International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 8 Issue 2, 2025

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

On a version of strong convergence theorems for the Jungck Picard-Multistep hybrid iterative scheme for a general class of functions

S.A. Raji*

Abstract

This paper establishes a version of strong convergence results for the Jungck Picard-Multistep hybrid iterative scheme. The results improve, generalize and extend some of the known ones in literature especially those of Olatinwo and Imoru (2008), Berinde (2001), and Akewe and Olaluwa (2017).

Keywords and phrases: strong convergence,convergence theorems; general class of function; Jungck Picard-multistep hybrid iteration; iterative scheme

1 Introduction

Suppose (X, d) be a complete metric space, let $T : X \to X$ be a selfmap of X. The set $F_T = \{p \in X : Tp = p\}$ is the set of fixed points of T $\in$ X. **Definition 1** [20]: Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving the operator T, this implies that,

$$f(T, x_n) = x_{n+1}, \quad n = 0, 1, 2 \ldots \tag{1.0}$$

where $x_0 \in X$ is the initial approximation and f is some function.
Suppose in (1.0),

$$f(T, x_n) = Tx_n \qquad (1.1)$$
$$x_{n+1} = Tx_n, \quad n = 0, 1, 2 \ldots \tag{1.2}$$

Eqn. (1.2), is the Picard iterative scheme.
Definition 2 [12]: For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck iteration defined by,

$$Sx_{n+1} = Tx_n, \quad n \geq 0 \tag{1.3}$$

Equation (1.3), is the Jungck iterative scheme.
Definition 3 [8] :
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Mann iteration defined by,

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \quad n = 0, 1, 2 \ldots \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is the sequence of real numbers in $[0, 1]$, then Eqn (1.4) is the Jungck-Mann Iterative scheme.
Definition 4[13]:
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Picard-Mann hybrid iteration defined by,

$$\begin{aligned} Sx_{n+1} &= Ty_n \\ Sy_n &= (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \quad n = 0, 1, 2 \ldots \end{aligned} \tag{1.5}$$

*Department of Mathematics, Lagos State University of Education (LASUED), Lagos, Nigeria; Email: sunwaraj123@yahoo.com

where $\{\alpha_n\}_{n=0}^{\infty}$ is the sequence of real numbers in $[0,1]$, then Eqn (1.5) is the Jungck Picard-Mann hybrid Iterative scheme.

Definition 5 [8]:
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration defined by

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n Tx_n, \quad n = 0,1,2\ldots \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1], then Eqn. (1.6) is the Jungck Ishikawa iterative scheme.

Definition 6[13]:
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Picard-Ishikawa hybrid iteration defined by

$$\begin{aligned} Sx_{n+1} &= Ty_n \\ Sy_n &= (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n Tx_n, \quad n = 0,1,2\ldots \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$, then Eqn. (1.7) is the Jungck Picard-Ishikawa hybrid iterative scheme.

Definition 7 [13] :
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Noor- iteration defined by

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n &= (1-\beta_n)Sx_n + \beta_n Tz_n \\ Sz_n &= (1-\gamma_n)Sx_n + \gamma_n Tx_n, \quad n = 0,1,2\ldots \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1], then (1.8) is Jungck Noor iterative scheme.

Definition 8 [13] :
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Picard-Noor iteration defined by

$$\begin{aligned} Sx_{n+1} &= Ty_n \\ Sy_n &= (1-\alpha_n)Sx_n + \alpha_n Tq_n \\ Sq_n &= (1-\beta_n)Sx_n + \beta_n Tz_n \\ Sz_n &= (1-\gamma_n)Sx_n + \gamma_n Tx_n, \quad n = 0,1,2\ldots \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1], then (1.9) is Jungck Picard-Noor iterative scheme.

Definition 9[1] :
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Multistep iteration defined by

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha T y_n^1 \\ Sy_n^1 &= (1-\beta_n^i)Sx_n + \beta_n^i T y_n^{i+1}, \quad i = 1,2,3,\ldots,k-2 \\ Sy_n^{k-1} &= (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} Tx_n,) \quad k \geq 2 \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, i=1,2,3 ...k-1 are sequences of real numbers in (0,1], then (1.10) is Jungck Multi-Step iterative scheme.

Definition 10[13]:
For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Picard-Multistep iteration defined by

$$
\begin{aligned}
Sx_{n+1} &= Tz_n \\
Sz_n &= (1-\alpha_n)Sx_n + \alpha Ty_n^1 \\
Sy_n^1 &= (1-\beta_n^i)Sx_n + \beta_n^i Ty_n^{i+1}, \quad i = 1,2,3,\ldots,k-2 \\
Sy_n^{k-1} &= (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1}Tx_n,) \quad k \geq 2
\end{aligned} \tag{1.11}
$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, i=1,2,3 ... k-1 are sequences of real numbers in [0,1], then (1.11) is Jungck Picard MultiStep iterative scheme.

In 1972, the following results were proved by Zamfirescu [22].

Theorem 1: Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping for which

there exist real numbers α, β and γ satisfying $0 \leq \alpha < 1, 0 \leq \beta, \gamma < 0.5$ such that, for each $x, y \in X$,

at least one of the following is true:

(z_1) $\quad d(Tx,Ty) \leq \alpha d(x,y)$,

(z_2) $\quad d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$,

(z_3) $\quad d(Tx,Ty) \leq \gamma[d(x,Ty) + d(y,Tx)]$.

An operator T satisfying the contractive conditions $(z_1),(z_2)and(z_3)$ in Theorem 1 above is called Zamfirescu

operator.

2 Preliminaries

The Zamfirescu condition has been employed by Rhoades [20],Berinde[2], Bosede [5], and many other reseachers

to establish several interesting convergence results.

Jungck (1976) introduce the following Iterative scheme to approximate common fixed point of Jungck

contraction map.

Let (X,d) be a metric space and $S,T : Y \to X$ be two non-selfmappings such that $T(Y) \subseteq S(Y)$,

$S(Y)$ is a complete subspace of X. For any $x_0 \in Y$, the sequece $\{Sx_n\}_{n=1}^{\infty}$ is the Jungck iterative scheme

defined by

$$Sx_{n+1} = Tx_n, \quad n = 0,1,2\ldots \tag{2.1}$$

Singh et al estabish the following general iteration process:Suppose $S, T : Y \to X$ and $T(Y) \subseteq S(Y)$. For any

given $x_0 \in Y$,suppose

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, 2 \dots \tag{2.2}$$

Jungck-Krasnoselskij iteration process is a special case of Mann-Iteration process. For any $x_0 \in Y$,the

Jungck-Krasnoselskij iteration scheme is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = (1 - \lambda)Sx_n + \lambda Tx_n, \quad n = 0, 1, 2 \dots \tag{2.3}$$

where $\alpha_n = \lambda$ for some $o < \lambda < 1$ is called the Jungck- Krasnoseskij Iteration process.

In (2005) Singh et al [21] introduced the Jungck-Mann Iterative process to obtain some common fixed points and

stability results.For any given $x_0 \in Y$ the Jungck-Mann Iterative scheme $\{Sx_n\}_{n=0}^{\infty}$ is defined by

$$Sx_{n+1} = (1-\alpha_n)Sx_n+\alpha_n Tx_n, \quad n = 0, 1, 2 \dots \tag{2.4}$$

where f is some function. If $f(T, x_n) = Tx_{n+1}$, thus (2.3) becomes Jungck-type iteration process of Singh et al.

The work of Singh et al (2005) was further worked upon by Olatinwo and Imoru (2008) [14] to obtain the

Jungck-Ishikawa and Jungck-Noor iterative schemes, the convergence obtain were used to approximate the

coincidence points of some pairs of generalised contractive -like operators. Suppose each pairs of maps

is injective. For $x_0 \in Y$, the Jungck- Ishikawa iterative scheme is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tx_n, \end{aligned} \tag{2.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Jungck-Ishikawa

Iteration process.

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tz_n \\ Sz_n &= (1 - \gamma_n)Sx_n + \gamma_n Tx_n. \end{aligned} \tag{2.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Jungck-Noor Iteration process.

Considering the Zamfirescu from Theorem 1, there shall be an extension for continuation:

For two nonself mappings $S, T : Y \to X$ with $T(Y) \subseteq S(Y)$, there exist real numbers α, β and γ

satisfying $0 \le \alpha < 1$, $0 \le \beta, \gamma < 0.5$ such that, for each $x, y \in Y$, at least one of the following is true:

$$
\begin{array}{llll}
(gz_1) & d(Tx, Ty) & \le & \alpha d(Sx, Sy), \\
(gz_2) & d(Tx, Ty) & \le & \beta[d(Sx, Tx) + d(Sy, Ty)], \\
(gz_3) & d(Tx, Ty) & \le & \gamma[d(Sx, Ty) + d(Sy, Tx)].
\end{array}
$$

The contractive conditions (gz_1), (gz_2) and (gz_3) will be called the generalized Zamfirescu condition.

In the prrocess the researchers shall consider a normed linear space for the contractive definition by

Bosede [5]: Let $(X, \|.\|)$ be a Banach space, $T : X \to X$ a selfmap of X, with a fixed point p such

that for each $y \in X$ and $0 \le a < 1$,we have,

$$\|p - Ty\| \le a\|p - y\| \tag{2.7}$$

The contractive definition (2.7) by Bosede was extended

$$\|p - Ty\| \le e^{L\|x - Tx\|} a\|p - y\|. \tag{2.8}$$

such that there exist $a \in [0, 1)$, $L \ge 0$ and for every $x, y \in X$.

Suppose $(X, \|.\|)$ is a normed linear space or Banach space. Therefore, using contractive condition (2.8).

Let $S, T : Y \to X$ be two nonself mappings such that $T(Y) \subseteq S(Y)$. $S(Y)$ is a complete subspace of X.Suppose also that z is a coincidence point of S and T, (that is Sz = Tz = p).

Then (2.8) becomes

$$\|p - Ty\| \le e^{L\|Sx - Tx\|} a\|p - Sy\|. \tag{2.9}$$

where $L(0) = 0$ and $\forall\, x, y \in X$.

Remark 2.1: The contractive definition (2.9) is more general than (2.8) in

the sense that if S=I in (2.9), we get (2.8).If $L = 0$ in (2.8), we get (2.7). This is an extended result of Bosede on general class of function.

Our aim in this paper is to establish some strong convergence results for Jungck-Noor, Jungck-Ishikawa, and

Jungck-Mann iteration processes for a general class of function to include some fixed point theorems for Jungck

-Noor three step iteration process associated with generalised Zamfirescu mappings in an arbitrary Banach

space.

Our results extend, improve and unify a multitude of fixed points in literature.

It is clear that the inequality (2.9) holds.Hence,we shall use condition (2.9) in the rest of the proof.

3 Main results

Theorem 3.1:
Let $(X, \|.\|)$ be a Banach Space, $T : X \to X$ a selfmap of X with a fixed point p, satisfying (2.9), (that is, $Tp = p$). For some $0 \leq a < 1$ and for each $y \in X$. Then the Jungck-Picard multistep hybrid iterative scheme defined by (1.11) converges strongly to p.
Proof:

$$\begin{aligned} \|Sx_{n+1} - p\| &= \|Tv_n - Tp\| \\ &\leq ae^{L\|p-Tp\|}[\|Sv_n - p\|] \\ &= ae^{L(0)}[\|Sv_n - p\|] \\ &= a\|Sv_n - p\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|Sv_n - p\| &= \|(1-\alpha_n)Sx_n + \alpha_n Ty_n^1 - (1-\alpha_n+\alpha_n)p\| \\ &\leq (1-\alpha_n)\|Sx_n - p\| + \alpha_n\|Ty_n^1 - p\| \end{aligned} \tag{3.2}$$

Using (2.9), and suppose $y = y_n^1$, gives

$$\|p - Ty_n^1\| \leq e^{L\|Sx-Tx\|}a\|p - Sy\| \tag{3.3}$$

substituting (3.3) in (3.2)

$$\begin{aligned} \|Sv_n - p\| &\leq (1-\alpha_n)\|Sx_n - p\| + \alpha_n[e^{L\|Sx-Tx\|}a\|p - Sy_n^1\|] \\ &\leq (1-\alpha_n)\|Sx_n - p\| + \alpha_n a\|p - Sy_n^1\| \end{aligned} \tag{3.4}$$

suppose $\beta_n^i \in [0, 1)$, $i = 1, 2, 3, \ldots, k-1$ and $n = 1, 2, \ldots$

$$
\begin{aligned}
\|Sy_n^1 - p\| &= \|(1-\beta_n^1)Sx_n + \beta_n^1 Ty_n^2 - (1-\beta_n^1+\beta_n^1)p\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + \|\beta_n^1\|Ty_n^2 - p\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1\|Sy_n^2 - p\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1[(1-\beta_n^2)\|Sx_n - p\| + \beta_n^2\|Ty_n^3 - p\|] \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1(1-\beta_n^2)\|Sx_n - p\| + a^2\beta_n^1\beta_n^2\|Sy_n^3 - p\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1(1-\beta_n^2)\|Sx_n - p\| + a^2\beta_n^1\beta_n^2[(1-\beta_n^3)\|Sx_n - p\| + \beta_n^3\|Ty_n^4 - p\|] \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1(1-\beta_n^2)\|Sx_n - p\| + a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|Sx_n - p\| + a^3\beta_n^1\beta_n^2\beta_n^3(1-\beta_n^3)\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta - n^1(1-\beta_n^2)\|Sx_n - p\| + a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|Sx_n - p\| \\
&+ \cdots + a^{k-2}\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2}\|Sy_n^{k-1} - p\| \\
&\leq (1-\beta_n^1)\|Sx_n - p\| + a\beta_n^1(1-\beta_n^2)\|Sx_n - p\| \\
&+ a^2\beta_n^1\beta_n^2(1-\beta_n^3)\|Sx_n - p\| + \ldots \\
&+ a^{k-2}\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2}[(1-\beta_n^{k-1})\|Sx_n - p\| + a\beta_n^{k-1}\|Sx_n - p\|] \\
&\leq [1-(1-a)\beta_n^1 - (1-a)a\beta_n^1\beta_n^2 - (1-a)a^2\beta_n^1\beta_n^2\beta_n^3 - \ldots \\
&- (1-a)a^{k-2}\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-1}]\|Sx_n - p\| \\
&\leq [1-(1-a)\beta_n^1]\|Sx_n - p\| \qquad (3.5)
\end{aligned}
$$

substituting (3.5) in (3.4), gives

$$
\begin{aligned}
\|Sv_n - p\| &\leq (1-\alpha_n)\|Sx_n - p\| + a\alpha_n[1-(1-a)\beta_n^1]\|Sx_n - p\| \\
&= [1-(1-a)\alpha_n]\|Sx_n - p\| \qquad (3.6)
\end{aligned}
$$

By induction (3.6) gives

$$\|Sv_n - p\| \leq \prod_{r=0}^{n}[1-(1-a)\alpha_r]\|Sx_0 - p\| \qquad (3.7)$$

substituting (3.7) in (3.1), gives

$$\|Sx_{n+1} - p\| \leq a(\prod_{r=0}^{n}[1-(1-a)\alpha_r]\|Sx_0 - p\|) \qquad (3.8)$$

By considering the fact that $\sum_{n=0}^{\infty}\alpha_n = \infty$,$0 \leq a < 1$, $\alpha_r, \beta_n^i \in [0,1]$ for for $i = 1,2,3\ldots,k-1$
From lemma 2.3, we obtain $\lim_{n\to\infty}\prod_{r=0}^{n}[1-(1-a)\alpha_r] = 0$
Therefore,
$\lim_{n\to\infty}\|x_{n+1} - p\| = 0$
That is $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 3.2: Let $(X, \|.\|)$ be a Banach space, and X an arbitrary set. Suppose that $S, T : Y \to X$ are two nonself mappings such that $T(Y) \subseteq S(Y)$. Suppose that z is a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose also that S and T satisfy the contractive condition (2.9). For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck Picard-Noor Iteration process defined by (2.6), where $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, the Jungck Picard-Noor Iterative process converges to p.

Proof:

Therefore, using the Jungck Picard-Noor Iterative scheme (1.9) and the triangle inequality (2.9), we obtain:

$$\begin{aligned} \|Su_{n+1} - p\| &= \|Tv_n - Tp\| \\ &\leq ce^{L\|Sp-Tp\|}[\|Sv_n - p\|] \\ &\leq ce^{L\|Sp-Tp\|}[\|Sv_n - p\|] \\ &= ce^{L(0)}[\|Sv_n - p\|] \\ &= c\|Sv_n - p\| \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \|Sv_n - p\| &= \|(1-\alpha_n)Su_n + \alpha_n Tw_n - p\| \\ &= \|(1-\alpha_n)Su_n + \alpha_n Tw_n - ((1-\alpha_n)+\alpha_n)p\| \\ &= \|(1-\alpha_n)(Su_n - p) + \alpha_n(Tw_n - p)\| \\ &\leq (1-\alpha_n)\|Su_n - p\| + \alpha_n\|Tw_n - p\| \\ &= (1-\alpha_n)\|Su_n - p\| + \alpha_n\|Tp - Tw_n\| \\ &\leq (1-\alpha_n)\|Su_n - p\| + \alpha_n ce^{L\|Sp-Tp\|}[\|Sw_n - Sx\|] \\ &\leq (1-\alpha_n)\|Su_n - p\| + \alpha_n ce^{L\|Sp-Tp\|}[\|Sw_n - p\|] \\ &= (1-\alpha_n)\|Su_n - p\| + \alpha_n ce^{L(0)}[\|Sw_n - p\|] \\ &= (1-\alpha_n)\|Su_n - p\| + \alpha_n c\|Sw_n - p\| \\ &= (1-\alpha_n)\|Su_n - p\| + \alpha_n c\|Sw_n - p\| \end{aligned} \tag{3.10}$$

The estimate of $\|Sw_n - p\|$ in (3.10), gives

$$\begin{aligned} \|Sw_n - p\| &= \|(1-\beta_n)Su_n + \beta_n Tx_n - p\| \\ &= \|(1-\beta_n)Su_n + \beta_n Tx_n - ((1-\beta_n)+\beta_n)p\| \\ &= \|(1-\beta_n)(Su_n - p) + \beta_n(Tx_n - p)\| \\ &\leq (1-\beta_n)\|Su_n - p\| + \beta_n\|Tx_n - p\| \\ &= (1-\beta_n)\|Su_n - p\| + \beta_n\|Tp - Tx_n\| \\ &\leq (1-\beta_n)\|Su_n - p\| + \beta_n ce^{L\|Sp-Tp\|}[\|Sx_n - p\|] \\ &= (1-\beta_n)\|Su_n - p\| + \beta_n ce^{L(0)}[\|Sx_n - p\|] \\ &= (1-\beta_n)\|Su_n - p\| + \beta_n c\|Sx_n - p\|. \end{aligned} \tag{3.11}$$

Substitute (3.11) into (3.10), gives

$$\begin{aligned} \|Sv_n - p\| &\leq (1-\alpha_n)\|Su_n - p\| + \alpha_n c[(1-\beta_n)\|Su_n - p\| + \beta_n c\|Sx_n - p\|] \\ &= [(1-\alpha_n) + c\alpha_n(1-\beta_n)]\|Su_n - p\| + c^2\alpha_n\beta_n\|Sx_n - p\| \end{aligned} \tag{3.12}$$

Finally, the estimate of $\|Sx_n - p\|$ in (3.11), gives

$$\begin{aligned}
\|Sx_n - p\| &= \|(1-\gamma_n)Su_n + \gamma_n Tu_n - p\| \\
&= \|(1-\gamma_n)Su_n + \gamma_n Tu_n - ((1-\gamma_n)+\gamma_n)p\| \\
&= \|(1-\gamma_n)(Su_n - p) + \gamma_n(Tu_n - p)\| \\
&\leq (1-\gamma_n)\|Su_n - p\| + \gamma_n\|Tu_n - p\| \\
&= (1-\gamma_n)\|Su_n - p\| + \gamma_n\|Tp - Tu_n\| \\
&\leq (1-\gamma_n)\|Su_n - p\| + \gamma_n c e^{L\|Sp-Tp\|}[\|Su_n - p\|] \\
&= (1-\gamma_n)\|Su_n - p\| + \gamma_n c e^{L(0)}[\|Su_n - p\|] \\
&= (1-\gamma_n)\|Su_n - p\| + \gamma_n c\|Su_n - p\|. \qquad (3.13)
\end{aligned}$$

Substitute (3.13) into (3.12), gives

$$\begin{aligned}
\|Sv_n - p\| &= [(1-\alpha_n) + c\alpha_n(1-\beta_n)]\|Su_n - p\| + c^2\alpha_n\beta_n[(1-\gamma_n)\|Su_n - p\| + c\gamma_n\|Su_n - p\|] \\
&= [(1-\alpha_n) + c\alpha_n(1-\beta_n) + c^2\alpha_n\beta_n(1-\gamma_n) + c^3\alpha_n\beta_n\gamma_n)]\|Su_n - p\| \\
&= [1 - \alpha_n + c\alpha_n - c\alpha_n\beta_n + c^2\alpha_n\beta_n - c^2\alpha_n\beta_n\gamma_n + c^3\alpha_n\beta_n\gamma_n]\|Su_n - p\| \qquad (3.14)
\end{aligned}$$

substitute (3.14) in (3.9) gives

$$\begin{aligned}
\|Su_{n+1} - p\| &\leq c[(1-(1-c)\alpha_n - c\alpha_n\beta_n)\|Sx_n - p\| + c^2\alpha_n\beta_n(1-\gamma_n + c\gamma_n)]\|Sx_n - p\| \\
&= c[(1-(1-c)\alpha_n - (1-c)c\alpha_n\beta_n - (1-c)c^2\alpha_n\beta_n\gamma_n]\|Sx_n - p\| \\
&\leq c[(1-(1-c)\alpha_n)]\|Sx_n - p\| \\
&\leq c\prod_{k=0}^{\infty}(1-(1-c)\alpha_k)\|Sx_0 - p\| \\
&\leq c\prod_{k=0}^{\infty} e^{-(1-(1-c)\alpha_k)}\|Sx_0 - p\| \\
&= ce^{-(1-c)}\sum_{k=0}^{\infty}\alpha_k\|Sx_0 - p\| \to 0 \qquad (3.15)
\end{aligned}$$

as $n \to \infty$. Since, $\sum_{k=0}^{\infty}\alpha_k = \infty$, $\delta \in [0,1)$. Therefore from (3.15), we obtain $\|Sx_n - p\| \to 0$ as $n \to \infty$, this

implies that the Jungck-Noor iteration converges strongly to p.

To prove the uniqueness, suppose $z_1, z_2 \in C(S,T)$, where C (S,T) is the set of coincidence points of S and

T such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$

Suppose on the contrary that $p_1 \neq p_2$. Now, considering the contractive condition,

$$\begin{aligned}
\|p_1 - p_2\| &= \|Tp_1 - Tp_2\| \\
&= \|Tp_2 - Tp_1\| \\
&\leq ce^{L\|Sp_1 - Tp_1\|}[\|Sp_2 - Sp_1\|] \\
&= ce^{L\|p_1 - p_1\|}[\|p_2 - p_1\|] \\
&= ce^{L(0)}[\|p_2 - p_1\|] \\
&= c\|p_2 - p_1\| \\
&= c\|p_1 - p_2\| < \|p_1 - p_2\|
\end{aligned}$$

is a contradiction. Hence, $p_1 = p_2$.

This completes the proof.

Consequently, we have the following corollaries, since the Jungck-Picard Ishikawa and Jungck Picard-Mann

iteration processes are special cases of Jungck Picard-Noor hybrid iteration process:

Corollary 1: Let X,Y,S,T,z and p be as defined in Theorem 3.2. For arbitrary $x_0 \in X$,let $\{Sx_n\}_{n=0}^{\infty}$ be

the Jungck-Picard Ishikawa hybrid iteration process defined by (1.7), where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are

sequences of real numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Hence, the Jungck Picard-Ishikawa hybrid Iteration process converges strongly to p.

Corollary 2: Let X,Y,S,T,z and p be as defined in Theorem 3.2. For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$

be the Jungck Picard-Mann hybrid iteration process defined by (1.5), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Hence, the Jungck-Mann iteration process converges strongly to p.

A special case of Jungck Picard-Mann hybrid iteration process is that of Jungck-Picard Krasnoselkij hybrid

iteration process which is Jungck Picard-Mann iteration, with each $\alpha_n = \lambda$ such that $0 < \lambda < 1$.

For arbitrary $x_0 \in Y$, the sequence $Sx_{n n=0}^{\infty}$ defined by

$$\begin{aligned}
Sx_{n+1} &= Ty_n \\
y_n &= (1-\lambda)Sx_n + \lambda Tx_n,\ n = 0, 1, 2, \ldots
\end{aligned} \tag{3.16}$$

Eqtn(3.16) is called the Jungck Picard-Kranoselkij hybrid iteration scheme.

Corollary 3: Let X,Y,S,T,z and p be defined in Theorem 3.2. For arbitrary $x_0 \in X$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Krasnoselkij iteration process defined by (3.7) for some $0 < \lambda < 1$.

Hence, the Jungck-Krasnoselskij iteration process converges strongly to p.

References

[1] H. Akewe and H. Olaluwa (2017). Approximation of the Jungck-Multistep iteration process generated by a generalised contractive-like operator in cone Banach spaces, J. Sci. Res. Dev., 17(1): 73-79.

[2] V. Berinde (2001). Iterative approximation of fixed points for pseudo-contractive operators, Seminar on Fixed Point Theory, 3, 210-216.

[3] V. Berinde (2002). Iterative Approximation of Fixed Points, Editura Efemeride.

[4] V.Berinde (2004). On the convergence of Ishikawa iteration in the class of quaasi-contractive operators, Acta Math.Univ.Comenianae, LXXIII(1),119-126.

[5] A.O. Bosede (2009). Noor iterations associated with Zamfirescu mappings in uniformly convex Banach spaces, Fasciculli Mathematici, 42, 29-38.

[6] A.O. Bosede (2011). Strong convergence of Noor iteration for a class of general class of function, Bulletin of Mathematical Analysis and Applications, 140-145.

[7] A.O. Bosede and B.E. Rhoades (2010). Stability of Picard and Mann iterations for a general class of functions, Journal of advaced Mathematical studies, 3(2),1-3.

[8] A.O. Bosede (2010). Strong convergence results for the Jungck-Ishikawa and Jungck-Mann iteration processes, Bulletin of Mathematical Analysis and Applications, 2(3): 65-73.

[9] L.B. Ciric (1990). Fixed point theorems in Banach spaces, Publ. Inst. Math.,(Beograd), 47(6), 85-87.

[10] C.O. Imoru, G. Akinbo and A.O. Bosede (2006). On the fixed points for weak compatible type and and parametrically $\phi(\in, \lambda; a)$-contraction mappings, Math. Sci. Res.Journal, 10(10), 259-267.

[11] S. Ishikawa (1974). Fixed point by a new iteration method, Proc. Amer. Math. Soc., 44, 147-150.

[12] G. Jungck (1976). Commuting mappings and fixed points, Amer. Math. Soc., 83(4),261-263.

[13] Z.Z. Jamil and A.K. Abdullateef (2015). On rate of convergence of Jungck-Picard S. Iterative Scheme, International Journal of Science and research (IJSR), 2319-7064.

[14] W.R. Mann (1953). Mean value iteration, Proc. Amer.Math.Soc., 4, 506-510.

[15] M.A Noor (2000). New approximations schemes for general variational inequalities, J. Math. Anal. Appl, 251, 217-299.

[16] M.O. Olatinwo (2008). Some stabbility and strong convergence results for the Jungck-Ishikawa iteration process, Creat. Math. Inform, 17,33-42.

[17] M.O. Olatinwo and C.O. Imoru (2008). Some convergence results for the Jungck-Mann and Jungck-Ishikawa iteration process in the class of generalized Zanfirescu operators, Acta Math.Univ. Comenianae, 2,299-304.

[18] O.V. Olisama, A.A. Mogbademu and J.O. Olaleru (2014a). Convergence of modified Multistep iteration scheme for p-nearly uniformly L-Lipschitzian Asymptotically Pseuddocontractive Maps, International Journal of Analysis and Applications, 4(2), 192-200

[19] O.V. Olisama, A.A. Mogbademu and J.O. Olaleru (2014b). Modified multi-step Iterative Processses for a finite family of Asymptotically Pseudo-contractive Maps, Applied Mathematical Sciences, 4(19), 901-910

[20] E. Picard (1893). Sur l'application des methodes d'approximations successives a l'etude de certaines equations differentielles ordinaires, J. Math. 9, 217-271.

[21] A. Rafiq (2006). A convergence theorem for Mann fixed point iteration procedure. Applied Mathematics E-Notes, 6,289-293.

[22] A. Rafiq (2006). On the convergence of the three step iteration process in the class of quasi-contractive operators, Acta Math. Acad. Paedagog Nyiregyhaziensis, 22, 305-309.

[23] B.E. Rhoades (1990). Fixed point theorems and stability results for fixed point iteration prcedures, Indian J.Pure Appl. Math., 21(1), pp,1-9.

[24] S.L. Singh, C. Bhatmagar, and S.N Mishra (2005). Stability of Jungck-type iteratve procedures, International J. Math. and Math. Sc., 19, 3035-3043.

[25] T. Zamfirescu (1972). Fix point theorems in metric spaces, Arch. Math., 23, 292-298.