International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 5, Issue 1 (May), 2022

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 - 5708

International Journal of Mathematical Analysis and Modelling
Volume 5, Issue 1, May 2022, Pages 56 – 68

A Fermat polynomial method for solving optimal control problems

M. U. Nduka*† and B. I. Oruh*‡

Abstract

In this paper, we propose an efficient computational approach to solve optimal control problems subject to state constraints and boundary conditions. The idea is to obtain an approximate numerical value of the objective function. The method is based on state parameterization. The state variable is considered as a linear combination of Fermat polynomials with unknown coefficients such that the state differential equations are satisfied. The optimal control problem is converted to quadratic programming problem which we easily solved by Lagrange method. Some numerical examples are considered to illustrate the reliability of the method. The numerical results obtained are accurate when compared with the analytic solution and results obtained from literatures.

Keywords: optimal control; parameterization; Fermat polynomials; optimization

1. Introduction

Control theory is a branch of optimization theory that deals with the minimization or maximization of an objective function subject to some state equations and constraints. It addresses the problem of how best to optimize or operate a dynamic system to optimize some given criteria or performance. From a general viewpoint, an optimal control problem is an optimization problem. Ever since its emergence, optimal control theories have been applied to a wide range of areas and disciplines such as finance, for biological models and computational biology, Economics for economic models and economics of recycling, Engineering for aerospace and robotics [14], [3], [9]. Optimal control problems consider the optimization of dynamic system characterized by state and control variables.

The goal of solving optimal control problems is to determine the control $U(t)$ and the state $X(t)$ of a dynamic system that either minimizes or maximizes a performance index or functional $J\big(u(t), x(t)\big)$ and at the same instant satisfying the dynamic constraints and boundary conditions [12]. The solutions to optimal control problems can be found using two major techniques, analytic and numerical techniques. Analytic techniques obtain the necessary optimality

* Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria
† Corresponding Author; modesty1799@gmail.com
‡ ifeanyichi@mouau.edu.ng

conditions through the maximum principle which is a generalization of classical calculus of variation.

However, it is often difficult to obtain analytic solution of optimal control problems. Furthermore, as the complexity of optimal control problems increases it becomes almost impossible to solve optimal control problems analytically. As a result, numerical techniques for solving optimization problems are employed to solve optimal control problems. The availability of numerous nonlinear programming softwares has made numerical techniques a better alternative. Numerical techniques have presented an attractive field for researchers of mathematical sciences, which have produced different numerical computational techniques and efficient algorithms in solving optimal control problems. Numerical techniques are achieved by discretization or parameterization.

Parameterization methods give a robust technique for solving optimal control problems in finite dimensional spaces. The method expresses the state, and/or control of a system as a function of some independent quantities known as parameters. State parameterization converts an optimal control problem to a nonlinear optimization problem and finds $(n+1)$ unknown polynomial coefficients of degree, at most$n$, in the form $\sum_{i=0}^{n} a_i t^i$. Polynomials have been used by many authors [13], [2], [15], [11], [4], [8] as approximating functions to the state and/or control variables of an optimal control problem, which result to different approximate values of the performance index. In most cases, Chebyshev polynomials and Legendre polynomials are used. [6], [5] solved Duffing Oscillator problem. The solution is based on state parameterization using linear combination of Chebyshev polynomials with unknown coefficients. This approximation converts an $(n+1)$ dimensional optimization problem into one-dimensional optimization problem which can then be solved easily. [10] proposed the use of Chebyshev wavelets, the state variable is approximated by a new hybrid based on second kind Chebyshev wavelets. subsequent iterations result to better accuracy. [5] proposed a numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme. [1] presented a technique to solve nonlinear optimal control problems. The method is based on using Legendre scaling functions to parameterize the state variable. This approach replaces the nonlinear optimal control problem by a sequence of time-varying linear quadratic optimal control problems which gives a better result when compared with some other methods. In spite of the many advantages that the state parameterization offer, the method has not been used extensively as there are numerous functions that have not been used which can give a better approximation and result than the ones in the literature.

In this paper we present the Fermat polynomial as the approximating function. The state variable of an optimal control problem is considered as a linear combination of Fermat polynomials with unknown coefficients such that the state differential equations are satisfied. The optimal control problem is converted to a quadratic programming problem resulting from a linear quadratic optimal control problem which can easily be solved by existing optimization methods.

2. Problem statement

Find the optimal control u^* and the optimal trajectory x^* that optimizes the performance index;

$$J = \int_{t_0}^{t_1} L(x(\tau), u(\tau), \tau)dt, \quad (1)$$

subject to the state equations:

$$\dot{x}(\tau) = f(x(\tau), u(\tau), \tau), \quad (2)$$

and boundary conditions:

$$x(t_0) = x_0, x(t_1) = x_1 \tau \in [t_0, t_1], \quad (3)$$

where f is a real-valued continuously differentiable function. $x(\tau): I \to \mathbb{R}^n$, $u(\tau): I \to \mathbb{R}^m$, $I = [t_0, t_1]$, L is assumed to be continuously differentiable and equation (1.1) is known as *Lagrangian* problem. For easy reference we shall denote equations (1)-(3).

2.1. Material and methods

We state that the parameterization involves the approximation of the state variable(s) of an optimal control problem using a given trial function or polynomials.

2.2.1. Fermat polynomials

Polynomial expansions are broadly used in many mathematical fields to obtain results for both numerical and analytical analysis. Many of these polynomials are obtained from difference equations. Fermat polynomials are among the polynomial sequences that are generated by difference equations.

Fermat polynomials can be generated by the difference equation:

$$f_{i+2}(t) = 3tf_{i+1}(t) - 2f_i(t), \qquad f_0(t) = 0, f_1(t) = 1, \qquad i \geq 0.$$

Fermat polynomials can also be obtained by setting $p(t) = 3t$, and $q(t) = -2$ in the Lucas polynomial sequence defined by:

$$F_n(t) = p(t)F_{n-1}(t) + q(t)F_{n-2}(t).$$

Fermat polynomials can be explicitly written as:

$$\mathcal{F}(t) = \begin{cases} \frac{\left(3t+\sqrt{9t^2-8}\right)^k - \left(3t-\sqrt{9t^2-8}\right)^k}{2^k\sqrt{9t^2-8}}, & t \neq \frac{2}{3} \\ 2^{\frac{k}{2}} sin\left(\frac{\pi}{4}k\right), & t = \frac{2}{3} \end{cases} \qquad k = 1,2 \ldots$$

The first few Fermat polynomials are:

$$\mathcal{F}_1(t) = 1,\ \mathcal{F}_2(t) = 3t, \mathcal{F}_3(t) = 9t^2 - 2,\ \mathcal{F}_4(t) = 27t^3 - 12t, \mathcal{F}_5(t) = 81t^4 - 54t^2 + 4.$$

It is necessary to note that $\mathcal{F}_k(t)$ is a polynomial of degree $(k-1)$ with integer coefficients.

2.2.2. State parameterization using Fermat polynomials

We approximate the system state variable by a sequence of Fermat polynomials with unknown coefficients. Let $Q \subset C^1([t_0, t_1])$ be set of all functions satisfying equation (3). The performance index in equation (1) can be expressed as a function of$x(t)$. Then optimal control problem (1)-(3) may be considered as a minimization of equation (1) on the set Q.

2.2.3. Approximation of state and control variables

Let Q_n be a subset of Q consisting of Fermat polynomials of degree up to n as

$$x_n = \sum_{i=1}^{n} a_i\, \mathcal{F}_i(t)\, n = 1,2 \ldots \tag{4}$$

Now we consider the minimization of J on Q_n with $\{a_i\}_{i=0}^{n}$ as unknowns.

Equation (2) can be re-written as

$$u(t) = \phi(x(t), \dot{x}(t), t). \tag{5}$$

The control variable is then written as a function of the unknown parameters of the state variables by substituting equation (4) into equation (5)

$$u_n(t) = \phi\left(\sum_{i=0}^{n} a_i \mathcal{F}_i\,(t), \sum_{i=0}^{n} a_i \dot{\mathcal{F}}_i\,(t), t\right). \tag{6}$$

2.2.4 Approximation of the performance index

By substituting these approximations of state variables (4) and control variables (6) into the objective function (1) we get

$$\hat{J}(a_0, a_1, \ldots a_n) = \int_{t_0}^{t_1} L\left(\sum_{i=1}^{n} a_i \mathcal{F}_i(t)\,, \phi\left(\sum_{i=1}^{n} a_i \mathcal{F}_i\,(t), \sum_{i=1}^{n} a_i \dot{\mathcal{F}}_i(t), t\right)\right) dt. \tag{7}$$

The boundary conditions become

$$x_n(t_0) = \sum_{i=1}^{n} a_i \mathcal{F}_i(t_0) = x_0,$$

$$x_n(t_1) = \sum_{i=1}^{n} a_i \mathcal{F}_i(t_1) = x_1.$$

The integration of equation (7) yields an expression in terms of the unknown parameters we seek. The optimal control problem is converted into a mathematical programming problem of unknown parameters with equality constraints.

The resulting problem can be stated in the form

$$minimize \quad \hat{J}(a_1, a_2, \dots a_n) = a'Ma, \tag{8}$$
$$subject\ to{:} \quad g(a_1, a_2 \dots a_3) = b, \tag{9}$$

where $a\epsilon\mathbb{R}^n$, $\hat{J}$ is the approximate value of J, M is an $n \times n$ matrix. We obtain the optimal control vector a^* by using the Lagrangian multiplier method.

3. Algorithm

Input: Optimal Control Problem (1)-(3).
Output: The approximate optimal control, optimal trajectory and approximate performance index.
Step 1. Approximate the state variable by nth Fermat series.
Step 2. Express the control variable as a function of the approximated state variable.
Step 3. Substitute the expressions in steps 1 and 2 into the performance index to obtain the approximate performance index $\hat{J}$.
Step 4. Find the set of equality constraints from the initial and final conditions and calculate the matrix M.
Step 5. Find the optimal parameters a^* by solving the optimization problem (8)-(9).
Step 6. Substitute a^* into equations (4), (6), and (7) to find the approximate optimal trajectory, approximate optimal control and the performance index respectively.

4. Numerical results

4.1. Example 1

We consider the minimization problem:

$$minimize\ J = \frac{1}{2}\int_0^1 (3x(t)^2 + u(t)^2)\, dt, \tag{10}$$
$$Subject\ to \quad u = \dot{x} + x \tag{11}$$
$$with\ conditions\ x(0) = 0, x(1) = 2. \tag{12}$$

4.1.1. Analytic Approach

First, we attempt the problem analytically using Pontryagin-Minimum-Principle. We construct the Hamiltonian of the problem:

$$H(x,u,\lambda) = 3x^2 + u^2 + \lambda(u - x), \tag{13}$$

$$\frac{\partial H}{\partial u} = 0 \Longrightarrow 2u + \lambda = 0,$$

$$u = -\frac{\lambda}{2}. \tag{14}$$

Substitute equation (13) into equation (14) and simplify

$$H = 3x^2 - \frac{\lambda^2}{2} - \lambda x. \tag{15}$$

Differentiate equation (15) with respect to x and λ

$$\frac{\partial H}{\partial x} = -\dot{\lambda} \Longrightarrow 6x - \lambda = -\dot{\lambda}, \qquad \frac{\partial H}{\partial \lambda} = \dot{x} \Longrightarrow -\frac{2\lambda}{4} - x = \dot{x}.$$

Simplifying we have

$$\ddot{x} - 4x = 0. \tag{16}$$

Hence, the solution to equation (16) is

$$x(t) = Acosh(2t) + Bsinh(2t).$$

Apply the boundary conditions to obtain the optimal state function

$$x^*(t) = \frac{2\sinh(2t)}{\sinh(2)}. \tag{17}$$

And the optimal control function

$$u^*(t) = \frac{4\cosh(2t) + 2\sinh(2t)}{\sinh(2)}. \tag{18}$$

Substitute (17) and (18) into equation (10)

$$J = \frac{1}{2}\int_0^1 \left(3\left(\frac{4sinh^2(2t)}{sinh^2(2)}\right) + \left(\frac{16cosh^2(2t) + 16\cosh(2t)\sinh(2t) + 4sinh^2(2t)}{sinh^2(2)}\right)\right)dt$$

$$J = \frac{8}{sinh^2(2)}\int_0^1 (sinh^2(2t) + cosh^2(2t) + \cosh(2t)\sinh(2t))dt$$

$$J = 6.14925721.$$

Hence $J = 6.14925721$ is the exact numerical solution of the optimal control problem (10)-(12).

4.1.2. The Proposed Approach

We approximate the state variable with 5th order Fermat polynomial

$$x \approx \sum_{i=1}^{5} a_i \mathcal{F}_i = a_1\mathcal{F}_1 + a_2\mathcal{F}_2 + a_3\mathcal{F}_3 + a_4\mathcal{F}_4 + a_5\mathcal{F}_5,$$

$$\hat{x}(t) = 81a_5t^4 + 27a_4t^3 + (9a_3 - 54a_5)t^2 + (3a_2 - 12a_4)t + a_1 - 2a_3 + 4a_5 \tag{19}$$

and the approximate control function

$$\hat{u}(t) = 81a_5t^4 + (27a_4 + 324a_5)t^3 + (9a_3 + 81a_4 - 54a_5)t^2 + (3a_2 + 18a_3 - 12a_4 - 108a_5)t + a_1 + 3a_2 - 2a_3 - 12a_4 + 4a_5 \tag{20}$$

The squares of equations (19) and (20) are substituted into equation (10) to obtain the approximate objective function in terms of the unknown parameters
Integrating and simplifying we have

$$\hat{J} = 2{a_1}^2 + 9a_1a_2 + 13a_1a_3 + 18a_1a_4 + \frac{179a_1a_5}{5} + 15{a_2}^2 + 63a_2a_3 + \frac{534a_2a_4}{5} + 198a_2a_5 + \frac{929{a_3}^2}{10} + \frac{819a_3a_4}{2} + \frac{5611a_3a_5}{7} + \frac{19659{a_4}^2}{35} + \frac{4923a_4a_5}{2} + \frac{213119{a_5}^2}{70} \tag{21}$$

The boundary conditions are substituted into equation (19) to obtain algebraic equations

$$a_1 - 2a_3 + 4a_5 = 0,$$

$$a_1 + 3a_2 + 7a_3 + 15a_4 + 31a_5 - 2 = 0. \tag{22}$$

The optimal control problem is finally reduced to an optimization problem. Our task is to minimize equation (21) subject to equation (22).

The solution of the system of equations yields $a^* = \left(\frac{307}{3913}, \frac{2939}{7206}, \frac{751}{1488}, \frac{137}{12060}, \frac{67}{11917}\right)$, and $\lambda^* = \left(\frac{1190}{1079}, -\frac{5067}{824}\right)$ which is the value that minimizes J.

Hence, $J(a^*) = 6.14927031$ is the approximate numerical solution of the optimal control problem, which is an acceptable approximation for the exact numerical solution $J^* = 6.14925721$.

The approximate optimal state and control functions are obtained by substituting the value of a^* into equations (19) and (20) as

$$x(t) = \frac{5427}{11917}t^4 + \frac{411}{1340}t^3 + \frac{27136}{18014}t^2 + \frac{12241}{11259}t + \frac{15433}{34694}$$
$$u(t) = \frac{5427}{11917}t^4 + \frac{47925}{22518}t^3 + \frac{48224}{45036}t^2 + \frac{31267}{22518}t + \frac{10118}{93056}$$

The graph of the state and control function are plotted below.

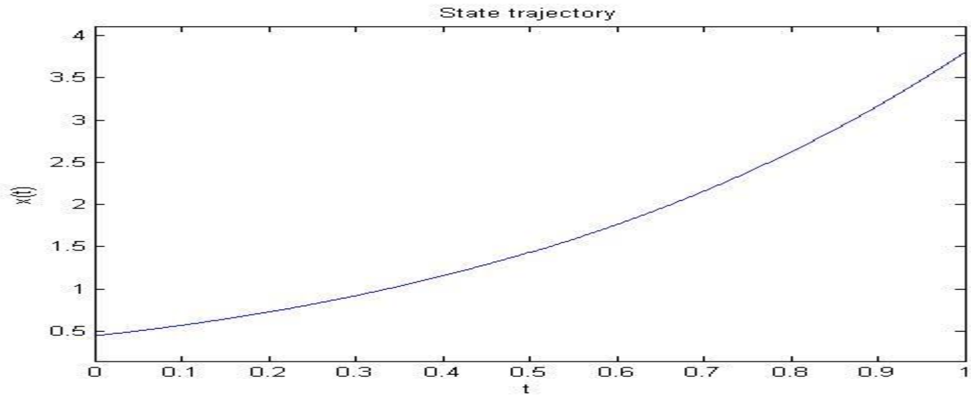

Figure 1a.

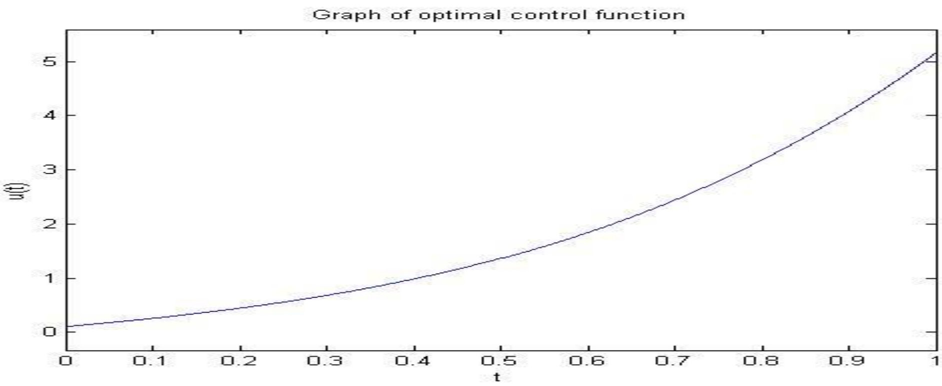

Figure 1b.

4.2. Example 2

$$Minimize \quad J = \int_0^1 \left(x^2(t) + \frac{1}{2}u^2(t)\right)dt, \tag{23}$$

$$Subject\ to\colon\ \dot{x}(t) = \frac{1}{2}x(t) + u(t),\ \ x(0) = 1. \tag{24}$$

The optimal solution is:

$$x^*(t) = \frac{2e^{3t} + e^3}{2e^{3t/2}(2 + e^3)}, \qquad u^*(t) = \frac{2e^{3t} - e^3}{2e^{3t/2}(2 + e^3)}.$$

And the exact solution for the performance index is:

$$J = \frac{e^6 + e^3 - 2}{e^6 + 4e^3 + 4} = 0.8641644978.$$

Using our approach, we approximate $x(t)$ by Fermat polynomial of order 7

$$x \approx \sum_{i=1}^{5} a_i \mathcal{F}_i,$$

$$\begin{aligned} x = {} & 729a_7t^6 + 243a_6t^5 + (81a_5 + 810a_7)t^4 + (27a_4 - 216a_6)t^3 \\ & + (9a_3 - 54a_5 + 216a_7)t^2 + (3a_2 - 12a_4 + 36a_6)t + a_1 - 2a \\ & + 4a_5 - 8a_7. \end{aligned} \tag{25}$$

From equation (24) we have:

$$\begin{aligned} u = {} & -\frac{729a_7}{2}t^6 + \left(4374a_7 - \frac{243a_6}{2}\right)t^5 + \left(1215a_6 - \frac{81a_5}{2} - 405a_7\right)t^4 \\ & + \left(324a_5 - \frac{27a_4}{2} + 108a_6 + 3240a_7\right)t^3 \\ & + \left(81a_4 - \frac{9a_3}{2} + 27a_5 - 648a_6 - 108a_7\right)t^2 \\ & + \left(18a_3 - \frac{3a_2}{2} + 6a_4 - 108a_5 - 18a_6 + 432a_7\right)t - \frac{a_1}{2} + 3a_2 \\ & + a_3 - 12a_4 - 2a_5 + 36a_6 + 4a_7 \end{aligned} \tag{26}$$

Then substituting equations (25) and (26) into equation (23) gives:

$$
\begin{aligned}
J(a_1, a_2, \dots a_7) = {} & \frac{9}{8}a_1{}^2 + \frac{15}{8}a_1a_2 - \frac{9}{4}a_1a_3 - \frac{93}{16}a_1a_4 - \frac{171}{20}a_1a_5 - \frac{171}{8}a_1a_6 \\
& - \frac{3771}{28}a_1a_7 + \frac{45}{8}a_2{}^2 + \frac{399}{16}a_2a_3 + \frac{639}{20}a_2a_4 + 48a_2a_5 \\
& + \frac{16551}{140}a_2a_6 + \frac{144267}{32}a_2a_7 + \frac{2079}{40}a_3{}^2 + 231a_3a_4 \\
& + \frac{61407}{140}a_3a_5 + \frac{28179}{32}a_3a_6 + \frac{454311}{20}a_3a_7 + \frac{104499}{280}a_4{}^2 \\
& + \frac{54579}{32}a_4a_5 + \frac{477873}{140}a_4a_6 + \frac{5424159}{80}a_4a_7 + \frac{127665}{56}a_5{}^2 \\
& + \frac{805509}{80}a_5a_6 + \frac{251430813}{1540}a_5a_7 + \frac{38100861}{3080}a_6{}^2 \\
& + \frac{28615023}{80}a_6a_7 + \frac{134572923999}{40040}a_7{}^2.
\end{aligned}
$$

Minimize $J(a_1, a_2, \dots a_7)$ subject to $x(t_0, a_1, a_2, \dots a_7) = x_0$. The solution is

$a^* = \left(\frac{5509605}{43424277}, -\frac{62695946}{13027283}, \frac{20145943}{14474759}, -\frac{24125360}{13027283}, \frac{35048278}{14474759}, -\frac{24431921}{13027283}, \frac{59827196}{43424277}\right)$ and $\lambda^* = -\frac{70048008}{40529325}$.

The approximate solution is $J(a^*) = 0.864164498$ which agrees with the exact solution.

	Method	J	Error
Exact Value		0.8641644978	0
Kafash, B. and Delavarkhalafi, A.	Restarted State Parameterization	0.8643546452	$1.9e^{-04}$
Eisa, B.M. and Awadalla, T.A.	Legendre	0.8641645132	$1.5e^{-09}$
This Research	Fermat	0.8641644978	0

Table 1. Comparison of the optimal value with other methods

We calculate the state and control variables as:

$$
x(t) = \frac{2895}{2882}t^6 - \frac{1642}{3603}t^5 + \frac{7468}{3603}t^4 - \frac{5174}{1126}t^3 + \frac{5066}{4504}t^2 - \frac{1729}{1407}t + 1
$$
$$
u(t) = -\frac{2895}{5765}t^6 + \frac{1169}{1407}t^5 - \frac{3732}{1126}t^4 + \frac{4769}{4504}t^3 - \frac{8741}{4504}t^2 + \frac{6449}{2252}t - \frac{2551}{1476}.
$$

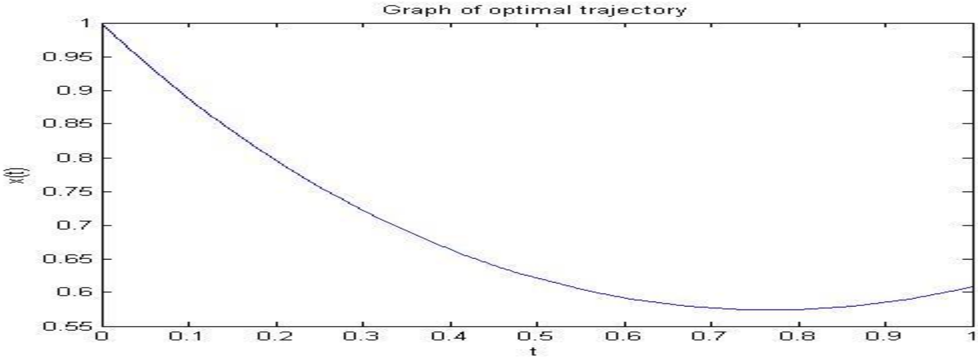

Figure 2a.

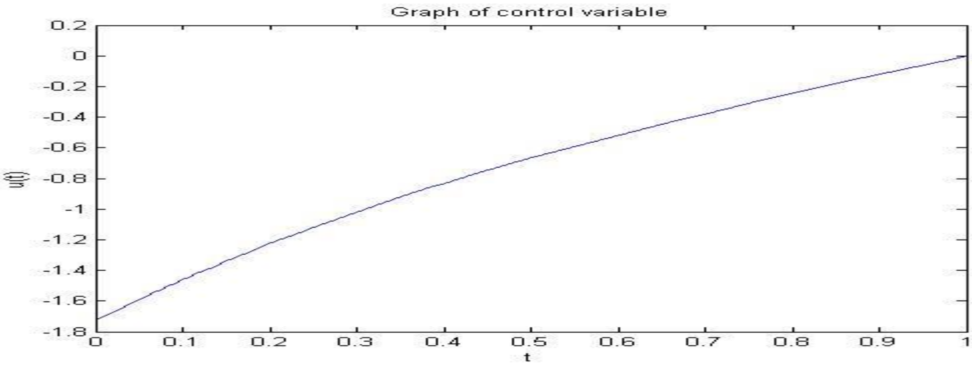

Figure 2b.

5. Convergence analysis

The convergence of parameterization method is generally based on Weierstrass Approximation Theorem, which have been rigorously proved by many researchers.

Theorem 1. (Weierstrass Approximation Theorem). Let $f \epsilon C([a,b],\mathbb{R})$. Then, there is a sequence of polynomials, $P_n(x)$, that converges uniformly to $f(x)$ on $[a,b]$.

Proof: See [6], [15].

Theorem 2. (Kafash and Delavarkhalafi, 2014). If, $\alpha_n = inf_{Q_n} J$, for $n = 1,2, \ . \ . \ .$, then $lim_{n\to\infty}\alpha_n$, where $\alpha = inf_Q \ J$.

Proof. Let $\hat{\alpha} > J(X(.))$ be the limit of non-increasing sequence $\{\alpha_n\}$. If $\hat{\alpha} > \alpha$, then $\varepsilon = \frac{\hat{\alpha}-\alpha}{2} > 0$, so; there exist $X(.)\epsilon Q$, $J\big(X(.)\big) < \varepsilon + \alpha = \frac{\hat{\alpha}-\alpha}{2} + \alpha = \frac{\hat{\alpha}+\alpha}{2} < \hat{\alpha}$ such that $\hat{\alpha} > J(X(.))$, which contradicts the continuity of J and Theorem 1.

6. Conclusion

This paper proposed numerical approach to solve optimal control problems based on state parameterization. We showed that Fermat polynomials can be used to approximate the state variables of optimal control problems which convert optimal control problems into quadratic programming problem. From the numerical results obtained, we can conclude that our proposed approach gives better and reliable result when compared with other methods.

References

[1] Eisa B.M. and Awadalla T.A. (2017). Optimal state and control variables parameterization. International Journal of Trend in Research and Development, 4(2): 2394-2433.

[2] El-Kady M. and Elbarbary E.M.E. (2002). A Chebyshev expansion method for solving nonlinear optimal control problems. Journal of Computational and Applied Mathematics, 129: 171-182.

[3] Highfill J. and McAsey M. (2001). An application of optimal control to the economics of recycling. Society for Industrial and Applied Mathematics, 43(4): 679-693.

[4] Jaddu H.M. (2002). Direct solution of nonlinear optimal control problems using quasilinearization and Chebyshev polynomials. Journal of Franklin Institute, 339: 479-498.

[5] Kafash B. and Delavarkhalafi A. (2014). Restarted state parameterization method for optimal control problems. Journal of Mathematics and Computer Science, 14: 151-161.

[6] Kafash B., Delavarkhalafi A. and Karbassi S.M. (2012a). Application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems. Journal of Scientifica Iranica, 19(3): 795-805.

[7] Kafash B., Delavarkhalafi A. and Karbassi S.M. (2012b). Numerical solution of nonlinear optimal control problems based on state parameterization. Iranian Journal of Science and Technology, 36(3): 331-340.

[8] Kafash B., Delavarkhalafi A., Karbassi S.M. and Boubaker K. (2014). A numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme. Journal of Interpolation and Approximation in Scientific Computing, 2014: 1-18.

[9] Lenhart S. and John T.W. (2007). Optimal control applied to biological models. Mathematical and Computational Biology. Chapman and Hall, CRC press, London.

[10] Mehdi R. (2016). Numerical solution of optimal control problems using a new second kind Chebyshev wavelet. Journal of Computational Methods for Differential Equations, 4(2), 162-169.

[11] Mehne H.H. and Borzabadi A.H. (2006). A numerical method for solving optimal control problems using state parameterization. Journal of Numerical Algorithm, 42: 165-169.

[12] Patterson M. (2013). Efficient solutions to nonlinear optimal control problems using adaptive mesh orthogonal collocation methods. Ph.d. Dissertation. University of Florida.

[13] Phelps C. and Kaminer I. (2014). Consistent approximation of a nonlinear optimal control problem with uncertain parameters. Automatic, 50(12): 2987-2997.

[14] Ping C and Sardar M.N.I. (2005). Optimal control models in finance: a new computational approach. Springer.

[15] Tohidi E., Samadi O.R.N., and Farah M.H. (2011). Legendre approximation for solving class of nonlinear optimal control problems. Journal of Mathematical Finance, 1: 8-13.