Ulam-Hyers stability Analysis of Caputo fractional order derivative

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Abstract

This study investigates the Ulam-Hyers stability of fractional derivatives for a given fractional-order system. As a theoretical foundation, fundamental definitions of various fractional derivatives are introduced. By applying the Gronwall inequality, sufficient conditions ensuring stability in the sense of Ulam-Hyers are rigorously established. The derived results constitute notable extensions and refinements of existing findings in the literature.

Keywords and phrases: Ulam-Hyers stability; Caputo derivative; Lyapunov function; fractional differential equation; singular kernel; non-local conditions.

1 Introduction

In recent years, the Ulam–Hyers stability theory for differential equations has received considerable attention due to its ability to ensure the existence of analytical solutions for problems exhibiting Ulam-type stability under appropriate conditions. As a branch of stability theory, it focuses on the stability of functional equations and plays a significant role in addressing practical problems in fields such as economics, biology, and particularly numerical analysis, where explicit solutions are often difficult to obtain (see [1, 2]). Historically, in 1940, S. M. Ulam, during an address at the University of Wisconsin, posed a fundamental question concerning the stability of functional equations and the conditions under which a linear mapping can exist in proximity to an approximately linear mapping (see [3]). As stated in [4], Ulam's stability problem can be formulated as follows: "When does the assertion of a theorem remain true, or nearly true, despite small perturbations in its hypotheses?" In 1941, Hyers in [5] provided an affirmative answer to Ulam's question by proving the stability of the additive functional equation in Banach spaces. This result was subsequently extended by Rassias in [6], leading to a surge of interest in the generalization of Hyers' theorem through various control conditions defining approximate solutions (see [7] and the references therein). As stated earlier, the Ulam–Hyers stability framework has since found applications in a wide range of practical contexts, and comprehensive studies addressing the existence, uniqueness, and Ulam stability of solutions have been presented in [8, 9, 10, 11]. Furthermore, fundamental results concerning Ulam stability, Ulam–Hyers stability, and Ulam–Hyers–Rassias stability for differential equations can be found in [12, 13, 14, 15]. Also, the theory of fractional calculus is nearly as old as the classical theory of differential equations, spanning more than three centuries [16, 17]. As discussed in [18, 19, 20], fractional calculus naturally arises in diverse scientific and engineering applications such as electrical circuits, chemical kinetics, control

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theory, biophysics, and data fitting, serving as a powerful modeling framework for a variety of complex physical phenomena [21, 22, 23]. More recently, its influence has expanded further into areas including mathematical physics, engineering, and control systems. The earliest systematic studies of fractional calculus can be traced back to the 19th century, notably to the works of Liouville, Riemann, Leibniz, and Caputo [24]. In the qualitative theory of fractional differential equations (FrDEs), the stability of solutions remains one of the central properties of interest. Although the study of stability in FrDEs is relatively recent, notable progress has been made in [25, 26, 27, 28]. Fundamental results on the existence and uniqueness of solutions to FrDEs using scalar Lyapunov functions were established in [29, 30, 31], where a definition of the Caputo fractional Dini derivative of a Lyapunov function was also introduced. Due to the challenges associated with this formulation, as noted in [32], a revised definition was proposed in [33, 34, 35], leading to sufficient conditions for the stability of nonlinear systems via scalar Lyapunov-like functions with weakly singular kernels and non-local conditions, where the non-locality property is an emphasis on the fact that, the properties of the fractional derivative is dependent on both the order of the derivative and the initial conditions. Further qualitative results for scalar FrDEs employing Lyapunov functionals and matrix inequalities were obtained in [36, 37]. Moreover, as emphasized in [38, 39, 40], while several Lyapunov-based methods for analyzing the stability of fractional-order systems have been developed, many of them remain constrained by technical difficulties and restrictive assumptions. Motivated by the studies in [41, 42, 43, 44], this paper establishes sufficient conditions for Ulam–Hyers stability of Caputo-type fractional differential equations by means of the Gronwall inequality. The results presented herein extend and refine existing findings in the literature.

2 Basic notations and definitions

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space with norm $\|\cdot\|$. Let Ω be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0,\infty),\ \mathbb{R} = (-\infty,\infty),\ t_0 \in \mathbb{R}_+, t > 0$.
Fractional calculus is widely recognized as a natural extension of the traditional integer-order calculus, allowing the concepts of differentiation and integration to be generalized to non-integer orders. Through this extension, fractional-order operators offer a more flexible and realistic approach to modeling various real-world systems and processes (see [7, 45, 46, 47]). Numerous definitions of fractional derivatives and fractional integrals have been proposed to capture these generalized notions.

2.1 General case

Let the number $n-1 < \beta < n,\ \beta > 0$ be given, where n is a natural number, and denotes the Gamma function.

Definition 2.1. *The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [48]).*

$$ {}_{t_0}^{RL}D_t^{\beta}\gamma(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t \geq t_0 $$

Definition 2.2. *The Caputo fractional derivative of order β of $x(t)$ is defined by (see [48])*

$$ {}_{t_0}^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t \geq t_0 $$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3. *The Grunwald-Letnikov fractional derivative of order β of $\gamma(t)$ is given by (see [28])*

$$ D_0^{\beta}\gamma(t) = \lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{[\frac{(t-t_0)}{h}]}(-1)^r({}^{\beta}C_r)\gamma(t-rh), t \geq t_0 $$

Definition 2.4. *The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (see [48])*

$$D_0^{\beta}\gamma(t) = \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^{\beta}C_r)\gamma(t-rh), t \geq t_0$$

where ${}^{\beta}C_r$ are the binomial coefficients and $[\frac{(t-t_0)}{h}]$ denotes the integer part of $\frac{(t-t_0)}{h}$.

Particular case

(when n=1). In most applications, the order of β is often less than 1, so that $\beta \in (0,1)$. For simplicity of notation, we will use ${}^{C}D^{\beta}$ instead of ${}_{t_0}^{C}D^{\beta}$ and the Caputo fractional derivative of order q of the function $\gamma(t)$ is

$$^{C}D^{\beta}\gamma = \frac{1}{\Gamma(1-\beta)} \int_{t_0}^{t} (t-s)^{-\beta}\gamma' ds, t \geq t_0 \tag{1}$$

3 Fractional Inequalities

Let $\beta \in (0,1)$ and ϵ be positive real number, $f : [t_0, \infty) \times E \to E$ be a continuous operator and $\sigma : [t_0, \infty) \times E \to \mathbb{R}_+$ be a continuous function.
We consider the following differential equations,

$$^{C}D^{\beta}\gamma(t) = f(t, \gamma(t)),\ t \in [t_0, \infty), \tag{2}$$

and the following inequalities,

$$|^{C}D^{\beta}\omega(t) - f(t, \omega(t))| \leq \epsilon,\ t \in [t_0, \infty) \tag{3}$$

$$|^{C}D^{\beta}\omega(t) - f(t, \omega(t))| \leq \alpha(t),\ t \in [t_0, \infty) \tag{4}$$

$$|^{C}D^{\beta}\omega(t) - f(t, \omega(t))| \leq \epsilon\alpha(t),\ t \in [t_0, \infty) \tag{5}$$

Definition 3.1. *[48] The equation* (2) *is Ulam-Hyers stable if there exists a real number $\wp > 0$ such that for each $\epsilon > 0$ and for each solution $\omega \in C^1([a,b), E)$ of the inequality* (3) *there exists a solution $\gamma \in C^1([a,b), E)$ of the equation* (2) *with*

$$|\omega(t) - \gamma(t)| \leq \wp_0\epsilon,\ t \in [a,b).$$

Definition 3.2. *[48] The equation* (2) *is generalized Ulam- Hyers stable if there exists a real number $\lambda_0 \in C(\mathbb{R}_+, \mathbb{R}_+),\ \lambda_0(0) = 0$ such that for each solution $\omega \in C^1([a,b), E)$ of the inequality* (3) *there exists a solution $\gamma \in C^1([a,b), E)$ of the equation* (2) *with*

$$|\omega(t) - \gamma(t)| \leq \lambda_0\epsilon,\ t \in [a,b)$$

Definition 3.3. *[48] The equation* (2) *is Ulam-Hyers-Rassias stable with respect to α if there exists $\wp_{0,\alpha} > 0$ such that for each $\epsilon > 0$ and for each solution $\omega \in C^1([a,b), E)$ of the inequality* (5) *there exists a solution $\gamma \in C^1([a,b), E)$ of the equation* (2) *with*

$$\|\omega(t) - \gamma(t)\| \leq \wp_{0,\alpha}\epsilon\alpha(t), \quad t \in [a,b). \tag{6}$$

Definition 3.4. *[48] The equation* (2) *is generalized Ulam-Hyers-Rassias stable with respect to α if there exists $\wp_{0,\alpha} > 0$ such that for each solution $\omega \in C^1([a,b), E)$ of the inequality* (4) *there exists a solution $\gamma \in C^1([a,b), E)$ of the equation* (2) *with*

$$\|\omega(t) - \gamma(t)\| \leq \wp_{0,\alpha}\alpha(t), \quad t \in [a,b). \tag{7}$$

Lemma 3.5. *[48] A function $\omega \in C^1([a,b),E)$ is a solution of the inequality (3) if and only if there exists a function $K_\omega \in C^1([a,b),E)$ (which depends on ω) such that,*

(i) $\|K_\omega\| \leq \epsilon, \quad t \in [a,b),$

(ii) ${}^CD^\beta\omega(t) = f(t,\omega(t)) + K_\omega, \quad t \in [a,b).$

Theorem 3.6. *Let $0 < \beta < 1$, if $\omega \in C^1([a,b),E)$ is a solution of the inequality (3) then $\omega(t)$ is a solution of the following integral equation,*

$$\left|\omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds\right| \leq \frac{\epsilon}{\Gamma(\beta+1)}\cdot(t-t_0)^\beta. \tag{8}$$

Proof. From Lemma 3.5,

$${}^CD^\beta\omega(t) = f(t,\omega(t)) + K_\omega, \quad t \in [a,b). \tag{9}$$

Integrating both sides in the Caputo sense,

$$\begin{aligned}
I^{-\beta}({}^CD^\beta\omega(t)) &= I^(-\beta f(t,\omega(t))) + I^{-\beta}(K_\omega),\ t \in [a,b)\\
\omega(t) - \omega(t_0) &= \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}K_\omega(s)ds,\\
\omega(t) &= \omega(t_0) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}K_\omega(s)ds,
\end{aligned}$$

Taking the norm of both sides,

$$\begin{aligned}
\left|\omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds\right| &= \left|\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}K_\omega(s)ds\right|\\
&\leq \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\left|K_\omega(s)\right|ds\\
&\leq \frac{\epsilon}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}ds\\
&\leq \frac{\epsilon}{\Gamma(\beta)}\cdot\frac{(t-t_0)^\beta}{\beta}.
\end{aligned}$$

Hence,

$$\left|\omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds\right| \leq \frac{\epsilon}{\beta\Gamma(\beta)}\cdot(t-t_0)^\beta = \frac{\epsilon}{\Gamma(\beta+1)}\cdot(t-t_0)^\beta. \tag{10}$$

□

Theorem 3.7. *Let $0 < \beta < 1$ be continuous and assume that:*

(i) $f \in C([t_0,\infty) \times E, E.$

(ii) The function f is locally Lipschitz with respect to the second argument υ, that is, there exists $l > 0$ such that,

$$\|f(t,\upsilon_1) - f(t,\upsilon_2)\| \leq l\|\upsilon_1 - \upsilon_2\|,\ \forall\, t \in [t_0,\infty),\ \textit{and}\ \upsilon_1,\upsilon_2 \in E$$

(iii) Let $\alpha \in C([t_0, \infty), \mathbb{R}_+)$ be an increasing function. If there exists $\epsilon > 0$ such that

$$\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \alpha(s) ds \leq \epsilon, \ \text{for each} \ t \in [t_0, \infty)$$

Then equation (2) is Ulam-Hyers stable.

Proof. Let $\omega \in C^1([a, b), E)$ be a solution of the inequality (4), and let $\gamma(t)$ be the unique solution of the initial value problem (IVP),

$$\begin{cases} {}^C D^\beta \gamma(t) = f(t, \gamma(t)), & t \in [t_0, \infty), \\ \gamma(t_0) = \omega(t_0). \end{cases} \tag{11}$$

Integrating both sides in the Caputo sense,

$$\gamma(t) = \gamma(0) + I^\beta f(s, \gamma(s))|_{s=t}$$

$$\gamma(t) = \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(s)) ds, \ \ t \in [t_0, \infty) \tag{12}$$

Using the inequality (4), we have

$$\omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \leq \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \alpha(s) ds.$$

$$\omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \leq \epsilon \tag{13}$$

Combining equations (12) and (13) we have that

$$\begin{aligned}
|\omega(t) - \gamma(t)| &\leq \left| \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(s)) ds \right| \\
&\leq \left| \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \right. \\
&\quad + \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \\
&\quad \left. - \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(s)) ds \right| \\
&\leq \left| \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \right| \\
&\quad + \left| \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) - f(s, \gamma(s)) \right| ds
\end{aligned} \tag{14}$$

Substituting (13) into (14) gives,

$$\leq \omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} |\omega(s) - \gamma(s)| \, ds.$$

Then there exists a constant $l^* > 0$ independent of $\wp_{0,\alpha}\alpha(t)$ such that
$|\omega(t) - \gamma(t)| \leq l^* \epsilon := M^*$, where $M^* > 0$ and $t \in [t_0, \infty)$.
Hence, (2) is Ulam-Hyers stable.

4 Application

Let $0 < \alpha < 1$. We consider the equation

$$^{C}D^{\alpha}x(t) = 0, \ \ t \in [a, b), \tag{15}$$

and the equation

$$|D^{\alpha}x(t)| \leq \epsilon, \ t \in [a, b). \tag{16}$$

Let $y \in C^1[a, b)$ be a solution of $|D^{\alpha}x(t)| \leq \epsilon$, $t \in [a, b)$. Then there exists $g \in C[a, b)$.

(i) $|g(t)| \leq \epsilon, \ t[a, b)$.

(ii) $D^{\alpha}x(t) = g(t), \ t \in [a, b)$.

Panel	Title in the Plot	Function Used	Description
(a)	Solution of fractional equation	$x(t) = 1$	A constant function representing a simple solution of a fractional differential equation.
(b)	Inequality bound	$y(t) = \pm\epsilon, \ \epsilon = 0.5$	Upper and lower bounds forming a symmetric band around the zero axis.
(c)	Example $y(t)$	$y(t) = \cos(t)$	Smooth oscillatory reference function.
(d)	$g(t)$	$g(t) = \epsilon$	A constant comparison function representing the tolerance magnitude $\|\epsilon\|$.
(e)	Comparison	$y(t) = \cos(t), \ g(t) = 0.3\sin(t)$	Side-by-side comparison showing different oscillation amplitudes and phase behavior.
(f)	Additional panel	$\sin(t)$	An auxiliary sine function used to complete the six-panel layout.

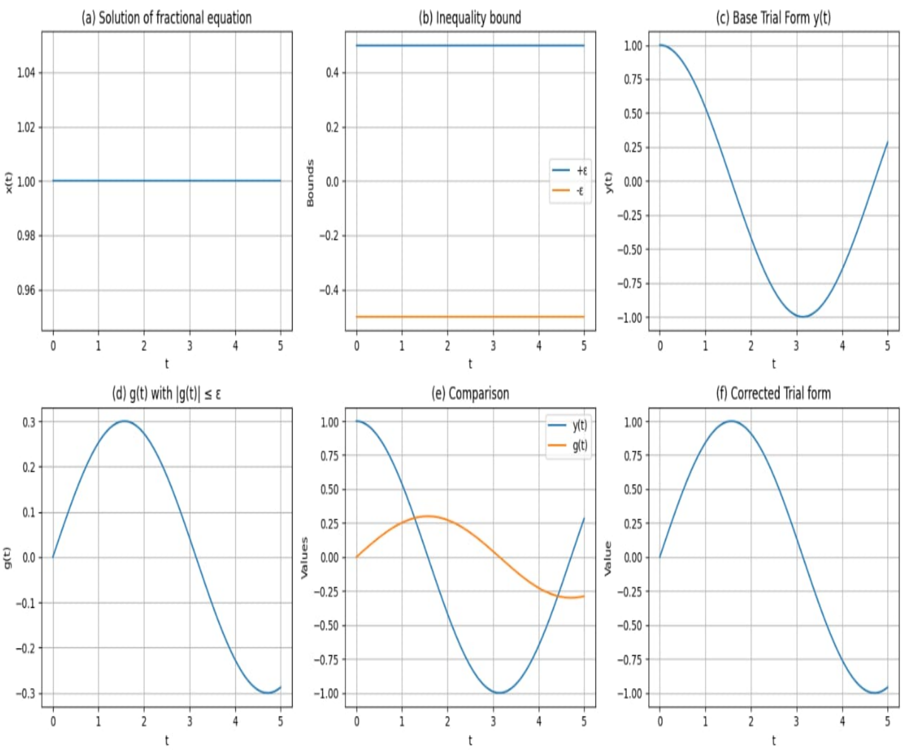

Figure 1: Qualitative illustration of solution stability and bounded correction in a fractional-order framework.

4.1 Discussion

Panel (a) depicts a constant solution over the entire interval. This behaviour reflects the presence of a steady-state solution of the underlying fractional equation. The absence of oscillatory or divergent features indicates that the solution remains invariant with respect to the evolution variable, providing a stable reference state.
Panel (b) illustrates the imposed inequality bounds, represented by constant upper and lower limits. These bounds define the admissible region within which auxiliary or corrective terms must remain. The symmetry and uniformity of the bounds across the domain indicate a globally enforced tolerance level. This panel plays a central role in the stability analysis, as it visually enforces the constraint conditions required for well-posedness and controlled solution behavior.
Panel (c) shows the base trial function used to approximate the solution structure. The smooth oscillatory profile indicates good regularity and continuity, properties that are desirable for both analytical treatment and numerical implementation. Panel (d) presents a bounded auxiliary function whose magnitude remains strictly within the prescribed bounds. The oscillatory nature of this function mirrors that of the base trial form but with significantly reduced amplitude. This behaviour demonstrates that corrective terms can be constructed to preserve smoothness while satisfying the stability constraints.
Panel (e) provides a direct comparison between the base trial function and the bounded auxiliary function. The contrast between the two curves highlights the hierarchical structure of the approximation: the trial function captures the primary behavior of the solution, while the auxiliary function introduces a secondary, bounded correction. This separation of scales is indicative of a stable approximation process and supports convergence arguments.
Panel (f) displays the corrected trial form obtained by incorporating bounded adjustments into the base trial function. The resulting profile retains the qualitative structure of the original trial form while exhibiting improved compliance with the admissibility conditions. Compared to panel (c), the corrected form shows moderated extrema and enhanced stability, indicating that the correction process effectively controls deviations without distorting the essential solution features.
The plots collectively demonstrate a coherent framework for constructing stable approximations in fractional-order problems subject to inequality constraints. The results confirm that smooth trial functions, when supplemented by appropriately bounded auxiliary terms, yield corrected solutions that remain stable, admissible, and structurally consistent across the domain. This approach provides a practical pathway for both theoretical analysis and numerical implementation of constrained fractional systems.

5 Conclusion

In this paper, the Ulam-Hyers stability for a given fractional order system is investigated. By adopting the simple mathematical concept of Gronwall inequality, sufficient conditions for the stability of the given fractional order system in the sense of Ulam-Hyers is established. Numerical simulations are conducted as part of this research to validate the stability results in the context of Ulam-Hyers stability. Furthermore, it is anticipated that this study will provide a compelling impetus for exploring the application of Ulam-Hyers stability in related domains, including impulsive systems and time-delay systems.

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