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Approximate solution to nonlinear system of equation: Pourreza transform variational iteration method

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Abstract

Novel mathematical methods were established to investigate the numerical results of a coupled Burger's equation (CBE) and a shallow water- coupled system of fractional order with Caputo, Caputo Fabrizio (CF), and Atangana Baleanu Caputo (ABC) derivatives under initial conditions. The method was obtained by coupling the Pourreza transform with the variational iteration method. The stability analysis of PVIM$_{ABC}$ was conducted to show the effectiveness of the scheme, and the comparison of the numerical results of PVIM$_C$, PVIM$_{CF}$, and PVIM$_{ABC}$ with existing methods was presented to show the validity and accuracy of this proposed numerical technique. The 3D graphical results were used to compare the exact and the approximate solution at the classical order of the illustrated examples, and 2D graphical results are obtained to show the numerical results of the illustrated examples at different fractional orders. Also, the tables obtained show the reliability and the convergence of the proposed method. For calculations, the Maple software package was utilized.

Keywords and phrases: Pourreza transform; variational iteration method; coupled Burger's equation; shallow water coupled system; Caputo

1 Introduction

Fractional differential equations have recently attracted the attention of numerous scientists due to their ability to handle a wide range of intricate phenomena in the fields of science and engineering. Fractional derivatives like Caputo, Caputo Fabrizio, and Atangana Baleanu derivative are processes that described the memory and non-local properties of fractional differential equations more precisely by fractional calculus[1]. One of the fractional derivatives used by researchers is Caputo derivative, which analyzes problems with initial conditions[2]. Additionally, a growing number of researchers from different fields such as applied mathematics, physics and engineering are becoming interested in fractional models with non-singular kernels because its provides a more realistic fading memory effect. To get over the singular kernel issue from the Caputo and Riemann-Liouville, Caputo Fabrizio employed an exponential function in fractional derivative [3]. Atangana Baleanu derivative is a recent fractional derivative with a non-singular Mittag-Leffler function as its kernel, its unique feature is that it eliminates singularity problems in the Caputo derivative by using the non-local and non-singular Mittag-Leffer function as its kernel [4, 5, 6, 7, 8].

Diverse numerical and analytical methods for solving fractional differential equations have increasingly gained the interest of scientists, which includes; Homotopy perturbation method (HPM) [9], Adomian decomposition method (ADM) [10], variational iteration method (VIM) [11], Homotopy analysis method (HAM) [12] and Differential transform method [13] which are frequently coupled with integral transform such as, Elzaki transform [14, 15], Laplace transform [16, 17] Sumudu transform [18] ,Yang transform [19], and Natural transform [20] so as to increase its efficacy in handling different types of fractional order differential equations [1].

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Coupled Burger's equations is a system of partial differential equations that simulates a number of phenomena, such as turbulence, shock waves, traffic flow, and are also useful in fields like fluid mechanics, nonlinear acoustics, and gas dynamics. They are based on the Burgers' equation, a fundamental nonlinear partial differential equation that models a balance between convection and diffusion. Previously, the Elzaki decomposition method was used to analyze the convergence and efficiency of fractional coupled Burger's equations in the Caputo Fabrizio sense [14]. Sawi transform coupled homotopy perturbation method showed high precision in solving coupled Burger's equations [21].

Coupled Burgers' equation is represented as [14, 21]:

$$\left.\begin{aligned} {D_t}^{\alpha} u - u_{xx} - 2uu_x + (uv)_x = 0 \\ {D_t}^{\beta} v - v_{xx} - 2vv_x + (uv)_x = 0 \end{aligned}\right\} 0 < \alpha \leq 1 \tag{1.1}$$

The fractional derivatives of $u(x,t)$ and $v(x,t)$ are represented by $D_t^{\alpha} u$ and $D_t^{\beta} v$, respectively; u and v are the dependent variables, and t and x are the time and space variables. This fractional derivative is important in modeling viscoelastic fluids because it can incorporate the history of the system state and has the ability to provide insights into the dynamics of interacting nonlinear systems, leading to a wider range of applications in various scientific and engineering disciplines and the study of more complex physical phenomena compared to the classical Burgers equation.

Another coupled system examined is the fractional shallow water coupled system (FSWCS), which illustrates a thin layer of fluid consisting of a constant density in hydrostatic equilibrium. Several researchers have recently examined the solution of FSWCS by investigating the solution of the FSWCS equation emanating in oceanic or sea environments for the purpose of elucidating the dynamics of water motion [22]. Both singular and non-singular kernels were utilized, and the Banach fixed point theorem was employed to demonstrate the convergence and uniqueness of the solution, and the method of solution utilized was the natural transform decomposition method. Also, three new methods were used by Ganie *et al.* [23], which include the variational iteration transform method, the homotopy perturbation transform method and the Yang transform decomposition method in Caputo derivative. The FSWCS [22, 23, 24] can be expressed as:

$$\left.\begin{aligned} {}^{C}{D_t}^{\alpha} u + uv_x + vu_x = 0 \\ {}^{C}{D_t}^{\alpha} v + vv_x + u_x = 0 \end{aligned}\right\} 0 < \alpha \leq 1 \tag{1.2}$$

where $u(x,t)$ represent the free shallow and $v(x,t)$ denotes the parallel velocity factor [25].

The Pourreza transform is one of the integral transforms with a unique kernel and was developed by Ahmadi *et al.* [26], which offers a novel approach to solving differential equations. Using this transformation, the differential equations can be simplified to sets of easily solvable algebraic equations. Basically, Pourreza transforms is limited to solving linear differential equations unless they are modified and as a result of that, this work is therefore aimed at solving nonlinear systems of fractional order differential equations using the scheme of the Pourreza transform that is capable of solving coupled nonlinear fractional-order differential equations for both Caputo, CF, and ABC derivatives; therefore, the approximate solution of (1.1) and (1.2) will be presented using PVIM, and through different plots and tables, the results obtained will be analyzed.

The remaining sections of this article include: Section 2 contains some definitions and properties of the Pourreza transform that are useful for this work, the correction functional that was deduced from VIM, and the stability concept. The formulation of PVIM schemes was outlined in Section 3, and the stability analysis of the PVIM scheme was presented in Section 4. The proposed technique is used to analyze the approximate solution of CBE and FSWCS in section 5. Numerical results are presented and discussed in section 6, and the last section contains the conclusion.

2 Preliminaries and notations

2.1 Properties and definition of Pourreza transform

From the set of functions given as [26, 27]

$$S = \left\{ f(t) : c_1, c_2 > 0,\ |f(t)| \ < \ Ne^{|t|/k}, t \in (-1) \times (0, \infty) \right\}, \tag{2.1}$$

The Pourreza transform over (2.1) is obtained. where S denotes the set, N the constant with finite number, c_1, and c_2 are the finite and the infinite constant respectively. The operator $P(\cdot)$ defines the Pourreza transform by the integral equation given as:

$$P\left\{f(t)\right\} = w\int_0^{\infty} e^{-w^2 t} f(t)dt = F(w) \qquad t > 0\ ,\ w > 0. \tag{2.2}$$

2.2 Some definitions of Pourreza transforms and fractional calculus

Definition 1: The Pourreza transform for a function $f(t)$ in Caputo derivative is given as [28];

$$P\left\{{}^{C}D_x^{\alpha} f(t)\right\} = w^{2\alpha}F(w) - \sum_{k=0}^{n-1} w^{2\alpha-2k-1} f^k\left(0\right). \tag{2.3}$$

Definition 2: The Pourreza transform for a function $f(t)$ in Caputo Fabrizio derivative is given as [29];

$$P\left\{{}^{CF}D_x^{\alpha} f(t)\right\} = \frac{F(w) - \frac{f(0)}{w}}{1 - \alpha + \alpha w^{-2}}. \tag{2.4}$$

Definition 3: The Pourreza transform for a function $f(t)$ in ABC derivative is given as [29];

$$P\left\{{}^{CF}D_x^{\alpha} f(t)\right\} = \frac{F(w) - \frac{f(0)}{w}}{1 - \alpha + \alpha w^{-2\alpha}}. \tag{2.5}$$

Definition 4: The correction functional from the Variational iteration method is constructed as [11, 16];

$$f_{\eta+1}(t) = f_{\eta}(t) + \int_0^t \Psi(r)\left(D_x^{\alpha} f_{\eta}(r) + Lf_{\eta}(r) + Nf_{\eta}(r) - g(r)\right) d(r). \quad \eta = 0, 1, 2, ..., \tag{2.6}$$

the general Lagrange multiplier is Ψ, which can be optimally identified utilizing the variational theory. The nth approximation is represented by Subscript η, and the restricted variation by f_{η} .

2.3 Some stability concepts

Definition 6: Let a metric space be $((\Omega),|.|)$ and A: $\Omega \to \Omega$ be a mapping which is said to be a contraction, if for all $\phi_1,\ \phi_2 \in \Omega$ and a real positive constant $\gamma < 1$ then, [16]:

$$|A\phi_1 - A\phi_2| \leq \gamma|\phi_1 - \phi_2|. \tag{2.7}$$

This means that any pair of points $\phi_1,\ \phi_2 \in \Omega$ have images closer than the points $\phi_1,\ \phi_2$ or in other words, the ratio in the equation below does not go beyond a positive constant γ, which is $< 1.$,

$$\frac{|A\phi_1 - A\phi_2|}{|\phi_1 - \phi_2|}. \tag{2.8}$$

By uniqueness theorem and Picard's existence for differential equations, the initial value problem of first order to be considered is given as;

$$\dot{u}(t) = A(t; u), \quad u(t_0) = u_0 \tag{2.9}$$

Taking t_0 and u_0 to be two real numbers and A to be a continuous mapping on the rectangle, then

$$\bar{R} = \{(t;u) : |t - t_0| \leq a,\ |u - u_0| \leq b\} \tag{2.10}$$

Hence, A is bounded on $\bar{R}$. Then, for all $(t;u) \in \bar{R}$, $|A(t;u)| \leq c$.
In its second argument, if A satisfies the Lipschitz condition on $\bar{R}$. Then, $\exists$ a Lipschitz constant γ, $\ni$ for all $(t;u),(t;v) \in \bar{R}$,

$$|A(t;u) - A(t,v)| \leq \gamma|u - v|. \tag{2.11}$$

With the stated conditions, the initial value problem in (2.9) is with a unique solution in the interval $(t_0 - \psi, t_0 + \psi)$, where $\psi < \{a, \frac{b}{c}, \frac{1}{\gamma}\}$.
In order to apply the Pourreza transform to solve differential equations, some formula for the Pourreza transform of derivatives needs to be established.

3 Methodology

This study aims at solving the fractional coupled Burgers' equation and fractional shallow water coupled system in the Caputo, CF, and ABC senses. Pourreza transform schemes were derived by incorporating the correction functional into the scheme of the Pourreza transform.

3.1 Formulation of Pourreza transform of Caputo, CF and ABC derivative

New Pourreza transform schemes are obtained by incorporating the correction functional into the scheme of the Pourreza transform.
Suppose the differential equation representing fractional order is written as,

$$^{C}D^{\alpha}f(t) + Lf(t) + Nf(t) = g(t), \tag{3.1}$$

where $^{C}D^{\alpha}f(t)$ is the fractional order derivative in the Caputo sense, $Lf(t)$ represents the linear function, $Nf(t)$ denotes the nonlinear function, and $g(t)$ denotes the source function.
Applying Pourreza transform to (3.1), gives:

$$w^{2\alpha}F(w) - \sum_{k=0}^{\eta-1} w^{2\alpha-2k-1} f^{k}\left(0\right) = P\left[-Lf(t) - Nf(t) + g(t)\right], \tag{3.2}$$

Simplifying,

$$F(w) = \sum_{k=0}^{\eta-1} w^{-2k-1} f^{k}(0) + \frac{1}{w^{2\alpha}} P\left[-Lf(t) - Nf(t) + g(t)\right], \tag{3.3}$$

Applying inverse Pourreza transform to (3.3), gives;

$$f(t) = P^{-1}\left[\sum_{k=0}^{\eta-1} w^{-2k-1} f^{k}(0) + \frac{1}{w^{2\alpha}} P\left[-Lf(t) - Nf(t) + g(t)\right]\right], \tag{3.4}$$

then the initial approximation is,

$$f_0(t) = P^{-1}\left[\sum_{k=0}^{\eta-1} w^{-2k-1} f^{k}(0)\right]. \tag{3.5}$$

Hence,

$$f_0(t) = \sum_{k=0}^{\eta-1} \frac{t^k}{\Gamma(k+1)} f^{k}(0). \tag{3.6}$$

From the correction functional,

$$f_{\eta+1}(t) = f_\eta(t) + \int_0^t \psi(r)({}^C D_x^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r). \tag{3.7}$$

Taking Pourreza transform of (3.7) then,

$$P\{f_{\eta+1}(t)\} = P\{f_\eta(t)\} + \int_0^t \psi(r)({}^C D_x^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r), \tag{3.8}$$

simplifying,

$$P\{f_{\eta+1}(t)\} = P\{f_\eta(t)\} + \int_0^t \psi(r)(w^{2\alpha}F(w) - \sum_{k=0}^{\eta-1} w^{2\alpha-2k-1} f^k(0) + P[Lf_\eta(r) + Nf_\eta(r) - g(r)d(r)]), \tag{3.9}$$

Regarding $Lf_\eta(r)+Nf_\eta(r)-g(r))d(r)$ as restricted term, differentiating W.R.T. $P\{f_\eta(t)\}$ and setting $\frac{P\{f_{\eta+1}(t)\}}{P\{f_\eta(t)\}} = 0$ gives;

$$0 = 1 + \left[\psi(r)w^{2\alpha}\right]_0^t, \tag{3.10}$$

Which gives,

$$\psi(t) = \frac{-1}{w^{2\alpha}}. \tag{3.11}$$

Therefore (3.8) becomes,

$$P\{f_{\eta+1}(t)\} = P\{f_\eta(t)\} - \frac{1}{w^{2\alpha}} P[{}^C D_x^\alpha f_\eta(t) + Lf_\eta(t) + Nf_\eta(t) - g(t)], \tag{3.12}$$

Taking the inverse Pourreza transform, gives;

$$f_{\eta+1}(t) = f_\eta(t) - P^{-1}\left[\frac{1}{w^{2\alpha}} P[{}^C D_x^\alpha f_\eta(t) + Lf_\eta(t) + Nf_\eta(t) - g(t)]\right]. \tag{3.13}$$

Which is the Pourreza transform scheme of the Caputo derivative when $\eta = 0, 1, 2, ...$, the successive approximations for $u_1, u_2, ...u_\eta$, can be obtained, which converges to the exact solution.
Moreso, the general fractional order differential equation of Caputo Fabrizio derivative is stated as,

$${}^{CF}D^\alpha f(t) + Lf(t) + Nf(t) = g(t). \tag{3.14}$$

where ${}^{CF}D^\alpha f(t)$ defines the fractional derivative in the CF sense, $Lf(t)$ represents the linear function, $Nf(t)$ denotes the nonlinear function, and $g(t)$ denotes the source function.
Also, from (3.14) and applying (2.4), gives;

$$\frac{F(w) - \frac{f(0)}{w}}{1 - \alpha + \alpha w^{-2}} = P[-Lf(t) - Nf(t) + g(t)]. \tag{3.15}$$

Simplifying,

$$F(w) = \frac{f(0)}{w} + (1 - \alpha + \alpha w^{-2})P\{-Lf(t) - Nf(t) + g(t)\}, \tag{3.16}$$

Applying inverse Pourreza transform to (3.16), we get;

$$f(t) = P^{-1}\left\{\frac{f(0)}{w} + (1 - \alpha + \alpha w^{-2})P[-Lf(t) - Nf(t) + g(t)]\right\}, \tag{3.17}$$

Then the initial approximation is,

$$f_0(t) = f(0). \tag{3.18}$$

From the correction functional,

$$f_{\eta+1}(t) = f_\eta(t) + \int_0^t \psi(r)({}^{CF}D_t^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r). \tag{3.19}$$

Taking Pourreza transform of (3.19) then,

$$P\{f_{\eta+1}(t)\} = P\left\{f_\eta(t) + \int_0^t \psi(r)({}^{CF}D_t^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r)\right\}, \tag{3.20}$$

Simplifying,

$$P\{f_{\eta+1}(t)\} = P\{f_n(t)\} + \int_0^t \psi(r)\frac{F(w) - \frac{f(0)}{w}}{1-\alpha+\alpha w^{-2}} + P(Lf_\eta(r) + Nf_\eta(r) - g(r))d(r). \tag{3.21}$$

Regarding $Lf_\eta(r)+Nf_\eta(r)-g(r))d(r)$ as restricted term, differentiating W.R.T. $P\{f_\eta(t)\}$ and setting $\dfrac{P\{f_{\eta+1}(t)\}}{P\{f_\eta(t)\}} = 0$ gives;

$$0 = 1 + \left[\frac{\psi(r)}{1-\alpha+\alpha w^{-2}}\right]_0^t, \tag{3.22}$$

which gives,

$$\psi(t) = -(1-\alpha+\alpha w^{-2}). \tag{3.23}$$

Therefore (3.21) becomes,

$$P\{f_{\eta+1}(t)\} = P(f_\eta(t)) - (1-\alpha+\alpha w^{-2})\left[\frac{F(w) - \frac{f(0)}{w}}{1-\alpha+\alpha w^{-2}} + P(Lf_\eta(t) + Nf_\eta(t) - g_\eta(t))\right]. \tag{3.24}$$

Using the inverse Pourreza transform, gives;

$$f_{\eta+1}(t) = f(0) - P^{-1}[(1-\alpha+\alpha w^{-2})P[Lf_\eta(t) + Nf_\eta(t) - g(t)]]. \tag{3.25}$$

which is the Pourreza transform scheme of CF derivative. Substituting $\eta = 0, 1, 2, 3, ...$, the successive approximations given as $u_1, u_2, ...u_\eta$ can be obtained, which converge to the exact solution.
Moreover, the general fractional order differential equation of the ABC derivative is stated as,

$${}^{ABC}D^\alpha f(t) + Lf(t) + Nf(t) = g(t). \tag{3.26}$$

where ${}^{ABC}D^\alpha f(t)$ defines the fractional derivative in the ABC sense, $Lf(t)$ represents the linear function, $Nf(t)$ denotes the nonlinear function, and $g(t)$ denotes the source function.
Also, from (3.26) and applying (2.5), gives;

$$\frac{F(w) - \frac{f(0)}{w}}{1-\alpha+\alpha w^{-2\alpha}} = P[-Lf(t) - Nf(t) + g(t)]. \tag{3.27}$$

Simplifying,

$$F(w) = \frac{f(0)}{w} + (1-\alpha+\alpha w^{-2\alpha})P\{-Lf(t) - Nf(t) + g(t)\}, \tag{3.28}$$

Applying inverse Pourreza transform to (3.28), we get;

$$f(t) = P^{-1}\left\{\frac{f(0)}{w} + (1-\alpha+\alpha w^{-2\alpha})P[-Lf(t) - Nf(t) + g(t)]\right\}, \tag{3.29}$$

Then the initial approximation is,

$$f_0(t) = f(0). \tag{3.30}$$

From the correction functional,

$$f_{\eta+1}(t) = f_\eta(t) + \int_0^t \psi(r)({}^{ABC}D_t^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r). \tag{3.31}$$

Taking Pourreza transform of (3.31) then,

$$P\{f_{\eta+1}(t)\} = P\left\{f_\eta(t) + \int_0^t \psi(r)({}^{ABC}D_t^\alpha f_\eta(r) + Lf_\eta(r) + Nf_\eta(r) - g(r))d(r)\right\}, \tag{3.32}$$

Simplifying,

$$P\{f_{\eta+1}(t)\} = P\{f_n(t)\} + \int_0^t \psi(r)\frac{F(w) - \frac{f(0)}{w}}{1-\alpha+\alpha w^{-2\alpha}} + P(Lf_\eta(r) + Nf_\eta(r) - g(r))d(r). \tag{3.33}$$

Regarding $Lf_\eta(r)+Nf_\eta(r)-g(r))d(r)$ as restricted term, differentiating W.R.T. $P\{f_\eta(t)\}$ and setting $\dfrac{P\{f_{\eta+1}(t)\}}{P\{f_\eta(t)\}} = 0$ gives;

$$0 = 1 + \left[\frac{\psi(r)}{1-\alpha+\alpha w^{-2\alpha}}\right]_0^t, \tag{3.34}$$

which gives,

$$\psi(t) = -(1-\alpha+\alpha w^{-2\alpha}). \tag{3.35}$$

Therefore (3.33) becomes,

$$P\{f_{\eta+1}(t)\} = P(f_\eta(t)) - (1-\alpha+\alpha w^{-2\alpha})\left[\frac{F(w) - \frac{f(0)}{w}}{1-\alpha+\alpha w^{-2\alpha}} + P(Lf_\eta(t) + Nf_\eta(t) - g_\eta(t))\right]. \tag{3.36}$$

Using the inverse Pourreza transform, gives;

$$f_{\eta+1}(t) = f(0) - P^{-1}[(1-\alpha+\alpha w^{-2\alpha})P[Lf_\eta(t) + Nf_\eta(t) - g(t)]]. \tag{3.37}$$

which is the Pourreza transform scheme of ABC derivative. Substituting $\eta = 0, 1, 2, 3, ...$, the successive approximations given as $u_1, u_2, ...u_\eta$ can be obtained, which converge to the exact solution.

4 Stability analysis Of the PVIM scheme

Theorem 2: Suppose $f(t)$ is a continuous function of exponential order and γ is a positive real constant such that $\dfrac{|A(f_{\eta+1}) - A(f_{\xi+1})|}{|f_\eta - f_\xi|} \leq \gamma$. Therefore, the scheme (3.37) is unconditionally stable.

Proof: Considering the PVIM scheme (3.37) of ABC derivative.

$$f_{\eta+1}(t) = f_\eta(t) - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[Lf_\eta(t) + Nf_\eta(t) - g(t)\right]\right]. \tag{4.1}$$

Checking the stability of PVIM as follows:

$$f_{\eta+1}(t) = A\{f_{\eta+1}\} = f_\eta(t) - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[Lf_\eta(t) + Nf_\eta(t) - g(t)\right]\right]. \tag{4.2}$$

Which later gives;

$$|A(f_{\eta+1}(t)) - A(f_{\xi+1}(t))| \leq |f_\eta(t) - f_\xi(t)| - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P[L(f_\eta(t) - f_\xi(t)) + N(f_\eta(t) - f_\xi(t))]\right], \tag{4.3}$$

Which implies,

$$\frac{|A(f_{\eta+1}(t)) - A(f_{\xi+1}(t))|}{|f_\eta(t) - f_\xi(t)|} \leq 1 - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P[L(f_\eta(t) - f_\xi(t)) + N(f_\eta(t) - f_\xi(t))]\right] = \gamma \tag{4.4}$$

Then,

$$\frac{|A(f_{\eta+1}(t)) - A(f_{\xi+1}(t))|}{|f_\eta(t) - f_\xi(t)|} \leq \gamma, \tag{4.5}$$

which implies that $|A(f_{\eta+1}(t)) - A(f_{\xi+1}(t))| \leq \gamma |f_\eta(t) - f_\xi(t)|$. Hence, the proposed scheme (3.13) is unconditionally stable by (2.11)

The same approach for proving the stability of PVIM$_{ABC}$ is also applied for PVIM$_C$ and PVIM$_{CF}$ derivatives.

5 Applications

5.1 Application of Pourreza transform schemes to solve nonlinear system of time Fractional Burger's equation

$$\left.\begin{aligned} {}^{C}D_t{}^{\alpha}u &= u_{xx} + 2uu_x - (uv)_x \\ {}^{C}D_t{}^{\beta}v &= v_{xx} + 2vv_x - (uv)_x \end{aligned}\right\} 0 < \alpha \leq 1 \tag{5.1}$$

with the initial conditions given as;

$$\left.\begin{aligned} u(x,0) &= \sin(x) \\ v(x,0) &= \sin(x) \end{aligned}\right\} \tag{5.2}$$

and the exact solution at $\alpha = \beta = 1$ is given by,

$$\left.\begin{aligned} u(x,t) &= \sin(x)\, e^{-t} \\ v(x,t) &= \sin(x)\, e^{-t}. \end{aligned}\right\} \tag{5.3}$$

Applying PVIM scheme to solve equation (5.1) involves taking Pourreza transform. Using initial condition, becomes

$$\left.\begin{aligned} U(w) &= \textstyle\sum_{k=0}^{n-1} w^{-2k-1}u^k(0) + \frac{1}{w^{2\alpha}}P\left[u_{xx} + 2uu_x - (uv)_x\right] \\ V(w) &= \textstyle\sum_{k=0}^{n-1} w^{-2k-1}v^k(0) + \frac{1}{w^{2\beta}}P\left[v_{xx} + 2vv_x - (uv)_x\right], \end{aligned}\right\} \tag{5.4}$$

Taking inverse Pourreza transform of equation (5.4), becomes

$$\left.\begin{aligned} u(x,t) &= \sin(x) + P^{-1}\left[\tfrac{1}{w^{2\alpha}}P\left[u_{xx} + 2uu_x - (uv)_x\right]\right] \\ v(x,t) &= \sin(x) + P^{-1}\left[\tfrac{1}{w^{2\beta}}P\left[v_{xx} + 2vv_x - (uv)_x\right]\right], \end{aligned}\right\} \tag{5.5}$$

From equation (3.13), we can construct the correction functional for equation (5.1) as follow:

$$\left.\begin{aligned} u_{n+1} &= u_n - P^{-1}\left[\tfrac{1}{w^{2\alpha}}P\left[D_t^{\alpha}u - u_{xx} - 2uu_x + (uv)_x\right]\right] \\ v_{n+1} &= v_n - P^{-1}\left[\tfrac{1}{w^{2\beta}}P\left[D_t^{\beta}v - v_{xx} - 2vv_x + (uv)_x\right]\right], \end{aligned}\right\} \tag{5.6}$$

Starting with the initial approximations then,

$$\begin{aligned} u_0 &= \sin(x) \\ v_0 &= \sin(x) \end{aligned},$$

$$\begin{aligned} u_1 &= \sin(x) - \tfrac{\sin(x)t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1 &= \sin(x) - \tfrac{\sin(x)t^{\beta}}{\Gamma(\beta+1)}, \end{aligned}$$

Where u_2 and v_2 is the solution to equation (5.1).

Consider nonlinear system of time fractional Burger's equation in Caputo Fabrizio derivative, equation (5.1) becomes,

$$\begin{aligned} {}^{CF}D_t{}^{\alpha}u - u_{xx} - 2uu_x + (uv)_x &= 0 \\ {}^{CF}D_t{}^{\beta}v - v_{xx} - 2vv_x + (uv)_x &= 0 \end{aligned} \tag{5.7}$$

Applying PVIM scheme to solve equation (5.7) involves taking Pourreza transform. Using the initial condition, it becomes

$$\begin{aligned} U(w) &= \tfrac{u(0)}{w} + (1-\alpha+\alpha w^{-2})P\left[u_{xx}+2uu_x-(uv)_x\right] \\ V(w) &= \tfrac{v(0)}{w} + (1-\beta+\beta w^{-2})P\left[v_{xx}+2vv_x-(uv)_x\right], \end{aligned} \tag{5.8}$$

Taking inverse Pourreza transform of equation (5.8), becomes

$$\begin{aligned} u(x,t) &= \sin(x) + P^{-1}\left[(1-\alpha+\alpha w^{-2})P\left[u_{xx}+2uu_x-(uv)_x\right]\right] \\ v(x,t) &= \sin(x) + P^{-1}\left[(1-\beta+\beta w^{-2})P\left[v_{xx}+2vv_x-(uv)_x\right]\right], \end{aligned} \tag{5.9}$$

From equation (3.25), we can construct the correction functional for equation (5.7) as follow:

$$\begin{aligned} u_{n+1} &= u_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2})P\left[-u_{xx}-2uu_x+(uv)_x\right]\right] \\ v_{n+1} &= v_0 - P^{-1}\left[(1-\beta+\beta w^{-2})P\left[-v_{xx}-2vv_x+(uv)_x\right]\right], \end{aligned} \tag{5.10}$$

Starting with the initial approximations then,

$$\begin{aligned} u_0 &= \sin(x) \\ v_0 &= \sin(x) \end{aligned}$$

$$\begin{aligned} u_1 &= \sin(x)\,\alpha - \sin(x)\,\alpha t \\ v_1 &= \sin(x)\,\beta - \sin(x)\,\beta t, \end{aligned}$$

Where u_2 and v_2 is the solution to equation (5.7).
Consider nonlinear system of time fractional Burger's equation in Atangana Baleanu Caputo derivative, equation (5.1) becomes,

$$\begin{aligned} {}^{ABC}D_t{}^{\alpha}u - u_{xx} - 2uu_x + (uv)_x &= 0 \\ {}^{ABC}D_t{}^{\beta}v - v_{xx} - 2vv_x + (uv)_x &= 0. \end{aligned} \tag{5.11}$$

Applying PVIM scheme to solve equation (5.11) involves taking Pourreza transform. Using initial condition, becomes

$$\begin{aligned} U(w) &= \tfrac{u(0)}{w} + (1-\alpha+\alpha w^{-2\alpha})P\left[u_{xx}+2uu_x-(uv)_x\right] \\ V(w) &= \tfrac{v(0)}{w} + (1-\beta+\beta w^{-2\beta})P\left[v_{xx}+2vv_x-(uv)_x\right], \end{aligned} \tag{5.12}$$

Taking inverse Pourreza transform of equation (5.12), becomes

$$\begin{aligned} u(x,t) &= \sin(x) + P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[u_{xx}+2uu_x-(uv)_x\right]\right] \\ v(x,t) &= \sin(x) + P^{-1}\left[(1-\beta+\beta w^{-2\beta})P\left[v_{xx}+2vv_x-(uv)_x\right]\right], \end{aligned} \tag{5.13}$$

From equation (3.37), we can construct the correction functional for equation (5.11) as follow:

$$\begin{aligned} u_{n+1} &= u_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[-u_{xx}-2uu_x+(uv)_x\right]\right] \\ v_{n+1} &= v_0 - P^{-1}\left[(1-\beta+\beta w^{-2\beta})P\left[-v_{xx}-2vv_x+(uv)_x\right]\right], \end{aligned} \tag{5.14}$$

Starting with the initial approximations then,

$$\begin{aligned} u_0 &= \sin(x) \\ v_0 &= \sin(x) \end{aligned}$$

$$\begin{aligned} u_1 &= \sin(x)\,\alpha - \frac{\sin(x)\,\alpha t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1 &= \sin(x)\,\beta - \frac{\sin(x)\,\beta t^{\beta}}{\Gamma(\beta+1)}, \end{aligned}$$

Where u_2 and v_2 is the solution to equation (5.11).

5.2 Application of Pourreza transform schemes to fractional Shallow Water coupled systems.

$$\left.\begin{array}{l} {}^{C}D_t{}^{\alpha}u + uv_x + vu_x = 0 \\ {}^{C}D_t{}^{\alpha}v + vv_x + u_x = 0 \end{array}\right\} 0 < \alpha \leq 1 \tag{5.15}$$

Subject to the initial conditions

$$\left.\begin{array}{l} u\left(x,0\right) = \frac{x^2-2x+1}{9} \\ v\left(x,0\right) = \frac{2}{3}\left(1-x\right) \end{array}\right\} \tag{5.16}$$

The exact solution at $\beta = 1$ is given by,

$$\left.\begin{array}{l} u\left(x,t\right) = \frac{(x-1)^2}{9(t-1)^2} \\ v\left(x,t\right) = \frac{2(x-1)}{3(t-1)} \end{array}\right\} \tag{5.17}$$

Applying the PVIM scheme to solve (5.15) requires the use of the Pourreza transform. Retaining the same initial condition, the process results in

$$\left.\begin{array}{l} U(w) = \sum_{k=0}^{n-1} w^{-2k-1}u^k(0) + \frac{1}{w^{2\alpha}}P\left[-vu_x - uv_x\right] \\ V(w) = \sum_{k=0}^{n-1} w^{-2k-1}v^k(0) + \frac{1}{w^{2\alpha}}P\left[-vv_x - u_x\right], \end{array}\right\} \tag{5.18}$$

Taking inverse Pourreza transform of (5.18), becomes

$$\left.\begin{array}{l} u\left(x,t\right) = \frac{x^2-2x+1}{9} + P^{-1}\left[\frac{1}{w^{2\alpha}}P\left[-vu_x - uv_x\right]\right] \\ v\left(x,t\right) = \frac{2}{3}\left(1-x\right) + P^{-1}\left[\frac{1}{w^{2\alpha}}P\left[-vv_x - u_x\right]\right] \end{array}\right\} \tag{5.19}$$

From equation (3.13), we can construct the correction functional for equation (5.15) as follow:

$$\left.\begin{array}{l} u_{n+1} = u_n - P^{-1}\left[\frac{1}{w^{2\alpha}}P\left[D_t^{\alpha}u + vu_x + uv_x\right]\right] \\ v_{n+1} = v_n - P^{-1}\left[\frac{1}{w^{2\alpha}}P\left[D_t^{\alpha}v + vv_x + u_x\right]\right], \end{array}\right\} \tag{5.20}$$

Starting with the initial approximations then,

$$\begin{array}{l} u_0 = \frac{1}{9}\left[x^2 - 2x + 1\right] \\ v_0 = \frac{2}{3}\left[1 - x\right], \end{array}$$

$$\begin{array}{l} u_1 = \frac{1}{9}\left[x^2 - 2x + 1 - \frac{4xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^2t^{\alpha}}{\Gamma(\alpha+1)}\right] \\ v_1 = \frac{2}{3}\left[1 - x + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{xt^{\alpha}}{\Gamma(\alpha+1)}\right], \end{array}$$

Where u_2 and v_2 is the solution to equation (5.15).
Consider the Shallow Water coupled systems in Caputo Fabrizio derivative, equation (5.15) becomes,

$$\left.\begin{array}{l} {}^{CF}D_t{}^{\alpha}u + uv_x + vu_x = 0 \\ {}^{CF}D_t{}^{\alpha}v + vv_x + u_x = 0 \end{array}\right\} \tag{5.21}$$

Applying the PVIM scheme to solve (5.21) requires the use of the Pourreza transform. Retaining the same initial condition, the process results in

$$\left.\begin{array}{l} U(w) = \frac{u(0)}{w} + (1 - \alpha + \alpha w^{-2})P\left[-vu_x - uv_x\right] \\ V(w) = \frac{v(0)}{w} + (1 - \alpha + \alpha w^{-2})P\left[-vv_x - u_x\right], \end{array}\right\} \tag{5.22}$$

Taking inverse Pourreza transform of equation (5.22), it becomes

$$\left.\begin{array}{l} u\left(x,t\right) = \frac{x^2-2x+1}{9} + P^{-1}\left[\left(1 - \alpha + \alpha w^{-2}\right)P\left[-vu_x - uv_x\right]\right] \\ v\left(x,t\right) = \frac{2}{3}\left(1-x\right) + P^{-1}\left[\left(1 - \alpha + \alpha w^{-2}\right)P\left[-vv_x - u_x\right]\right] \end{array}\right\} \tag{5.23}$$

From equation (3.25), we can construct the correction functional for equation (5.21) as follow:

$$\begin{aligned} u_{n+1} &= u_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2})P\left[vu_x + uv_x\right]\right] \\ v_{n+1} &= v_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2})P\left[vv_x + u_x\right]\right], \end{aligned} \tag{5.24}$$

Starting with the initial approximations then,

$$\begin{aligned} u_0 &= \tfrac{1}{9}\left[x^2 - 2x + 1\right] \\ v_0 &= \tfrac{2}{3}\left[1 - x\right], \end{aligned}$$

$$\begin{aligned} u_1 &= \tfrac{1}{3}\left[x^2 - 2x + 1 + \tfrac{4}{3}\alpha x - \tfrac{2}{3}\alpha - \tfrac{2}{3}\alpha x^2 - \tfrac{4}{3}\alpha xt + \tfrac{2}{3}\alpha t + \tfrac{2}{3}\alpha x^2 t\right] \\ v_1 &= \tfrac{2}{3}\left[2 - 2x - \alpha + \alpha x + \alpha t - \alpha xt\right], \end{aligned}$$

Where u_2 and v_2 is the solution to equation (5.21).

Consider the Shallow Water coupled systems in Atangana Baleanu Caputo derivative, equation (5.15) becomes,

$$\left.\begin{aligned} {}^{ABC}D_t^{\alpha}u + uv_x + vu_x = 0 \\ {}^{ABC}D_t^{\alpha}v + vv_x + u_x = 0 \end{aligned}\right\} \tag{5.25}$$

Applying the PVIM scheme to solve (5.25) requires the use of the Pourreza transform. Retaining the same initial condition, the process results in

$$\begin{aligned} U(w) &= \tfrac{u(0)}{w} + (1-\alpha+\alpha w^{-2\alpha})P\left[-vu_x - uv_x\right] \\ V(w) &= \tfrac{v(0)}{w} + (1-\alpha+\alpha w^{-2\alpha})P\left[-vv_x - u_x\right], \end{aligned} \tag{5.26}$$

Taking inverse Pourreza transform of equation (5.26), it becomes

$$\left.\begin{aligned} u(x,t) &= \tfrac{x^2-2x+1}{9} + P^{-1}\left[\left(1-\alpha+\alpha w^{-2\alpha}\right)P\left[-vu_x - uv_x\right]\right] \\ v(x,t) &= \tfrac{2}{3}(1-x) + P^{-1}\left[\left(1-\alpha+\alpha w^{-2\alpha}\right)P\left[-vv_x - u_x\right]\right] \end{aligned}\right\} \tag{5.27}$$

From equation (3.37), we can construct the correction functional for equation (5.25) as follow:

$$\begin{aligned} u_{n+1} &= u_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[vu_x + uv_x\right]\right] \\ v_{n+1} &= v_0 - P^{-1}\left[(1-\alpha+\alpha w^{-2\alpha})P\left[vv_x + u_x\right]\right]. \end{aligned} \tag{5.28}$$

Starting with the initial approximations then,

$$\begin{aligned} u_0 &= \tfrac{1}{9}\left[x^2 - 2x + 1\right] \\ v_0 &= \tfrac{2}{3}\left[1 - x\right], \end{aligned}$$

$$\begin{aligned} u_1 &= \tfrac{1}{3}\left[x^2 - 2x + 1 + \tfrac{4}{3}\alpha x - \tfrac{2}{3}\alpha - \tfrac{2}{3}\alpha x^2 - \tfrac{4}{3}\tfrac{\alpha x t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{2}{3}\tfrac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{2}{3}\tfrac{\alpha x^2 t^{\alpha}}{\Gamma(\alpha+1)}\right] \\ v_1 &= \tfrac{2}{3}\left[2 - 2x - \alpha + \alpha x + \tfrac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} - \tfrac{\alpha x t^{\alpha}}{\Gamma(\alpha+1)}\right], \end{aligned}$$

where u_2 and v_2 is the solution to equation (5.25)

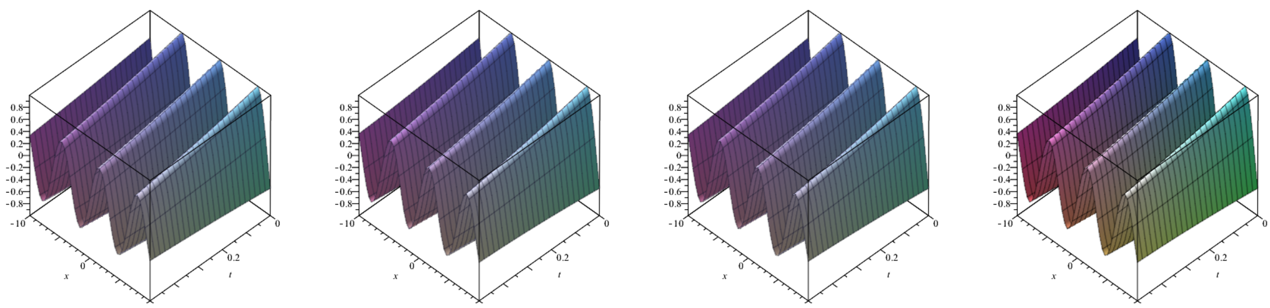
6 Results and discussion

PVIM schemes have been applied to two coupled system of equations which are non-linear system of time Fractional Burger's equation and fractional shallow water systems. The numerical results were presented and the approximate solution is derived to the second order by the proposed method.

Figures 1 and 2 show the agreement between the shape of the exact and the approximate solution at classical order for Caputo, CF, and ABC for the first example. Figure 1A displays the exact solution, and 1B, 1C, and 1D show the approximate solution using PVIM in Caputo, CF, and ABC derivatives, respectively. Figure 1 shows the non-local wave propagation or oscillating fluid flow characteristic solutions of the time fractional Burgers equation, which shows stable solutions over the used parameters.

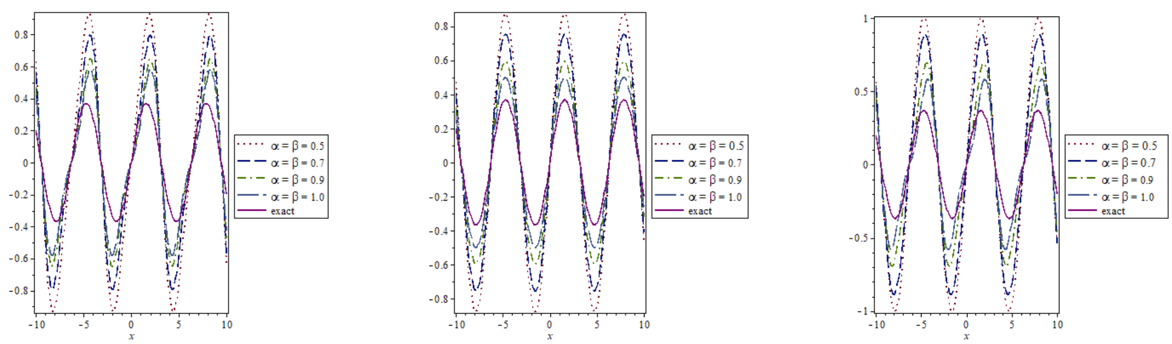
As time evolves, the nonlinear convective terms in the coupled Burgers equation tend to steepen the wave. This steepening behavior indicates the onset of shock formation, especially near the wave crest; the slope is most significant. Figure 2 displays the two-dimensional images of the application one at different fractional orders for a fixed value of $t = 1.0$. It was observed that as the values of α increase from 0.5 to 1.0, the amplitude decreases and the approximation improves, demonstrating that the PVIM approximation scheme effectively captures the physical behavior of the CBE, meaning less deviation, smoother behavior, and better convergence. Also, as the values of α decrease from 1.0 to 0.5 the amplitude increases, which shows noticeable discrepancies and less accurate approximations of the expected physical behavior.

Table 1 shows the absolute error of the coupled Burgers equation in classical order for PVIM in Caputo, CF, and ABC derivatives. The result was compared with EADM [14] in CF derivative, and the accuracy of the proposed method was revealed. The mean absolute error (MAE) obtained for PVIM$_C$, PVIM$_{CF}$, and PVIM$_{ABC}$ are 0.01166117931, 0.01248873241, and 0.01166117953, respectively and the mean absolute error obtained for EADM [14] is 0.0125. It was then observed that the mean absolute error obtained from PVIM schemes are found to be smaller when campared with that of EADM [14]



(A) (B) (C) (D)

Figure 1: (A) is the Exact solution of $u(x,t), v(x,t)$. (B), (C) and (D) are the Approximate solution of CBE in Caputo, CF and ABC derivative respectively at $\alpha = 1.0$ under the same initial conditions.



(A) (B) (C)

Figure 2: (A), (B) and (C) are the numerical result of CBE at different fractional order of $u(x,t), v(x,t)$ in Caputo, CF and ABC respectively when $t = 1.0$

Table 1: Comparison of the Absolute error Results u, v for CBE when $\alpha = \beta = 1$ in PVIM schemes and EADM derivative

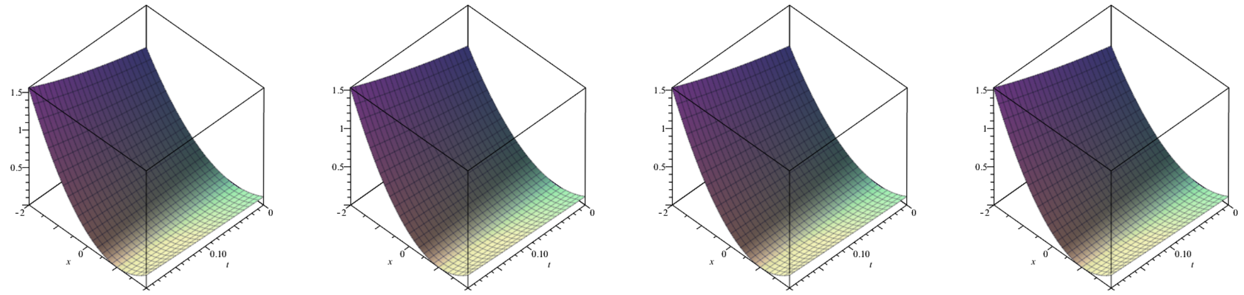
x	t	exact	|exact − approximate| PVIM$_C$	|exact − approximate| PVIM$_{CF}$	|exact − approximate| PVIM$_{ABC}$	|exact − approximate| EADM$_{CF}$[14]
0.25	0.25	0.1926783972	0.0006425580	0.0006059460	0.000642558086	0.0006
0.50	0.25	0.3733769849	0.0010171135	0.0011742172	0.001017113320	0.0012
0.75	0.25	0.5308608001	0.0009281621	0.0016694817	0.000928162313	0.0017
0.25	0.50	0.1500580866	0.0054186441	0.0045693880	0.005418644090	0.0046
0.50	0.50	0.2907862882	0.0086759722	0.0088546735	0.008675971950	0.0089
0.75	0.50	0.4134348068	0.0081917273	0.0125894184	0.008191728290	0.0126
0.25	0.75	0.1168653554	0.0191416101	0.0145679980	0.019141610220	0.0146
0.50	0.75	0.2264645889	0.0309357002	0.0282302289	0.030935701100	0.0282
0.75	0.75	0.3219833512	0.0299991263	0.0401372400	0.029999126400	0.0401
MAE			0.01166117931	0.01248873241	0.01166117953	0.0125

Figures 3A, 3B, 3C, and 3D represent the exact and the approximate solution $u(x,t)$ for PVIM in Caputo, CF, and ABC derivatives, respectively, for FSWCS. It is observed from the figures of $u(x,t)$ that as the wave propagates along the spatial domain, its amplitude continuously diminishes due to the dissipation effects inherent in the classical shallow water model. The reduction in height of the wave profile represents the gradual loss of wave energy over space and time, which is characteristic of shallow water systems at classical order.

Figures 4A, 4B, 4C, and 4D represent the exact and the approximate solution $v(x,t)$ for PVIM in Caputo, CF, and ABC derivative, respectively, for FSWCS. The graph of $v(x,t)$ displays a linear decline in the amplitude of the solution along the spatial coordinate, which confirms the dissipation property of the system. The uniform downward slope of the surface implies that the system exhibits stable and predictable damping behavior as time progresses.

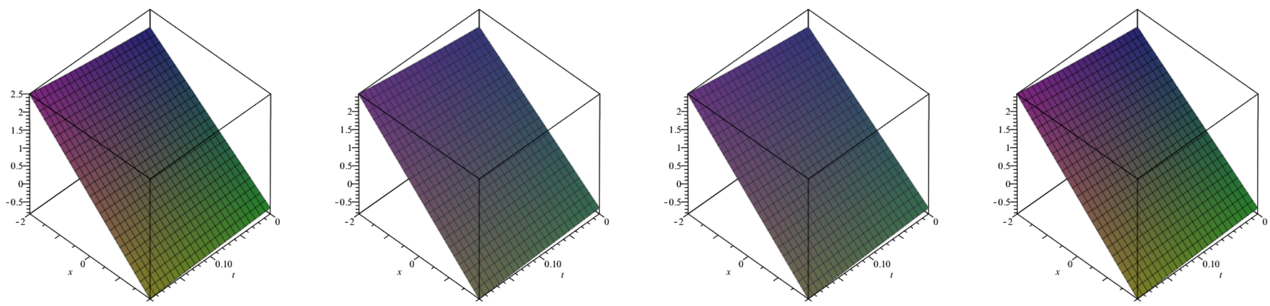
Figures 5 and 6 compare the approximate solution of FSWCS at different fractional orders of $u(x,t)$ and $v(x,t)$ at $\alpha = 0.5, 0.7, 0.9, 1.0$ and the exact solution in Caputo, CF, and ABC derivatives, respectively. The plot shows the agreements in different fractional derivatives presented. Figure 5 shows the evolution of $u(x,t)$, which represent the water surface elevation or horizontal velocity in the fractional shallow water system. The parabolic shape across all α for $u(x,t)$ shows a diffusive and symmetric behavior of the wave or fluid profile and also reveals a smooth monotonically decaying profile as both time and space increase, which indicates the influence of fractional-order diffusion in the system and inherently incorporates a stronger memory effect in the fluid behavior. Unlike classical integer order that dissipates energy more abruptly, the fractional terms spread the energy dissipation over a longer period, resulting in a more gradual energy loss throughout the domain. Figure 6 shows the evolution of $v(x,t)$, which may represent velocity flux or momentum distribution in the shallow water system. For lower fractional order, the curves are steeper and deviate more due to a stronger memory effect, which shows that the fractional order induces enhanced transport or sharper gradients in the solution. The inclination of the curve suggests a non-symmetric propagation of flow or momentum, where lower α causes overestimation of the transport effects.

Tables 2 and 3 show the absolute error of the fractional shallow water equation of $u(x,t)$ and $v(x,t)$ in classical order using PVIM in Caputo, CF, and ABC derivatives. The mean absolute error of PVIM$_C$, PVIM$_{CF}$, and PVIM$_{ABC}$ of $u(x,t)$ are 0.004857799744, 0.004857804575, and 0.004857804325, respectively. Also, the mean absolute error PVIM$_C$, PVIM$_{CF}$, and PVIM$_{ABC}$ of $v(x,t)$ are 0.001196627938, 0.001196626125, and 0.001196625562. The results were compared with NDTM [25] in CF derivative and it is observed that the errors obtained from the proposed method are found to be small when compared to those of NDTM which verifies the accuracy of the proposed methods. Generally, the plots and the tables show the comprehensive performance of the approach under different fractional orders to reveal the effectiveness of the scheme in solving nonlinear fractional-order differential equations.



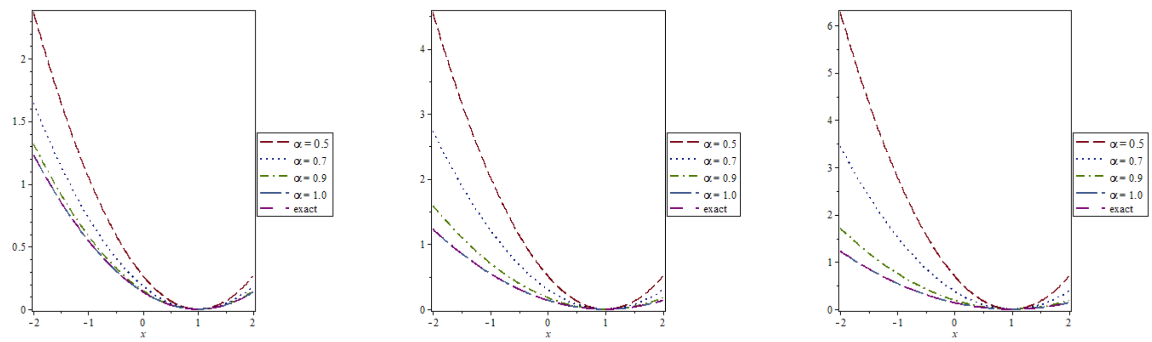
(A) (B) (C) (D)

Figure 3: (A) is the Exact solution of $u(x,t)$, (B),(C) and (D) are the Approximate solution in Caputo, Caputo Fabrizio and Atangana Baleanu derivative respectively at $\alpha = 1.0$ under the same initial conditions.



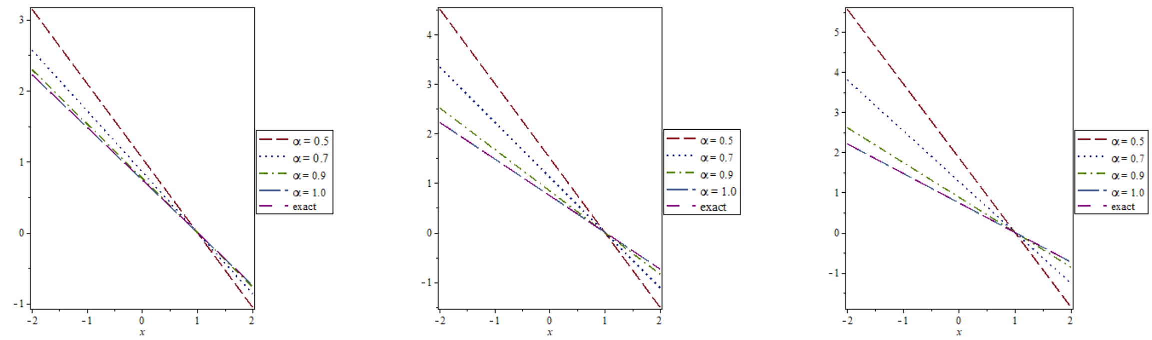
(A) (B) (C) (D)

Figure 4: (A) is the Exact solution of $v(x,t)$, (B),(C) and (D) are the Approximate solution in Caputo, Caputo Fabrizio and Atangana Baleanu derivative respectively at $\alpha = 1.0$ under the same initial conditions.



(A) (B) (C)

Figure 5: (A),(B) and (C) are the numerical result at different fractional order of $u(x,t)$ in Caputo, CF and ABC respectively when t=0.1



(A) (B) (C)

Figure 6: (A),(B) and (C) are the numerical result at different fractional order of $v(x,t)$ in Caputo, CF and ABC respectively when t=0.1

Table 2: Comparison of the Absolute error Results $u(x,t)$ for FSWCS when $\alpha = 1$ in PVIM schemes and NDTM derivative

x	t	exact	$\vert$exact − approximate$\vert$ PVIM$_C$	$\vert$exact − approximate$\vert$ PVIM$_{CF}$	$\vert$exact − approximate$\vert$ PVIM$_{ABC}$	$\vert$exact − appro NDTM$_C$
2.5	0.025	0.262984878	0.000010920	0.000010920	0.000010920	0.0003025867
2.5	0.050	0.277008310	0.000091643	0.000091644	0.000091644	0.0012791440
2.5	0.075	0.292184076	0.000324700	0.000324701	0.000324701	0.0030434510
2.5	0.100	0.308641975	0.000808642	0.000808642	0.000808642	0.0057253090
5.0	0.025	1.870114691	0.000077654	0.000077656	0.000077656	0.0022546080
5.0	0.050	1.969836873	0.000651687	0.000651689	0.000651690	0.0095076550
5.0	0.075	2.077753429	0.002308984	0.002308987	0.002308987	0.0225682400
5.0	0.100	2.194787380	0.005750342	0.005750346	0.005750345	0.0423594000
7.5	0.025	4.938271604	0.000205053	0.000205057	0.000205058	0.0035085520
7.5	0.050	5.201600492	0.001720863	0.001720868	0.001720868	0.0153260600
7.5	0.075	5.486567649	0.006097163	0.006097168	0.006097168	0.0375890600
7.5	0.100	5.795610426	0.015184501	0.015184507	0.015184507	0.0727349100
10.0	0.025	9.467455621	0.000393122	0.000393135	0.000393135	0.0031693790
10.0	0.050	9.972299169	0.003299170	0.003299182	0.003299177	0.0102008300
10.0	0.075	10.51862673	0.011689230	0.011689240	0.011689240	0.0169982700
10.0	0.100	11.11111111	0.029111120	0.029111130	0.029111130	0.0188888900
MAE			0.004857799744	0.004857804575	0.004857804325	

Table 3: Comparison of the Absolute error Results $v(x,t)$ for FSWCS when $\alpha = 1$ in PVIM schemes and NDTM derivative

x	t	exact	$\vert$exact − approximate$\vert$ PVIM$_C$	$\vert$exact − approximate$\vert$ PVIM$_{CF}$	$\vert$exact − approximate$\vert$ PVIM$_{ABC}$	$\vert$exact − appro NDTM$_C$
2.5	0.025	-1.025641026	0.000012554	0.000012553	0.000012552	0.00001602564
2.5	0.050	-1.052631579	0.000103802	0.000103801	0.000103800	0.00013157890
2.5	0.075	-1.081081081	0.000362331	0.000362330	0.000362330	0.00045608110
2.5	0.100	-1.111111111	0.000888889	0.000888888	0.000888887	0.00111111100
5.0	0.025	-2.735042736	0.000033478	0.000033477	0.000033476	0.00004273504
5.0	0.050	-2.807017544	0.000276803	0.000276802	0.000276801	0.00035087720
5.0	0.075	-2.882882882	0.000966215	0.000966214	0.000966214	0.00121621600
5.0	0.100	-2.962962962	0.002370370	0.002370368	0.002370369	0.00296296300
7.5	0.025	-4.444444444	0.000054399	0.000054397	0.000054396	0.00006944444
7.5	0.050	-4.561403508	0.000449805	0.000449803	0.000449802	0.00057017540
7.5	0.075	-4.684684684	0.001570101	0.001570099	0.001570099	0.00197635100
7.5	0.100	-4.814814814	0.003851852	0.003851850	0.003851850	0.00481481500
10.0	0.025	-6.153846154	0.000075321	0.000075318	0.000075317	0.00009615385
10.0	0.050	-6.315789474	0.000622808	0.000622805	0.000622803	0.00078947370
10.0	0.075	-6.486486486	0.002173986	0.002173983	0.002173983	0.00273648600
10.0	0.100	-6.666666666	0.005333333	0.005333330	0.005333330	0.00666666700
MAE			0.001196627938	0.001196626125	0.001196625562	

7 Conclusion

In this paper, PVIM was successfully used to get the approximate results of CBE and SWCS of fractional order. The exact solution at classical order was compared with the approximate solution in Caputo, CF, and ABC derivatives through the illustrated three dimensional graph. From the result obtained, the numerical comparison is made with EADM for example one and NDTM for example two. The comparison highlights the effectiveness of PVIM in yielding results that closely align with the exact solutions, demonstrating its reliability in solving fractional order problems. Additionally, the findings suggest that PVIM can serve as a powerful tool for researchers tackling similar mathematical challenges. Furthermore, the analysis underscores the versatility of PVIM in addressing a variety of fractional order differential equations, providing both accuracy and efficiency. This positions PVIM as an essential method for advancing research in fractional calculus and its applications across different fields of study. By leveraging its capabilities, researchers can explore new avenues in modeling complex systems, thereby enhancing our understanding of dynamic processes in engineering, physics, and beyond.

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