International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 1 (May), 2022

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 - 5708

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 1, May 2022, Pages 77 – 89

Numerical solutions of higher-order second kind Fredholm integro-differential equations with modified homotopy analysis method

M. D. Aloko*, J. A. Osilagun†, and Johnson D. Audu‡

Abstract

We present a modified Homotopy Analysis Method (MHAM) for solving higher-order nonlinear Fredholm integro-differential equations. This approach utilizes the homotopy principles but introduces two auxiliary parameters to construct a modified homotopy. The Parameters make it easier to analyze linear and strongly nonlinear problems without linearizing the problems directly. The series solution including the recurrence relations are carefully developed and presented explicitly without the need of applying any discretization. The error estimates are tabulated while the solution plots are graphically displayed. When compared with exact solutions and other traditional methods, our scheme proved reliable, efficient, and cost-effective.

Keywords: homotopy; higher order; Fredholm- integro-differential equations; decomposition.

1. Introduction

Nonlinear Fredholm equations naturally serve as model equations for many memory-dependent dynamical systems. These include population dynamics, epidemiology, and diffusion [20]. As a result, a lot of research has been directed towards approximating different kinds of Nonlinear Fredholm equations. Cubic spline collocation methods have recently been used for Integro-Differential equation systems [18], convolution integral approach [12], application of Legendre polynomials [18], Taylor collocation [18,6], and method of Chebyshev's polynomials [14]. Maleknejadel et al applied Berstein Operation Matrix to solve higher order-Volterra-Fredholm Integral-Differential equations [17]. More recent, high-order approximations techniques are employed in [4, 5]. Parameter variation approach for different problems related to Integro-differential equation systems is presented in [16]. In [18], a

* National Agency for Science and Engineering Infrastructure, FMST, Abuja, Nigeria
† Department of Mathematics, University of Lagos, Lagos, Nigeria
‡ Corresponding Author: King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia; Johnsonaudu@gmail.com

new modified Adomian method is presented while a numerical solution for the Taylor expansion method for Fredholm integral equations is contained in [19].

In this paper, we propose and implement an iterative scheme modified Homotropy Analysis Method (MHAM) to approximate the following integro-differential equation.

$$u^{(m)}(x) = \beta(x) + \mu \int_a^{g(x)} G(x,s)u^{(n)}(s)\,ds, \tag{1}$$

subject to the initial conditions

$u^{(k)}(0) = c_k,$
where $c_k, k = 0, 1, \dots (m-1)$, k, n , and m are integers with $n < m$.

It is worthy to note that some complicated problems that cannot be handled with our classical method are covered in [13, 3, 9].

In the remainder of this section, we present a general framework of the problem. In Section 2, the main ideas of homotopy including the series solutions and recurrence equations are presented while our modified homotopy scheme is provided in Section 3. In Section 4, we numerically implemented our scheme on some higher-order integro-differential equations and compared our approximate solution with the exact solution and conventional HAM.

2. Homotopy analysis method (HAM)

The Homotopy analysis method employs the concepts of homotopy from topology to generate a convergent series solution from nonlinear systems, however, instead of using the traditional homotopy, a nonzero auxiliary parameter h, referred to as the convergence –control parameter was introduced and nonzero auxiliary function $H(x)$ is used to construct the homotopy Analysis Method. We consider the following differential equation

$$\aleph u(x) = 0, \tag{2}$$

where, $\aleph$ is non-linear operator, $u(x)$ is an unknown function of independent variable x. Let $u_0(x)$ denotes an initial approximation of $u(x)$ and $\mathcal{L}$ denotes an auxiliary linear operator with the property:

$$\mathcal{L}[\beta(x)] = 0 \text{ when } \beta(x) = 0. \tag{3}$$

Using equation (1), we define the non-linear operator as

$$\aleph[u(x)] = u^{(m)}(x) - \beta(x) - \mu \int_a^{g(x)} G(x,s)u^{(n)}(s)\,ds\ . \tag{4}$$

Let $\mathcal{L} = I\ and\ H(x) = 1$, then to construct the so- called homotopy method, the following family of equation is used

$$(1-r)\mathcal{L}[\varphi(x;r,h) - u_0(x)] = rhH(x)\aleph[\varphi(x;r,h)], \tag{5}$$

where, $r \in [0,1]$ is an embedding parameter, $\varphi(x,r)$ is a function of $x\ and\ r$, h is a non-zeros auxiliary parameter and $H(x)$ is a non-zero auxiliary function. When $r = 0,$ from equation (5), we have

$$\mathcal{L}[\varphi(x;0,h) - u_0(x)] = 0. \tag{6}$$

It follows that

$$\left.\begin{array}{l} \phi(x;0,h) = u_0(x), \\ r = 1, \\ \aleph[(\phi(x;1,h)] = 0; x \geq 0, \\ subject \\ \phi(x;1,h) = 0. \end{array}\right\} \tag{7}$$

Since $h \neq 0, H(x) \neq 0$ and from the definition in equation (4), equations (7) are equivalent provided $\varphi(x;1,h) = u(x).$

As the embedding parameter r increases from 0 to 1, the solution $\varphi(x;r)$ of equation (2) depends upon the embedding parameter r and varies from the initial approximation $u_0(x)$ to the solution $u(x)$ of equation (2), see [3]. In topology, such a kind of continuous variation is called deformation. The zeroth-order deformation for equation (1) is constructed as

$$\aleph[u(x)] = u^{(n)}(x) - \beta(x) - \mu \int_a^{h(x)} k(x,s)u^{(m)}(s)\, ds\,. \tag{8}$$

$$\left.\begin{array}{c} (1-r)[\varphi(x;r,h) - u(x;0,h)] = rh(x)\left[\frac{\partial^n \varphi(x;r,h)}{\partial x^n} - \beta(x) - \mu \int_a^{h(x)} k(x,s)u^{(n)}(s)\, ds\right], \\ or \\ (1-r)[\varphi(x;r,h) - u_0(x)] = rh(x)\aleph[\varphi(x;r,h)]. \end{array}\right\} \tag{9}$$

Expand $\varphi(x;r,h)$ in Taylor series with respect to r , we get

$$\varphi(x;r,h) = \varphi(x;0,h) + \sum_{m=1}^{\infty} \frac{r^m}{m!}\frac{\partial^m}{\partial r^m}\varphi(x;r,h)\bigg|_{r=0}. \tag{10}$$

Let $u_m(x,h) = \frac{1}{m!}\frac{\partial^m \varphi(x;r,h)}{\partial r^m}|_{r=0}$. Using equations (5) and (6), if the auxiliary linear operator, the initial approximation, the auxiliary parameter h and the auxiliary function are properly chosen, the series equation (9) converges at r=1, then we have

$$\phi(x;r,h) = u_0(x) + \sum_{m=1}^{\infty} u_m(x,h)r^m\,. \tag{11}$$

Equation (11) provide a relationship between the initial approximation $u_0(x)$) and the exact solution $u(x)$ by means of the terms $u_m(x)$ $(m = 1, 2, \dots)$ which are unknown and defining the vector $\vec{u}_n = \{u_0(x), u_1(x), \dots u_n(x)\}$.

Differentiating equation (9) with respect to r yields

$$(1-r)\frac{\partial\varphi(x;r,h)}{\partial \mathrm{r}} - \varphi(x;r,h) - u_0(x) = h\aleph[\varphi(x;r,h)] + rh\frac{\partial\aleph(\varphi(x;r,h))}{\partial \mathrm{r}}. \quad (12)$$

Setting $r = 0$, equation (12) yields

$$\frac{\partial\varphi(x;r)}{\partial \mathrm{r}}|_{r=0} = h\aleph[\varphi(x;r)]|_{r=0}. \quad (13)$$

Differentiating equation (11) with respect to r, gives

$$\frac{\partial\varphi(x;r,h)}{\partial r} = \sum_{m=1}^{\infty} m u_m(x,h) r^{m-1},$$

$$= u_1(x,h) + \sum_{m=2}^{\infty} m u_m(x,h) r^{m-1}. \quad (14)$$

Using equation (14) and equation (13), the deformation equation of first-order is obtained as:

$$\left.\begin{array}{c} u_1(x,h) = h\aleph(u_0(x,h)), \\ u_1(x,h) = h\left(u_0(x) - \beta(x) - \mu\int_{\mathrm{a}}^{\mathrm{h(x)}}(\mathrm{k}(x,\mathrm{s}) u_0{}^{(n)}(\mathrm{s})\, ds\right). \end{array}\right\} \quad (15)$$

Differentiating equation (12) with respect to r and setting $r = 0$,

$$-2\frac{\partial\varphi(x;r,h)}{\partial \mathrm{r}} + (1-r)\frac{\partial\varphi(x;r,h)}{\partial \mathrm{r}} = 2h\frac{\partial\aleph(\varphi(x;r,h))}{\partial \mathrm{r}} + rh\frac{\partial^2\aleph(\varphi(x;r,h))}{\partial \mathrm{r}^2}.$$

Using equation (14) and equation (15), to obtain

$$-2u_1(x,h) + \sum_{m=2}^{\infty} m u_m(x,h) r^{m-1} + (1-r)[2u_2(x,h) + \sum_{m=2}^{\infty} m(m-1) u_m(x,h) r^{m-2}$$
$$= -2h[u_1(x,h) + \sum_{m=2}^{\infty} m u_m(x,h) r^{m-1}] - 2h\aleph(u_0(x,h) + qh\frac{\partial^2\aleph(\varphi(x,r)}{\partial r^2}.$$

Setting $r = 0$, the second-order deformation equation is also obtained as

$$u_2(x,h) = (1+h)u_1(x,h) - \mathrm{h}\mu(\int_{\mathrm{a}}^{\mathrm{h(x)}}(\mathrm{k}(x,\mathrm{s}) u_0{}^{(n)}(\mathrm{s}))ds).$$

To determine the m^{th} -order deformation, differentiating equation (11) m –times with respect to r and then setting $r = 0$, using equations (13)-(15), we have

$$\frac{\partial^m}{\partial r^m}(1-r)[\varphi(x;r,h)-u_0(x)] = \frac{\partial^m}{\partial r^m}\left(rh\aleph[\varphi(x;r,h)]-\beta(x)-\mu\int_{a}^{h(x)}k(x,s)u^{(n)}(x;r,h)\right)ds.$$

Substituting the homotopy series and dividing by $m!$, then $m-order$ deformation equation is obtained as

$$u_m(x,h) = (1+h)u_{m-1}(x) - \frac{h}{(m-1)!}\int_{a}^{h(x)}\frac{\partial^{m-1}\varphi(x;r)}{\partial r^{m-1}}k(x,s,)u^{(n)}(x;r)|_{r=0}ds. \quad (16)$$

A partition of $[a, h(x)]$, can be chosen as follows:

$a = x_1 < x_2 < \cdots < x_n = h(x) = b$ and $u_m(x_i)$ is computed as

$$u_m(x_i,h) = (1+h)u_{m-1}(x_i,h) - \frac{1}{(m-1)!}\int_{a}^{h(x_i)}\frac{\partial^{m-1}\varphi(x;r,h)}{\partial r^{m-1}}k(x,s)u^{(n)}(x;r,h)|_{r=0,\ x=x_i}ds.$$

Using equations (6) and (7), taking inverse $\mathcal{L}^{-1}$, (9) yields

$$\left.\begin{matrix} u_0 = u(x;0,h) = \mathcal{L}^{-1}(\beta(x)) = b_i \\ and \\ u(x;1,h) = u(x) \end{matrix}\right\}. \quad (17)$$

Also equation (16) can be expressed as

$$\mathcal{L}[u_m(x) - x_m u_{m-1}(x)] = hR_m\vec{u}_{m-1}(x). \quad (18)$$

where $R_m\vec{u}_{m-1} = \frac{1}{(m-)!}\frac{\partial^{m-1}\aleph[\varphi(x;r,h)]}{\partial r^{m-1}}|_{r=0}$ and $x_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$. (19)

Using equation (18)

$$u_m(x) = (1-x_m)u_{m-1}(x) + h\mathcal{L}^{-1}H(x)R_m\vec{u}_{m-1}(x). \quad (20)$$

3. Derivation of modified homotopy analysis method (MHAM)

A new recursion algorithm is obtained for $u_m(x)$ such that the initial approximant $u_0(x) = b_i$ obtained is decomposed into two parts $b_i(x) = b_{0i}(x) + b_{1i}(x)$ and replacing the two components by a series say b_l of infinite component expressed as: $b_l(x) = \sum_{n=0}^{\infty} b_{in}(x)$.

Taking $\mathcal{L}^{-1}$ of equation (18) and using equation (17).

The scheme for the Modified Homotopy Analysis Method (MHAM) is constructed as

$$\left.\begin{aligned} u_0 &= b_{10}(x), \\ u_1 &= b_{11}(x) + h\mathcal{L}^{-1}R_m\vec{u}_0(x), \\ u_2 &= b_{12}(x) + h\mathcal{L}^{-1}R_m\vec{u}_1(x), \\ u_3 &= b_{13}(x) + h\mathcal{L}^{-1}R_m\vec{u}_2(x), \\ &:: \\ &:: \\ u_m &= b_{i,m}(x) - h\mathcal{L}^{-1}R_m\vec{u}_{m-1}(x), m = 2,3, \dots \end{aligned}\right\} \tag{21}$$

It easy to obtain where the $m - th$ approximation of $u(x)$ for $m \geq 1$, that is given by

$$u(x) \approx u_0 + \sum_{m=1}^{k} u_m(x). \tag{22}$$

Equation (21) reduces the number of terms involved in each component, and hence the size of calculations is minimized compared to the conventional Homotopy Analysis method only. When $k \to \infty$, an accurate approximate of equation (21) can be obtained with the series solution of equation (2) by using the initial conditions [7, 8, 1, 6]. Thus, the Modified Homotopy Analysis Method (MHAM) provides a numerical method for solving integro-differential classes of problems and overcome the difficulty of decomposing $u(x)$ with an efficient improvement over the traditional Homotopy Analysis Method.

4. Numerical experiment

In order to analyze the results, three numerical examples are given to illustrate the accuracy and effective properties of the method and MAPLE- 17 package on Window 2010 is used to carry-out the calculation. Results are displayed in tables and graphical plots.

Example 1 [21].

Consider the third order Non-linear Fredholm integro differential equation of the second kind.

$$u'''(x) = xe^x + \frac{1079}{360}e^x + \frac{1}{120}\int_0^1 e^{(x-2s)}\mathrm{u}^2(\mathrm{s})\mathrm{ds}, \tag{23}$$

for $\mathrm{x} \in [0,1]$ with the initial condition $u(0) = 2, u'(0) = 1, u''(0) = 2$, and the exact solution $u(x) = xe^x$.

Using the MHAM algorithm (21), the zero deformation equation, and the Taylor series expansion:

$$b_l(x) = x + x^2 + \frac{1}{360} - \frac{359}{360}x - \frac{719}{720}x^2 + (x - \frac{1}{360})e^x \equiv x + x^2 + \frac{1}{360} - \frac{359}{360}x - \frac{719}{720}x^2 + \left(x - \frac{1}{360}\right)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^2 + 0(x^5\right) \equiv x + x^2 + \frac{1079}{2160}x^3 + \frac{1439}{8640}x^4 + \frac{1}{24}x^5$$

Let $b_{10}(x) = x + x^2 = u_0(x)$, $b_{11}(x) = +\frac{1079}{2160}x^3 + \frac{1439}{8640}x^4$, $b_{12}(x) = \frac{1}{24}x^5$

Setting $h = -1$ for each approximation,

$$u_1(x) = b_{11}(x) - h\frac{1}{120}\int_0^x\int_0^x\int_0^x\left(\int_0^1 exp(\xi - 2r)\, u^2{}_0(r)dr\right)d\xi d\xi d\xi,$$

$$u_1(x) = \frac{7}{480}x^2 - \frac{3}{32}e^{-2}x^2 - \frac{7}{480}e^x x^2 + \frac{3}{32}e^x e^{-2}x^2 + x^3$$

$$\begin{aligned} u_2(x) = & -\frac{922699}{36864000}e^x e^{-2}x^2 - \frac{6130109}{552960000}x^2 + \frac{922699}{36864000}e^{-2}x^2 \\ & + \frac{2943}{163840}e^{-4}x^2 - \frac{15739}{30720}e^{-3}x^2 - \frac{63}{163840}e^{-6}x^2 \\ & + \frac{219709}{2764800}e^{-1}x^2 + \frac{39}{4096}e^{-5}x^2 + \frac{15739}{30720}e^{-3}e^x x^2 \\ & + \frac{63}{163840}e^x e^{-6}x^2 + \frac{6130109}{552960000}e^x x^2 - \frac{2943}{163840}e^x e^{-4}x^2 \\ & - \frac{219709}{2764800}e^{-1}e^x x^2 - \frac{39}{4096}e^{-5}e^x x^2 - \frac{3}{8}e^x e^{-2}x^3 \\ & + \frac{7}{120}e^x x^3 - 6x^3 + \frac{7}{480}e^x x^4 - \frac{3}{32}e^x e^{-2}x^4 + \frac{1}{24}x^5 \end{aligned}$$

:::

$$u_n(x) = b_{i,n}(x) - h\int_0^x\int_0^x\int_0^1(\xi r)u_{n-1}^2(r)drd\xi d\xi$$

The approximation solution by MHAM for the same Example 1 is written in the form:

$$\begin{aligned} u(x) = & \ u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x) = \\ & x - 5x^3 + \frac{1}{24}x^5 + \frac{7}{120}e^x x^3 + \frac{2943}{163840}e^{-4}x^2 - \frac{15739}{30720}e^{-3}x^2 \\ & + \frac{219709}{2764800}e^{-1}x^2 - \frac{63}{163840}e^{-6}x^2 + \frac{39}{4096}e^{-5}x^2 \\ & + \frac{7}{480}e^x x^4 - \frac{3}{8}e^x e^{-2}x^3 + \frac{15739}{30720}e^{-3}e^x x^2 - \frac{39}{4096}e^{-5}e^x x^2 \\ & + \frac{2533301}{36864000}e^x e^{-2}x^2 + \frac{63}{163840}e^x e^{-6}x^2 + \frac{554893891}{552960000}x^2 \\ & - \frac{2533301}{36864000}e^{-2}x^2 - \frac{1933891}{552960000}e^x x^2 - \frac{219709}{2764800}e^{-1}e^x x^2 \\ & - \frac{3}{32}e^x e^{-2}x^4 - \frac{2943}{163840}e^x e^{-4}x^2 \end{aligned}$$

 IJMAM, Vol. 5, Issue 1 (2022) ©NSMB; www.tnsmb.org
(Formerly Journal of the Nigerian Society for Mathematical Biology)

x_i	Exact Sol	HAM		MHAM	
		$u^5{}_{MHAM}$- APR Sol	ABS-E	$u^5{}_{MHAM}$- APR Sol	ABS-E
0.1	0.110517	0.110517	1.51E-07	0.110517	5.12E-11
0.2	0.244281	0.244279	1.24E-06	0.244281	3.82E-10
0.3	0.404958	0.404953	4.29E-06	0.404958	1.31E-09
0.4	0.596730	0.596719	1.04E-05	0.596730	3.17E-09
0.5	0.824361	0.824340	2.09E-05	0.824361	6.36E-09
0.6	1.093271	1.093234	3.72E-05	1.093271	1.13E-08
0.7	1.409627	1.409566	6.06E-05	1.409627	1.84E-08
0.8	1.780433	1.780340	9.31E-05	1.780433	2.83E-08
0.9	2.213643	2.213506	1.36E-04	2.213643	4.14E-08
1.0	2.718282	2.718089	1.93E-04	2.718282	5.85E-08

Table 1: Comparison of Absolute Error Table for Example **1**

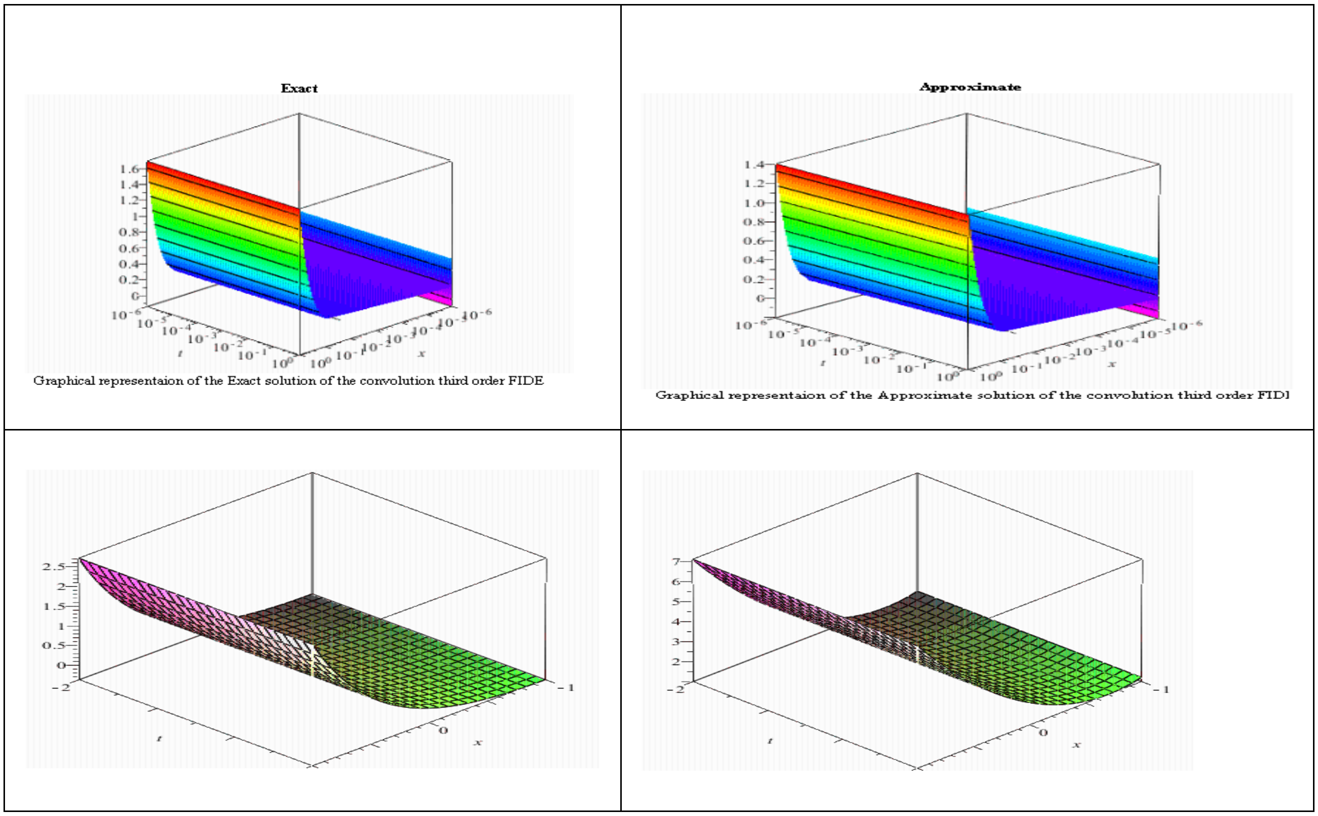

Figure 1: Graphical representation using 3-iterate of FIDE Example 1

Example 2: Consider the initial value problem [10]

$$u'''(x) = e^{x} - \frac{1}{4}(e^{x} + 1) + \int_0^1 xs(u'')^2(s)ds, \tag{24}$$

$$u(0) = u'(0) = u''(0) = 1.$$

Using the MHAM algorithm (21), the zero deformation using equation, and Taylor series expansion for

$$
\begin{aligned}
b_l(x) = \frac{1}{4} + \frac{x}{4} + \frac{x^2}{8} + \frac{3}{4}e^x - \frac{x^3}{24} \\
&\equiv 1 + x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{32} + \frac{x^5}{160} + \frac{x^6}{960} + \frac{x^7}{6720} + \frac{x^8}{53760} + \frac{x^9}{483840} + \frac{x^{10}}{4838400} \\
&+ \frac{x^{11}}{53222400} + \frac{x^{12}}{6386688000} + \frac{x^{13}}{8302694400} + \frac{x^{14}}{116237721600} + O(x^{15}) \\
&\equiv 1 + x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{32} + \frac{x^5}{160} + \frac{x^6}{960} + \frac{x^7}{6720} + \frac{x^8}{53760} + \frac{x^9}{483840} + \frac{x^{10}}{4838400} \\
&+ \frac{x^{11}}{53222400} + \frac{x^{12}}{6386688000} + \frac{x^{13}}{8302694400} + \frac{x^{14}}{116237721600}
\end{aligned}
$$

Let $b_{10}(x) = 1 + x = u_0(x)$, $b_{11}(x) = \frac{x^2}{2} + \frac{x^3}{12}$, $b_{12}(x) = \frac{x^5}{160} + \frac{x^6}{960}$, $b_{13}(x) = \frac{x^7}{6720} + \frac{x^8}{53760}$, $b_{14}(x) = \frac{x^9}{483840} + \frac{x^{10}}{4838400}, \ldots,$

Setting $\mathrm{h} = -1$ for each approximation,

$$u_1(x) = b_{11}(x) - h\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}\left(\int_0^1(\xi r)(u''_0)^2(\mathrm{r})dr\right)d\xi d\xi d\xi = \frac{1}{2}x^2 + \frac{1}{12}x^3$$

$$u_2(x) = b_{12}(x) - h\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}(\int_0^1(\xi r)\,(u''_1)^2(\mathrm{r})dr)d\xi d\xi d\xi = \frac{1}{32}x^4 + \frac{1}{160}x^5 - \frac{43}{1152}h\,x^4$$

$$u_3(x) = b_{13}(x) - h\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}(\int_0^1(\xi r)\,(u''_2)^2(\mathrm{r})dr)d\xi d\xi d\xi =$$
$$\frac{1}{960}x^6 + \frac{1}{6720}x^7 - \frac{139}{86016}h\,x^4 + \frac{43}{14336}h^2x^4 - \frac{1849}{1327104}h^3x^4$$

$$u_4(x) = b_{14}(x) - h\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}\int_0^{\mathrm{x}}(\int_0^1(\xi r)\,(u''_3)^2(\mathrm{r})dr)d\xi d\xi d\xi =$$
$$
\begin{aligned}
&\frac{1}{53760}x^8 + \frac{1}{483840}x^9 - \frac{461}{81100800}h\,x^4 + \frac{3015977}{187280916480}h^4x^4 \\
&\quad + \frac{7367}{990904320}h^2x^4 - \frac{202367}{12331253760}h^3x^4 \\
&\quad - \frac{3418801}{1761205026816}h^7x^4 - \frac{5393533}{399532621824}h^5x^4 \\
&\quad + \frac{79507}{9512681472}h^6x^4
\end{aligned}
$$

$u_5(x) = b_{15}(x) - h\int_0^{x}\int_0^{x}\int_0^{x}(\int_0^{1}(\xi r)\,(u''_4)^2(\mathrm{r})dr)d\xi d\xi d\xi =$

$$\frac{1}{4838400}x^{10} + \frac{1}{53222400}x^{11} - \frac{461}{81100800}h\,x^4 + \frac{3015977}{187280916480}h^4x^4 + \frac{7367}{990904320}h^2x^4 - \frac{202367}{12331253760}h^3x^4 - \frac{3418801}{1761205026816}h^7x^4 - \frac{5393533}{399532621824}h^5x^4 + \frac{79507}{9512681472}h^6x^4$$

::

$$u_m(x) = b_{im}(x) - h\int_0^{x}\int_0^{x}\int_0^{x}(\int_0^{1}(\xi r)\,(u''_{m-1})^2(\mathrm{r})dr)d\xi d\xi d\xi =$$

The approximation solution by MHAM for the same Example **2** is written in the form:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots + u_n(x)$$

x_i	Exact Sol	MHAM		[10]	
		APR Sol n=7	ABS-E	$u^7{}_{MHAM}$- APR Sol	ABS-E
0.0	1.00000	1.00000	0.00E+00	1.00000	0.00E+00
0.1	1.10517	1.10517	2.51E-13	1.10517	2.22E-08
0.2	1.22140	1.22140	6.49E-11	1.22140	7.58E-07
0.3	1.34986	1.34986	1.68E-09	1.34985	6.12E-06
0.4	1.49182	1.49182	1.70E-08	1.49180	2.74E-05
0.5	1.64872	1.64872	1.03E-07	1.64863	8.85E-05
0.6	1.82212	1.82212	4.46E-07	1.82189	2.33E-04
0.7	2.01375	2.01375	1.55E-06	2.01322	5.31E-04
0.8	2.22554	2.22554	4.56E-06	2.22445	1.09E-03
0.9	2.45960	2.45959	1.18E-05	2.45753	2.08E-03
1.0	2.71828	2.71825	2.79E-05	2.71458	3.70E-03

Table 2: Results and Comparison of Absolute Error Table for FIDE Example **2**

5. Conclusion

In this paper, we provided a modified Homotopy Analysis (MHAM) approach for approximating strongly nonlinear physical models in engineering and applied sciences that are governed by higher-order Fredholm integro -differential equation of the second kind.
The implementation and the numerical examples both show that our technique is efficient. The scheme is simple to implement, it reduces the computational difficulty associated with other traditional methods.

Acknowledgement: The authors would like to thank National Agency for Science and Engineering Infrastructure (NASENI), University of Lagos (UNILAG), and King Fahd University of Petroleum and Minerals (KFUPM) for their support**.**

References

[1] Al-Hayani W. (2013). Solving nth-order integro-differential equations using the combined Laplace transform- Adomian decomposition method. Applied Mathematics, 4: 882-886.

[2] Chandru M., Das P., and Ramos H. (2018). Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data. Mathematical Methods in the Applied Sciences, 41(14): 5359-5387.

[3] Das P. (2018). A higher order difference method for singularly perturbed parabolic partial differential equations. Journal of Difference Equations and Applications, 24(3): 452-477.

[4] Das P. (2019). An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh. Numerical Algorithms, 81(2): 465-487.

[5] Das P. and Natesan S. (2014). Optimal error estimate using mesh equidistribution technique for singularly perturbed system of reaction–diffusion boundary-value problems. Applied mathematics and computation, 249: 265-277.

[6] Das P. and Vigo-Aguiar J. (2019). Parameter uniform optimal order numerical approximation of a class of singularly perturbed system of reaction diffusion problems involving a small perturbation parameter. Journal of Computational and Applied Mathematics, 354: 533-544.

[7] Das P., Rana S., and Ramos H. (2020). A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis. International Journal of Computer Mathematics, 97(10): 1994-2014.

[8] Das P., Rana S., and Ramos H. (2020). On the approximate solutions of a class of fractional order nonlinear Volterra integro-differential initial value problems and boundary value problems of first kind and their convergence analysis. Journal of Computational and Applied Mathematics, 113116.

[9] Das P., Rana S., and Vigo-Aguiar J. (2020). Higher order accurate approximations on equidistributed meshes for boundary layer originated mixed type reaction diffusion systems with multiple scale nature. Applied numerical mathematics, 148: 79-97.

[10] Hemeda A. A. (2012). New iterative method: application to nth-order integro-differential equations. International Mathematical Forum, 47(7): 2317 – 2332.

[11] Jafarzadeh Y. and Keramti B. (2018). numerical method for a system of integro-differentials equations and convergence analysis by Taylor collocation, Ain Shams Engineering Journal,

9(4): 1433-1438.

[12] Katsidelis J. T. (2019). numerical solution of integro-differentials equations with convolution integrals. Archive of Applied Mechanics, 89(10): 2019–2032.

[13] Keshvaraz E. and Ordokhani Y. (2019). A fast numerical algorithm based on the Taylor wavelets for solving the fractional integro-differential equations with weakly singular kernels. Maths Meth Appl. Sci., 42: 4427-4443.

[14] Kurt N. and Sezer M. (2008). Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients. Journal of the Franklin Institute, 345(8): 839-850.

[15] Liao S. (2005). Comparison between the homotopy analysis method and homotopy perturbation method. Applied Mathematics and Computation, 169(2): 1186-1194.

[16] Noor M.A., Waheed K.I., and Al-Said E.A. (2016). Variation of parameters method for solving system of nonlinear system of Volterra integro-differential equations. Computer Mathematics Application, 62: 2126-2131.

[17] Maleknejad K., Basirat B., and Hashemizadeh E. (2011). Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra- Fredholm integro-differential equations. Computer and Mathematics with Applications, 61: 2821-2828.

[18] Memudu W.W. and Taiwo O.A. (2016). Exponentially fitted collocation approximation method for the numerical solutions of Higher Order Linear Fredholm Integro-differential-Difference Equations. Theoretical Mathematics and Applications, 6(1): 131-150.

[19] Sezer M. (1994). Taylor polynomial solutions of Volterra integral equations. International Journal of Mathematical Education in Science and Technology, 25(5): 625-633.

[20] Vahdati S., Abbas Z., and Ghasemi M. (2010). Math. Application of homotopy analysis method to Fredholm and Volterra integral equations. Mathematical Sciences Quarterly Journal.

[21] Wazwaz A. M. (2011). Linear and nonlinear integral equations: methods and applications, Springer.