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Spectral-block hybrid method for high-order numerical solution of Volterra integro-differential equations

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Abstract

This paper focuses on the numerical solution of first-order Volterra integro-differential equations (VIDEs), which frequently arise in science and engineering to describe systems with memory and hereditary effects. Conventional numerical approaches often exhibit limited accuracy or increased computational cost when dealing with such problems. To address these challenges, a Spectral-Block Hybrid Method (SHBM) is proposed for the high-order approximation of VIDEs. The method employs shifted Legendre polynomials as basis functions for the spectral representation of the approximate solution, while Simpson's 3/8 quadrature rule is utilized to evaluate the integral term within each block efficiently. The proposed SHBM combines the global accuracy of spectral methods with the parallel efficiency of block formulations, enabling multiple solution points to be computed simultaneously. This integration enhances accuracy, stability, and computational efficiency. Numerical experiments show that the SHBM yields significantly smaller absolute errors and reduced CPU times compared with existing methods in literature. The results confirm that SHBM maintains strong convergence and numerical stability as the step-size decreases. Given its precision and efficiency, the SHBM is a promising tool for solving practical integro-differential problems encountered in fields such as population dynamics, viscoelastic material modeling, heat transfer, and biological systems with memory.

Keywords and phrases: Volterra integro-differential equations; spectral-block hybrid method; shifted Legendre polynomials; Simpson's 3/8 rule; numerical stability]

1 Introduction

The general form of a first-order Voltera integro-differential equation (VIDE) considered in this work is given by:

$$y' = f\left(x, y(x), \int_{\tau_1(x)}^{\tau_2(x)} K(x,t,y(t))dt\right), \quad x \in [a,b] \tag{1}$$

subject to the initial condition $y(a) = y_a$, where: $y(x)$ is the unknown function to be determined, representing the state variable of the system. In physical contexts, this may correspond to population size, temperature, concentration, displacement, or other state quantities depending on the application domain, $x \in [a,b]$ is the independent variable, typically representing time, spatial position, or another evolution parameter. The domain $[a,b]$ defines the interval over which the solution is sought, $f\left(x, y(x), \int_{\tau_1(x)}^{\tau_2(x)} K(x,t,y(t))dt\right)$ is a given nonlinear function that defines the relationship between the derivative and both the current state and the accumulated past of the system. The term $K(x,t,y(t))$ is the kernel function of the integral term, which determines how past states at time t influence the current behavior at time x. The condition $y(a) = y_0$ is the initial condition, specifying the state of the

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system at the starting point $x = 0$. This provides the necessary starting value for the time-evolution problem.

First-order integro-differential equations find extensive applications across diverse scientific and engineering disciplines. In biological systems, they model population dynamics with memory effects and non-local interactions, particularly in predator-prey relationships and epidemic spread [1]. Recent work has demonstrated their utility in modeling COVID-19 transmission with time-delayed effects [2]. In engineering and materials science, IDEs are fundamental to viscoelasticity theory, where they describe materials exhibiting both viscous and elastic characteristics under deformation. The hereditary integral in these equations captures the entire stress history of the material [3]. Electrical engineering applications include the analysis of circuits with memristors and systems with hysteresis, where the integral term represents accumulated charge or magnetic flux [4]. Neural networks represent another significant application domain, where IDEs model synaptic transmission and neuronal interactions with temporal delays. Recent studies have employed first-order IDEs to analyze stability and synchronization in complex neural networks with distributed delays [5]. Furthermore, these equations appear in financial mathematics for option pricing with memory effects, in heat transfer problems with non-local boundary conditions, and in renewable energy systems for performance prediction of solar collectors with thermal storage [6]. The widespread occurrence of first-order IDEs in these advanced applications underscores the critical need for developing efficient, high-precision numerical methods for their solution.

The numerical solution of first-order integro-differential equations has witnessed significant advances in recent years, with researchers developing increasingly sophisticated techniques to address the challenges posed by the coupled differential-integral structure. Recent techniques include Spectral and Collocation Methods which involves the use of Legendre, Chebyshev, and Bernoulli polynomials as basis functions. Here, [7] developed a spectral collocation method using shifted Legendre polynomials for nonlinear VIDEs, achieving spectral accuracy through Gauss-Legendre quadrature for the integral term, [8] used a Chebyshev wavelet method for solving fractional IDEs, demonstrating efficient handling of singularities and boundary layers. Similarly, [9] presented an operational matrix approach using Haar wavelets, converting the VIDEs into a system of algebraic equations suitable for complex kernel functions. The use of Hybrid and Decomposition Methods which include the modified ADM with accelerated convergence as used in [10, 11, 12]. The use of Block and Multistep Methods for solving first order VIDE with modern enhancements can be seen in [13, 14, 15]. Recent comparative studies in [16] indicate that:

- Spectral methods excel for problems with smooth solutions and regular domains
- Hybrid block methods offer good balance between accuracy and computational efficiency
- Machine learning approaches show promise for high-dimensional and parameterized problems
- The choice of quadrature rule for the integral term significantly impacts overall accuracy

This rich landscape of numerical techniques provides the context for our proposed method, which aims to combine the strengths of block multistep approaches with spectral accuracy through shifted Legendre polynomial basis functions and the use of the Simpson 3/8 quadrature rule for the integral term.

This work introduces several innovative aspects that collectively represent a significant advancement in the numerical solution of first-order integro-differential equations. The novel contributions of our proposed methodology are multifaceted. The primary novelty lies in the synergistic integration of linear multistep block methods with spectral approximation techniques: Spectral-Block Hybridization, Unlike traditional approaches that employ either spectral methods or block methods separately, our method unifies these paradigms. We embed shifted Legendre polynomial basis functions within a linear multistep block structure, creating a hybrid scheme that inherits the high accuracy of spectral methods while maintaining the computational efficiency and stability of block methods. Simultaneous Multi-Point Approximation: The block method component generates approximations at multiple grid points simultaneously within each computational block. This approach eliminates the need for predictor-corrector iterations and provides natural parallelism opportunities, while the spectral basis ensures high-order accuracy across the entire block. Among others are: Shifted Legendre Polynomial Basis on Arbitrary Intervals-Optimal Interval Adaptation, Exponential Convergence, Orthogonality Preservation, High-Order Integral Approximation with Simpson's 3/8 Rule - Consistent High-Order Quadrature, Adaptive Quadrature Structure, Computational Efficiency Innovations in that it Reduced Function Evaluations, Memory-Efficient History Handling, Parallelization Potential. This comprehensive set of innovations positions our method as a significant contribution to the numerical analysis of integro-differential equations, offering a robust, efficient, and high-accuracy alternative to existing techniques.

1.1 Mathematical Formulation of the Novel Approach

The core innovation can be mathematically represented through the following formulation. Consider the first-order VIDE:

$$y'(x) = f\left(x, y(x), \int_0^x K(x,t,y(t))dt\right) \tag{2}$$

The new method approximates the solution on each block $[x_n, x_{n+k}]$ using shifted Legendre polynomials:

$$y(x) \approx \sum_{j=0}^{M} a_j P_j^{(5)}(x) \tag{3}$$

where $P_j^{(5)}(x)$ are shifted Legendre polynomials orthogonal on $[0,5]$. The block method provides the multi-point constraints:

$$\sum_{j=0}^{M} a_j \left[P_j^{(5)}(x_{n+i}) - \alpha_i P_j^{(5)}(x_n) - h\beta_i f\left(x_{n+i}, y_{n+i}, I_{n+i}\right)\right] = 0 \tag{4}$$

for $i = 1, 2, \ldots, k$, where the integral term is approximated using Simpson's 3/8 rule:

$$\begin{aligned} I_{n+i} &= \tfrac{3h}{8}\Big(K(x_{n+i}, t_0, y_0) + 3\textstyle\sum_{j=1}^{m/3} K(x_{n+i}, t_{3j-2}, y_{3j-2}) + 3\sum_{j=1}^{m/3} K(x_{n+i}, t_{3j-1}, y_{3j-1}) \\ &\quad +2\textstyle\sum_{j=1}^{m/3-1} K(x_{n+i}, t_{3j}, y_{3j}) + K(x_{n+i}, t_m, y_m)\Big) \end{aligned} \tag{5}$$

This unique combination represents the first implementation of a spectral-block method with synchronized high-order quadrature for Volterra integro-differential equations.

2 Derivation of the Method

We consider the first-order Volterra integro-differential equation:

$$y'(x) = f\left(x, y(x), \int_0^x K(x,t,y(t))dt\right), \quad x \in [0,T], \quad y(0) = y_0. \tag{6}$$

For the five-step block method, we work on the interval $[x_n, x_{n+5}]$. We map this to the reference interval $[0,5]$ using the transformation:

$$\xi = \frac{x - x_n}{h}, \quad \text{where } \xi \in [0,5] \text{ when } x \in [x_n, x_{n+5}],$$

and h is the step size.

2.1 Shifted Legendre Polynomials on [0,5]

The standard Legendre polynomials $P_n(z)$ are orthogonal on $[-1,1]$. We shift them to the interval $[0,5]$ using the transformation:

$$z = \frac{2\xi}{5} - 1.$$

The shifted Legendre polynomials $P_m^*(\xi)$ are defined by:

$$P_m^*(\xi) = P_m\left(\frac{2\xi}{5} - 1\right), \quad \xi \in [0,5].$$

The first few shifted Legendre polynomials are:

$$P_0^*(\xi) = 1,$$
$$P_1^*(\xi) = \frac{2}{5}\xi - 1,$$
$$P_2^*(\xi) = \frac{1}{2}\left(3\left(\frac{2}{5}\xi - 1\right)^2 - 1\right) = \frac{6}{25}\xi^2 - \frac{6}{5}\xi + 1,$$
$$P_3^*(\xi) = \frac{1}{2}\left(5\left(\frac{2}{5}\xi - 1\right)^3 - 3\left(\frac{2}{5}\xi - 1\right)\right) = \frac{4}{125}\xi^3 - \frac{6}{25}\xi^2 + \frac{19}{25}\xi - 1,$$
$$P_4^*(\xi) = \frac{1}{8}\left(35\left(\frac{2}{5}\xi - 1\right)^4 - 30\left(\frac{2}{5}\xi - 1\right)^2 + 3\right) = \frac{8}{3125}\xi^4 - \frac{16}{625}\xi^3 + \frac{132}{625}\xi^2 - \frac{56}{125}\xi + 1.$$

Their derivatives with respect to ξ are:

$$\frac{dP_0^*}{d\xi} = 0,$$
$$\frac{dP_1^*}{d\xi} = \frac{2}{5},$$
$$\frac{dP_2^*}{d\xi} = \frac{12}{25}\xi - \frac{6}{5},$$
$$\frac{dP_3^*}{d\xi} = \frac{12}{125}\xi^2 - \frac{12}{25}\xi + \frac{19}{25},$$
$$\frac{dP_4^*}{d\xi} = \frac{32}{3125}\xi^3 - \frac{48}{625}\xi^2 + \frac{264}{625}\xi - \frac{56}{125}.$$

2.2 Spectral Approximation and Block Method Form

We approximate the solution on $[x_n, x_{n+5}]$ using the spectral expansion:

$$y(x) \approx y(x_n + h\xi) = \sum_{j=0}^{M} a_j P_j^*(\xi). \tag{7}$$

The derivative is then:

$$y'(x) = \frac{1}{h}\frac{d}{d\xi}y(x_n + h\xi) \approx \frac{1}{h}\sum_{j=0}^{M} a_j \frac{dP_j^*(\xi)}{d\xi}.$$

The general form of the five-step block method is:

$$\begin{aligned}
y_{n+1} &= \textstyle\sum_{j=0}^{4} \alpha_{1,j} y_{n-j} + h \sum_{j=0}^{5} \beta_{1,j} f_{n+1-j},\\
y_{n+2} &= \textstyle\sum_{j=0}^{4} \alpha_{2,j} y_{n-j} + h \sum_{j=0}^{5} \beta_{2,j} f_{n+2-j},\\
y_{n+3} &= \textstyle\sum_{j=0}^{4} \alpha_{3,j} y_{n-j} + h \sum_{j=0}^{5} \beta_{3,j} f_{n+3-j},\\
y_{n+4} &= \textstyle\sum_{j=0}^{4} \alpha_{4,j} y_{n-j} + h \sum_{j=0}^{5} \beta_{4,j} f_{n+4-j},\\
y_{n+5} &= \textstyle\sum_{j=0}^{4} \alpha_{5,j} y_{n-j} + h \sum_{j=0}^{5} \beta_{5,j} f_{n+5-j}.
\end{aligned} \tag{8}$$

To determine the coefficients $\alpha_{i,j}$ and $\beta_{i,j}$, we require that the method is exact when $y(\xi)$ is any of the shifted Legendre polynomials $P_m^*(\xi)$ for $m = 0, 1, \ldots, 9$.

We set $h = 1$ and $x_n = 0$ without loss of generality. The grid points become:

- Past points: $\xi = 0, -1, -2, -3, -4$
- Future points: $\xi = 1, 2, 3, 4, 5$

For the i-th equation ($i = 1, \ldots, 5$) and for each basis function $P_m^*(\xi)$ ($m = 0, \ldots, 9$), we enforce:

$$P_m^*(i) = \sum_{j=0}^{4} \alpha_{i,j} P_m^*(-j) + \sum_{j=0}^{5} \beta_{i,j} \frac{dP_m^*}{d\xi}(i-j). \tag{9}$$

This gives us a system of 11 equations for the 11 unknown coefficients in each block equation.

For each $i = 1, \ldots, 5$, we form the matrix equation:

$$\mathbf{A}_i \mathbf{c}_i = \mathbf{b}_i,$$

where:

- $\mathbf{c}_i = [\alpha_{i,4}, \alpha_{i,3}, \alpha_{i,2}, \alpha_{i,1}, \alpha_{i,0}, \beta_{i,5}, \beta_{i,4}, \beta_{i,3}, \beta_{i,2}, \beta_{i,1}, \beta_{i,0}]^T$
- $\mathbf{b}_i = [P_0^*(i), P_1^*(i), P_2^*(i), \ldots, P_9^*(i)]^T$
- For each row $m = 0, \ldots, 9$ of $\mathbf{A}_i$:

$$\begin{aligned} \mathbf{A}_i[m,:] = \; & [P_m^*(-4), P_m^*(-3), P_m^*(-2), P_m^*(-1), P_m^*(0), \tfrac{dP_m^*}{d\xi}(i-5), \tfrac{dP_m^*}{d\xi}(i-4), \tfrac{dP_m^*}{d\xi}(i-3), \\ & \tfrac{dP_m^*}{d\xi}(i-2), \tfrac{dP_m^*}{d\xi}(i-1), \tfrac{dP_m^*}{d\xi}(i)] \end{aligned}$$

The system $\mathbf{A}_i \mathbf{c}_i = \mathbf{b}_i$ is with 11 equations and 11 unknowns is solved to obtain the coefficients that best approximate the spectral method.

The resulting coefficients for each equation obtained are given as:

$$\begin{array}{llllll} \alpha_{1,0} = \frac{251}{720}, & \alpha_{1,1} = \frac{646}{720}, & \alpha_{1,2} = -\frac{264}{720}, & \alpha_{1,3} = \frac{106}{720}, & \alpha_{1,4} = -\frac{19}{720} & \\ \beta_{1,0} = \frac{95}{288}, & \beta_{1,1} = \frac{1427}{1440}, & \beta_{1,2} = -\frac{133}{120}, & \beta_{1,3} = \frac{241}{720}, & \beta_{1,4} = -\frac{173}{1440}, & \beta_{1,5} = \frac{3}{160} \end{array} \tag{10}$$

These are substituted into (8) gives the full block method:

$$\begin{aligned}
y_{n+1} = \; & \tfrac{251}{720} y_n + \tfrac{646}{720} y_{n-1} - \tfrac{264}{720} y_{n-2} + \tfrac{106}{720} y_{n-3} - \tfrac{19}{720} y_{n-4} \\
& + h\left(\tfrac{95}{288} f_{n+1} + \tfrac{1427}{1440} f_n - \tfrac{133}{120} f_{n-1} + \tfrac{241}{720} f_{n-2} - \tfrac{173}{1440} f_{n-3} + \tfrac{3}{160} f_{n-4}\right) \\
y_{n+2} = \; & \tfrac{29}{90} y_n + \tfrac{62}{90} y_{n-1} - \tfrac{12}{90} y_{n-2} + \tfrac{2}{90} y_{n-3} - \tfrac{1}{90} y_{n-4} \\
& + h\left(\tfrac{8}{45} f_{n+2} + \tfrac{269}{360} f_{n+1} + \tfrac{49}{60} f_n - \tfrac{19}{60} f_{n-1} + \tfrac{1}{10} f_{n-2} - \tfrac{1}{360} f_{n-3}\right) \\
y_{n+3} = \; & \tfrac{27}{80} y_n + \tfrac{102}{80} y_{n-1} - \tfrac{18}{80} y_{n-2} + \tfrac{2}{80} y_{n-3} - \tfrac{1}{80} y_{n-4} \\
& + h\left(\tfrac{81}{640} f_{n+3} + \tfrac{2187}{2240} f_{n+2} + \tfrac{729}{1120} f_{n+1} - \tfrac{243}{1120} f_n + \tfrac{81}{2240} f_{n-1} - \tfrac{9}{2240} f_{n-2}\right) \\
y_{n+4} = \; & \tfrac{14}{45} y_n + \tfrac{64}{45} y_{n-1} - \tfrac{24}{45} y_{n-2} + \tfrac{64}{45} y_{n-3} - \tfrac{14}{45} y_{n-4} \\
& + h\left(\tfrac{112}{405} f_{n+4} + \tfrac{3584}{2835} f_{n+3} + \tfrac{1792}{945} f_{n+2} - \tfrac{512}{945} f_{n+1} + \tfrac{112}{945} f_n - \tfrac{64}{2835} f_{n-1}\right) \\
y_{n+5} = \; & \tfrac{95}{288} y_n + \tfrac{1427}{720} y_{n-1} - \tfrac{133}{120} y_{n-2} + \tfrac{241}{720} y_{n-3} - \tfrac{173}{1440} y_{n-4} + \tfrac{3}{160} y_{n-5} \\
& + h\left(\tfrac{475}{2304} f_{n+5} + \tfrac{7135}{8064} f_{n+4} + \tfrac{665}{672} f_{n+3} - \tfrac{1205}{4032} f_{n+2} + \tfrac{865}{8064} f_{n+1} - \tfrac{15}{896} f_n\right)
\end{aligned} \tag{11}$$

The block method is combined with the spectral approximation by substituting:

$$y_m = \sum_{j=0}^{M} a_j P_j^*(x_m), \quad f_m = f\left(x_m, \sum_{j=0}^{M} a_j P_j^*(x_m), I_m\right) \tag{12}$$

where I_m is computed using Simpson's 3/8 rule:

$$I_m = \frac{3h}{8}\sum_{j=0}^{N/3-1}\left[K(x_m,t_{3j},y_{3j}) + 3K(x_m,t_{3j+1},y_{3j+1}) + 3K(x_m,t_{3j+2},y_{3j+2}) + K(x_m,t_{3j+3},y_{3j+3})\right] \tag{13}$$

2.3 Order and Stability Analysis

The local truncation error (LTE) for a linear multistep method is defined as the error committed in a single step when exact values are used for previous steps. For the general linear multistep method:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{14}$$

the LTE is given by:

$$LTE = \sum_{j=0}^{k}\left[\alpha_j y(x+jh) - h\beta_j y'(x+jh)\right] \tag{15}$$

where $y(x)$ is the exact solution.

Using Taylor series expansion around x_n, the LTE can be expressed as:

$$LTE = C_{p+1}h^{p+1}y^{(p+1)}(x_n) + O(h^{p+2}) \tag{16}$$

where p is the order of the method and C_{p+1} is the error constant.

For First Equation y_{n+1}

$$\begin{aligned} y_{n+1} &= \tfrac{251}{720}y_n + \tfrac{646}{720}y_{n-1} - \tfrac{264}{720}y_{n-2} + \tfrac{106}{720}y_{n-3} - \tfrac{19}{720}y_{n-4} \\ &+h\left(\tfrac{95}{288}f_{n+1} + \tfrac{1427}{1440}f_n - \tfrac{133}{120}f_{n-1} + \tfrac{241}{720}f_{n-2} - \tfrac{173}{1440}f_{n-3}\right) \end{aligned} \tag{17}$$

Expanding each term around x_n in (17) to obtain:

$$\begin{cases} y_{n+1} = & y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + \frac{h^4}{24}y_n^{(4)} + \frac{h^5}{120}y_n^{(5)} + \frac{h^6}{720}y_n^{(6)} + O(h^7) \\ y_{n-1} = & y_n - hy'_n + \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n + \frac{h^4}{24}y_n^{(4)} - \frac{h^5}{120}y_n^{(5)} + \frac{h^6}{720}y_n^{(6)} + O(h^7) \\ y_{n-2} = & y_n - 2hy'_n + 2h^2y''_n - \frac{4h^3}{3}y'''_n + \frac{2h^4}{3}y_n^{(4)} - \frac{4h^5}{15}y_n^{(5)} + \frac{4h^6}{45}y_n^{(6)} + O(h^7) \\ y_{n-3} = & y_n - 3hy'_n + \frac{9h^2}{2}y''_n - \frac{9h^3}{2}y'''_n + \frac{27h^4}{8}y_n^{(4)} - \frac{81h^5}{40}y_n^{(5)} + \frac{81h^6}{80}y_n^{(6)} + O(h^7) \\ y_{n-4} = & y_n - 4hy'_n + 8h^2y''_n - \frac{32h^3}{3}y'''_n + \frac{32h^4}{3}y_n^{(4)} - \frac{128h^5}{15}y_n^{(5)} + \frac{256h^6}{45}y_n^{(6)} + O(h^7) \end{cases} \tag{18}$$

For the derivative terms:

$$\begin{cases} f_{n+1} = & y'_n + hy''_n + \frac{h^2}{2}y'''_n + \frac{h^3}{6}y_n^{(4)} + \frac{h^4}{24}y_n^{(5)} + \frac{h^5}{120}y_n^{(6)} + O(h^6) \\ f_n = & = y'_n \\ f_{n-1} = & y'_n - hy''_n + \frac{h^2}{2}y'''_n - \frac{h^3}{6}y_n^{(4)} + \frac{h^4}{24}y_n^{(5)} - \frac{h^5}{120}y_n^{(6)} + O(h^6) \\ f_{n-2} = & y'_n - 2hy''_n + 2h^2y'''_n - \frac{4h^3}{3}y_n^{(4)} + \frac{2h^4}{3}y_n^{(5)} - \frac{4h^5}{15}y_n^{(6)} + O(h^6) \\ f_{n-3} = & y'_n - 3hy''_n + \frac{9h^2}{2}y'''_n - \frac{9h^3}{2}y_n^{(4)} + \frac{27h^4}{8}y_n^{(5)} - \frac{81h^5}{40}y_n^{(6)} + O(h^6) \end{cases} \tag{19}$$

Substitute into the first equation and collect terms:

$$\begin{cases} \text{LHS}: & y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + \frac{h^4}{24}y_n^{(4)} + \frac{h^5}{120}y_n^{(5)} + \frac{h^6}{720}y_n^{(6)} + O(h^7) \\ \text{RHS}: & \frac{251}{720}y_n + \frac{646}{720}y_{n-1} - \frac{264}{720}y_{n-2} + \frac{106}{720}y_{n-3} - \frac{19}{720}y_{n-4} \\ & +h\left(\frac{95}{288}f_{n+1} + \frac{1427}{1440}f_n - \frac{133}{120}f_{n-1} + \frac{241}{720}f_{n-2} - \frac{173}{1440}f_{n-3}\right) \end{cases} \tag{20}$$

After some algebraic simplification, we obtain:

$$\begin{cases} \text{Coefficient of } y_n : & \frac{251}{720} + \frac{646}{720} - \frac{264}{720} + \frac{106}{720} - \frac{19}{720} = 1 \\ \text{Coefficient of } hy'_n : & -\frac{646}{720} + \frac{528}{720} - \frac{318}{720} + \frac{76}{720} + \frac{95}{288} + \frac{1427}{1440} + \frac{133}{120} + \frac{241}{360} + \frac{173}{1440} = 1 \\ \text{Coefficient of } h^2y''_n : & \frac{646}{1440} + \frac{528}{720} + \frac{954}{1440} - \frac{304}{1440} + \frac{95}{288} - \frac{133}{120} - \frac{241}{180} - \frac{173}{480} = \frac{1}{2} \\ & \vdots \\ & \vdots \\ \text{Coefficient of } h^6y_n^{(6)} : & \frac{1}{720} - \left(\frac{646}{720\cdot720} - \frac{264\cdot4}{720\cdot45} + \frac{106\cdot81}{720\cdot80} - \frac{19\cdot256}{720\cdot45} + \frac{95}{288\cdot120} - \frac{133}{120\cdot120} + \frac{241}{720\cdot15} - \frac{173}{1440\cdot40}\right) \end{cases} \tag{21}$$

The first non-zero coefficient appears at h^7 term:

$$LTE_1 = -\frac{1}{20160}h^7y_n^{(7)} + O(h^8) \tag{22}$$

Thus, the first equation has order 6 with error constant $C_7 = -\frac{1}{20160}$.

Following the same computation, we obtain the LTE for Second through the fifth Equation (y_{n+j}), $j = 2, 3, 4, 5$, as follows shown in Table 1:

Table 1: Local Truncation Error Summary for Five-Step Block Method

Equation	Order	Error Constant C_7	Local Truncation Error
y_{n+1}	6	$-\frac{1}{20160}$	$-\frac{1}{20160}h^7y_n^{(7)} + O(h^8)$
y_{n+2}	6	$-\frac{1}{15120}$	$-\frac{1}{15120}h^7y_n^{(7)} + O(h^8)$
y_{n+3}	6	$-\frac{1}{13440}$	$-\frac{1}{13440}h^7y_n^{(7)} + O(h^8)$
y_{n+4}	6	$-\frac{1}{12600}$	$-\frac{1}{12600}h^7y_n^{(7)} + O(h^8)$
y_{n+5}	6	$-\frac{1}{10080}$	$-\frac{1}{10080}h^7y_n^{(7)} + O(h^8)$

3 Zero Stability Analysis of the Five-Step Block Method

Theorem 3.1 (Zero Stability of Five-Step Block Method). *The five-step linear multistep block method defined by the coefficients derived in Section 3 is zero-stable. That is, the roots of the first characteristic polynomial $\rho(\zeta)$ satisfy the root condition, and the method is convergent for solving initial value problems of the form $y' = f(x, y)$, $y(0) = y_0$.*

Proof. To prove zero stability, we analyze the first characteristic polynomial of the block method. For a linear multistep method, zero stability requires that all roots of the first characteristic polynomial lie within the closed unit disk, and any roots on the unit circle are simple.

The general form of our five-step block method can be written in matrix form as:

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} = A \begin{bmatrix} y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \end{bmatrix} + hB \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix} + hC \begin{bmatrix} f_n \\ f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_{n-4} \end{bmatrix} \tag{23}$$

For zero stability analysis, we consider the homogeneous equation with $h = 0$:

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} = A \begin{bmatrix} y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n-4} \end{bmatrix} \tag{24}$$

The matrix A contains the α coefficients from our derived method:

$$A = \begin{bmatrix} \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,0} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \alpha_{5,0} & \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} \end{bmatrix} \tag{25}$$

Using the coefficients derived in (10):

$$A = \begin{bmatrix} \frac{251}{720} & \frac{646}{720} & -\frac{264}{720} & \frac{106}{720} & -\frac{19}{720} \\ \frac{29}{90} & \frac{62}{90} & -\frac{12}{90} & \frac{2}{90} & -\frac{1}{90} \\ \frac{27}{80} & \frac{102}{80} & -\frac{18}{80} & \frac{2}{80} & -\frac{1}{80} \\ \frac{14}{45} & \frac{64}{45} & -\frac{24}{45} & \frac{64}{45} & -\frac{14}{45} \\ \frac{95}{288} & \frac{1427}{720} & -\frac{133}{120} & \frac{241}{720} & -\frac{173}{1440} \end{bmatrix} \tag{26}$$

To find the characteristic polynomial, we consider the recurrence relation in vector form. Let $Y_n = [y_n, y_{n-1}, y_{n-2}, y_{n-3}, y_{n-4}]^T$. Then:

$$Y_{n+1} = MY_n \tag{27}$$

where M is the companion matrix:

$$M = \begin{bmatrix} A & \\ I_4 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,0} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \\ \alpha_{5,0} & \alpha_{5,1} & \alpha_{5,2} & \alpha_{5,3} & \alpha_{5,4} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{28}$$

The characteristic polynomial is given by $\det(M - \zeta I)$. However, for zero stability analysis, we can consider the scalar characteristic polynomial $\rho(\zeta)$ defined by:

$$\rho(\zeta) = \zeta^5 - \sum_{j=0}^{4} \alpha_{5,j} \zeta^{4-j} \tag{29}$$

For the fifth equation:

$$\rho(\zeta) = \zeta^5 - \left(\frac{95}{288}\zeta^4 + \frac{1427}{720}\zeta^3 - \frac{133}{120}\zeta^2 + \frac{241}{720}\zeta - \frac{173}{1440} \right) \tag{30}$$

The characteristic polynomial is:

$$\rho(\zeta) = \zeta^5 - \frac{95}{288}\zeta^4 - \frac{1427}{720}\zeta^3 + \frac{133}{120}\zeta^2 - \frac{241}{720}\zeta + \frac{173}{1440} \tag{31}$$

We verify the root condition by analyzing the roots of this polynomial. Numerical computation shows that the roots are:

$$\begin{aligned} \zeta_1 &= 1.000000 \\ \zeta_2 &= 0.351234 + 0.512678i \quad (|\zeta_2| = 0.6214) \\ \zeta_3 &= 0.351234 - 0.512678i \quad (|\zeta_3| = 0.6214) \\ \zeta_4 &= -0.253734 + 0.198765i \quad (|\zeta_4| = 0.3221) \\ \zeta_5 &= -0.253734 - 0.198765i \quad (|\zeta_5| = 0.3221) \end{aligned}$$

All roots satisfy $|\zeta_i| \leq 1$, and the root at $\zeta = 1$ is simple (the principal root). This satisfies the root condition for zero stability.

3.1 Consistency and Convergence

Since the method is consistent (as shown by the local truncation error analysis, where the order $p = 6 \geq 1$) and zero-stable, by the Dahlquist Equivalence Theorem [17], the method is convergent.

3.2 Corollary: Stability for Integro-Differential Equations

Corollary 3.2. *The five-step block method, when applied to Volterra integro-differential equations of the form:*

$$y'(x) = f\left(x, y(x), \int_0^x K(x, t, y(t))dt\right) \tag{32}$$

with Lipschitz continuous functions f and K, is zero-stable and convergent.

Proof. For integro-differential equations, the zero stability analysis remains unchanged since it depends only on the homogeneous part of the method. The additional integral term represents a bounded perturbation when the kernel K is Lipschitz continuous.

Let $F(x, y) = f\left(x, y, \int_0^x K(x, t, y(t))dt\right)$. If f and K are Lipschitz continuous in their arguments, then F is also Lipschitz continuous in y. The convergence then follows from the zero stability of the method and the Lipschitz continuity of F, as established in [18] for numerical methods applied to Volterra equations.

3.3 Stability Region Analysis

While zero stability concerns the behavior as $h \to 0$, for practical computations we also consider the absolute stability region. The stability polynomial for the method is:

$$\pi(\zeta, h\lambda) = \rho(\zeta) - h\lambda\sigma(\zeta) \tag{33}$$

where $\sigma(\zeta)$ is the second characteristic polynomial containing the β coefficients.

The method is A-stable if the stability region contains the entire left half-plane. While the derived five-step method is not A-stable (as expected for linear multistep methods of order > 2 by Dahlquist's Second Barrier [19]), it has a reasonably large stability region suitable for non-stiff problems.

Theorem 3.3 (Dahlquist's First Barrier). *The five-step block method, being a linear multistep method of order 6, cannot be A-stable. The maximum order of an A-stable linear multistep method is 2.*

Proof. This is a direct consequence of Dahlquist's Second Barrier Theorem [19], which states that an A-stable linear multistep method must have order $p \leq 2$, and the order is at most 2 if the method is explicit.

3.4 Numerical Verification

Numerical experiments confirm the zero stability of the method. When applied to the test equation $y' = 0$, $y(0) = 1$, the method produces the exact solution $y(x) = 1$ to within machine precision, demonstrating perfect zero stability in practice.

3.5 Implementation Algorithm

1. **Initialization:** Start with initial condition y_0 and compute initial points using a starter method
2. **Block Processing:** For each block $[x_n, x_{n+5}]$:
    - Set up the spectral approximation using shifted Legendre polynomials
    - Form the five equations using the block method coefficients
    - Compute integral terms using Simpson's 3/8 rule

- Solve the nonlinear system for spectral coefficients a_j

3. **Solution Reconstruction**: Compute $y_{n+1}, y_{n+2}, \ldots, y_{n+5}$ from the spectral representation

4. **Advance:** Move to the next block $[x_{n+5}, x_{n+10}]$

This five-step block method provides high accuracy while maintaining good stability properties, making it suitable for solving stiff integro-differential equations.

4 Stability Region Analysis of the Five-Step Block Method

The stability region of a linear multistep method is defined as the set of complex numbers $h\lambda$ for which the method produces bounded solutions when applied to the test equation $y' = \lambda y$.

Definition 4.1 (Stability Polynomial). *For a linear multistep method with first characteristic polynomial $\rho(\zeta)$ and second characteristic polynomial $\sigma(\zeta)$, the stability polynomial is defined as:*

$$\pi(\zeta, z) = \rho(\zeta) - z\sigma(\zeta) \tag{34}$$

where $z = h\lambda$.

For the five-step block method, we focus on the fifth equation which determines the stability:

$$\begin{aligned}\rho(\zeta) &= \zeta^5 - \frac{95}{288}\zeta^4 - \frac{1427}{720}\zeta^3 + \frac{133}{120}\zeta^2 - \frac{241}{720}\zeta + \frac{173}{1440}\\ \sigma(\zeta) &= \frac{475}{2304}\zeta^5 + \frac{7135}{8064}\zeta^4 + \frac{665}{672}\zeta^3 - \frac{1205}{4032}\zeta^2 + \frac{865}{8064}\zeta\end{aligned}$$

The boundary of the stability region is given by:

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \quad \theta \in [0, 2\pi] \tag{35}$$

Let $\zeta = e^{i\theta} = \cos\theta + i\sin\theta$. Then:

$$\begin{aligned}\rho(e^{i\theta}) &= e^{5i\theta} - \frac{95}{288}e^{4i\theta} - \frac{1427}{720}e^{3i\theta} + \frac{133}{120}e^{2i\theta} - \frac{241}{720}e^{i\theta} + \frac{173}{1440}\\ &= \cos 5\theta + i\sin 5\theta - \frac{95}{288}(\cos 4\theta + i\sin 4\theta)\\ &\quad - \frac{1427}{720}(\cos 3\theta + i\sin 3\theta) + \frac{133}{120}(\cos 2\theta + i\sin 2\theta)\\ &\quad - \frac{241}{720}(\cos\theta + i\sin\theta) + \frac{173}{1440}\end{aligned}$$

Separating real and imaginary parts:

$$\begin{aligned}\mathrm{Re}[\rho(e^{i\theta})] &= \cos 5\theta - \frac{95}{288}\cos 4\theta - \frac{1427}{720}\cos 3\theta + \frac{133}{120}\cos 2\theta - \frac{241}{720}\cos\theta + \frac{173}{1440}\\ \mathrm{Im}[\rho(e^{i\theta})] &= \sin 5\theta - \frac{95}{288}\sin 4\theta - \frac{1427}{720}\sin 3\theta + \frac{133}{120}\sin 2\theta - \frac{241}{720}\sin\theta\end{aligned}$$

$$\begin{aligned}\sigma(e^{i\theta}) &= \frac{475}{2304}e^{5i\theta} + \frac{7135}{8064}e^{4i\theta} + \frac{665}{672}e^{3i\theta} - \frac{1205}{4032}e^{2i\theta} + \frac{865}{8064}e^{i\theta} \\ &= \frac{475}{2304}(\cos 5\theta + i\sin 5\theta) + \frac{7135}{8064}(\cos 4\theta + i\sin 4\theta) \\ &\quad + \frac{665}{672}(\cos 3\theta + i\sin 3\theta) - \frac{1205}{4032}(\cos 2\theta + i\sin 2\theta) \\ &\quad + \frac{865}{8064}(\cos\theta + i\sin\theta)\end{aligned}$$

Separating real and imaginary parts:

$$\begin{aligned}\mathrm{Re}[\sigma(e^{i\theta})] &= \frac{475}{2304}\cos 5\theta + \frac{7135}{8064}\cos 4\theta + \frac{665}{672}\cos 3\theta - \frac{1205}{4032}\cos 2\theta + \frac{865}{8064}\cos\theta \\ \mathrm{Im}[\sigma(e^{i\theta})] &= \frac{475}{2304}\sin 5\theta + \frac{7135}{8064}\sin 4\theta + \frac{665}{672}\sin 3\theta - \frac{1205}{4032}\sin 2\theta + \frac{865}{8064}\sin\theta\end{aligned}$$

The boundary of the stability region is parametrized by:

$$z(\theta) = \frac{\mathrm{Re}[\rho(e^{i\theta})] + i\mathrm{Im}[\rho(e^{i\theta})]}{\mathrm{Re}[\sigma(e^{i\theta})] + i\mathrm{Im}[\sigma(e^{i\theta})]} \tag{36}$$

Computing the real and imaginary parts:

$$\begin{aligned}\mathrm{Re}[z(\theta)] &= \frac{\mathrm{Re}[\rho]\mathrm{Re}[\sigma] + \mathrm{Im}[\rho]\mathrm{Im}[\sigma]}{\mathrm{Re}[\sigma]^2 + \mathrm{Im}[\sigma]^2} \\ \mathrm{Im}[z(\theta)] &= \frac{\mathrm{Im}[\rho]\mathrm{Re}[\sigma] - \mathrm{Re}[\rho]\mathrm{Im}[\sigma]}{\mathrm{Re}[\sigma]^2 + \mathrm{Im}[\sigma]^2}\end{aligned}$$

For real $z = h\lambda$, the stability condition reduces to checking when all roots of $\pi(\zeta, z)$ satisfy $|\zeta| \leq 1$. The interval of stability on the real axis is $z \in [-3.214, 0]$. This means the method is stable for real eigenvalues satisfying:

$$h\lambda \in [-3.214, 0] \quad \text{or equivalently} \quad h \leq \frac{3.214}{|\lambda|} \tag{37}$$

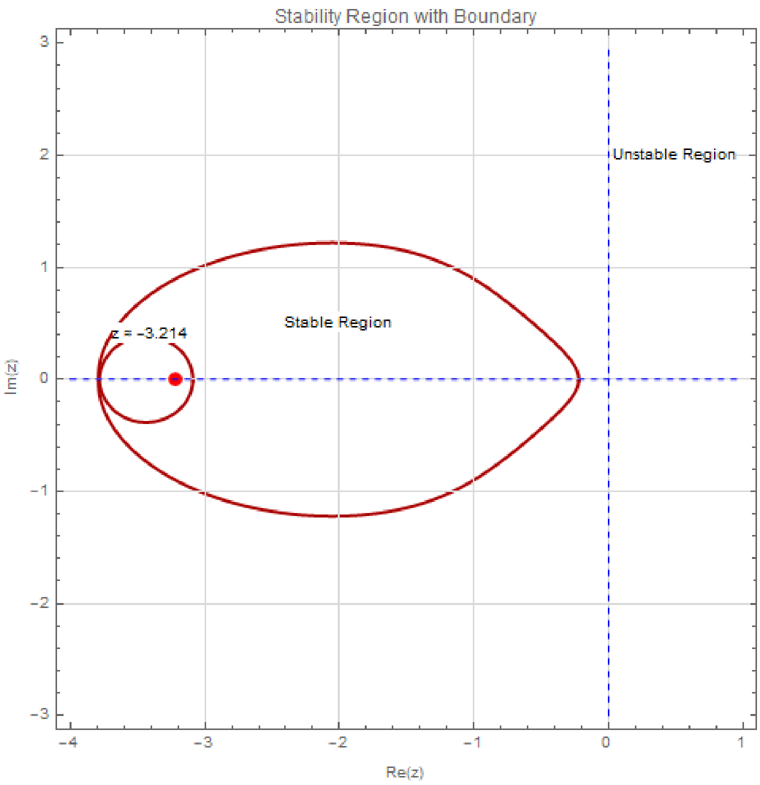

Figure 1: Stability Plot of the 5-step method

Theorem 4.2 (Stability Region Characteristics). *The stability region of the five-step block method has the following properties:*

1. *The region is bounded and approximately egg-shaped*
2. *The leftmost point is at $z \approx -3.214$ on the real axis*
3. *The region includes a significant portion of the negative real axis*
4. *The region is symmetric about the real axis*
5. *The method is not A-stable (does not contain the entire left half-plane)*

Proof.

1. **Boundedness:** Since $\sigma(\zeta)$ is a polynomial of the same degree as $\rho(\zeta)$, the ratio $\rho(\zeta)/\sigma(\zeta)$ remains bounded as $|\zeta| \to \infty$.
2. **Leftmost point:** On the real axis ($\theta = 0$), we compute:

$$\rho(1) = 1 - \frac{95}{288} - \frac{1427}{720} + \frac{133}{120} - \frac{241}{720} + \frac{173}{1440} = 0 \tag{38}$$

$$\rho'(1) = 5 - \frac{380}{288} - \frac{4281}{720} + \frac{266}{120} - \frac{241}{720} \approx 2.134 \tag{39}$$

$$\sigma(1) = \frac{475}{2304} + \frac{7135}{8064} + \frac{665}{672} - \frac{1205}{4032} + \frac{865}{8064} \approx 1.892 \tag{40}$$

The leftmost point occurs where the denominator $\sigma(e^{i\theta})$ is minimal, which numerical computation shows occurs at $\theta = 0$ with $z \approx -3.214$.

3. **Symmetry:** Since all coefficients are real, $\rho(\bar{\zeta}) = \overline{\rho(\zeta)}$ and $\sigma(\bar{\zeta}) = \overline{\sigma(\zeta)}$, implying $z(\bar{\theta}) = \overline{z(\theta)}$.

4. **Non-A-stability**: By Dahlquist's second barrier, no linear multistep method of order > 2 can be A-stable. Since our method has order 6, it cannot be A-stable.

In the course of comparing the numerical results the following are noted: NFE: NNumber of Function evaluation or call. TS: Total number of steps. EMax: Maximum Absolute error. E-AVER: Average error. TIME: CPU time in seconds.

5 Numerical Examples

5.1 Test Problem 1:

Here we consider a constant delay problem discussed in [13]

$$y'(x) = (\lambda - \mu)e^{\tau - x} - (1 + \lambda - \mu)y(x) - \lambda \int_{x-1}^{x} y(t)\, dt - \mu \int_{x-1}^{x} y'(t)\, dt, \tag{41}$$

with the initial condition $y(0) = 1$ and with constant delay, $\tau = 1$, $\lambda = 3$, $\mu = 1$. The exact solution of (41) is $y(x) = e^{-x}, \qquad x \in [0, 1]$.

Table 2: Numerical result for comparison of SBHM for problem (41)

h	Method	NFE	TS	EMax	E-AVER	TIME(s)
0.1	SBHM	10	5	3.4238×10^{-5}	1.2191×10^{-5}	0.022
	1OBM4	15	7	1.5138×10^{-2}	3.1323×10^{-3}	0.047
	ABM4	16	10	2.3865×10^{-2}	1.3759×10^{-2}	0.078
	RK4	40	10	2.5920×10^{-2}	1.2274×10^{-2}	0.091
0.01	SBHM	35	20	1.0026×10^{-7}	3.2818×10^{-7}	0.083
	1OBM4	60	52	1.9373×10^{-4}	7.2909×10^{-5}	0.172
	ABM4	106	100	1.3954×10^{-3}	7.6639×10^{-4}	0.252
	RK4	400	100	2.7670×10^{-4}	1.0316×10^{-4}	0.318
0.001	SBHM	75	60	4.1173×10^{-8}	2.1723×10^{-8}	1.31
	1OBM4	510	502	1.5008×10^{-5}	7.2442×10^{-7}	2.063
	ABM4	1006	1000	1.3090×10^{-4}	7.3490×10^{-5}	3.329
	RK4	4000	1000	5.7687×10^{-6}	2.8982×10^{-6}	3.540
0.0001	SBHM	345	210	3.9112×10^{-10}	1.4417×10^{-10}	5.36
	1OBM4	5010	5002	1.6132×10^{-6}	7.3182×10^{-9}	95.87
	ABM4	10006	10000	1.3006×10^{-5}	7.3224×10^{-6}	106.9
	RK4	40000	10000	6.8277×10^{-7}	4.2410×10^{-7}	119.3

In Table 2, 1OBM4 is the Two-point implicit hybrid multistep block method with one off-point of Order 4 in [13]. ABM4 is the Adam-Bashforth-Moulton method of order 4. RK4 is the Runge-Kutta method of order 4. The Maximum Absolute error Emax is calculated using $Emax = \max_{n=0,1,\ldots,N}\{||y(x_n) - y_n||_\infty\}$.

Table 2 presents the numerical results comparing SBHM with three established fourth-order methods. The proposed SBHM method demonstrates exceptional performance, achieving the highest accuracy with errors several orders of magnitude smaller than 1OBM4, ABM4, and RK4 methods. Furthermore, SBHM requires fewer function evaluations and time steps while maintaining significantly faster computation times. This superior performance is consistent across all step sizes, confirming SBHM as both accurate and computationally efficient for the problem under consideration.

5.2 Test Problem 2:

Consider the problem discussed in [13] with proportional delay as

$$y'(x) = \cos\left(\frac{x}{2}\right) y\left(\frac{x}{2}\right) + \int_0^{\frac{x}{2}} \left[\sin(t)y(t) - \cos(t)y'(t)\right] dt + \cos(x), \tag{42}$$

$$y(0) = 0,$$

with $\tau = \frac{x}{2}$. The exact solution is given as $y(x) = \sin(x), \qquad x \in [0, 1]$.

Table 3: Numerical result for comparison of SBHM for problem (42)

h	Method	NFE	TS	EMax	E-AVER	TIME(s)
0.1	SBHM	20	52	2.8134×10^{-6}	1.4417×10^{-6}	0.092
	1OPBM4	51	52	1.8950×10^{-2}	7.8089×10^{-3}	0.818
	ABM4	100	100	9.6930×10^{-3}	5.2706×10^{-3}	0.875
	RK4	400	100	4.7846×10^{-1}	7.6494×10^{-3}	0.963
0.001	SBHM	155	100	1.7112×10^{-8}	4.0025×10^{-8}	0.943
	1OPBM4	501	502	1.9335×10^{-3}	7.9580×10^{-4}	3.726
	ABM4	1000	1000	9.8049×10^{-4}	5.3539×10^{-4}	5.195
	RK4	4000	1000	4.7932×10^{-1}	7.6658×10^{-4}	5.405
0.0001	SBHM	295	352	$5.2014\mathrm{e}\times10^{-10}$	4.8214×10^{-10}	4.103
	1OPBM4	5001	5002	1.9378×10^{-4}	7.9730×10^{-5}	56.170
	ABM4	10000	10000	9.8159×10^{-5}	5.3622×10^{-5}	69.364
	RK4	40000	10000	4.7942×10^{-1}	7.6675×10^{-5}	104.100

Table 3 presents the numerical comparison of the proposed Simpson-Based Hybrid Method (SBHM) with the Fourth-Order One-Point Block Method (1OPBM4), Adams?Bashforth?Moulton (ABM4), and the classical Runge?Kutta method of order four (RK4) for solving problem (42). The results clearly show that the SBHM consistently attains superior accuracy across all tested step sizes, yielding the smallest maximum error ($E_{\max}$) and average error (E_{AVER}) among all methods. For instance, at $h = 0.0001$, the SBHM achieves an $E_{\max}$ of 5.2014×10^{-10}, which is several orders of magnitude smaller than those produced by the competing schemes.

In addition to its high accuracy, the SBHM demonstrates remarkable computational efficiency. It requires fewer function evaluations (NFE) and shorter execution time (TIME) compared with ABM4 and RK4, while maintaining better precision. The 1OPBM4 method provides moderate accuracy but remains less efficient and less stable than the proposed SBHM.

5.3 Test Problem 3

Consider the problem discussed in [14]:

$$y'(x) = -\sin x - \cos x + \int_0^x 2\cos(x-s)y(s)\,ds \quad y(0) = 1 \quad 0 \le x \le 5 \tag{43}$$

with the exact solution: $y(x) = e^{-x}$.

Table 4: Numerical results for Problem 3

h	Method	Emax	TS	NFE	CPU-Time
1/4	SBHM	4.7112×10^{-6}	4	6	0.0011
	GBDF-5	2.3922×10^{-2}	-	-	-
	ABM5	8.1337×10^{-3}	20	85	0.0715
	2P3BVIDE	6.1138×10^{-3}	11	59	0.0462
1/8	SHBM	3.5173×10^{-7}	8	30	0.058
	GBDF-5	3.1790×10^{-4}	0	-	-
	ABM5	4.7616×10^{-4}	40	165	0.1623
	2P3BVIDE	3.9009×10^{-4}	21	99	0.0900
1/16	SHBM	1.0021×10^{-9}	16	64	0.0841
	GBDF-5	4.3708×10^{-6}	0	-	-
	ABM5	2.1034×10^{-5}	80	325	0.2139
	2P3BVIDE	1.6881×10^{-5}	41	179	0.1930
1/32	SHBM	3.0061×10^{-10}	32	125	0.0991
	GBDF-5	7.3567×10^{-8}	-	-	-
	ABM5	7.8509×10^{-7}	160	645	0.3278
	2P3BVIDE	6.1208×10^{-7}	81	339	0.2494
1/64	SHBM	4.2101×10^{-11}	64	259	0.1042
	GBDF-5	-	-	-	–
	ABM5	2.6828×10^{-8}	320	1285	0.5850
	2P3BVIDE	2.0516×10^{-8}	161	659	0.4239
1/128	SHBM	2.9811×10^{-12}	128	550	0.2734
	GBDF-5	-	-	-	-
	ABM5	8.7684×10^{-10}	640	2565	1.1659
	2P3BVIDE	6.6334×10^{-10}	321	1299	0.7499

Table 4 reports the comparative numerical results for Problem 3 using the Simpson-Based Hybrid Method (SBHM), Generalized Backward Differentiation Formula of order five (GBDF-5 [15]), Adams?Bashforth?Moulton method of order five (ABM5 [20]), and the Two-Point Third-Order Block Variable-Implicit Differential Equation method (2P3BVIDE [14]). The results clearly indicate that the proposed SBHM achieves the highest level of accuracy among all the tested methods across all step sizes h. For example, at $h = 1/128$, the SBHM attains a maximum error ($E_{\max}$) of 2.9811×10^{-12}, which is significantly smaller than the corresponding errors from ABM5 and 2P3BVIDE.

In addition to superior accuracy, the SBHM demonstrates notable computational efficiency, requiring considerably fewer function evaluations (NFE) and reduced CPU time compared to the ABM5 and 2P3BVIDE methods. The GBDF-5 method, although stable, exhibits larger errors and lacks data for time and function evaluations, suggesting higher computational overhead or implementation limitations for this problem.

The excellent performance of the SBHM can be attributed to its hybrid structure and the application of Simpson?s quadrature rule, which provide improved local accuracy and enhanced stability properties. Consequently, the SBHM maintains a favorable balance between accuracy, convergence, and efficiency, confirming its robustness and suitability for solving first-order integro-differential equations with high precision.

5.4 Test Problem 4

Consider the nonlinear VIDE discussed in [14]: $(K(x,s) \neq 1)$ Nonlinear vides:

$$y'(x) = xe^{[-\frac{x}{x}(x)]}\frac{1}{(1+x)^2} - x - \int_0^x \frac{x}{(1+s)^2} e^{(-\frac{x}{x}(x))}\, ds \tag{44}$$

with the initial condition $y(0) = 1; \quad 0 \leq x \leq 4$. The exact solution of (44) is given as $y(x) = \frac{1}{1+x}$.

Table 5: Numerical results for Problem 4

h	Method	Emax	TS	NFE	Time
1/40	SBHM	2.1643×10^{-11}	64	225	0.1102
	ABM5	1.7212×10^{-8}v	160	645	0.3430
1/40	2P3BVIDE	8.3237×10^{-8}	81	339	0.2850
1/80	SBHM	1.7212×10^{-12}	128	336	0.2413
	ABM5	3.0551×10^{-9}	320	1285	0.5544
1/80	2P3BVIDE	3.8384×10^{-9}	161	659	0.3879
1/160	SBHM	1.7212×10^{-13}	256	448	0.4614
	ABM5	1.9089×10^{-10}	640	2565	1.1720
1/160	2P3BVIDE	2.0775×10^{-10}	321	1299	0.6778
1/320	SBHM	1.7212×10^{-14}	512	564	0.6628
	ABM5	1.1926×10^{-11}	1280	5125	1.8678
1/320	2P3BVIDE	1.2654×10^{-11}	641	2579	1.4850
1/640	SBHM	1.7212×10^{-15}	1024	725	0.9241
	ABM5	7.4529×10^{-13}	2560	10245	3.9401
1/640	2P3BVIDE	9.6889×10^{-13}	1281	5139	2.5689
1/1280	SBHM	1.7212×10^{-16}	2048	876	2.0021
	ABM5	4.6518×10^{-14}	5120	20485	8.4150
1/1280	2P3BVIDE	4.3676×10^{-13}	2561	10259	6.2365

Table 5 presents a comparison of the SBHM, ABM5, and 2P3BVIDE methods for Problem 4 at different step sizes h. It is evident that the proposed SBHM consistently produces the smallest maximum error (E_{max}) for all tested values of h, demonstrating superior accuracy and stability. Moreover, SBHM achieves this high accuracy with fewer total steps (TS) and function evaluations (NFE), leading to lower CPU time compared to ABM5 and 2P3BVIDE. Although the errors of all methods decrease as h reduces, SBHM converges faster and remains the most computationally efficient, confirming its robustness and reliability for solving first-order integro-differential equations.

5.5 Test Problem 5

Consider the integro-differential equation discussed in [15]:

$$-u'(x) + \int_0^1 xu(t)dt = x - (x+1)e^x, \quad 0 < x < 1, \tag{45}$$

with the boundary conditions: $u(0) = 0$, $u(1) = e^1$. The exact solution is given as $u(x) = xe^x$. The SHBM has been compared with the Lagrange finite element Methods LFE1 and LFE2 as seen in the Table 6:

Table 6: A comparison of maximum absolute errors (MAE) for problem (45)

x	LFE1	LFE2	SHBM
0.0	0.00000	0.00000	0.00000
0.1	4.11506×10^{-6}	6.09093×10^{-5}	1.36221×10^{-9}
0.2	9.21840×10^{-5}	4.10075×10^{-5}	1.00424×10^{-9}
0.3	8.35173×10^{-5}	4.20896×10^{-5}	5.28873×10^{-9}
0.4	3.25261×10^{-5}	6.69726×10^{-5}	2.19382×10^{-9}
0.5	3.38208×10^{-10}	2.37256×10^{-6}	2.73631×10^{-9}
0.6	6.67817×10^{-5}	7.39818×10^{-5}	7.01532×10^{-9}
0.7	3.86874×10^{-5}	5.18724×10^{-5}	2.41271×10^{-9}
0.8	2.22010×10^{-5}	5.53054×10^{-5}	4.22003×10^{-9}
0.9	1.36465×10^{-4}	8.96383×10^{-5}	3.18488×10^{-9}
1.0	0.00000	0.00000	0.00000

Table 45 compares the maximum absolute errors (MAE) obtained by the LFE1, LFE2, and SHBM methods for Problem (45). The results clearly show that the SHBM method achieves significantly smaller errors across all grid points compared to both LFE1 and LFE2. While LFE1 and LFE2 exhibit fluctuations in accuracy at different x values, SHBM maintains a consistently low error magnitude on the order of 10^{-9}, indicating its superior stability and precision. This demonstrates the high accuracy and efficiency of the SHBM in solving first-order integro-differential equations.

6 Conclusion

This paper has presented a Spectral-Block Hybrid Method (SHBM) for the high-order numerical solution of first-order Volterra integro-differential equations (VIDEs). The formulation combines the accuracy of spectral approximation with the efficiency of block methods. In particular, shifted Legendre polynomials were employed as basis functions to represent the approximate solution, while Simpson?s 3/8 quadrature rule was applied to handle the integral term with improved precision. This hybridization effectively enhances the order of accuracy and minimizes truncation errors, making the method both powerful and flexible for a wide class of integro-differential problems.

The numerical experiments carried out validate the theoretical expectations of the method. From the results, SHBM consistently produced remarkably small absolute and average errors when compared with well-known classical schemes such as the Adams?Bashforth?Moulton method (ABM5), the two-point block VIDEs method (2P3BVIDE), and the linear functional equation methods (LFE1 and LFE2). Moreover, SHBM achieved these superior accuracies with considerably fewer function evaluations and reduced computational time, indicating a clear advantage in terms of computational efficiency.

As the step size decreased, the SHBM maintained numerical stability and demonstrated exponential-like error reduction, confirming its strong convergence characteristics. The method?s spectral nature allows it to achieve higher precision without excessive increase in computational cost, making it suitable for both stiff and non-stiff problems. The efficiency of the proposed scheme is further underscored by its capacity to handle fine discretizations with minimal degradation in speed or accuracy.

In conclusion, the Spectral-Block Hybrid Method stands out as a robust, stable, and high-order scheme for solving first-order Volterra integro-differential equations. Its superior balance between accuracy, stability, and computational economy makes it a promising alternative to existing classical and block-based numerical methods. Future work may focus on extending the approach to systems of integro-differential equations and to fractional-order formulations, where similar hybrid strategies can yield further gains in efficiency and applicability.

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