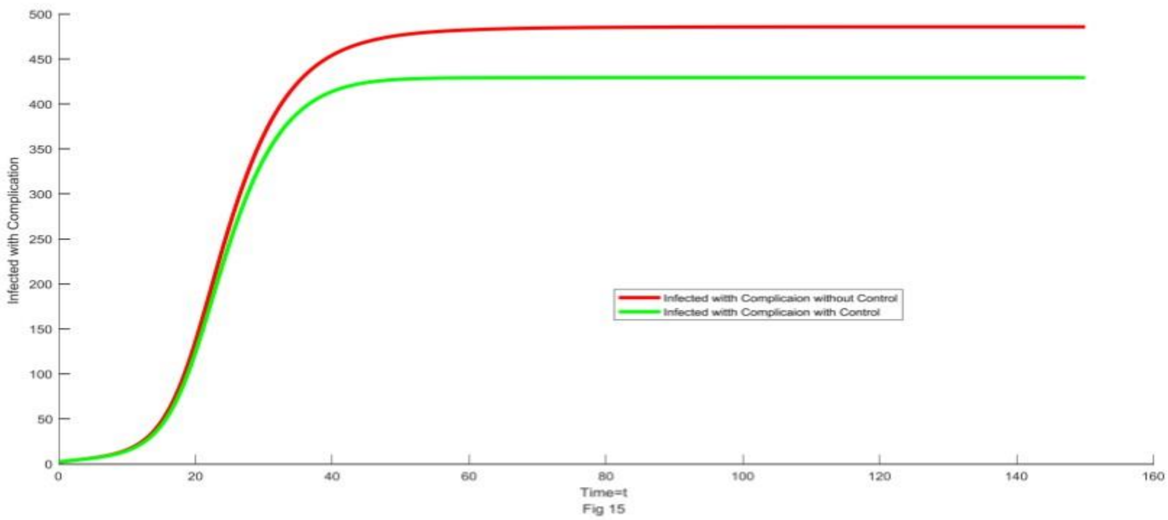

Figure 15: The Dynamics of Infected with Complication Smartphones with treatment control strategy only $(0 < \tau_3(t) < 1, and\ \tau_1(t) = \tau_2(t) = 0)$.

Without and with control $\tau_3(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	635.2779	0.87
Infected	79.7262	77.2635	3.09
Infected with Complication	637.8093	429.2418	1.16

Table 4: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_3(t)$

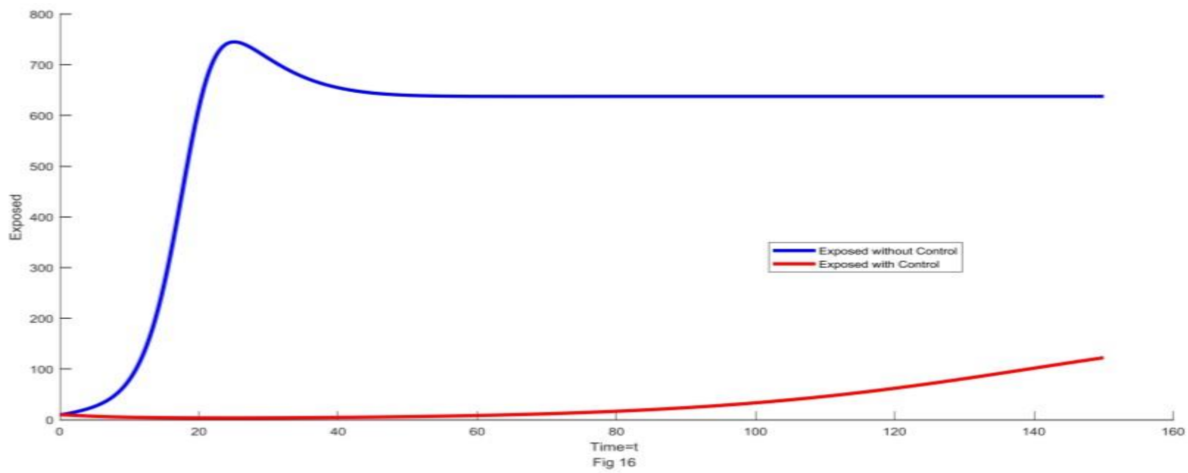

Figure 16: The Dynamics of Exposed Smartphones with vaccination and quarantine Control strategies only $(0 < \tau_1(t), \tau_2(t) < 1 \; and \; \tau_3(t) = 0)$.

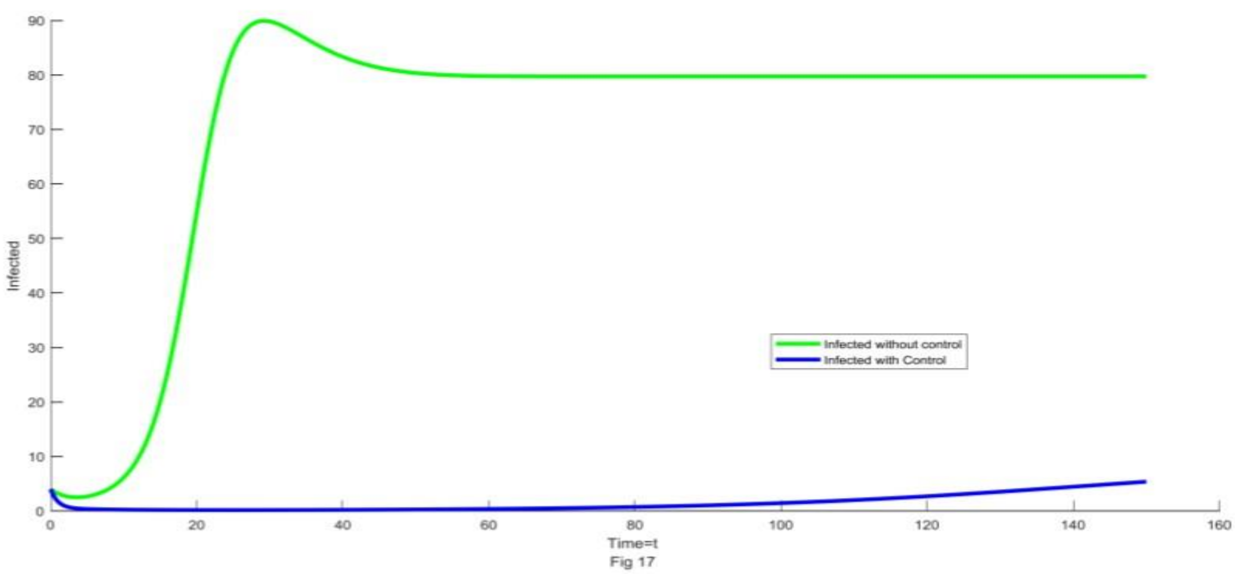

Figure 17: The Dynamics of Infected Smartphones with vaccination and quarantine Control strategies only $(0 < \tau_1(t), \tau_2(t) < 1 \; and \; \tau_3(t) = 0)$.

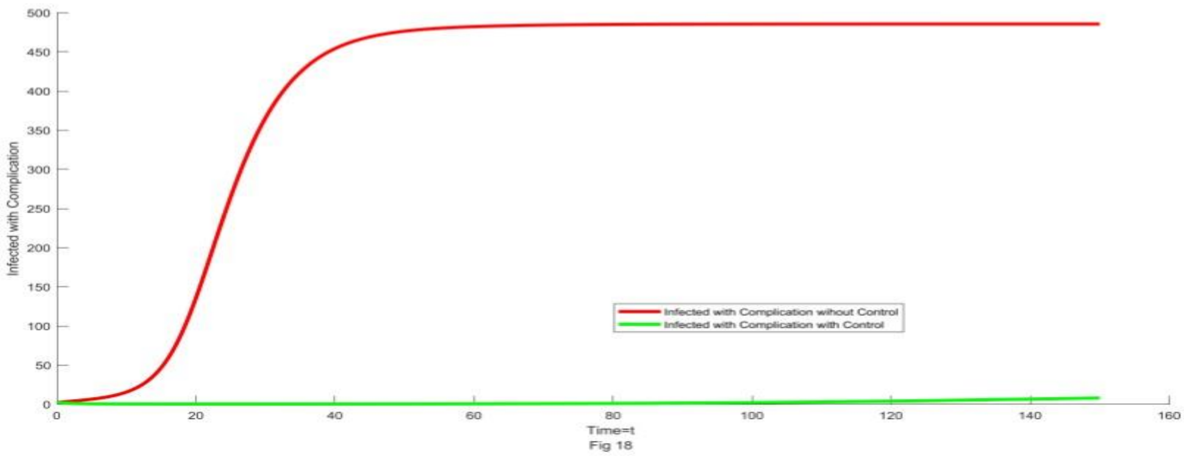

Figure 18: The Dynamics of Infected with Complication Smartphones with vaccination and quarantine Control strategies only $(0 < \tau_1(t), \tau_2(t) < 1 \ and\ \tau_3(t) = 0\ )$.

Without and with control $\tau_1(t) and\ \tau_2(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	122.5242	80.79
Infected	79.7262	5.3701	93.26
Infected with Complication	637.8093	7.9962	98.35

Table 5: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_1(t) and\ \tau_2(t)$.

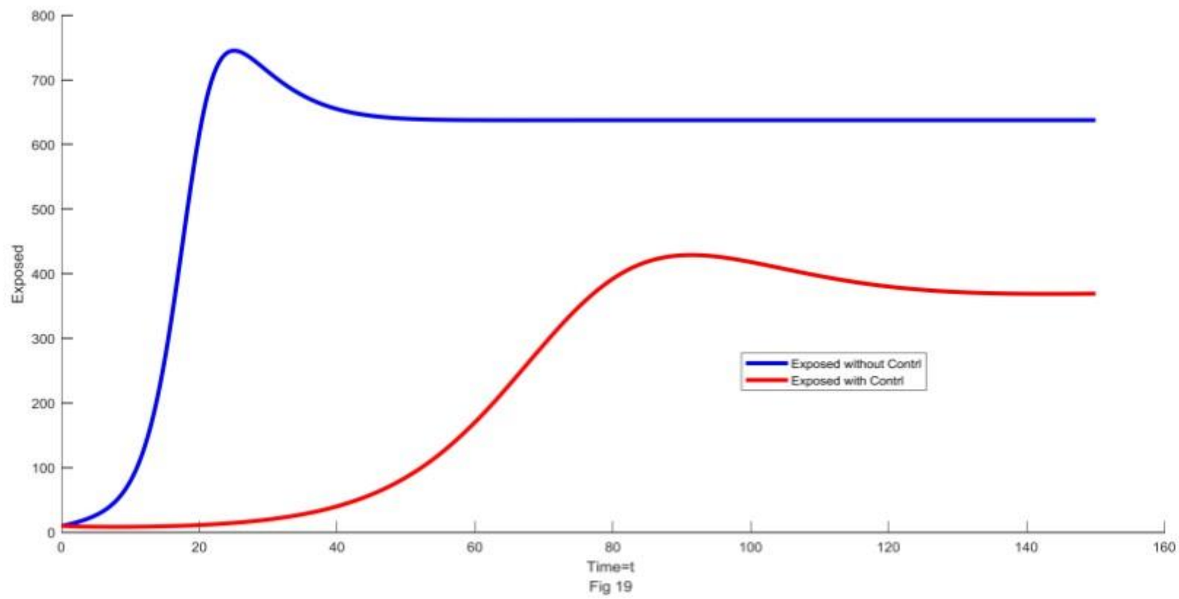

Figure 19: The Dynamics of Exposed Smartphones with vaccination and treatment Control strategies only $(0 < \tau_1(t), \tau_3(t) < 1 \; and \; \tau_2(t) = 0 \,)$.

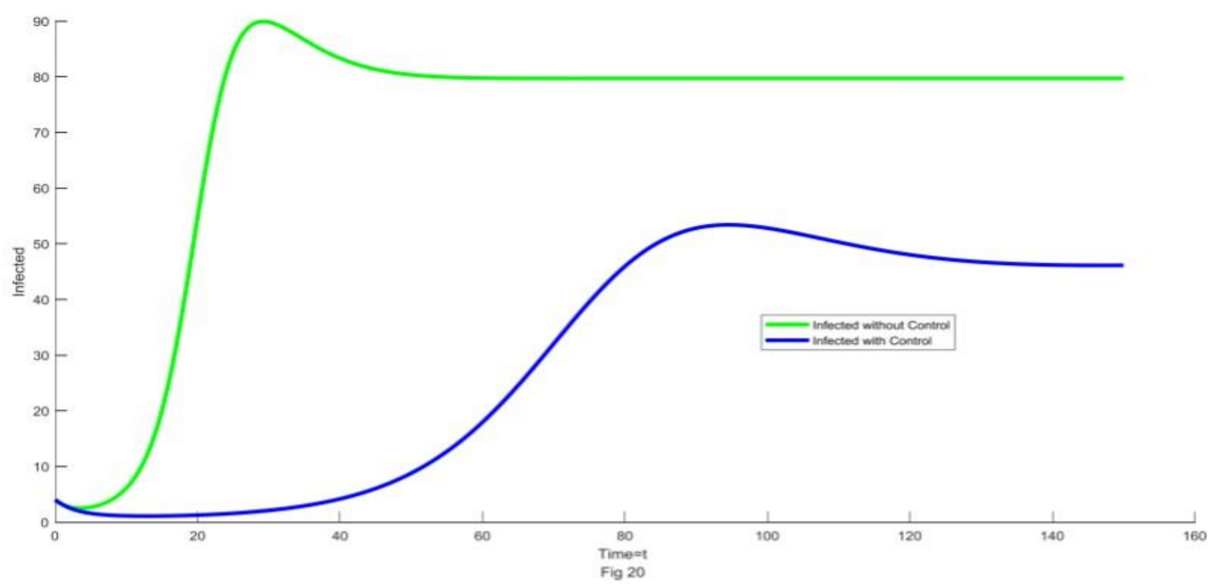

Figure 20: The Dynamics of Infected Smartphones with vaccination and treatment Control strategies only $(0 < \tau_1(t), \tau_3(t) < 1 \; and \; \tau_2(t) = 0 \,)$.

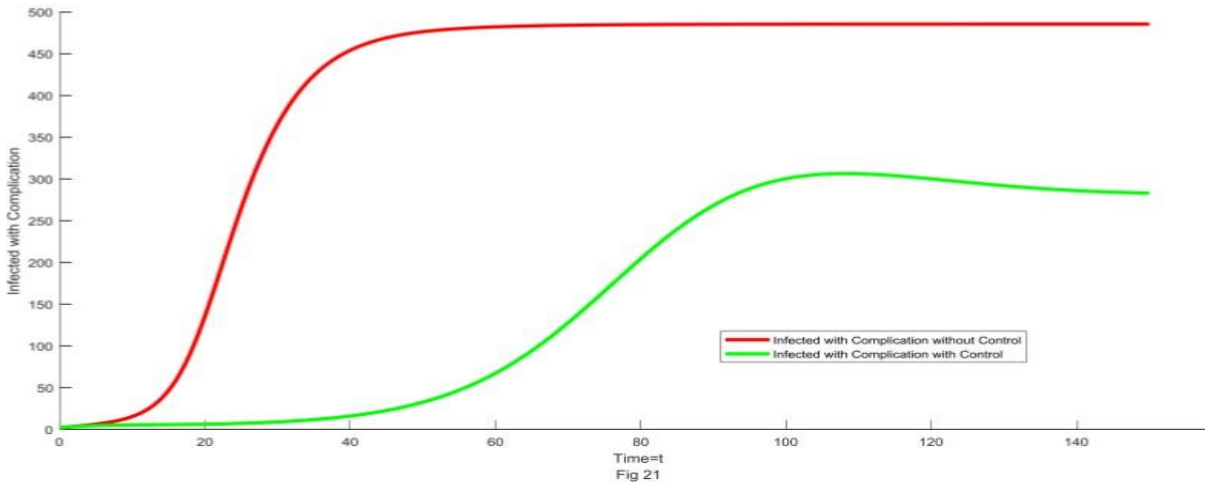

Figure 21: The Dynamics of Infected with Complication Smartphones with vaccination and treatment Control strategies only $(0 < \tau_1(t), \tau_3(t) < 1 \; and \; \tau_2(t) = 0 \,)$.

Without and with control $\tau_1(t) and \; \tau_3(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	369.1904	42.12
Infected	79.7262	46.0457	42.14
Infected with Complication	637.8093	283.1645	41.70

Table 6: Percentage of the decreasing number of Exposed, Infected and Infected with Complication Without and with control $\tau_1(t) and \; \tau_3(t)$.

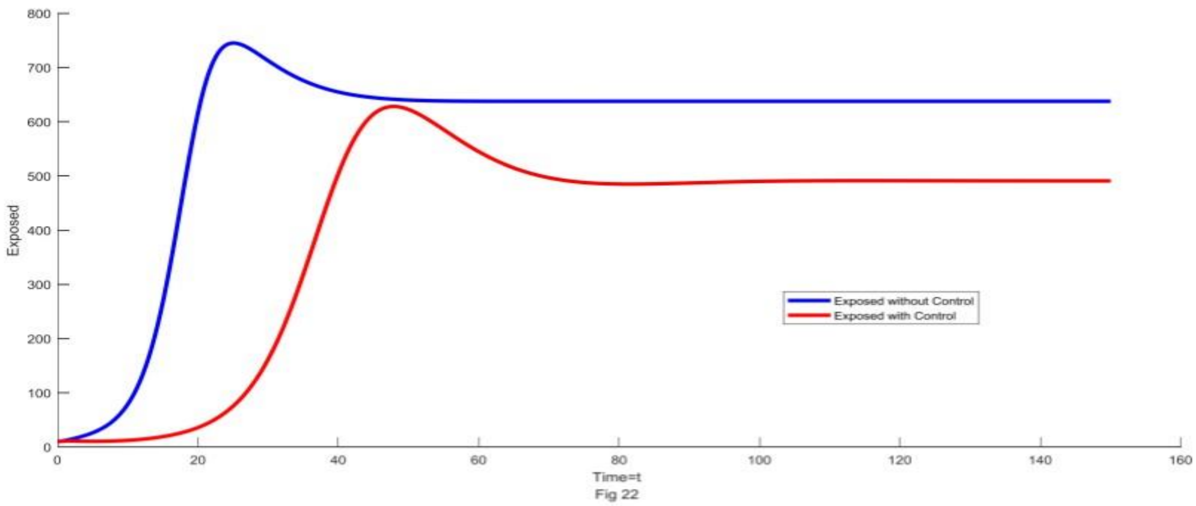

Figure 22: The Dynamics of Exposed Smartphones with quarantine and treatment Control strategies only $(0 < \tau_2(t), \tau_3(t) < 1 \; and \; \tau_1(t) = 0)$.

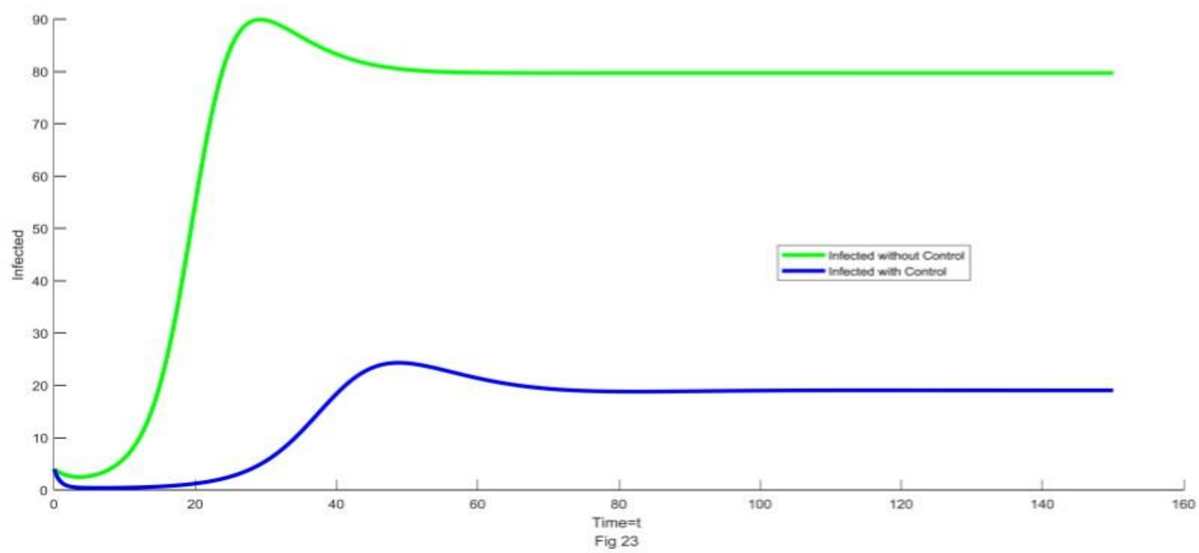

Figure 23: The Dynamics of Infected Smartphones with quarantine and treatment Control strategies only $(0 < \tau_2(t), \tau_3(t) < 1 \; and \; \tau_1(t) = 0)$.

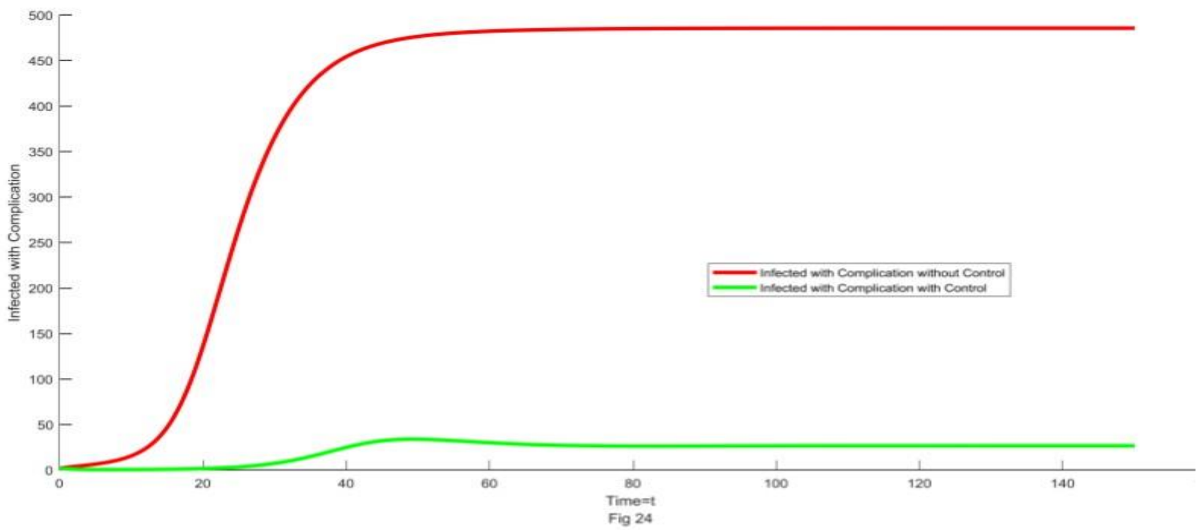

Figure 24: The Dynamics of Infected with Complication Smartphones with quarantine and treatment Control strategies only $(0 < \tau_2(t), \tau_3(t) < 1 \; and \; \tau_1(t) = 0 \,)$.

Without and with control $\tau_2(t) and \; \tau_3(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	490.9569	23.02
Infected	79.7262	19.0457	76.11
Infected with Complication	637.8093	26.6492	94.51

Table 7: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_2(t) and \; \tau_2(t)$.

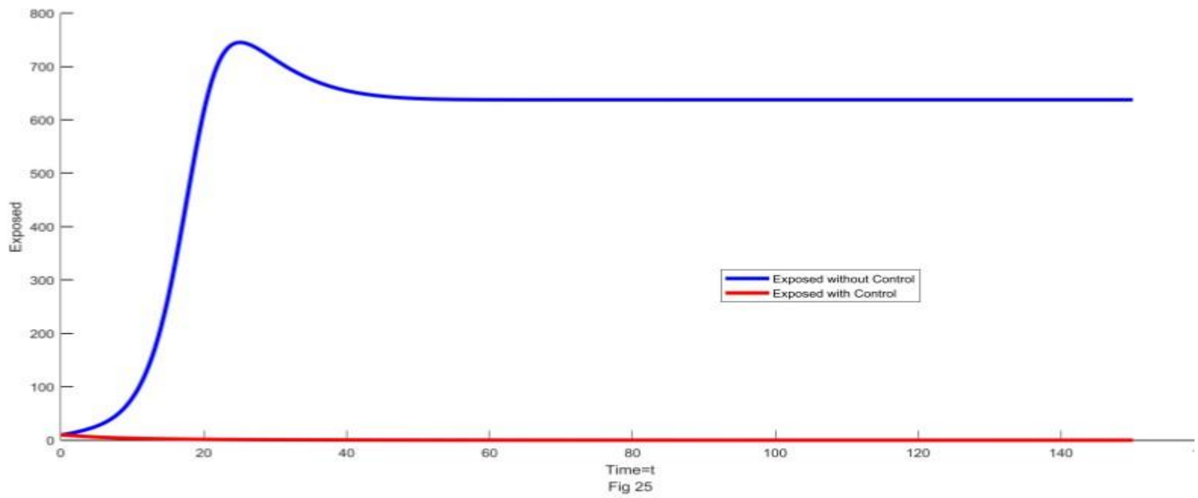

Figure 25: The Dynamics of Exposed Smartphones with vaccination, quarantine and treatment Control strategies $(0 < \tau_1(t), \tau_2(t), \tau_3(t) < 1)$.

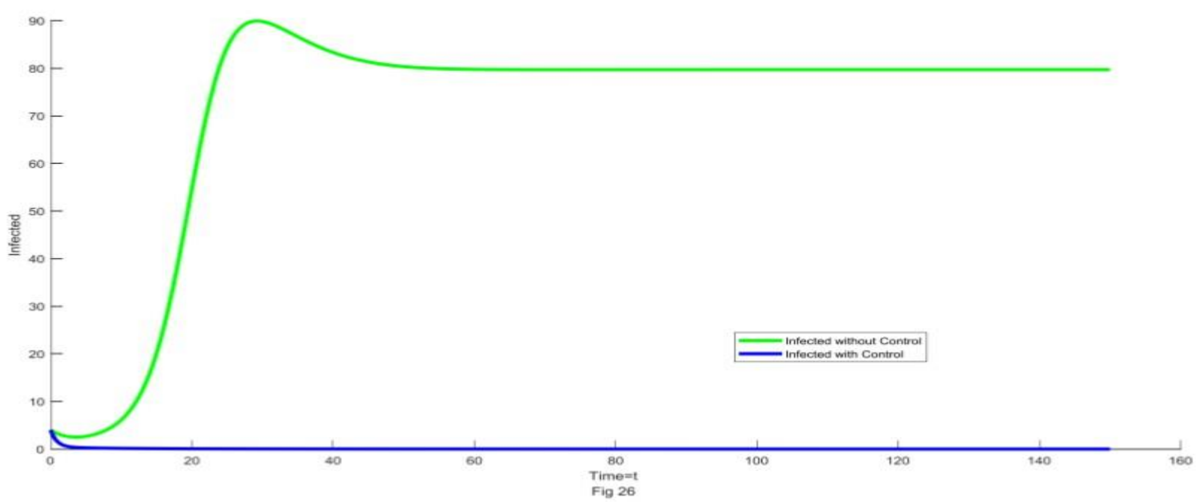

Figure 26: The Dynamics of Infected Smartphones with vaccination, quarantine and treatment Control strategies $(0 < \tau_1(t), \tau_2(t), \tau_3(t) < 1)$.

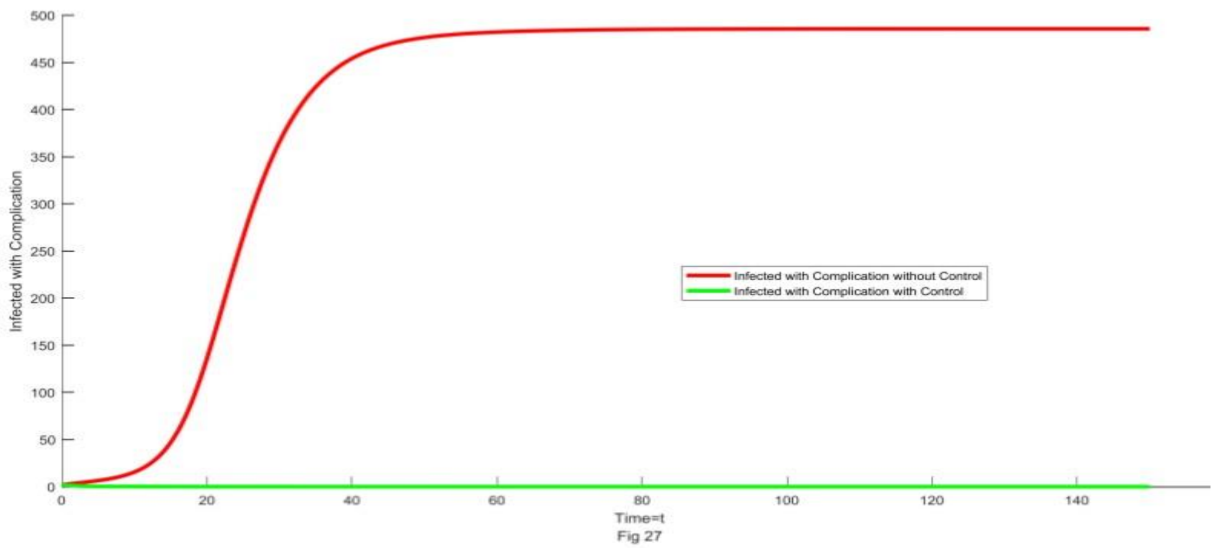

Figure 27: The dynamics of Infected with Complication Smartphones with vaccination, quarantine and treatment Control strategies $(0 < \tau_1(t), \tau_2(t), \tau_3(t) < 1)$.

Without and with control $\tau_1(t), \tau_2(t) and\ \tau_3(t)$ after $t = 150$	Without control	With control	Percentage of decreasing
Exposed	637.8093	0.0029371	99.9995
Infected	79.7262	0.00011848	99.9995
Infected with Complication	637.8093	0.00016898	99.9999

Table 8: Percentage of the decreasing number of Exposed, Infected and Infected with Complication without and with control $\tau_1(t), \tau_2(t) and\ \tau_2(t)$.

Figures 2 and 3 reveal the dynamics of the six compartments without and with optimal control respectively. From Figure 2, the population of exposed, infected and infected with complication were seen to be increasing because of lack of optimal control strategies. Also, we shall observe from figure 3 that the population of exposed, infected and infected with complication were drastically reduced whereas susceptible and recovered population increases due to the effectiveness of the optimal control strategies. The result of the individual compartments of exposed, infected and infected with complications were respectively displayed in Figures 4, 5 and 6 and it shows that without optimal control strategies, the Smartphone virus infection will continuously be transmitted from one Smartphone to another.

When we used only the optimal control $\tau_1(t)$, (Vaccination), there is an increase in the population of Smartphones protected from virus and the result were shown in Figures 7 (Exposed Smartphones), figure 8 (Infected Smartphones) and figure 9 (Infected with Complication Smartphones). This simply means that if Smartphones are adequately and effectively installed with antivirus, there will be a decrease in the exposed, infected and infected with complication Smartphones in the population. The percentage decrease is represented in table 2. In Figures 10, 11 and 12, we use only the optimal control $\tau_2(t)$. Here, the quarantine strategy was adopted for all Smartphones in the compartments of infected and infected with complication. The result of this optimal control strategy, indicates a slight decrease in the population of the exposed as in Figure 10, and a greater percentage decrease in the population of the infected and infected with complication Smartphones as shown in Figure 11 and Figure 12 respectively, which implies that the strategy is effective. Also, the percentage decreases in the required compartments were given in Table 3. Table 4 displayed the result of only the optimal control $\tau_3(t)$, which is treatment (repair). We observe that treatment only as an optimal control strategy for Smartphone virus is not strong enough to reduce the population of exposed, infected and infected with complication Smartphones. From Figure 13, we observe that treatment (repair) does not have effect on exposed Smartphones, and similar results were displayed in Figure 14 and Figure 15.

In Table 5, we show the result of combined optimal control strategies for $\tau_1(t)\ and\ \tau_2(t)$, (vaccination and quarantined). Figure 16 displayed the effect of vaccination and quarantine on the exposed Smartphone population, Figure 17 shows the effect of vaccination and quarantine on the infected Smartphone population whereas Figure 18 displayed the effect of vaccination and quarantine on the infected with complication. From these figures, we observe that vaccination and quarantine have a great effect by decreasing the population of the exposed, infected and infected with complication Smartphones. Also, table 6 presents the effect of combined optimal control strategies for $\tau_1(t)\ and\ \tau_3(t)$ (Vaccination and treatment). The outcome of vaccination and treatment on the exposed Smartphones is demonstrated in Figure 19. More so, the outcome of vaccination and treatment on the infected Smarphone population is presented in Figure 20 while the outcome of the same vaccination and treatment on infected with complication is shown in Figure 21. Examining these figures, we notice that the combined optimal control strategies of vaccination and treatment does not have much effect on the exposed, infected and infected with complication Smartphones. In the same way, we displayed the result of the combined optimal control strategies for $\tau_2(t)\ and\ \tau_3(t)$ (quarantine and treatment) on the exposed, infected and infected with complication Smartphones in table 7. In Figure 22, we demonstrated that quarantine and treatment does not have much effect on the exposed Smartphone population. it was presented in Figure 23, that quarantine and treatment have much effect on infected Smartphone population and also Figure 24 showed that it has great effect on infected with complication. From the results of these combinations, we observed that $\tau_1(t)\ and\ \tau_2(t)$, (vaccination and quarantined) optimal control strategy is more effective than the other two combinations. Since the main objective of this model is to minimize Smartphone virus infection in the population, we merged the three optimal control strategies $\tau_1(t), \tau_2(t), and\ \tau_3(t)$, that is, vaccination, quarantine and treatment. Based on Table 8 which shows the percentage of the decreasing number of exposed, infected and infected with complication Smartphones, we observe that the combination of the three optimal control strategies (vaccination, quarantine and treatment) has the greatest decreasing effect on the exposed Smartphones as in Figure 25, infected Smartphones as in Figure 26 and infected with complication Smartphones as in Figure 27. From these outcomes, we noticed that combined intervention of $\tau_1(t), \tau_2(t), and\ \tau_3(t)$ is a better control strategy for Smartphone virus infection and need to be deployed early in order to reduce Smartphone virus infection to the barest minimum.

6. Conclusion

In this study, we proposed the optimal control strategies to mitigate Smartphone virus infection transmission dynamics. In order to minimize the number of infected and infected with complication, we introduced three control strategies $\tau_1(t), \tau_2(t), and\ \tau_3(t)$, which respectively represents Vaccination, quarantined and treatment (repair). We also obtained the characterization of the optimal controls by applying the results of the control theory. The model was solved numerically by Runge – Kutta method of order four in MATLAB. Numerical simulation of the model shows the effectiveness of vaccination, quarantine and treatment, combination of any of the two control strategies and even the three control strategies. However, the results revealed that a combination of vaccination, quarantine and treatment is a better control strategy for Smartphone virus infection.

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