# Approximation of four step iteration scheme with errors on closed convex subsets of normed linear spaces

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**Abstract**

This paper investigates the convergence of four-step iteration schemes with error terms defined on closed convex subsets of normed linear spaces. Sufficient conditions are established for convergence to a fixed point of certain classes of norm linear operators. The results extend and unify several known theorems in fixed point approximation theory such as Berinde [9], Agarwal [11], Bosede [3] and Wasfi [6].

**Keywords:** Four-step iteration, Iteration scheme with errors, Closed convex subset and Normed linear spaces.

## 1 Introduction

A pair $(E, |.|)$ is called a normed linear space if the mapping $\|.\| : E \to [0, \infty)$ for all $x, y \in E$ and all scalars $\alpha$, satisfies:
$i.$ $|x| \geq 0$, with $|x| = 0$ if and only if $x = 0$;
$ii.$ $|\alpha x| = |\alpha|, |x|$;
$iii.$ $|x + y| \leq |x| + |y|$.
The pair $(E, |.|)$ is then called a normed linear space.

The set of fixed points of $T$ is defined by

$$F_T = \{p \in E : Tp = p\}, \tag{1.0}$$

such that,$T : E \to E$, be a self-map on $E$ and $(E, d)$ represents a non empty closed convex subset

of a normed linear space E.

**Definition 1** [1]:

Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving the operator T, this implies that,

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2 \ldots \tag{1.1}$$

where $x_0 \in X$ is the initial approximation and f is some function.
Suppose in (1.1),

$$\begin{aligned} f(T, x_n) &= Tx_n & (1.2)\\ then, \qquad x_{n+1} &= Tx_n, \quad n = 0, 1, 2 \ldots & (1.3) \end{aligned}$$

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Eqn. (1.3), is the Picard iterative scheme.
**Definition 2** [2]
For arbitrary $x_0 \in E$, Mann Iteration process is defined as

$$x_{n+1} = (1-\tau_n)x_n + \tau_n T x_n, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

where $\{\tau_n\}$ is a sequence of real numbers in $[0, 1]$, then we have the Mann iteration process. [2].
**Definition 3**[3]
For arbitrary $x_0 \in E$, Mann Iteration process with errors is defined as

$$x_{n+1} = (1-\tau_n)x_n + \tau_n T x_n + r_n, \quad n = 0, 1, 2, \ldots \tag{1.5}$$

where $\{\tau_n\}$ is a sequence of real numbers in $[0, 1]$, $r_n$ satisfies $\sum_{n=0}^{\infty} \|r_n\| < \infty$ then we have the Mann iteration process with errors. [3].
**Definition 4**[4]
For arbitrary $x_0$ in $E$, Ishikawa iteration process is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T x_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{1.6}$$

where $\{\tau_n\}$ and $\{\tau_n'\}$ are sequences of real numbers in $[0, 1]$, then we have the Ishikawa iteration process. [4].
**Definition 5**[3]
For arbitrary $x_0$ in $E$, Ishikawa iteration process with errors is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n + r_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T x_n + s_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{1.7}$$

where $\{\tau_n\}$ and $\{\tau_n'\}$ are sequences of real numbers in $[0, 1]$, $r_n$ and $s_n$ satisfy $\sum_{n=0}^{\infty} \|r_n\| < \infty$ and $\sum_{n=0}^{\infty} \|s_n\| < \infty$ then we have the Ishikawa iteration process with errors. [3].
**Definition 6**[5]
For arbitrary $x_0$ in $E$, Noor iteration process is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T z_n \\ z_n &= (1-\tau_n'')x_n + \tau_n'' T x_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{1.8}$$

where $\{\tau_n\}$, $\{\tau_n'\}$ and $\{\tau_n''\}$ are sequences of real numbers in $[0, 1]$, then we have the Noor iteration process. [5].
**Definition 7**[3]
For arbitrary $x_0$ in $E$, Noor iteration process with errors is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n + r_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T z_n + s_n \\ z_n &= (1-\tau_n'')x_n + \tau_n'' T x_n + t_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{1.9}$$

where $\{\tau_n\}$, $\{\tau_n'\}$ and $\{\tau_n''\}$ are sequences of real numbers in $[0, 1]$, $r_n$ , $s_n$ and $t_n$ satisfy $\sum_{n=0}^{\infty} \|r_n\| < \infty$, $\sum_{n=0}^{\infty} \|s_n\| < \infty$ and $\sum_{n=0}^{\infty} \|t_n\| < \infty$ then we have the Noor iteration process with errors. [3].

**Definition 8**[6]
For arbitrary $x_0$ in $E$, four step iteration process is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T z_n \\ z_n &= (1-\tau_n'')x_n + \tau_n'' T v_n, \\ v_n &= (1-\tau_n''')x_n + \tau_n''' T x_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{1.10}$$

where $\{\tau_n\}$, $\{\tau_n'\}$, $\{\tau_n''\}$ and $\{\tau_n'''\}$ are sequences of real numbers in $[0,1]$, then we have the Four step iteration process [6].
**Definition 9**[3]
For arbitrary $x_0$ in $E$, Four step iteration process with errors is defined as

$$\begin{aligned} x_{n+1} &= (1-\tau_n)x_n + \tau_n T y_n + r_n \\ y_n &= (1-\tau_n')x_n + \tau_n' T z_n + s_n \\ z_n &= (1-\tau_n'')x_n + \tau_n'' T v_n + t_n, \\ v_n &= (1-\tau_n''')x_n + \tau_n''' T x_n + u_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{1.11}$$

where $\{\tau_n\}$, $\{\tau_n'\}$, $\{\tau_n''\}$ and $\{\tau_n'''\}$ are sequences of real numbers in $[0,1]$, $r_n$ , $s_n$, $t_n$ and $u_n$ satisfy $\sum_{n=0}^{\infty}\|r_n\| < \infty$, $\sum_{n=0}^{\infty}\|s_n\| < \infty$, $\sum_{n=0}^{\infty}\|t_n\| < \infty$ and $\sum_{n=0}^{\infty}\|u_n\| < \infty$ then we have the Four step
iteration process with errors [3].

**Definition 10**[6]
For arbitrary $x_0$ in $E$, $(SBT_n)$ iteration process is defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)T y_n + \alpha_n T v_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ v_n &= (1-\tau_n)T y_n + \tau_n T z_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{1.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ are sequences of real numbers in $[0,1]$, then we have the $(SBT_n)$ iteration process [6].
**Definition 11**[3]
For arbitrary $x_0$ in $E$, $(SBT_n)$ iteration process with errors is defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)T y_n + \alpha_n T v_n + r_n \\ y_n &= (1-\beta_n)x_n + \beta_n z_n + s_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n + t_n \\ v_n &= (1-\tau_n)T y_n + \tau_n T z_n + u_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ are sequences of real numbers in $[0,1]$, $r_n$ , $s_n$, $t_n$ and $u_n$ satisfy $\sum_{n=0}^{\infty}\|r_n\| < \infty$, $\sum_{n=0}^{\infty}\|s_n\| < \infty$, $\sum_{n=0}^{\infty}\|t_n\| < \infty$ and $\sum_{n=0}^{\infty}\|u_n\| < \infty$. Then we have the $(SBT_n)$
iteration process with errors [3].

**Theorem 1.1.**[7] Let $(E,d)$ be a complete metric space and $T : E \to E$ be a mapping for which there exist real numbers $\alpha$, $\beta$ and $\gamma$ satisfying $0 \le \alpha < 1$ , $0 \le \beta < 0.5$ and $0 \le \gamma < 0.5$ such that for each $x, y \in E$ at least one of the following is true:

$(Z_1)$ $d(Tx,Ty) \le \alpha d(x,y)$,

$(Z_2)$ $d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$,

$(Z_3)$ $d(Tx,Ty) \leq \gamma[d(x,Ty) + d(y,Tx)]$.

From the above theorem 1. The contractive conditions $(Z_1)$, $(Z_2)$ and $(Z_3)$ is satisfied by an operator $T$ called Zamfirescu operator.Then, $T$ is a Picard Operator, since its iterate converges to a unique fixed point.

## 2 Preliminaries

This section presents the fundamental definitions, notations and auxillary results that will be employed throughout this paper. These concepts are standard in normlinear functional analysis and fixed point theory and are included to ensure completeness and clarity of exposition.
In mathematics , a weak contraction is a mapping that brings any two distinct points strictly closer together, but without a uniform “scalling factor” as required for a standard contraction.

Many researchers have proposed numerous extensions of the Banach contraction principle. In particular,
[8] modified the classical Banach contraction by introducing the idea of a weak contraction in metric spaces. The notion of weak contraction in the sense of Berinde [9] is stated below.
**Definition 1**
A self-map $T$ on a metric space $(E,d)$ is called a weak contraction, if there exists a function $m \in [0,1)]$ and $n \geq 0$ satisfying the inequality:

$$d(Tx,Ty) \leq md(x,y) + n(d(x,Ty)) \tag{2.1}$$

for all $x, y \in E$.
**Definition 2**
Let $E$ be a real normed linear space endowed with a norm $\|.\|$. A non-empty subset $C \subset E$ is said to be closed if it contains all its limit points, and convex if for any $x, y \in C$ and for all $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y \in C \tag{2.2}$$

Equation $(2,2)$ is a closed convex subsets of normed linear spaces play a central role in fixed point approximation theory as they provide a stable domain for iterative processes.
**Definition 3**
Let $T : C \to C$ be a self-mapping defined on a non-empty closed convex subset $C$ of a normed linear space $E$. A point $x* \in C$ is called a fixed point of $T$ if

$$Tx^* = x^* \tag{2.3}$$

**Definition 4**
A mapping $T : C \to C$ is called a contraction if there exists a constant $k \in (0,1)$ such that

$$\|Tx - Ty\| \quad \leq \quad k\|x - y\|, \tag{2.4}$$

for all $x, y \in C$
Our aim in this paper is to extend known results by incorporating the convergence of two four-step iterative schemes with error terms in normed linear spaces, using the notion of weak contraction (2.1) using iteration (1.13) first and then iteration (1.11). However, in the proofs of our results, this paper

employ the method of Bosede [3].

**Remark 1:** It is observe that (1.9) contains (1.7) and (1.5).
**Remark 2:** It is observe that if $\tau'' = 0$ in (1.8) for each $n$, then (1.8) reduces to (1.6).
**Remark 3:** It is observe that if $\tau' = 0$ in (1.6) for each $n$, then (1.6) reduces to (1.4).
**Remark 4:** It is observed that if the error term in (1.11) and (1.13) are removed, then it reduces to (1.10) and (1.12)

This paper apply Lemma 1 as contain in Liu [10], in the following sequel.
**Lemma 1**
Let $a_n, b_n, c_n$ and $d_n$ be sequences of non negative numbers satisfying

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n + d_n, \quad n = 1, 2, 3, \ldots$$

Suppose,
i. $\sum_{n=0}^{\infty} b_n = \infty$ then $lim_{n\to\infty} c_n = 0$
ii. $\sum_{n=0}^{\infty} d_n < \infty$ then $lim_{n\to\infty} a_n = 0$.

## 3 Main Results

**Theorem 2**
Let $E$ be a non empty closed convex subset of a normed linear space. Consider the self-mapping $T : E \to E$ satisfying the contractive requirement (2.1). Assume that the sequence $\{x_n\}_{n=0}^{\infty}$ is generated by the $(SBT_n)$ iteration process with error defined in (1.13) and suppose

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \|r_n\| < \infty, lim_{n\to\infty}\|s_n\| = 0\ , \ lim_{n\to\infty}\|t_n\| = 0 \ and \ lim_{n\to\infty}\|u_n\| = 0.$$

Then $(SBT_n)$ iteration process with error converges to p.
**Proof**
Using triangle inequality (1.13) and contraction mapping (2.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Ty_n + \alpha_n Tv_n + r_n - p\| \\ &\leq (1-\alpha_n)\|Ty_n - p\| + \alpha_n\|Tv_n - p\| + \|r_n\| \\ &\leq \delta(1-\alpha_n)\|y_n - p\| + \delta\alpha_n\|v_n - p\| + \|r_n\| \end{aligned} \quad (3.1)$$

Considering the estimate of $\|y_n - p\|$ in (3.1), gives

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n z_n + s_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| + \|s_n\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| + \|s_n\| \end{aligned} \quad (3.2)$$

Considering the estimate of $\|z_n - p\|$ in (3.2), gives

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n + t_n - p\| \\ &\leq \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p) + t_n\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \delta\gamma_n\|x_n - p\| + \|t_n\| \\ &= (1-\gamma_n(1-\delta))\|x_n - p\| + \|t_n\| \end{aligned} \quad (3.3)$$

From (3.2), substitute (3.3) to obtain,

$$\begin{aligned}
\|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| + \|s_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n[1-\gamma_n(1-\delta))\|x_n - p\| + \|t_n\|] + \|s_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1-\gamma_n+\delta\gamma_n)\|x_n - p\| + \|t_n\|] + \|s_n\| \\
&\leq (1-\beta_n\gamma_n+\delta\beta_n\gamma_n)\|x_n - p\| + \beta_n\|t_n\| + \|s_n\| \\
&\leq (1-\beta_n\gamma_n(1-\delta))\|x_n - p\| + \beta_n\|t_n\| + \|s_n\| \qquad (3.4)
\end{aligned}$$

Considering the estimate of $\|v_n - p\|$ in (3.1), gives

$$\begin{aligned}
\|v_n - p\| &\leq \|(1-\tau_n)Ty_n + \tau_n Tz_n + u_n - p\| \\
&\leq \|(1-\tau_n)(Ty_n - p) + \tau_n(Tz_n - p) + u_n\| \\
&= (1-\tau_n)\delta\|y_n - p\| + \delta\tau_n\|z_n - p\| + \|u_n\| \\
&\leq \delta(1-\tau_n)\|y_n - p\| + \delta\tau_n\|z_n - p\| + \|u_n\| \qquad (3.5)
\end{aligned}$$

By (3.4) and (3.3), (3.5) becomes

$$\begin{aligned}
\|v_n - p\| &\leq \delta(1-\tau_n)[(1-\beta_n\gamma_n(1-\delta))\|x_n - p\| + \beta_n\|t_n\| + \|s_n\|] + \delta\tau_n[(1-\gamma_n(1-\delta))\|x_n - p\| \\
&+ \|t_n\|] + \|u_n\| \\
&= [\delta(1-\tau_n)(1-\beta_n) + \delta(1-\tau_n)\beta_n(1-(1-\delta)\gamma_n\beta_n) + \delta_n\tau_n(1-\gamma_n(1-\delta))]\|x_n - p\| \\
&+ \delta(1-\tau_n)\|s_n\| + [\delta\beta_n - \delta\beta_n\tau_n + \delta\tau_n]\|t_n\| + \|u_n\| \qquad (3.6)
\end{aligned}$$

In Conclusion, substitute (3.6) and (3.4) into (3.1), gives

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \delta(1-\alpha_n)\|y_n - p\| + \delta\alpha_n\|v_n - p\| + \|r_n\|| \\
&\leq \delta(1-\alpha_n)[(1-\beta_n)\|x_n - p\| + \beta_n(1-\gamma_n\beta_n(1-\delta))\|x_n - p\| + \beta_n\|t_n\| + \|s_n\|] \\
&+ \delta\alpha_n[(\delta(1-\tau_n)(1-\beta_n) + \delta(1-\tau_n)\beta_n(1-\gamma_n\beta_n(1-\delta)) + \delta\tau_n(1-\gamma_n(1-\delta)))\|x_n - p\| \\
&+ \delta(1-\tau_n)\|s_n\| + (\delta\beta_n - \delta\beta_n\tau_n + \delta\tau_n)\|t_n\| + \|u_n\|] + \|r_n\| \\
&= [\delta(1-\alpha_n)(1-\beta_n) + \delta(1-\alpha_n)\beta_n(1-\gamma_n\beta_n(1-\delta)) + \delta^2\alpha_n(1-\tau_n)(1-\beta_n) \\
&+ \delta^2\alpha_n(1-\tau_n)\beta_n(1-\gamma_n\beta_n(1-\delta)) + \delta^2\alpha_n\tau_n(1-\gamma_n(1-\delta))]\|x_n - p\| + \delta\alpha_n\|u_n\| \\
&+ [\delta(1-\alpha_n)\beta_n + \delta\alpha_n(\delta\beta_n - \delta\beta_n\tau_n + \delta\tau_n)]\|t_n\| + [\delta(1-\alpha_n) + \delta^2\alpha_n(1-\tau_n)]\|s_n\| + \|r_n\| \qquad (3.7)
\end{aligned}$$

By applying Lemma 1 and considering the fact that

$$0 \leq \alpha_n \leq 1, \quad 0 \leq \delta < 1, \quad 0 \leq [\delta(1-\alpha_n)(1-\beta_n) + \delta(1-\alpha_n)\beta_n(1-\gamma_n\beta_n(1-\delta)) + \delta^2\alpha_n(1-\tau_n)(1-\beta_n) + \delta^2\alpha_n(1-\tau_n)\beta_n(1-\gamma_n\beta_n(1-\delta)) + \delta^2\alpha_n\tau_n(1-\gamma_n(1-\delta))] < 1$$

$$\sum_{n=0}^{\infty}\alpha_n = \infty, \quad \sum_{n=0}^{\infty}\|r_n\| < \infty, lim_{n\to\infty}\|s_n\| = 0\ ,\ lim_{n\to\infty}\|t_n\| = 0\ and\ lim_{n\to\infty}\|u_n\| = 0. \qquad (3.8)$$

we obtain,

$$lim_{n\to\infty}\|x_n - p\| = 0 \qquad (3.9)$$

Therefore, $(SBT_n)$ iteration process with errors converges to a fixed point of p.
This completes the proof.

**Theorem 3**
Let $E$ be a non empty closed convex subset of a normed linear space. Consider the self-mapping $T : E \to E$ satisfying the contractive condition (2.1). Assume that the sequence $\{x_n\}_{n=0}^{\infty}$ is generated by the Four - step iteration process with errors defined in (1.11) and suppose

$$\sum_{n=0}^{\infty} \tau_n = \infty, \quad \sum_{n=0}^{\infty} \|r_n\| < \infty, lim_{n\to\infty}\|s_n\| = 0\ ,\ lim_{n\to\infty}\|t_n\| = 0\ and\ lim_{n\to\infty}\|u_n\| = 0.$$

Then, the Four-step iteration process converges to p.

**Proof**
Using triangle inequality (1.11) and contraction mapping (2.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\tau_n)x_n + \tau_n T y_n + r_n - p\| \\ &\leq (1-\tau_n)\|x_n - p\| + \tau_n\|Ty_n - p\| + \|r_n\| \\ &= (1-\tau_n)\|x_n - p\| + \tau_n\|Ty_n - p\| + \|r_n\| \\ &= (1-\tau_n)\|x_n - p\| + \delta\tau_n\|y_n - p\| + \|r_n\| \end{aligned} \tag{3.10}$$

Considering the estimate of $\|y_n - p\|$ in (3.10), gives

$$\begin{aligned} \|y_n - p\| &= \|(1-\tau_n')x_n + \tau_n' T z_n + s_n - p\| \\ &\leq (1-\tau_n')\|x_n - p\| + \tau_n'\|Tz_n - p\| + \|s_n\| \\ &= (1-\tau_n')\|x_n - p\| + \delta\tau_n'\|z_n - p\| + \|s_n\| \end{aligned} \tag{3.11}$$

Substitute (3.11) into (3.10), we obtain

$$\|x_{n+1} - p\| \leq (1-(1-\delta)\tau_n - \delta\tau_n\tau_n')\|x_n - p\| + \delta^2\tau_n\tau_n'\|z_n - p\| + \tau_n\delta\|s_n\| + \|r_n\| \tag{3.12}$$

The estimate of $\|z_n - p\|$ in (3.11), is obtained as follows:

$$\begin{aligned} \|z_n - p\| &= \|(1-\tau_n'')x_n + \tau_n'' T v_n + t_n - p\| \\ &\leq (1-\tau_n'')\|x_n - p\| + \tau_n''\|Tv_n - p\| + \|t_n\| \\ &= (1-\tau_n'')\|x_n - p\| + \tau_n''\delta\|v_n - p\| + \|t_n\| \end{aligned} \tag{3.13}$$

Substitute (3.13) into (3.12), gives

$$\begin{aligned} \|x_{n+1} - p\| &= (1-\tau_n(1-\delta) - \delta\tau_n\tau_n'(1-\delta(1-\tau_n'')))\|x_n - p\| + \delta^3\tau_n\tau_n'\tau_n''\|v_n - p\| \\ &+ \delta^2\tau_n\tau_n'\|t_n\| + \tau_n\delta\|s_n\| + \|r_n\| \end{aligned} \tag{3.14}$$

Consider $\|v_n - p\|$ in (3.13), gives

$$\begin{aligned} \|v_n - p\| &= \|(1-\tau_n''')x_n + \tau_n''' T x_n + u_n - p\| \\ &\leq (1-\tau_n''')\|x_n - p\| + \tau_n'''\|Tx_n - p\| + \|u_n\| \\ &= (1-\tau_n''')\|x_n - p\| + \delta\tau_n'''\|x_n - p\| + \|u_n\| \end{aligned} \tag{3.15}$$

Substitute (3.15) into (3.14), gives

$$\begin{aligned} \|x_{n+1}-p\| &= (1-\tau_n(1-\delta)-\delta\tau_n\tau_n'(1-\delta(1-\tau_n''))+\delta^3\tau_n\tau_n'\tau_n''(1-\tau_n'''(1+\delta)))\|x_n-p\| \\ &+ \delta^3\tau_n\tau_n'\tau_n''\|u_n\|+\delta^2\tau_n\tau_n'\|t_n\|+\tau_n\delta\|s_n\|+\|r_n\| \end{aligned} \tag{3.16}$$

By applying Lemma 1 and considering the fact that

$$0 \leq \tau_n \leq 1, \quad 0\leq\delta<1, \;\; 0\leq(1-\tau_n(1-\delta)-\delta\tau_n\tau_n'(1-\delta(1-\tau_n''))+\delta^3\tau_n\tau_n'\tau_n''(1-\tau_n'''(1+\delta))) < 1$$

$$\sum_{n=0}^{\infty}\tau_n=\infty, \quad \sum_{n=0}^{\infty}\|r_n\|<\infty, \; lim_{n\to\infty}\|s_n\|=0\;, \lim_{n\to\infty}\|t_n\|=0 \; and \; lim_{n\to\infty}\|u_n\|=0. \tag{3.17}$$

we obtain,

$$lim_{n\to\infty}\|x_n-p\| = 0 \tag{3.18}$$

Therefore, Four-step iteration process with error converges to a fixed point of p.
This ends the proof.

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