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# On Bohr and Bohr-Rogosinski phenomena for subclasses of univalent functions

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**Abstract**

In this paper, we establish sharp Bohr and Bohr–Rogosinski-type inequalities for several subclasses of univalent functions in the unit disk, including starlike, convex, odd, and even functions, as well as Mocanu's class of analytic functions. Using refined coefficient estimates and growth bounds, we determine exact Bohr and Rogosinski radii for each class, with sharpness verified through extremal examples such as the Koebe function and its variants. We also derive the Bohr radii for the convolution analogues of univalent and convex functions. Our results refine and extend existing inequalities, provide new radii for symmetric subclasses, and contribute to a deeper understanding of the Bohr phenomenon in geometric function theory.

*Keywords and phrases*: Bohr inequalty; Bohr–Rogosinski inequality; Convex; Mocanu's class; Univalent
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## 1 Introduction and preliminaries

Let $f(z)$ be an analytic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We say that $f(z)$ satisfies the *Bohr phenomenon* if, for some $R \in (0,1)$, the inequality

$$\sum_{n=1}^{\infty} |a_n| r^n \le \operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big) \tag{1}$$

holds for all $|z| = r \le R$. Here, $\operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big)$ denotes the Euclidean distance between the point $f(0)$ and the boundary $\partial$ of the domain $f(\mathbb{D})$. The largest such $R \in (0,1)$ for which (1) holds is called the *Bohr radius*.
Denote by $\mathcal{B}$ the class of analytic functions in $\mathbb{D}$ satisfying $|f(z)| \le 1$. The classical Bohr inequality for functions $f \in \mathcal{B}$ is stated as follows:

**Theorem 1.1.** *Suppose* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *with* $f(z) \in \mathcal{B}$. *Then*

$$\sum_{n=1}^{\infty} |a_n| r^n \le dist\,(f(0), \partial\mathbb{D}) = 1 - |a_0| \quad for \quad |z| = r \le \frac{1}{3}, \tag{2}$$

*where the constant* $1/3$ *cannot be improved.*

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In 1914, Harald Bohr [7] first established inequality (2) for $r \le R = 1/6$. Subsequently, M. Riesz, I. Schur, and N. Wiener independently improved this result and proved that it holds for $r \le R = 1/3$ (see [1]). The constant $R = 1/3$ is therefore regarded in the literature as the *classical Bohr radius* for the class $\mathcal{B}$.

In recent years, the study of Bohr inequalities has emerged as a vibrant area of research in analytic function theory. The Bohr phenomenon has since seen significant extensions and refinements across a wide array of function classes and analytic settings. Notably, there has been considerable progress in establishing Bohr-type inequalities for various subclasses of univalent functions, including starlike, convex etc (see [1, 16]). These developments have also extended to harmonic mappings (see [5, 6, 10, 12, 15]), functions in several complex variables (see [3]), certain integral operators (see [13]) and in abstract settings (see [4]). Recent work has also introduced Bohr–Rogosinski-type inequalities, providing sharper bounds by combining classical coefficient inequalities with modulus constraints. Such results not only generalize the classical Bohr inequality but also contribute deeper insights into the structural properties of analytic and univalent functions in the unit disk. The notion of Rogosinski radius is described as follows:

If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, then for $N \ge 1$, we have $|S_N(z)| < 1$ in the disk $|z| < 1/2$ and the radius is sharp, where $S_N(z) = \sum_{n=0}^{N-1} a_n z^n$ denotes the partial sum of $f$ . In 2021, Kayumov et al. [13] introduced Bohr-Rogosinski sum, and define it as follows:

$$R_N^f(z) = |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n, \ \ |z| = r. \tag{3}$$

If $N = 1$ in (3), the result will be related to the classical Bohr sum (2) with $a_0 = f(0)$ replaced as $f(z)$. Thus, the Bohr-Rogosinski's radius is defined to be the largest number $\rho > 0$ such that $R_N^f(z) \le 1$ for $|z| \le \rho$ (see [1, 13]). For other classes of functions, such as univalent function, convex function, harmonic univalent and subordinate classes, Bohr-Rogosinski's radius is defined to be the largest number $\rho > 0$ such that (see [13, 15])

$$R_N^f(z) = |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \le |f(0)| + dist(f(0), \partial f(\mathbb{D})).$$

For a comprehensive account of improvements and generalizations of the Bohr inequality, we refer the reader to [1, 2, 5, 11, 12].

Now, let $\mathcal{S}$ denote class of univalent (one-one) functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{4}$$

normalized by $f(0) = f'(0) - 1 = 0$ in $\mathbb{D}$. It is well-known that the Bieberbach's conjecture (see [9] page 37) asserts that the coefficient of each $f \in \mathcal{S}$ satisfy $|a_n| \le n$ for $n \ge 2$ and the strict inequality holds for all $n$, unless $f$ is the Koebe function $k(z) = z/(1-z)^2$ or one of its rotations. This conjecture was eventually proved by De Branges [8]. The following theorem is well-known in the literature.

**Theorem 1.2.** *Let $f \in \mathcal{S}$, then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D}. \tag{5}$$

A function $f(z)$ is starlike if it is analytic in $\mathbb{D}$, $f(0) = 0$, $f'(0) \ne 0$ and satisfies $\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$. The subclass of $\mathcal{S}$ consisting of the starlike functions is denoted by $\mathcal{S}^*$, the same coefficient bound and growth theorem are applicable to starlike functions (see [9]). A univalent function is convex if $\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The subclass of $\mathcal{S}$ consisting of the convex functions is denoted by $\mathcal{C}$ (see [9]). For convex function, the following theorems hold:

**Theorem 1.3** ([9]). *If $f \in \mathcal{C}$, then $|a_n| \leq 1$ for $n \geq 1$. Strict inequality holds for $n$ unless $f$ is a rotation of the function defined by $f(z) = z/(1-z)$.*

**Theorem 1.4** ([9]). *Let $f \in \mathcal{C}$. Then*

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|}, \quad z \in \mathbb{D}. \tag{6}$$

Another interesting concept in geometric function theory is the study of odd functions that are univalent and convex in the unit disk. The subclass of odd univalent functions is denoted by $\mathcal{S}^2$, while the subclass of odd convex functions is denoted by $\mathcal{C}^2$. The following preliminary theorems are well-known

**Theorem 1.5** (see [17]). *Let $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{S}^2$. Then*

1. $|a_{2n-1}| \leq 1, \;\; n \geq 1.$
2. $\dfrac{|z|}{1+|z|^2} \leq |f(z)| \leq \dfrac{|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$

**Theorem 1.6** (see [17]). *Let $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{C}^2$. Then*

1. $|a_{2n-1}| \leq \dfrac{1}{2n-1}, \;\; n \geq 1.$
2. $tan^{-1}|z| \leq |f(z)| \leq \dfrac{1}{2} \ln \dfrac{1+|z|}{1-|z|}, \;\; z \in \mathbb{D}.$

Recently, Abu-Muhanna *et al.* [1] established the following Bohr-type inequality for functions subordinate to a univalent and convex functions.

**Theorem 1.7** ([1]). *Let $f$ and $g$ be analytic in $\mathbb{D}$, with $g$ univalent in $\mathbb{D}$, and suppose that $f \prec g$ (i.e., $f$ is subordinate to $g$). Then the inequality (2) holds for $r \leq R = 3 - 2\sqrt{2}$. The sharpness of this radius is demonstrated by the Koebe function $g(z) = z/(1-z)^2$. Moreover, if $f$ is convex in $\mathbb{D}$, then the Bohr radius improves to $R = 1/3$.*

Building upon the results of Abu-Muhanna *et al.* [1], who employed the subordination principle to establish Bohr inequalities for univalent and convex functions, this paper aims to both refine and extend those results. In particular, we focus on deriving Bohr and Bohr–Rogosinski inequalities for the subclasses of odd and even univalent and convex functions. Additionally, we obtain Bohr inequalities for Mocanu's class of analytic functions, and finally, we establish the Bohr radii for the convolution analogue of univalent functions.

## 2 Main results

Our main results are organized into three sections. Section 2.1 addresses the Bohr phenomenon for the classes of univalent and convex functions. Section 3 establishes the Bohr phenomenon for a Mocanu-type class of analytic functions. Section 4 is devoted to the Bohr radius for the convolution (Hadamard) analogue of univalent and convex functions.

### 2.1 Bohr and Bohr-Rogosinski inequality for the classes $\mathcal{S}$ and $\mathcal{C}$

**Theorem 2.1.** *Let $f \in \mathcal{S}$, then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq dist(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R = 3 - 2\sqrt{2}, \tag{7}$$

*where the number $R = 3 - 2\sqrt{2}$ is best possible. Moreover,*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq 1 \quad for \quad r \leq \frac{3 - \sqrt{5}}{2}. \tag{8}$$

*The constant $(3 - \sqrt{5})/2$ is best possible.*

*Proof.* From (5), we have $|f(z)| \geq \dfrac{|z|}{(1 + |z|)^2}$. Hence

$$dist\,(f(0), \partial f(\mathbb{D})) = \lim_{|z| \to 1} ||f(z)| - |f(0)|| \geq \lim_{|z| \to 1} \frac{|z|}{(1 + |z|)^2} = \frac{1}{4}. \tag{9}$$

Since $f \in \mathcal{S}$, then $|a_n| \leq n$, $n \geq 2$. Thus, a sufficient condition for (7) to hold in $\mathbb{D}$ is

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + \sum_{n=2}^{\infty} n r^n = \frac{r}{(1 - r)^2} \leq \frac{1}{4}. \tag{10}$$

Solving the inequality $r/(1 - r)^2 \leq 1/4$ for $r \in (0, 1)$, we have $r \leq 3 - 2\sqrt{2}$. To complete the proof, it remains to show that the radius $R = 3 - 2\sqrt{2}$ is sharp. Consider the Koebe function

$$k(z) = \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D}.$$

It is straightforward to verify that $|k(-r)| = \dfrac{r}{(1 + r)^2}$ and $|a_n| = n$ for $n \geq 1$. Therefore,

$$\mathrm{dist}(k(0), \partial k(\mathbb{D})) = \lim_{|z| \to 1} ||k(-r)| - |k(0)|| = \lim_{r \to 1} \frac{r}{(1 + r)^2} = \frac{1}{4}.$$

Furthermore, define

$$\phi(r) := r + \sum_{n=2}^{\infty} |a_n| r^n = r + \sum_{n=2}^{\infty} n r^n = \frac{r}{(1 - r)^2}.$$

For $r = R = 3 - 2\sqrt{2}$,

$$\phi(R) = \frac{R}{(1 - R)^2} = \frac{1}{4} = \mathrm{dist}(k(0), \partial k(\mathbb{D})).$$

Hence, the equality is satisfied. Differentiating $\phi$, we get

$$\phi'(r) = \frac{1 + r}{(1 - r)^3}.$$

Clearly, $\phi'(r) > 0$ for all $r \in (0, 1)$, implying that $\phi(r)$ is an increasing function. Since $\phi(r)$ is increasing, it follows that for all $r > R$, $\phi(r) > \phi(R) = 1/4$. Thus, the inequality (7) fails beyond $R = 3 - 2\sqrt{2}$. This establishes the sharpness of the radius $R = 3 - 2\sqrt{2}$ and completes the proof of the first part of the theorem. For the second part, the constant $\frac{1}{4}$ in (10) is replaced by 1, and upon evaluation, the desired result follows.
We will now state and prove the corresponding Bohr-Rogosinski inequality for Theorem 2.1.

**Theorem 2.2.** *Suppose $f \in \mathcal{S}$, then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq dist(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R_{s,1}^N, \tag{11}$$

*where $R_{s,1}^N$ is the minimum positive root of $r^2 - 6r + 4r^N(Nr - N - r) + 1 = 0$. Moreover,*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \quad for \quad r \leq R_{s,2}^N, \tag{12}$$

*and $R_{s,2}^N$ is the minimum positive root $r^2 - 3r + r^N(Nr - N - r) + 1 = 0$. The constants $R_{s,1}^N$ and $R_{s,2}^N$ cannot be improved.*

*Proof.* Let $z = re^{i\theta}$, from (5), $|f(z)| \leq r/(1-r)^2$. It follows from (9), (11) that

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq \frac{r}{(1-r)^2} + \sum_{n=N}^{\infty} nr^n \leq \frac{1}{4}. \tag{13}$$

Evaluating (13) yields

$$\frac{r - r^N(Nr - N - r)}{(1-r)^2} \leq \frac{1}{4},$$

and simplifying gives $r^2 - 6r + 4r^N(Nr - N - r) + 1 \geq 0$. This holds for $r \in (0,1)$ if $r \leq R_{s,1}^N$, where $R_{s,1}^N$ is the minimum positive root of the equation $r^2 - 6r + 4r^N(Nr - N - r) + 1 = 0$. To demonstrate the sharpness of the constant $R_{s,1}^N$, we consider the Koebe function $f(z) = z/(1-z)^2$ and follow the same procedure adopted in Theorem 2.1, so, we omit the proof. To proof the second part of the theorem, $1/4$ in (13) will be replaced by 1, and adopting the style of proof of the first part, the result follows.

| $N$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $R_{s,1}^N$ | 0.1010 | 0.1444 | 0.1634 | 0.1695 | 0.1711 |

Table 1: Computed values of Bohr radii $R_{s,1}^N$ for selected $N$-values.

**Remark 2.3.** Theorems 2.1 and 2.2 also hold for starlike function, since it has the same coefficient estimate and growth theorem with the univalent function.

**Theorem 2.4.** *Suppose that $\sum_{n=1}^{\infty} a_n z^n$ with $f \in \mathcal{C}$. Then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq dist(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq \frac{1}{3}, \tag{14}$$

*where the number $1/3$ is best possible. Moreover,*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq 1 \quad for \quad r \leq \frac{1}{2}, \tag{15}$$

*The constant $1/2$ cannot be improved.*

*Proof.* Using the Theorem 1.4, we have $|f(z)| \geq \dfrac{|z|}{1+|z|}$.

$$\therefore dist(f(0), \partial f(\mathbb{D})) = \lim_{|z| \to 1} \big| |f(z)| - |f(0)| \big| \geq \lim_{|z| \to 1} \frac{|z|}{1+|z|} = \frac{1}{2}. \tag{16}$$

Since $f \in \mathcal{C}$, by Theorem 1.3, we have $|a_n| \leq 1$ for all $n \geq 1$. Then, from (16), inequality (14) becomes

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + \sum_{n=2}^{\infty} r^n = \frac{r}{1-r} \leq \frac{1}{2}, \tag{17}$$

which holds if $r \leq 1/3$. To prove that the radius $1/3$ is sharp, consider the function

$$f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \qquad z \in \mathbb{D}.$$

Clearly, $|f(-r)| = \dfrac{r}{1+r}$ and $|a_n| = 1$ for all $n \geq 1$. Therefore,

$$\operatorname{dist}(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} \big| |f(-r)| - |f(0)| \big| = \lim_{r \to 1} \frac{r}{1+r} = \frac{1}{2}.$$

Hence, the Bohr sum (14) becomes

$$r + \sum_{n=2}^{\infty} |a_n| r^n = r + \sum_{n=2}^{\infty} r^n.$$

Now, let

$$P(r) = \frac{r}{1-r}, \qquad \Rightarrow \qquad P'(r) = \frac{1}{(1-r)^2}.$$

It is clear that $P'(r) > 0$ for all $r \in (0,1)$, which implies that $P(r)$ is an increasing function.

At $r = 1/3$, we have

$$P(1/3) = \frac{1/3}{1-1/3} = \frac{1}{2} = \operatorname{dist}(f(0), \partial f(\mathbb{D})),$$

which shows that the equality is attained at $1/3$. Since $P(r)$ is an increasing function, then for $r > 1/3$, we get $P(r) > P(1/3) = 1/2$, and thus $\dfrac{r}{1-r} > \dfrac{1}{2}$. Therefore, the inequality fails for $r > 1/3$, showing that the radius $1/3$ is best possible. This completes the proof of the first part of the theorem. For the second part, replacing $1/2$ in (17) by 1 yields $r \leq 1/2$. Hence, the proof of Theorem 2.4 is complete.

We will also state and prove the corresponding Bohr-Rogosinski inequality for Theorem 2.4.

**Theorem 2.5.** *Let $f \in \mathcal{C}$, then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq dist(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R_{c,1}^N, \tag{18}$$

*where $R_{c,1}^N$ is the minimum positive root of $2r^N + 3r - 1 = 0$. Moreover,*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \quad for \quad r \leq R_{c,2}^N, \tag{19}$$

*and $R_{c,2}^N$ is the minimum positive root $r^N + 2r - 1 = 0$. The constants $R_{c,1}^N$ and $R_{c,2}^N$ cannot be improved.*

*Proof.* Since $f \in \mathcal{C}$, it follows from Theorem 1.4 that, $|f(z)| \leq \dfrac{r}{1-r}$, where $z = re^{i\theta}$. Using (16), (18) becomes

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq \frac{r}{1-r} + \sum_{n=N}^{\infty} r^n \leq \frac{1}{2}. \tag{20}$$

Evaluating (20), we get $2r^N + 3r - 1 \leq 0$. This holds for $r \in (0,1)$ if $r \leq R^N_{c,1}$, where $R^N_{c,1}$ is the minimum positive root of the equation $2r^N + 3r - 1 = 0$. The sharpness of the radius $R^N_{c,1}$ can be proved by considering the function $f(z) = z/(1-z)$ and follow the same procedure adopted in Theorem 2.4. The proof the second part of the theorem follows from (20) by replacing $1/2$ on the R.H.S. by 1. This completes the proof of Theorem 2.5.

| $N$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $R^N_{s,1}$ | 0.2500 | 0.3028 | 0.3222 | 0.3294 | 0.3320 |

Table 2: Computed values of Bohr radii $R^N_{c,1}$ for selected $N$-values.

**Theorem 2.6.** *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1}$ *is an odd function with* $f \in \mathcal{S}^2$. *Then*

$$r + \sum_{n=2}^{\infty} |a_{2n-1}|r^{2n-1} \leq dist\,(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq \sqrt{2} - 1, \tag{21}$$

*where the number* $\sqrt{2} - 1$ *is best possible. Moreover,*

$$r + \sum_{n=2}^{\infty} |a_{2n-1}|r^{2n-1} \leq 1 \quad for \quad r \leq \frac{1}{2}(\sqrt{5} - 1). \tag{22}$$

*The constant* $(\sqrt{5} - 1)/2$ *is best possible.*

*Proof.* Let $f \in \mathcal{S}^2$. By Theorem 1.5, $|a_{2n+1}| \leq 1$, $n \geq 1$. Therefore

$$r + \sum_{n=2}^{\infty} |a_{2n-1}|r^{2n-1} \leq r + \sum_{n=2}^{\infty} r^{2n-1} = \frac{r}{1-r^2}. \tag{23}$$

Lower bound Theorem 1.5 (2) gives $|f(z)| \geq \dfrac{|z|}{1+|z|^2}$, so that

$$dist\,(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} ||f(z)| - |f(0)|| \geq \lim_{|z|\to 1} \frac{|z|}{1+|z|^2} = \frac{1}{2}. \tag{24}$$

In order for (21) to hold in $\mathbb{D}$, equations (23) and (24) reduce to the inequality

$$\frac{r}{1-r^2} \leq \frac{1}{2}. \tag{25}$$

Inequality (25) holds only if $r \leq \sqrt{2} - 1$ for $r \in (0,1)$. We now proceed to show that the radius $\sqrt{2} - 1$ is sharp. To this end, consider the function

$$f(z) = \frac{z}{1-z^2} = z + \sum_{n=2}^{\infty} z^{2n-1}, \quad z \in \mathbb{D}.$$

For this function, we observe that

$$|f(ir)| = \frac{r}{1+r^2} \quad \text{and} \quad |a_{2n-1}| = 1, \; n \geq 1.$$

Hence,

$$\text{dist}(f(0), \partial f(\mathbb{D})) = \lim_{r\to 1} \big||f(ir)| - |f(0)|\big| = \lim_{r\to 1} \frac{r}{1+r^2} = \frac{1}{2}.$$

Therefore,

$$r + \sum_{n=2}^{\infty} |a_{2n-1}|r^{2n-1} = \frac{1}{r}\sum_{n=1}^{\infty} r^{2n} = \frac{r}{1-r^2} =: H(r).$$

If we require that $H(r) \leq \mathrm{dist}(f(0), \partial f(\mathbb{D}))$, then $r/(1-r^2) \leq 1/2$, which simplifies to

$$r \leq \sqrt{2}-1 \quad \text{or} \quad r \geq -(\sqrt{2}+1).$$

Since $r \in (0,1)$, the inequality holds precisely for $r \leq \sqrt{2}-1$. Furthermore, it is straightforward to verify that $H'(r) > 0$, implying that $H(r)$ is an increasing function. For $r = \sqrt{2}-1$, we have $H(\sqrt{2}-1) = \frac{1}{2}$, and for $r > \sqrt{2}-1$, it follows that $H(r) > \frac{1}{2}$. This confirms that $\sqrt{2}-1$ is indeed the largest possible Bohr radius for which inequality (21) holds. Finally, if in (25) the constant $\frac{1}{2}$ is replaced by 1, the sharp Bohr radius becomes $(\sqrt{5}-1)/2$. This completes the proof of Theorem 2.6.

**Theorem 2.7.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$ with $f(z) \in \mathcal{S}^2$. Then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_{2n-1}| r^{2n-1} \leq dist\,(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R_N, \tag{26}$$

*where $R_N$ is the positive real root of $2r^{2N-1} + r^2 + 2r = 1$ Moreover,*

$$|f(z)| + \sum_{n=N}^{\infty} |a_{2n-1}| r^{2n-1} \leq 1 \quad for \quad r \leq K_N. \tag{27}$$

*The number $K_N$ is the positive real root of $r^{2N-1} + r^2 + r = 1$.*

*Proof.* Let $z = re^{i\theta}$, the upper bound of Theorem 1.5 (2) gives $|f(z)| \leq r/(1-r^2)$. Hence

$$\begin{aligned} |f(z)| + \sum_{n=N}^{\infty} |a_{2n-1}| r^{2n-1} &\leq \frac{r}{1-r^2} + \sum_{n=N}^{\infty} r^{2n-1} \\ &= \frac{r}{1-r^2} + \frac{r^{2N-1}}{1-r^2}. \end{aligned} \tag{28}$$

Using (24), equation (26) holds in $\mathbb{D}$ if

$$\frac{r + r^{2N-1}}{1-r^2} \leq \frac{1}{2},$$

which is true for $r \in (0,1)$ only if $r \leq R_N$, where $R_N$ is given by (27). The proof of the second part of Theorem 2.7 follows directly from (28) with $1/2$ replaced by 1. This completes the proof of Theorem 2.7. The sharpness of radius $R_N$ can be shown can considering the function $f(z) = z/(1-z^2)$ and follow the same style of proof in Theorem 2.6.

| $N$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $R_N$ | 0.2361 | 0.3761 | 0.4064 | 0.4128 | 0.4140 |

Table 3: Computed values of Bohr radii $R_N$ for selected $N$-values.

**Theorem 2.8.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{C}^2$. Then*

$$r + \sum_{n=2}^{\infty} |a_{2n-1}| r^{2n-1} \leq dist\,(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq 0.6557..., \tag{29}$$

*where the constant 0.6557... is best possible. Moreover,*

$$r + \sum_{n=2}^{\infty} |a_{2n-1}| r^{2n-1} \leq 1 \quad for \quad r \leq 0.7615. \tag{30}$$

*The constant 0.7615... is is best possible.*

*Proof.* Let $f \in \mathcal{C}^2$. By Theorem 1.6, $|a_{2n-1}| \leq 1/(2n-1)$, $n \geq 1$. Hence

$$r + \sum_{n=2}^{\infty} |a_{2n-1}| r^{2n-1} \leq \sum_{n=1}^{\infty} \frac{1}{2n-1} r^{2n-1} = \frac{1}{2} \ln \frac{1+r}{1-r}. \tag{31}$$

Applying the lower bound Theorem 1.6 (2), we obtain

$$dist\,(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} |\,|f(z)| - |f(0)|\,| \geq \lim_{|z|\to 1} tan^{-1}|z| = \frac{\pi}{4}. \tag{32}$$

Combining (31) and (32), (29) becomes

$$\frac{1}{2} \ln \frac{1+r}{1-r} \leq \frac{\pi}{4}, \tag{33}$$

and this is possible for $r \in (0,1)$ only if $r \leq 0.6557....$ Specifically, solving $(1/2)\ln((1+r)/(1-r)) = \pi/4$ yields $r = \tanh(\pi/4) \approx 0.6557$. We now proceed to prove that the radius $0.6557\ldots$ is sharp. Consider the function defined by

$$f(z) = \frac{1}{2} \ln \frac{1+z}{1-z} = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1}, \quad z \in \mathbb{D}.$$

It follows that

$$|f(ir)| = \tan^{-1} r, \quad |a_{2n-1}| = \frac{1}{2n-1}, \quad n \geq 1.$$

Hence,

$$\operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big) = \lim_{|z|\to 1} \big|\,|f(ir)| - |f(0)|\,\big| = \lim_{r\to 1^-} \tan^{-1} r = \frac{\pi}{4}.$$

On the other hand, we have

$$\sum_{n=1}^{\infty} |a_{2n-1}|\, r^{2n-1} = \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} = \frac{1}{2} \ln \frac{1+r}{1-r} = \operatorname{arctanh} r.$$

Therefore, the Bohr-type inequality

$$\sum_{n=1}^{\infty} |a_{2n-1}|\, r^{2n-1} \leq \operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big)$$

reduces for this extremal function to arctanh$r \leq \pi/4$. Equality holds when

$$r = \tanh\left(\frac{\pi}{4}\right) = 0.6557942026\ldots \approx 0.6557\ldots.$$

For $r > \tanh(\pi/4)$, the inequality fails as arctanh$r > \pi/4$. Hence, the Bohr radius $R = \tanh(\pi/4) = 0.6557\ldots$ is sharp for the class of odd convex functions. This completes the proof of the first part of the theorem. The proof of the second part follows the same style of proof of the first part. Thus, the proof of Theorem 2.8 is complete.

**Theorem 2.9.** *Suppose that $f(z) = z^2 + \sum_{n=2}^{\infty} a_{2n} z^{2n}$ is an even univalent function. Then*

$$\sum_{n=1}^{\infty} |a_{2n}| r^{2n} \leq dist(f(0),\ \partial f(\mathbb{D})), \quad for \quad r \leq \sqrt{2} - 1, \tag{34}$$

*where the constant $\sqrt{2}-1$ cannot be improved. Moreover, if $f$ is an even convex function, then inequality (34) holds for $r \leq 1/\sqrt{3}$.*

*Proof.* Since $f$ is an even univalent function, then adopting the lower bound of (5), we have

$$|f(z)| \geq \frac{|z|^2}{(1+|z|^2)^2},$$

so that

$$dist(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} |\,|f(z)| - |f(0)|\,| \geq \lim_{|z|\to 1} \frac{|z|^2}{(1+|z|^2)^2} = \frac{1}{4}. \tag{35}$$

By De-Brange's theorem, $|a_{2n}| \leq n$, $n \geq 1$. Hence

$$\sum_{n=1}^{\infty} |a_{2n}| r^{2n} \leq \sum_{n=1}^{\infty} n z^{2n} = \frac{r^2}{(1-r^2)^2}. \tag{36}$$

Applying (35) and (36), inequality (34) holds in $\mathbb{D}$ if

$$\frac{r^2}{(1-r^2)^2} \leq \frac{1}{4}.$$

Solving for $r \in (0,1)$, we get $\sqrt{2}-1$. To complete the proof we show that the constant $\sqrt{2}-1$ is sharp. Consider the even Koebe-like function

$$f(z) = \frac{z^2}{(1-z^2)^2} = z^2 + \sum_{n=2}^{\infty} n z^{2n}, \qquad z \in \mathbb{D}.$$

Clearly $f$ is an even function and each coefficient satisfies $|a_{2n}| = n$ for $n \geq 1$. For $r \in (0,1)$ we compute the coefficient series

$$\sum_{n=1}^{\infty} |a_{2n}|\, r^{2n} = \sum_{n=1}^{\infty} n r^{2n} = \frac{r^2}{(1-r^2)^2} := Q(r),$$

As observed, $|f(ir)| = r^2/(1+r^2)^2$, so that

$$\operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big) = \lim_{r\to 1^-} \frac{r^2}{(1+r^2)^2} = \frac{1}{4}.$$

Hence, for this extremal function the Bohr-type inequality for even coefficients becomes

$$Q(r) = \frac{r^2}{(1-r^2)^2} \leq \frac{1}{4}.$$

The inequality holds in $(0,1)$ exactly when

$$r^2 \leq 3 - 2\sqrt{2} \qquad \therefore r \leq \sqrt{3-2\sqrt{2}} = \sqrt{2}-1.$$

Equality is attained at $r = \sqrt{2}-1$, and for any $r > \sqrt{2}-1$ the inequality fails as $Q(r) > 1/4$. Consequently the radius $\sqrt{2}-1$ is best possible (sharp) for the class of even univalent functions. This completes the proof of the first of the theorem.

Now, suppose that $f$ is an even convex function, then usng the lower bound of (6), we have

$$|f(z)| \geq \frac{|z|^2}{1+|z|^2}.$$

Therefore,

$$dist(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} |\,|f(z)| - |f(0)|\,| \geq \lim_{|z|\to 1} \frac{|z|^2}{1+|z|^2} = \frac{1}{2}. \tag{37}$$

Using Theorem 1.3 and (37), we have

$$\sum_{n=1}^{\infty}|a_{2n}|r^{2n} \leq \sum_{n=1}^{\infty} r^{2n} = \frac{r^2}{1-r^2} \leq \frac{1}{2}, \tag{38}$$

which holds in $(0,1)$ for $r \leq 1/\sqrt{3}$. To prove the sharpness, consider the function

$$f(z) = \frac{z^2}{1-z^2} = \sum_{n=1}^{\infty} z^{2n}, \qquad z \in \mathbb{D}.$$

It is easy to see that $a_{2n} = 1$ for all $n \geq 1$. Therefore

$$J(r) = \sum_{n=1}^{\infty}|a_{2n}|r^{2n} = \sum_{n=1}^{\infty} r^{2n} = \frac{r^2}{1-r^2}.$$

At $r = 1/\sqrt{3}$,

$$J(r) = J(1/\sqrt{3}) = \frac{(1/\sqrt{3})^2}{1-(1/\sqrt{3})^2} = \frac{1}{2} = \operatorname{dist}(f(0), \partial f(\mathbb{D})).$$

Thus, equality is attained at $r = 1/\sqrt{3}$. After few calculations, we find that $J'(r) > 0$ for all $r \in (0,1)$. Hence, at $r > 1/\sqrt{3}$, $J(r) > J(1/\sqrt{3}) = 1/2$, implying that inequality (34) does not exists beyond $1/\sqrt{3}$ for even convex function in $\mathbb{D}$. This completes the proof of Theorem 2.9.

**Theorem 2.10.** *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. *Then*

$$\sum_{n=1}^{\infty} \frac{|f^{(n)}(z)|}{n!} r^n \leq \operatorname{dist}\big(f(0),\, \partial f(\mathbb{D})\big), \qquad \textit{for } r \leq 0.1173\ldots, \tag{39}$$

*where the constant* $0.1173\ldots$ *is the best possible radius. Moreover, if* $f(z) \in \mathcal{C}$, *then* (39) *holds for* $r \leq 0.2192\ldots$. *The constant* $0.2192\ldots$ *is best possible.*

*Proof.* Let $r = |z|$.

**Case 1:** $f \in \mathcal{S}$. By the known derivative estimate for the class $\mathcal{S}$ (see [9], p. 70),

$$|f^{(n)}(z)| \leq n!\, \frac{n+r}{(1-r)^{n+2}}, \qquad n \geq 1.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|f^{(n)}(z)|}{n!} r^n &\leq \sum_{n=1}^{\infty} \frac{n+r}{(1-r)^{n+2}} r^n = \frac{1}{(1-r)^2}\sum_{n=1}^{\infty} n\, t^n + \frac{r}{(1-r)^2}\sum_{n=1}^{\infty} t^n, \qquad t = \frac{r}{1-r}\\ &= \frac{1}{(1-r)^2}\frac{t}{(1-t)^2} + \frac{1}{(1-r)^2}\frac{t}{1-t} = \frac{r(1-2r^2)}{(1-r)^2(1-2r)^2}.\end{aligned}$$

Since $f \in \mathcal{S}$, (9) gives $\operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big) \geq 1/4$. Thus a sufficient condition for (39) yields

$$\frac{r(1-2r^2)}{(1-r)^2(1-2r)^2} \leq \frac{1}{4},$$

which simplifies to the polynomial inequality $4r^4 - 4r^3 + 13r^2 - 10r + 1 \geq 0$. Solving for $r \in (0,1)$, we get $r \leq r_0 = 0.1173\ldots$, where $r_0$ is the smallest positive root of $4r^4 - 4r^3 + 13r^2 - 10r + 1 = 0$.

Hence the constant $0.1173\dots$ is best possible. The sharpness can be demonstrated by considering the Koebe function (or its appropriate rotation): beyond $r_0$ the function

$$\frac{r(1-2r^2)}{(1-r)^2(1-2r)^2}$$

exceeds $1/4$, so the inequality fails.

**Case 2:** $f \in \mathcal{C}$ (convex functions). For $f \in \mathcal{C}$ we have the sharper derivative estimate

$$|f^{(n)}(z)| \le \frac{n!}{(1-r)^{n+1}}, \qquad n \ge 1.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{|f^{(n)}(z)|}{n!} r^n \le \sum_{n=1}^{\infty} \frac{1}{(1-r)^{n+1}} r^n = \frac{r}{(1-r)(1-2r)}.$$

Since $f \in \mathcal{C}$, then by (16), we have $\operatorname{dist}\big(f(0), \partial f(\mathbb{D})\big) \ge 1/2$. Thus it suffices to solve

$$\frac{r}{(1-r)(1-2r)} \le \frac{1}{2}.$$

This holds for $r \in (0,1)$ if $r \le r_1 = (5-\sqrt{17})/4$. Hence (39) holds for every $r \le (5-\sqrt{17})/4$. To prove sharpness of $r_1$, consider the extremal convex function $f(z) = \dfrac{z}{1-z}$. The nth derivative of the geometric series gives

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad n \ge 1. \qquad \therefore \frac{|f^{(n)}(r)|}{n!} = \frac{1}{(1-r)^{n+1}}, \quad r \in (0,1).$$

and therefore the left-hand side of (39) for this function equals

$$G(r) := \sum_{n=1}^{\infty} \frac{1}{(1-r)^{n+1}} r^n = \frac{r}{(1-r)(1-2r)}, \qquad 0 < r < \frac{1}{2}.$$

After a straightforward computation, one finds that $G(r)$ is strictly increasing for $r \in (0, 1/2)$. The radius $r_1 = \frac{5-\sqrt{17}}{4}$ satisfies $G(r_1) = 1/2$. Hence, for any $r > r_1$ we have $G(r) > 1/2$, and consequently, the inequality (39) fails for the extremal function $f(z) = z/(1-z)$ when $r > r_1$. Therefore, the radius $r_1 = (5-\sqrt{17})/4$ is best possible.

## 3 Bohr Phenomenon for Mocanu's Class

In this section, we shall consider the following subclass of $\mathcal{S}$ defined by:

$$\mathcal{B}(M) := \{f \in \mathcal{A} : \ |zf''(z)| \le M, \quad z \in \mathbb{D}\}, \ \ M > 0 \tag{40}$$

where $\mathcal{A}$ is asummed to be the class of analytic functions. For $f \in \mathcal{B}(M)$, then $f$ is starlike if $0 < M \le 1$ and convex if $0 < M \le 1/2$ (see [16]). The class $\mathcal{B}(M)$ was first introduced by Mocanu [16] and later studied by some authors for several properties. Ghosh and Vasudevarao [10] studied the stable harmonic mapping relating to the class (40) in which their style of proof will be adopted to obtain the coefficient estimate and the growth theorem for the class $\mathcal{B}(M)$.

**Lemma 3.1.** *Let $f \in \mathcal{B}(M)$. Then $|a_n| \le \dfrac{M}{n(n-1)}$ for $n \ge 2$.*

*Proof.* Since $f \in \mathcal{B}(M)$, then $|zf''(z)| \leq M$. But $zf''(z)$ is analytic in $\mathbb{D}$. Hence, by Cauchy's integral formula, we get

$$n(n-1)\frac{f^{(n)}(0)}{n!} = n(n-1)a_n = \frac{1}{2\pi i}\int_c \frac{zf''(z)}{z^n}\,dz, \tag{41}$$

where $c$ is the circle $|z| = r$, $r \in (0,1)$ and $0 < \theta < 2\pi$, for $z = re^{i\theta}$. Thus, (41) yields

$$n(n-1)|a_n| = \frac{1}{2\pi}\int_0^{2\pi} \frac{|zf''(z)|}{r^n} r d\theta \leq \frac{M}{2\pi r^{n-1}}\int_0^{2\pi} d\theta = \frac{M}{r^{n-1}}. \tag{42}$$

Then the result follows as $r \to 1^-$.

**Lemma 3.2.** *Suppose that $f \in \mathcal{B}(M)$. Then*

$$|z| - \frac{M}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{M}{2}|z|^2. \tag{43}$$

*Proof.* From the fact that $|zf''(z)| \leq M$, there exists a Schwarz function $\varphi(z)$ in $\mathbb{D}$ such that $zf''(z) = \varphi(z)M$ with $|\varphi(z)| < 1$. We have

$$\int_0^z f''(z)dz = \int_0^z \frac{\varphi(z)M}{z}\,dz. \tag{44}$$

Since $f \in \mathcal{B}(M)$ is a normalized function with $f'(0) = 1$, the LHS integral of (44) yields $f'(z) - 1$. Also, by Schwarz lemma, $|\varphi(z)| \leq |z|$, and thus, (44) becomes

$$|f'(z) - 1| \leq \int_0^z \frac{|\varphi(z)|M}{|z|}\,dz \quad \text{or} \quad |f'(z) - 1| \leq M|z|, \tag{45}$$

Using the reverse triangle inequality ($\big|\,|a| - |b|\,\big| \leq |a - b|$), we have that

$$\big|\,|f'(z)| - |1|\,\big| \leq |f'(z) - 1| \leq M|z|.$$

Thus

$$|\,|f'(z)| - 1\,| \leq M|z| \quad \implies \quad -M|z| \leq |f'(z)| - 1 \leq M|z|,$$

which can be written as $1 - M|z| \leq |f'(z)| \leq 1 + M|z|$ (the distortion theorem), which . Now, from $|f'(z)| \leq 1 + M|z|$, we integrate along the radial segment from 0 to $z$. On letting $z = re^{i\theta}$, we have

$$|f(z)| = \left|\int_0^z f'(z)dz\right| \leq \int_0^{|z|} f'(re^{i\theta})dr \leq \int_0^z (1 + Mr)dr = |z| + \frac{M}{2}|z|^2,$$

which gives the upper part of the theorem. To obtained the lower part, we proceed from (45). Since $|f'(z) - 1| \leq M|z|$, we have $\text{Re}f'(z) \geq 1 - M|z|$. We know that $|f(re^{i\theta})| \geq \text{Re}(e^{-i\theta}f(re^{i\theta}))$, let $z = re^{i\theta}$ and integrating $\text{Re}(e^{-i\theta}f(re^{i\theta}))$ along the radial path, we have

$$\text{Re}(e^{-i\theta}f(re^{i\theta})) = \int_0^r \text{Re}f'(re^{i\theta})dr \geq \int_0^r (1 - Mr)dr = r - \frac{M}{2}r^2.$$

Hence inequality $|f(re^{i\theta})| \geq \text{Re}(e^{-i\theta}f(re^{i\theta}))$ yields

$$|f(re^{i\theta})| \geq r - \frac{M}{2}r^2 \quad \text{or} \quad |f(z)| \geq |z| - \frac{M}{2}|z|^2.$$

This completes the proof of Lemma 3.2.
We will now state and prove several Bohr phenomenon for the class of $f \in \mathcal{B}(M)$.

**Theorem 3.3.** *Let $f \in \mathcal{B}(M)$. Then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leq dist\,(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R_M, \tag{46}$$

*where $R_M$ is the positive root of the equation $2r + M\,(2r + 1 + 2(1-r)\ln(1-r)) = 2$. Moreover,*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq dist\,(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq R_M^N, \tag{47}$$

*and the constant $R_M^N$ is the positive root of the equation*

$$2r + M(1 + r^2) + 2Mr^N[\Phi(r, 1, N-1) - \Phi(r, 1, N)] = 2.$$

*Proof.* Since $f \in \mathcal{B}(M)$, then using the lower bound of (43), we have

$$dist\,(f(0), \partial f(\mathbb{D})) = \lim_{|z|\to 1} ||f(z)| - |f(0)|| \geq \lim_{|z|\to 1} \left(|z| - \frac{M}{2}|z|^2\right) = 1 - \frac{M}{2}. \tag{48}$$

Hence, applying Lemma 3.1 and (48) in (46), we have

$$\begin{aligned} r + \sum_{n=2}^{\infty} |a_n| r^n &\leq r + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} r^n \\ &= r + M\,(2r + (1-r)\ln(1-r)) \leq 1 - \frac{M}{2}. \end{aligned} \tag{49}$$

On simplifying (49), we obtain

$$2r + M\,(2r + 1 + 2(1-r)\ln(1-r)) - 2 \leq 0. \tag{50}$$

For $r \in (0,1)$, clearly, (50) is possible only if $r \leq R_M$ where $R_M$ is the positive root

$$2r + M\,(2r + 1 + 2(1-r)\ln(1-r)) = 2.$$

This completes the proof of (46).

**Proof of the second part of Theorem 3.3**

Let $z = re^{i\theta}$, it then follows the upper bound of (43) that $|f(z)| \leq r + \dfrac{M}{2} r^2$. Hence, applying (48) and Lemma 3.1 in (47) gives

$$\begin{aligned} |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n &\leq r + \frac{M}{2} r^2 + M \sum_{n=N}^{\infty} \frac{1}{n(n-1)} r^n \\ &= r + \frac{M}{2} r^2 + Mr^N\,[\Phi(r, 1, N-1) - \Phi(r, 1, N)] \\ &\leq 1 - \frac{M}{2}, \end{aligned} \tag{51}$$

where $\Phi(z, a, b)$ is the Lerch transcendent function defined by

$$\Phi(z, a, b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^a}.$$

Thus, simplifying (51) to get

$$2r + M(1 + r^2) + 2Mr^N\,[\Phi(r, 1, N-1) - \Phi(r, 1, N)] - 2 \leq 0.$$

The last inequality holds for $r \in (0,1)$ if $r \leq R_M^N$ where $R_M^N$ is defined as in (3.8). This completes proof of Theorem 3.3. Some values are taken for $M$ and $N$ in (46) and (47), we deduce various sharp Bohr radii $R_M$ and $R_M^N$ and are given in Tables 4 and 5 respectively.

Table 4: The value of Bohr radii $R_M$ for some values of $M$.

| M | 0.1 | 0.3 | 0.4 | 0.5 | 0.6 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|
| $R_M$ | 0.8861 | 0.7351 | 0.6758 | 0.6225 | 0.5738 | 0.4883 | 0.4441 | 0.4043 |

Table 5: The values of Bohr radii $R_M^N$ for some values of $M$ and $N$.

| M | 0.1 | 0.3 | 0.5 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $R_M^2$ | 0.8557 | 0.6839 | 0.5672 | 0.519 | 0.4347 | 0.3605 |
| $R_M^3$ | 0.8861 | 0.7351 | 0.6225 | 0.5738 | 0.4883 | 0.4043 |
| $R_M^4$ | 0.8959 | 0.7501 | 0.6370 | 0.5871 | 0.4958 | 0.4119 |
| $R_M^5$ | 0.9005 | 0.7562 | 0.6420 | 0.5914 | 0.4987 | 0.4137 |

**Theorem 3.4.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{B}(M)$. If $S_r$ denotes the area of the image of the subdisk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ under the mapping $f$, then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n + \frac{S_r}{\pi} \leq \operatorname{dist}(f(0), \partial f(\mathbb{D})) \quad for \quad r \leq K_M, \tag{52}$$

*where $K_M$ is the positive root of the equation*

$$2M^2 \left[r^2(Li_2(r^2) - 1) - (1 - r^2)\ln(1 - r^2)\right] + 2r^2 + 2r + M(2r + 1 + 2(1 - r)\ln(1 - r)) = 2. \tag{53}$$

*Proof.* It is well-known that $S_r = \int\int_{|z|<r} |f'(z)|^2 dxdy$, transforming to polar coordinate and with $f$ defined as in (4) we have

$$S_r = \int_0^{2\pi}\int_0^r \rho d\rho d\theta + \sum_{n=2}^{\infty} n^2|a_n|^2 \int_0^{2\pi}\int_0^r \rho^{2n-1} d\rho d\theta = \pi\left(r^2 + \sum_{n=2}^{\infty} n|a_n|^2 r^{2n}\right)$$

$$\therefore \frac{S_r}{\pi} = r^2 + \sum_{n=2}^{\infty} n|a_n|^2 r^{2n}. \tag{54}$$

Now, using Lemma 3.1, (48) and (54) in (52), we have

$$\begin{aligned} r + \sum_{n=2}^{\infty} |a_n| r^n + \frac{S_r}{\pi} &= r^2 + r + \sum_{n=2}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} n|a_n|^2 r^{2n} \\ &\leq r^2 + r + M\sum_{n=2}^{\infty} \frac{1}{n(n-1)} r^n + M^2 \sum_{n=2}^{\infty} \frac{1}{n(n-1)^2} r^{2n} \\ &= r^2 + r + M\left(r + (1-r)\ln(1-r)\right) \\ &\quad + M^2\left[r^2\left(\mathrm{Li}_2(r^2) - 1\right) - (1 - r^2)\ln(1 - r^2)\right] \\ &\leq 1 - \frac{M}{2}, \end{aligned} \tag{55}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm. Simplifying (55) gives

$$2r^2 + 2r + M(2r + 1 + 2(1 - r)\ln(1 - r)) + 2M^2\left[r^2\left(\mathrm{Li}_2(r^2) - 1\right) - (1 - r^2)\ln(1 - r^2)\right] - 2 \leq 0.$$

This inequality holds in $(0, 1)$ only if $r \leq K_M$, where $K_M$ is the root of the equation (53). To complete the proof, we need to show that (55) has root in $(0, 1)$. To show this, consider from (55).

$$f(r) = r^2 + r + M\sum_{n=2}^{\infty} \frac{1}{n(n-1)} r^n + M^2 \sum_{n=2}^{\infty} \frac{1}{n(n-1)^2} r^{2n} + \frac{M}{2} - 1. \tag{56}$$

After few computations, we find that

$$f(0) = \frac{M}{2} - 1 < 0 \quad \text{and} \quad f(1) = M^2\left(-1 + \frac{\pi^2}{6}\right) + \frac{3M}{2} + 1 > 0$$

for $0 < M \leq 1$. Clearly, $f(0).f(1) < 0$. Hence, by the intermediate value theorem, $f(r)$ has at least one root in $(0, 1)$. This completes the proof.

## 4 Convolution analogue of Bohr radius in classes $\mathcal{S}$ and $\mathcal{C}$

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be either univalent ($\mathcal{S}$) or convex ($\mathcal{C}$), and if $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is an analytic function in $\mathbb{D}$, the convolution (Hadamard product) operator is given (see [14]) as

$$H(z) = (f * g)(z) := \sum_{n=1}^{\infty} a_n b_n z^n, \qquad z \in \mathbb{D}. \tag{57}$$

In this section, we shall consider the Bohr radius for the convolution operator of the form (57), where $g(z)$ is a hypergeometric function defined by

$$g(z) = {}_2F_1(a; b; c; z) - 1 = z\, {}_2F_1(a; b; c; z) := \sum_{n=1}^{\infty} b_n z^n, \quad b_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!},$$

and $(k)_n = \dfrac{\Gamma(n+k)}{\Gamma(k)}$ is the well-known Pochhammer symbol. In particular, we shall only consider the case where

$$g(z) = {}_2F_1(\alpha; 1; 1; z) - 1 = z\, {}_2F_1(\alpha; 1; 1; z) = \frac{z}{(1-z)^{\alpha}}, \quad \alpha \geq 1. \tag{58}$$

It is easy to find that

$$b_n = \frac{(\alpha)_{n-1}}{(n-1)!} = \binom{\alpha+n-2}{n-1}. \tag{59}$$

**Theorem 4.1.** *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is defined as in (58). Then,*

$$\sum_{n=1}^{\infty} |a_n||b_n| r^n \leq dist(f(0), \partial f(\mathbb{D})), \quad r \leq R_{\alpha} \tag{60}$$

*where $R_{\alpha}$ is the minimal positive root of* $(1-r)^{\alpha+1} - 4r + (\alpha-1)4r^2 = 0$.

*Proof.* Let $f \in \mathcal{S}$. Then by de Brange's theorem, $|a_n| \leq n$ for $n \geq 1$. Thus, using (59), we obtain

$$\sum_{n=1}^{\infty} |a_n||b_n| r^n \leq \sum_{n=1}^{\infty} \binom{\alpha+n-2}{n-1} n r^n = \frac{r + (\alpha-1)r^2}{(1-r)^{\alpha+1}}. \tag{61}$$

Combining (61) and (9), we have

$$\frac{r + (\alpha-1)r^2}{(1-r)^{\alpha+1}} \leq \frac{1}{4}. \tag{62}$$

Solving (62) for $r \in (0, 1)$, we get $r \leq R_{\alpha}$, where $R_{\alpha}$ is the minimal positive root of the equation $4(r + (\alpha-1)r^2) - (1-r)^{\alpha+1} = 0$. The sharpness of the radius $R_{\alpha}$ can be shown by considering the Koebe function. This completes the proof.

Table 6: The values of Bohr radii $R_\alpha$ for the convolution $H(z) = f * g$, with $\alpha = 1, 2, 3, 4, 5$.

| $\alpha$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $R_\alpha$ | 0.1716 | 0.13968 | 0.12052 | 0.10726 | 0.09734 |

**Remark 4.2.** Setting $\alpha = 1$ in Theorem 4.1, the Bohr radius for the convolution $H(z) = f * g$ is given by $R_1 = 3 - 2\sqrt{2}$, which coincides with the classical result in Theorem 2.1. Moreover, Table 7 shows that as $\alpha \to \infty$, the Bohr radius $R$ approaches zero.

**Theorem 4.3.** *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$ *and* $g(z) = \sum_{n=1}^{\infty} b_n z^n$ *be given as in (58). Then*

$$\sum_{n=1}^{\infty} |a_n||b_n| r^n \leq dist(f(0), \partial f(\mathbb{D})), \quad r \leq P_\alpha \tag{63}$$

*where* $P_\alpha$ *is the minimal positive root of* $(1-r)^\alpha - 2r = 0$.

*Proof.* Let $f \in \mathcal{S}$, Theorem 1.3 yields $|a_n| \leq 1,\ n \geq 1$. Using (59), we have

$$\sum_{n=1}^{\infty} |a_n||b_n| r^n \leq \sum_{n=1}^{\infty} \binom{\alpha + n - 2}{n - 1} r^n = \frac{r}{(1-r)^\alpha}. \tag{64}$$

on applying (17) and (64), we get

$$\frac{r}{(1-r)^\alpha} \leq \frac{1}{2}, \tag{65}$$

which holds for $r \in (0, 1)$ only if $r \leq P_\alpha$, where $P_\alpha$ is the minimal positive root of $(1-r)^\alpha = 2r$. The sharpness of the radius $P_\alpha$ can be demonstrated by considering $f(z) = z/(1-z)$. This completes the proof.

Table 7: The values of Bohr radii $P_\alpha$ for the convolution $H(z) = f * g$, with $\alpha = 1, 2, 3, 4, 5$.

| $\alpha$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P_\alpha$ | 0.13333 | 0.26795 | 0.22908 | 0.20238 | 0.18283 |

**Remark 4.4.** Setting $\alpha = 1$ in Theorem 4.3, we derive $P_1 = 1/3$ of the classical case Theorem 2.4.

## 5 Conclusion

We have derived exact Bohr and Bohr–Rogosinski-type inequalities for key subclasses of univalent functions, covering starlike, convex, odd, even, and Mocanu's class of analytic functions. For each class, the Bohr and Rogosinski radii were obtained in closed form and shown to be sharp via extremal functions. These findings generalize several known results and offer a unified framework for studying the Bohr phenomenon across diverse subclasses of analytic functions. Future research may extend these techniques to higher-dimensional complex spaces, harmonic mappings, and operator-induced function classes.

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