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Mathematical model on impact of velocities for stratified deep water under modified gravity and coriolis effect with simulation

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Abstract

This study presents a mathematical model for simulating the dynamics of stratified deep water under the influence of modified gravity and the Coriolis effect. The model is developed to capture the complex interactions between density stratification, gravitational forces, and rotational effects in large-scale oceanic and geophysical fluid systems. The governing equations are derived from the principles of fluid dynamics, incorporating the effects of buoyancy, pressure gradients, and the Coriolis force, which arises due to the Earth's rotation. A modified gravity term is introduced to account for variations in gravitational acceleration, which is essential for modeling phenomena such as oceanic circulation, geostrophic flows, and internal wave propagation in stratified fluids. The mathematical formulation includes the Navier-Stokes equations, the continuity equation, and the equation of state for a stratified fluid. The Coriolis effect is incorporated through the Coriolis parameter, which depends on latitude and the Earth's angular velocity. The modified gravity term is introduced as a perturbation to the standard gravitational acceleration, allowing for the simulation of non-uniform gravitational fields in different regions of the ocean. To analyze the behavior of the system, numerical simulations were performed. The simulations are validated against known solutions and experimental data to ensure the accuracy and reliability of the model. The results demonstrate the influence of modified gravity and the Coriolis effect on the velocity structure, density distribution, and overall flow patterns in stratified deep water. The model provides a valuable tool for understanding and predicting the behavior of geophysical fluids in various environmental and climatic contexts.

Keywords and phrases: stratified flow; coriolis effect; modified gravity; internal waves; perturbation

1 Introduction

The study of fluid dynamics in stratified environments is a fundamental aspect of geophysical and oceanographic research [1], as it plays a critical role in understanding large-scale ocean currents, atmospheric circulation, and the transport of heat, mass, and momentum across the Earth's systems [2]. Stratified fluids, such as the ocean, are characterized by variations in density with depth [16], which arise primarily from differences in temperature, salinity, and pressure [3]. These density gradients give rise to complex flow patterns [4], including internal waves, geostrophic currents, and other phenomena that are essential for the global climate system [19].

In the context of deep water dynamics, the interplay between gravity, rotation, and stratification becomes particularly significant [5]. The Earth's rotation introduces the Coriolis effect, which influences the direction and magnitude of fluid motion, leading to the formation of large-scale circulation patterns [6]. Additionally, variations in gravitational acceleration [15], such as those caused by the Earth's shape or local geophysical conditions [7], can further modify the behavior of stratified fluids. These factors

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collectively determine the velocity fields and flow structures in deep water systems, making it essential to develop accurate mathematical models that incorporate these physical effects [17].

This study focuses on the development of a mathematical model for simulating the velocities and dynamics of stratified deep water under the influence of modified gravity and the Coriolis effect [18]. The model is based on the fundamental principles of fluid mechanics, including the Navier–Stokes equations, the continuity equation, and the equation of state for a stratified fluid [16][17][18]. By incorporating the effects of modified gravity and the Coriolis force, the model aims to capture the complex interactions that govern the behavior of deep water in geophysical settings [17]. The inclusion of modified gravity allows for the simulation of non-uniform gravitational fields, which are relevant in regions with varying topography, density stratification, or other geophysical anomalies [8]. The Coriolis effect, represented through the Coriolis parameter, is essential for accurately describing the rotational influence on fluid motion, particularly in large-scale systems where the Earth's rotation plays a dominant role [9].

Through numerical simulations, this model provides insights into the velocity structures, density distributions, and overall flow patterns in stratified deep water [10][11]. The results of the simulations are validated against known solutions and experimental data, ensuring the model's reliability and applicability in real-world scenarios [12]. This work contributes to the broader understanding of geophysical fluid dynamics and offers a valuable tool for predicting and analyzing the behavior of stratified fluids in various environmental and climatic contexts [13][14][20].

2 Model Assumption

When constructing a mathematical model for simulating velocities in stratified deep water under modified gravity, several key assumptions are typically made to simplify the physical system and enable the model to be computationally tractable. These assumptions are based on fluid dynamics, stratification, and gravity effects:

1. **Incompressible Fluid**: The fluid is incompressible, meaning that its density does not vary with pressure.

2. **Laminar Flow**: The flow is laminar. The fluid moves in smooth, parallel layers without turbulence.

3. **Hydrostatic Pressure Distribution**: The pressure distribution is hydrostatic, that is, pressure varies only with depth and not with horizontal position.

4. **Modified Gravity**: The gravitational acceleration is modified, possibly due to variations in the Earth's gravity field.

5. **Horizontal Flow with Negligible Vertical Velocity**: The flow is primarily horizontal, and the partial derivative in vertical velocity is negligible or zero.

3 Model formulation

3.1 Model geometry

Figure 1 displays three layers of stratified deep water with continuous densities ρ_1, ρ_2, ρ_3 and horizontal velocities u_1, u_2, u_3 with varying depths h_1, h_2 and h_3. The modified gravity which is $g^{'} = g\frac{\rho_{n+1}-\rho_n}{\rho_n}$ is gravity altered or modified in such a way that it affects the continuous stratification of deep water, where $n = 1, 2, 3, ...$

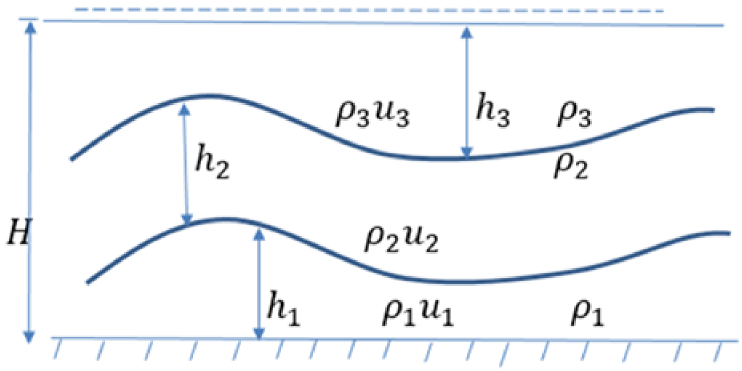

Figure 1: The geometry for stratification of three layers

3.2 Governing equations

The corresponding equations describing the motion of the stratified deep water are:

$$\frac{\partial(h_1u_1)}{\partial t}+\frac{\partial\left(h_1u_1^2+\frac{g\frac{\rho_1-\rho_0}{\rho_0}h_1^2}{2}\right)}{\partial x}=-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_2}{\partial x}-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x}+fu_1 \tag{1}$$

$$\frac{\partial(h_1v_1)}{\partial t}=-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x}-fu_1 \tag{2}$$

$$\frac{\partial h_1}{\partial t}+\frac{\partial(h_1u_1)}{\partial x}=0 \tag{3}$$

$$\frac{\partial(h_2u_2)}{\partial t}+\frac{\partial\left(h_2u_2^2+\frac{g\frac{\rho_2-\rho_1}{\rho_1}h_2^2}{2}+g\frac{\rho_2-\rho_1}{\rho_1}h_2h_1\right)}{\partial x}=-g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial x}-g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial x}+fu_2 \tag{4}$$

$$\frac{\partial(h_2v_2)}{\partial t}=-g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial y}-g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial y}-fu_2 \tag{5}$$

$$\frac{\partial h_2}{\partial t}+\frac{\partial(h_2u_2)}{\partial x}=0 \tag{6}$$

$$\frac{\partial(h_3u_3)}{\partial t}+\frac{\partial\left(h_3u_3^2+\frac{g\frac{\rho_3-\rho_2}{\rho_2}h_3^2}{2}+g\frac{\rho_3-\rho_2}{\rho_2}h_3h_2\right)}{\partial x}=-g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_3}{\partial x}-g\frac{\rho_3-\rho_2}{\rho_2}h_3\frac{\partial(\xi)}{\partial x}+fu_3 \tag{7}$$

$$\frac{\partial(h_3v_3)}{\partial t}=-g\frac{\rho_3-\rho_2}{\rho_2}h_1\frac{\partial h_2}{\partial y}-g\frac{\rho_3-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial y}-fu_3 \tag{8}$$

$$\frac{\partial h_3}{\partial t}+\frac{\partial(h_3u_3)}{\partial x}=0 \tag{9}$$

3.3 Continuity and momentum analysis

Consider the continuity and momentum equations of the model at first stratification under modified gravity and Coriolis effect [16][18]. This will help to establish procedure for understanding the general case in the overall analysis of the model equations.

For the continuity equation:

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1 u_1)}{\partial x} = 0 \tag{10}$$

$$\frac{\partial h_1}{\partial t} + h_1\frac{\partial u_1}{\partial x} + u_1\frac{\partial h_1}{\partial x} = 0 \tag{11}$$

Recall $\frac{DU}{Dt} = \frac{du_1}{dt} + u.\nabla U$

$$\Rightarrow \frac{\partial h_1}{\partial t} + u_1\frac{\partial h_1}{\partial x} = -h_1\frac{\partial u_1}{\partial x} \tag{12}$$

From the momentum equation:

$$\frac{\partial(h_1 u_1)}{\partial t} + \frac{\partial(h_1 u_1^2 + g' h_1^2/2)}{\partial x} = -g' h_1\frac{\partial h_1}{\partial x} - g' h_1\frac{\partial(\xi)}{\partial x} + f u_1 \tag{13}$$

This gives

$$\frac{\partial(h_1 u_1)}{\partial t} + \frac{\partial(h_1 u_1^2)}{\partial x} + g'\frac{\partial(h_1^2/2)}{\partial x} = -g' h_1\frac{\partial h_2}{\partial x} - g' h_1\frac{\partial \xi}{\partial x} + f u_1 \tag{14}$$

$\frac{Du_1}{Dt} = \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x}$, similarly for the second and third layer we have

$$\frac{Du_2}{Dt} = \frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} \tag{15}$$

$$\frac{Du_3}{Dt} = \frac{\partial u_3}{\partial t} + u_3\frac{\partial u_3}{\partial x} \tag{16}$$

For generality, let $u_1,\ h_1$ be $u,\ h$ respectively so we can develop governing equations for our series expansion in first to third stratified layers.

Equation (14), when expanded becomes

$$u\frac{\partial h_1}{\partial t} + h\frac{\partial u_1}{\partial t} + 2uh\frac{\partial u_1}{\partial x} + u^2\frac{\partial h}{\partial x} + g'\frac{\partial h}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{17}$$

Then

$$u\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right) + h\left(\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x}\right) + g'\frac{\partial h}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{18}$$

Substitute equation (12) into equation (16), which yields

$$-uh\frac{\partial u}{\partial x} + h\frac{\partial u}{\partial t} + 2uh\frac{\partial u}{\partial x} + g'\frac{\partial h_1}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{19}$$

$$h\frac{\partial u}{\partial t} + uh\frac{\partial u}{\partial x} + g' h\frac{\partial h}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{20}$$

$$h\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + g' h\frac{\partial h}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{21}$$

But $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{du}{dt}$, hence equation (9) becomes

$$h\frac{du}{dt} + g' h\frac{\partial h}{\partial x} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{22}$$

$$h\frac{du}{dt} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + fu \tag{23}$$

$$\frac{du}{dt} = -g' h\frac{\partial h}{\partial x} - g' h\frac{\partial \xi}{\partial x} + \frac{1}{h} fu \tag{24}$$

Suppose the solution (u_1, u_2, h_1, h_2) is

$$u_1(x,t) = u_0(x,t) + fu_1(x,t) + ... \tag{25}$$
$$u_2(x,t) = u_0(x,t) + fu_2(x,t) + ... \tag{26}$$
$$u_3(x,t) = u_0(x,t) + fu_3(x,t) + ... \tag{27}$$
$$h_1(x,t) = h_0(x,t) + fh_1(x,t) + ... \tag{28}$$
$$h_2(x,t) = h_0(x,t) + fh_2(x,t) + ... \tag{29}$$

Substituting equation (21) in (20), we get

$$\frac{d}{dt}(h_0 + fh_1 + ...) + (h_0 + fh_1 + ...)\left(\frac{\partial}{\partial x}(u_0 + fu_1 + ...) + \frac{\partial}{\partial x}(u_0 + fu_1 + ...)\right) = 0 \tag{30}$$
$$\frac{d}{dt}(h_0 + fh_2 + ...) + (h_0 + fh_2 + ...)\left(\frac{\partial}{\partial x}(u_0 + fu_1 + ...) + \frac{\partial}{\partial x}(u_0 + fu_2 + ...)\right) = 0 \tag{31}$$

3.4 Boundary and Initial Conditions

Consider the deep water flow model with initial condition and boundary condition

$$\left\{\begin{array}{ll} u_1(x,0) = u_0, & u_x(0,y,t) = 0, \quad u_x(l,t) = 0, \\ u_2(x,0) = u_0, & u_{(2)x}(x,t) = 0, \quad u_{(2)x}(l,t) = 0, \\ u_3(x,0) = u_0, & u_{(3)x}(x,t) = 0, \quad u_{(3)x}(l,t) = 0 \\ h(x,0) = me^{-s(x^2)} - \xi(x), & h_x(u,t) \end{array}\right\} \tag{32}$$

where $\xi(x,y) = \beta\sin(\alpha x), 0 \leq \alpha \leq 90$, and the values of β and α depends on the size of the stratified layer. Also, α is the measure of strength and β is the measure of the deep water system, $h(x,t=0) = me^{-s(x^2)} - \xi(x)$.

When boundary conditions are introduced in equation (24) and simplified as (Topman et al., 2025) we obtain:

$$\begin{aligned} h_1(x,t) = & -t^2 g\frac{\rho_1-\rho_0}{\rho_0}m^2 sxe^{-2sx^2} + t^2\beta sin\alpha xg\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} + \\ & \frac{2}{t}t^2\frac{\rho_1-\rho_0}{\rho_0}m^2s^2x^2e^{-2sx^2} - 2t^2\beta\sin(\alpha x)\,g\frac{\rho_1-\rho_0}{\rho_0}ms^2x^2e^{-sx^2} - \\ & \frac{1}{2}t^2me^{-sx^2}g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta\sin(\alpha x) + \frac{1}{2}t^2g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2\sin^2(\alpha x) + \\ & \frac{1}{2}t^2\frac{m^2\alpha\beta e^{-2sx^2}(2cos\alpha x - sin\alpha x) + me^{-sx^2}g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2}{(me^{-sx^2}-\beta sin\alpha x)^2} - \\ & \frac{1}{2}t^2\frac{m\alpha\beta^2\sin(\alpha x)\,e^{-sx^2}(2cos\alpha x - sin\alpha x) + g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^3\sin(\alpha x)}{(me^{-sx^2}-\beta sin\alpha x)^2} - \\ & \frac{1}{2}t^2u_0me^{-sx^2}\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{(me^{-sx^2}-\beta sin\alpha x)^2} + \frac{1}{2}t^2u_0\beta\sin(\alpha x)\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{(me^{-sx^2}-\beta sin\alpha x)^2} \end{aligned} \tag{33}$$

$$u_1(x,t) = +2gt\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} + gt\frac{\rho_1-\rho_0}{\rho_0}\alpha\beta tcos\alpha x - t\frac{g\frac{\rho_1-\rho_0}{\rho_0}\alpha\beta cos\alpha x}{me^{-sx^2}-\beta sin\alpha x} + t\frac{u_0}{me^{-sx^2}-\beta sin\alpha x} \tag{34}$$

Similarly, we can extend the results of equation (24) and (26) to obtain $h_2(x,t)$, $h_3(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ respectively.

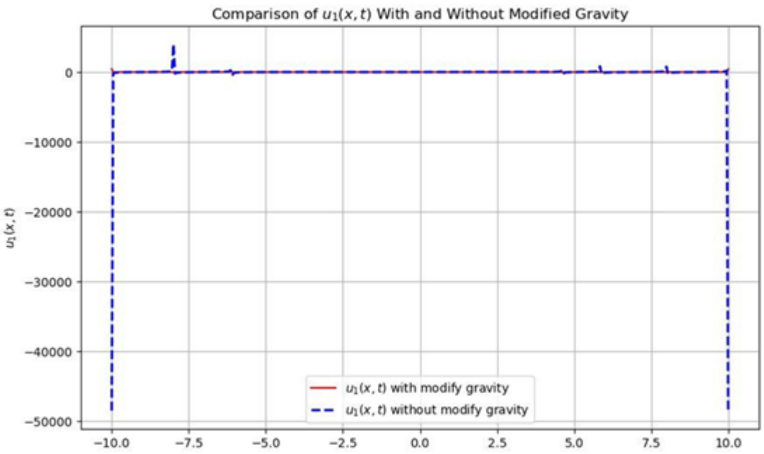

Figure 2: The velocity of the stratified layer is unaffected and tend to drag with the deep water parcel as it stratifies at the first layer. The speed of the water is faster at the first layer because the water surface is directly in contact with the air surface and the density gradient is of small value leading to high gravity value, therefore there is no consequential effect on the velocity at first layer. The overall take is that the velocity is almost the same in first stratified layer. Modified gravity refers to a scenario where gravity is altered or modified in such a way that it affects the stratification of deep water.

4 Results and discussion

4.1 Comparison of $u_1(x,t)$ with and without modified gravity

4.2 Key findings

The key findings from our analysis include:

1. **Stratified Flow Dynamics**: The model demonstrates that velocity profiles in stratified deep water are significantly influenced by density gradients, leading to distinct layers with varying flow characteristics.

2. **Modified Gravity Effects:** The inclusion of modified gravity introduces additional terms in the governing equations, which affect the vertical and horizontal flow structures, particularly in regions with high stratification.

3. **Coriolis Effect Influence**: The Coriolis force, arising from the Earth's rotation, modifies the direction and magnitude of fluid motion, leading to phenomena such as geostrophic balance and vorticity generation in stratified flows.

4. **Simulation Results**: Numerical simulations validate the model's predictions, showing that velocity variations are closely tied to the interplay between gravity, stratification, and rotational effects. The results highlight the importance of these factors in understanding large-scale oceanic and atmospheric flows.

5 Conclusion

Summarily, the mathematical model effectively captures the complex interactions between velocity, stratification, modified gravity, and the Coriolis effect in deep water systems. The simulation results

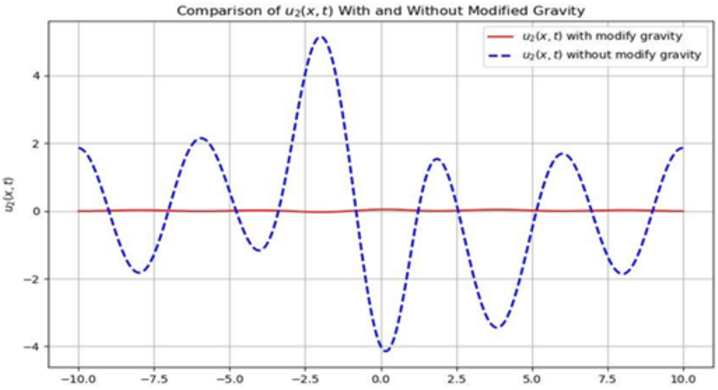

Figure 3: Without modified gravity the velocity is faster but with change in density gradient which leads to modified gravity the velocity is slower leading to stratification at second layer.

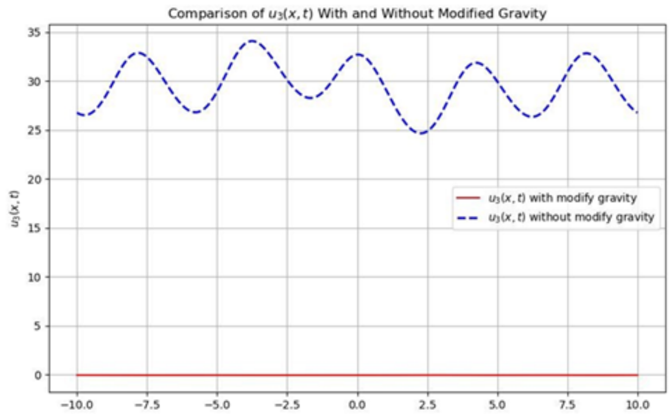

Figure 4: The velocity at third layer maintains varying amplitudes of oscillation at standard gravity but when the gravity is modified then stratification occurs leading to decrease in velocity of the deep water at third layer.

provide valuable insights into the behavior of stratified fluids under rotational and gravitational influences, supporting further research in fluid dynamics and environmental modeling.

Conflict of Interest: The authors declare no conflict of interest.

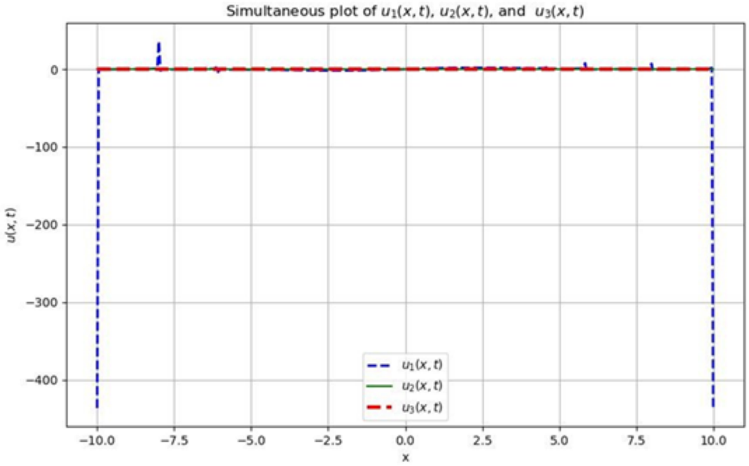

Figure 5: The simultaneous simulation of the three velocity shows that the velocity at first layer is the fastest due to it is free to air and the density gradient is drastically reduced. The velocity in the second layer of the stratified column is faster than the velocity in the third stratified layer. The simulation demonstrates how the velocities of stratified deep water interact under modified gravity.

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