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Numerical solutions of first order delay differential equations using second derivative block Adams Moulton Methods

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Abstract

In this paper, we formulated and applied Second Derivative Block Adams Moulton Methods in solving first order delay differential equations without the introduction of interpolation formula in evaluating the delay argument. The delay argument was evaluated by applying an acceptable idea of sequences. The continuous forms of these of block methods were worked-out through the use of linear multistep collocation procedure using matrix inversion approach. The resulting discrete schemes obtained from the various continuous expressions of the block methods were observed to be self-starting, consistent, zero-stable, convergent and satisfying region of $P-$and $Q-$stabilities. From the results obtained, the scheme for step number $k = 4$ performed better than the schemes for step numbers $k = 3$ and 2 in terms of accuracy when compared with their exact results.

Keywords and phrases: First order delay differential equations; second derivative Adams Moulton method; block method.

1 Introduction

In modern work of engineering, physics, applied mathematics, medicine and economics, the solutions of first order delay differential equations arise very frequently. To date, these equations have been extensively studied and books have been composed along the mathematical methods available for such equations. Delay Differential equations are one of the widely used and important tools in mathematical modeling and can be solved analytically or using numerical approaches. Most of the real-life situations will be more realistic when they are model using delay differential equations (DDEs) because its solution does not only require the knowledge of the current state but also of the history part of the system being modeled. Therefore, differential equations that involve both the current and the history or past state of the system are known as delay differential equations (DDEs). In accordance with reality in the business world, there is a certain time delay when a customer ordered for goods and the time in which the goods are being delivered due to road traffic or accident. Delays can also be experienced in other real life situations. Therefore, ignoring delays in real life situations means ignoring reality. Numerical methods are crucial tools for solution of delay differential equation which cannot be neglected. In literature, various types of numerical methods have been developed in treating the problems of the first order delay differential equations (DDEs) of the form:

$$
\begin{aligned}
y'(t) &= f(t, y(t), y(t-\tau)), \text{ for } t > t_0, \tau > 0 \\
y(t) &= c(t), \text{ for } t \leq t_0
\end{aligned} \tag{1.1}
$$

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where $c(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument and $y(t-\tau)$ is the solution of the delay argument. Delay Differential equations are one of the widely used and important tools in mathematical modeling and can be solved analytically or using numerical approaches. However a good number of researchers such as [1], used interpolation methods to examine the delay argument while solving delay differential equations in a predictor-corrector mode. [2], applied the linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays with Nordsieck's interpolation technique. [3], solved delay differential equations by the ?ve -point one-step block method using *Neville's* interpolation in approximating the delay term. [4], adopted the direct two-point fourth and fifth order multistep block method in the form of Adams- Moulton Method in predictor-corrected (PECE) mode to compute the numerical solutions at two points simultaneously for second order delay differential equations. [5], applied Butcher type two-step block hybrid multistep method to determine accurate and efficient parallel solutions of ODEs. [6],[7], applied linear multistep approach in solving delay differential equations with the aid of interpolation techniques in investigating the delay term. [8] used implicit 2-point Block Backward Differential Formulae to solve a set of delay differential equations with the help of interpolation technique in approximating the delay term. [9], applied the Zhou method to obtain approximate-analytical solutions of certain system of functional differential equations (SFDEs) induced by proportional delays. [10], procured the results of some system of time-fractional differential equations (TFDDEs) using proportional delays. [11], studied the result of fractional order delay differential equations by applying Shifted Legendre polynomials with Collocation and Galerkin settings. [12], applied Reformulated Block Backward Differentiation Formulae Methods to determine the numerical solutions of (1.1) without the use of interpolation method for the delay term.
The hindrance in using these interpolation techniques by [3] is that the numerical method use in solving DDEs should be the same with the interpolating polynomials which is very hard to achieve; otherwise, the accuracy of the method will not be preserved. It is expected that in the approximation of the delay term, using an accurate and efficient formula will be appropriate.
In order to deal with this hindrance, we constructed and applied second derivative block adams moulton methods to solve first order delay differential equations without the introduction of interpolation formula in estimating the delay argument. These block methods will be applied using constant step size r and the delay argument was evaluated by applying an acceptable idea of sequence developed by [13], without using any interpolation formula so as to maintain the required accurate results.

2 Methodology/Model formulation

2.1 Formulation of Second Derivative Block Adam Moulton Methods

The generalized form for k-step Adam Moulton Methods by [14], was derived as;

$$y_{v+k} - y_{v+k-1} = r\sum_{d=0}^{k}\beta_d(x)f_{v+d} \tag{2.1}$$

And its second derivative is expressed as;

$$y_{v+k} - y_{v+k-1} = r\sum_{d=0}^{k}\beta_d(x)f_{v+d} + r^2\sum_{d=0}^{k}\gamma_d(x)g_{v+d} \tag{2.2}$$

Using (2.2), we derived the continuous form of the discrete second order generalized Adams Moulton Methods:

$$y(x) = \alpha_{k-1}(x)y_{v+k-1} + r\sum_{d=0}^{w-1}\beta_d(x)f_{v+d} + r^2\sum_{d=0}^{w-1}\gamma_d(x)g_{v+d} \tag{2.3}$$

where $\alpha_d(x)$, $\beta_d(x)$ and $\gamma_d(x)$ are continuous coefficients of the method defined as;

$$\alpha_d(x) = \sum_{a=0}^{e+w-1} \alpha_{d,a+1} x^a \text{ for } d = \{0,1,...,e-1\} \tag{2.4}$$

$$r\beta_d(x) = \sum_{a=0}^{e+w-1} r\beta_{d,a+1} x^a \text{ for } d = \{0,1,...,w-1\} \tag{2.5}$$

$$r^2\gamma_d(x) = \sum_{a=1}^{e+w-1} r^2\gamma_{d,a+1} x^a \text{ for } d = \{0,1,...,w-1\} \tag{2.6}$$

where $x_0, \ldots, x_{w-1}$ are the w collocation points, $x_{v+d}, d = 0,1,2,...,e-1$ are the e arbitrarily chosen interpolation points and r is the constant step size.
To get $\alpha_d(x)$, $\beta_d(x)$ and $\gamma_d(x)$, [5] to developed a matrix equation of the form

$$EH = \mathrm{I} \tag{2.7}$$

where I represents the unit matrix of dimension $(e+w) \times (e+w)$ and E and H are matrices presented as:

$$E = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & . & \alpha_{e-1,1} & r\beta_{0,1} & . & r\beta_{w-1,1} & r^2\gamma_{0,1} & . & r^2\gamma_{w-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & . & \alpha_{e-1,2} & r\beta_{0,2} & . & r\beta_{w-1,2} & r^2\gamma_{0,2} & . & r^2\gamma_{w-1,2} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,e+w} & \alpha_{1,e+w} & . & \alpha_{e-1,e+w} & r\beta_{0,e+w} & . & r\beta_{w-1,e+w} & r^2\gamma_{0,e+w} & . & r^2\gamma_{w-1,e+w} \end{pmatrix} \tag{2.8}$$

$$H = \begin{pmatrix} 1 & x_v & x_v^2 & x_v^3 & x_v^4 & . & . & x_a^{u+v-1} \\ 1 & x_{v+1} & x_{v+1}^2 & x_{v+1}^3 & x_{v+1}^4 & . & . & x_{a+1}^{u+v-1} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & x_{v+e-1} & x_{v+e-1}^2 & x_{v+e-1}^3 & x_{v+e-1}^4 & . & . & x_{v+e-1}^{e+w-1} \\ 0 & 1 & 2x_v & 3x_v^2 & 4\,x_v^3 & . & . & (e+w-1)x_v^{e+w-2} \\ 0 & 1 & 2x_{v+1} & 3x_{v+1}^2 & 4x_{v+1}^3 & . & . & (e+w-1)x_{v+1}^{e+w-2} \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 2x_{v+e-1} & 3x_{v+e-1}^2 & 4x_{v+e-1}^3 & . & . & (e+w-1)x_{v+e-1}^{e+w-2} \\ 0 & 0 & 2 & 6x_v & 12x_v^2 & . & . & (e+w-1)(e+w-2)x_v^{e+w-3} \\ 0 & 0 & 2 & 6x_{v+1} & 12x_{v+1}^2 & . & . & (e+w-1)(e+w-2)x_{v+1}^{e+w-3} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 2 & 6x_{v+e-1} & 12x_{v+e-1}^2 & . & . & (e+w-1)(e+w-2)x_{v+w-1}^{e+w-3} \end{pmatrix} \tag{2.9}$$

From (2.7) the columns of $E = H^{-1}$ give the continuous coefficients of the continuous scheme (2.3).

2.2 Formulation of Second Derivative Block Adam Moulton Methods for $k = 2$

Here, also the number of interpolation points, $e = 1$ and the number of collocation points, $w = 4$. Therefore, (2.3) becomes

$$y(x) = \alpha_1(x)y_{v+1} + r\beta_0(x)f_v + r\beta_1(x)f_{v+1} + r\beta_2(x)f_{v+2} + r^2\gamma_2(x)g_{v+2} \tag{2.10}$$

The matrix H in (2.7) becomes

$$H=\begin{pmatrix} 1 & x_v+r & (x_v+r)^2 & (x_v+r)^3 & (x_v+r)^4 \\ 0 & 1 & 2x_v & 3x_v^2 & 4x_v^3 \\ 0 & 1 & 2x_v+2r & 3(x_v+r)^2 & 4(x_v+r)^3 \\ 0 & 1 & 2x_v+4r & 3(x_v+2r)^2 & 4(x_v+2r)^3 \\ 0 & 0 & 2 & 6x_v+12r & 12(x_v+2r)^2 \end{pmatrix} \tag{2.11}$$

The inverse of the matrix $E=H^{-1}$ is computed using Maple 18 from which the continuous form is obtained using (2.7) and evaluating it at $x=x_v$ and $x=x_{v+2}$, the following discrete schemes are obtained;

$$y_v=y_{v+1}-\frac{17}{48}rf_v-\frac{11}{12}rf_{v+1}+\frac{13}{48}rf_{v+2}-\frac{1}{8}r^2g_{v+2}$$

$$y_{v+2}=y_{v+1}-\frac{1}{48}rf_v+\frac{5}{12}rf_{v+1}+\frac{29}{48}rf_{v+2}-\frac{1}{8}r^2g_{v+2} \tag{2.12}$$

2.3 Formulation of Second Derivative Block Adam Moulton Methods for $k=3$

Here, also the number of interpolation points, $e=1$ and the number of collocation points, $w=5$. Therefore, (2.3) becomes

$$y(x)=\alpha_2(x)y_{v+2}+r\beta_0(x)f_v+r\beta_1(x)f_{v+1}+r\beta_2(x)f_{v+2}+r\beta_3(x)f_{v+3}+r^2\gamma_3(x)g_{v+3} \tag{2.13}$$

Also the matrix H in (2.7) becomes

$$H=\begin{pmatrix} 1 & x_v+2r & (x_v+2r)^2 & (x_v+2r)^3 & (x_v+2r)^4 & (x_v+2r)^5 \\ 0 & 1 & 2x_v & 3x_v^2 & 4x_v^3 & 5x_v^4 \\ 0 & 1 & 2x_v+2r & 3(x_v+r)^2 & 4(x_v+r)^3 & 5(x_v+r)^4 \\ 0 & 1 & 2x_v+4r & 3(x_v+2r)^2 & 4(x_v+2r)^3 & 5(x_v+2r)^4 \\ 0 & 1 & 2x_v+6r & 3(x_v+3r)^2 & 4(x_v+3r)^3 & 5(x_v+3r)^4 \\ 0 & 0 & 2 & 6x_v+18r & 12(x_v+3r)^2 & 20(x_v+3r)^3 \end{pmatrix} \tag{2.14}$$

The inverse of the matrix $E=H^{-1}$ is computed using Maple 18 from which the continuous form is also obtained using (2.7) and evaluating it at $x=x_v, x=x_{v+1}$ and $x=x_{v+3}$, the following discrete schemes are obtained:

$$y_v=y_{v+2}-\frac{43}{135}rf_v-\frac{7}{5}rf_{v+1}-\frac{1}{5}rf_{v+2}-\frac{11}{135}rf_{v+3}+\frac{2}{45}r^2g_{v+3}$$

$$y_{v+1}=y_{v+2}+\frac{23}{1080}rf_v-\frac{9}{20}rf_{v+1}-\frac{29}{40}rf_{v+2}+\frac{83}{540}rf_{v+3}-\frac{11}{180}r^2g_{v+3}$$

$$y_{v+3}=y_{v+2}+\frac{7}{1080}rf_v-\frac{1}{20}rf_{v+1}+\frac{19}{40}rf_{v+2}+\frac{307}{540}rf_{v+3}-\frac{19}{180}r^2g_{v+3} \tag{2.15}$$

2.4 Formulation of Second Derivative Block Adam Moulton Methods for $k=4$

Here, also the number of interpolation points, $e=1$ and the number of collocation points, $w=6$. Therefore, (2.3) becomes

$$y(x)=\alpha_3(x)y_{v+3}+r\beta_0(x)f_v+r\beta_1(x)f_{v+1}+r\beta_2(x)f_{v+2}+r\beta_3(x)f_{v+3}+r\beta_4(x)f_{v+4}+r^2\gamma_4(x)g_{v+4} \tag{2.16}$$

Also the matrix H in (2.7) becomes

$$H=\begin{pmatrix} 1 & x_v+3r & (x_v+3r)^2 & (x_v+3r)^3 & (x_v+3r)^4 & (x_v+3r)^5 & (x_v+3r)^6 \\ 0 & 1 & 2x_v & 3x_v^2 & 4x_v^3 & 5x_v^4 & 6x_v^5 \\ 0 & 1 & 2x_v+2r & 3(x_v+r)^2 & 4(x_v+r)^3 & 5(x_v+r)^4 & 6(x_v+r)^5 \\ 0 & 1 & 2x_v+4r & 3(x_v+2r)^2 & 4(x_v+2r)^3 & 5(x_v+2r)^4 & 6(x_v+2r)^5 \\ 0 & 1 & 2x_v+6r & 3(x_v+3r)^2 & 4(x_v+3r)^3 & 5(x_v+3r)^4 & 6(x_v+3r)^5 \\ 0 & 1 & 2x_v+8r & 3(x_v+4r)^2 & 4(x_v+4r)^3 & 5(x_v+4r)^4 & 6(x_v+4r)^5 \\ 0 & 0 & 2 & 6x_v+24r & 12(x_v+4r)^2 & 20(x_v+4r)^3 & 30(x_v+4r)^4 \end{pmatrix} \tag{2.17}$$

The inverse of the matrix $E=H^{-1}$ is computed using Maple 18 from which the continuous form is also obtained using (2.7) and evaluating it at $x=x_v, x=x_{v+1}, x=x_{v+2}$ and $x=x_{v+4}$, the following discrete schemes are obtained:

$$\begin{aligned} y_v &= y_{v+3}-\frac{201}{640}rf_v-\frac{7}{5}rf_{v+1}-\frac{99}{160}rf_{v+2}-\frac{9}{10}rf_{v+3}+\frac{149}{640}rf_{v+4}-\frac{3}{32}r^2g_{v+4} \\ y_{v+1} &= y_{v+3}+\frac{1}{90}rf_v-\frac{17}{45}rf_{v+1}-\frac{19}{15}rf_{v+2}-\frac{17}{45}rf_{v+3}+\frac{1}{90}rf_{v+4} \\ y_{v+2} &= y_{v+3}-\frac{11}{1920}rf_v+\frac{7}{135}rf_{v+1}-\frac{83}{160}rf_{v+2}-\frac{19}{30}rf_{v+3}+\frac{1831}{17280}rf_{v+4}-\frac{11}{288}r^2g_{v+4} \end{aligned}$$

$$y_{v+4}=y_{v+3}-\frac{17}{5760}rf_v+\frac{1}{45}rf_{v+1}-\frac{41}{480}rf_{v+2}+\frac{47}{90}rf_{v+3}+\frac{3133}{5760}rf_{v+4}-\frac{3}{32}r^2g_{v+4} \tag{2.18}$$

3 Convergence analysis

Here, the investigations of order, error constant, consistency, zero stability and region of the absolute stability of (2.12), (2.15) and (2.18) are worked-out.

3.1 Order and Error Constant

The order and error constants for (2.12) are obtained as follows:

$$C_0=\alpha_0+\alpha_1+\alpha_2=\begin{pmatrix}0\\0\end{pmatrix}, c_1=\alpha_1+2\alpha_2-\beta_0-\beta_1-\beta_2=\begin{pmatrix}0\\0\end{pmatrix} c_2=\frac{1}{2}\alpha_1+2\alpha_2-\beta_1-2\beta_2=\begin{pmatrix}-\frac{1}{8}\\-\frac{1}{8}\end{pmatrix}$$

Therefore, (2.12) has an order, $p=1$ and error constants, $-\frac{1}{8},-\frac{1}{8}$
Applying the same approach for (2.15) and can be represented as follows:

$$\begin{aligned} C_0 &= \alpha_0+\alpha_1+\alpha_2+\alpha_3=\begin{pmatrix}0\\0\\0\end{pmatrix} \\ C_1 &= \alpha_1+2\alpha_2+3\alpha_3-\beta_0-\beta_1-\beta_2-\beta_3=\begin{pmatrix}0\\0\\0\end{pmatrix} \\ C_2 &= \frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3-\beta_1-2\beta_2-3\beta_3=\begin{pmatrix}\frac{2}{45}\\-\frac{11}{180}\\-\frac{19}{180}\end{pmatrix} \end{aligned}$$

Therefore, (2.15) has order $p=1$ and error constants, $\frac{2}{45},-\frac{11}{180},-\frac{19}{180}$
With the way for (2.18) and can be represented as:

$$C_0 = \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},\; C_1 = \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4-\beta_0-\beta_1-\beta_2-\beta_3-\beta_4 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3+8\alpha_4-\beta_1-2\beta_2-3\beta_3-4\beta_4 = \begin{pmatrix}-\frac{3}{32}\\0\\-\frac{11}{288}\\-\frac{3}{32}\end{pmatrix}$$

Therefore, (2.18) has order $p=1$ and error constants, $-\frac{3}{32}, 0, -\frac{11}{288}, -\frac{3}{32}$

3.2 Consistency

Since $p=1$ in (2.12), (2.15) and (2.18) satisfying the condition for consistency of order $p \geq 1$, then the schemes are consistent.

3.3 Stability Analysis

The zero stability for (2.12) is evaluated as follows:

$$\begin{aligned}\begin{pmatrix}-1&0\\-1&1\end{pmatrix}\begin{pmatrix}y_{v+1}\\y_{v+2}\end{pmatrix} &= \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}y_{v-1}\\y_v\end{pmatrix}+r\begin{pmatrix}-\frac{11}{12}&\frac{13}{48}\\\frac{5}{12}&\frac{29}{48}\end{pmatrix}\begin{pmatrix}f_{v+1}\\f_{v+2}\end{pmatrix}\\ &+r\begin{pmatrix}0&-\frac{17}{48}\\0&-\frac{1}{48}\end{pmatrix}\begin{pmatrix}f_{v-1}\\f_v\end{pmatrix}+r^2\begin{pmatrix}0&-\frac{1}{8}\\0&-\frac{1}{8}\end{pmatrix}\begin{pmatrix}g_{v+1}\\g_{v+2}\end{pmatrix}\\ &+r^2\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}g_{v-1}\\g_v\end{pmatrix}\end{aligned}$$

$$\begin{aligned}P_2^{(1)} &= \begin{pmatrix}-1&0\\-1&1\end{pmatrix},\; P_1^{(1)}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\; Q_2^{(1)}=\begin{pmatrix}-\frac{11}{12}&\frac{13}{48}\\\frac{5}{12}&\frac{29}{48}\end{pmatrix},\; R_2^{(1)}=\begin{pmatrix}0&-\frac{17}{48}\\0&-\frac{1}{48}\end{pmatrix}\\ \text{and } S_2^{(1)} &= \begin{pmatrix}0&-\frac{1}{8}\\0&-\frac{1}{8}\end{pmatrix}\end{aligned}$$

Therefore, the root condition of the first characteristics polynomial of the discrete schemes in (2.12) is determined by:

$$\begin{aligned}\Omega(\varsigma) &= \det(\varsigma P_2^{(1)}-P_1^{(1)})\\ &= \left|\varsigma P_2^{(1)}-P_1^{(1)}\right| = 0\end{aligned} \tag{3.1}$$

Now we have,

$$\Omega(\varsigma) = \left|\varsigma\begin{pmatrix}-1&0\\-1&1\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}\right| = \begin{pmatrix}-\varsigma&0\\-\varsigma&\varsigma\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}, \Rightarrow \Omega(\varsigma) = \begin{pmatrix}-\varsigma&-1\\-\varsigma&\varsigma\end{pmatrix}$$

Using Maple (2.17) software, we obtain:
$\Omega(\varsigma) = -\varsigma^2-\varsigma, \Rightarrow -\varsigma^2-\varsigma = 0 \Rightarrow \varsigma_1=-1, \varsigma_2=0$. Since $|\varsigma_i|<1,\; i=1,2$, the discrete schemes in (2.12) is zero stable. Applying the same procedure for (2.15) and can be presented as follows:

$$
\begin{aligned}
\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{v-2} \\ y_{v-1} \\ y_{v} \end{pmatrix} + r\begin{pmatrix} -\frac{7}{5} & -\frac{1}{5} & -\frac{11}{135} \\ -\frac{9}{20} & -\frac{29}{40} & \frac{83}{540} \\ -\frac{1}{20} & \frac{19}{40} & \frac{307}{540} \end{pmatrix}\begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \end{pmatrix} \\
&+ r\begin{pmatrix} 0 & 0 & -\frac{43}{135} \\ 0 & 0 & \frac{23}{1080} \\ 0 & 0 & \frac{7}{1080} \end{pmatrix}\begin{pmatrix} f_{v-2} \\ f_{v-1} \\ f_{v} \end{pmatrix} + r^2\begin{pmatrix} 0 & 0 & \frac{2}{45} \\ 0 & 0 & -\frac{11}{180} \\ 0 & 0 & -\frac{19}{180} \end{pmatrix}\begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \end{pmatrix} \\
&+ r^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{v-2} \\ g_{v-1} \\ g_{v} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
P_2^{(2)} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \; P_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q_2^{(2)} = \begin{pmatrix} -\frac{7}{5} & -\frac{1}{5} & -\frac{11}{135} \\ -\frac{9}{20} & -\frac{29}{40} & \frac{83}{540} \\ -\frac{1}{20} & \frac{19}{40} & \frac{307}{540} \end{pmatrix}, \\
R_2^{(2)} &= \begin{pmatrix} 0 & 0 & -\frac{43}{135} \\ 0 & 0 & \frac{23}{1080} \\ 0 & 0 & \frac{7}{1080} \end{pmatrix} \text{ and } S_2^{(2)} = \begin{pmatrix} 0 & 0 & \frac{2}{45} \\ 0 & 0 & -\frac{11}{180} \\ 0 & 0 & -\frac{19}{180} \end{pmatrix}
\end{aligned}
$$

Therefore, the root condition of the first characteristics polynomial of the discrete schemes in (2.15) is determined by:

$$
\begin{aligned}
\Omega(\varsigma) &= \det(\varsigma P_2^{(2)} - P_1^{(2)}) \\
&= \left|\varsigma P_2^{(2)} - P_1^{(2)}\right| = 0
\end{aligned} \tag{3.2}
$$

Now we have,

$$
\begin{aligned}
\Omega(\varsigma) &= \left|\varsigma\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & -\varsigma & 0 \\ \varsigma & -\varsigma & 0 \\ 0 & -\varsigma & \varsigma \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right| \\
\Rightarrow \Omega(\varsigma) &= \begin{pmatrix} 0 & -\varsigma & -1 \\ \varsigma & -\varsigma & 0 \\ 0 & -\varsigma & \varsigma \end{pmatrix}
\end{aligned}
$$

Using Maple (2.17) software, we obtain:
$\Omega(\varsigma) = \varsigma^3 + \varsigma^2, \Rightarrow \varsigma^3 + \varsigma^2 = 0 \Rightarrow \varsigma_1 = -1, \varsigma_2 = 0, \varsigma_3 = 0$. Since $|\varsigma_i| < 1,\ i = 1, 2, 3$, the discrete schemes in (2.15) is zero stable. By the same approach (2.18) can be presented as follows:

$$
\begin{aligned}
\begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_{v} \end{pmatrix} + r\begin{pmatrix} -\frac{7}{5} & -\frac{99}{160} & -\frac{9}{10} & \frac{149}{640} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{7}{135} & -\frac{83}{160} & -\frac{19}{30} & \frac{1831}{17280} \\ \frac{1}{45} & -\frac{41}{480} & \frac{47}{90} & \frac{3133}{5760} \end{pmatrix}\begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \end{pmatrix} \\
&+ r\begin{pmatrix} 0 & 0 & 0 & -\frac{201}{640} \\ 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{1920} \\ 0 & 0 & 0 & -\frac{17}{5760} \end{pmatrix}\begin{pmatrix} f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_{v} \end{pmatrix} + r^2\begin{pmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{11}{288} \\ 0 & 0 & 0 & -\frac{3}{32} \end{pmatrix}\begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \\ g_{v+4} \end{pmatrix} \\
&+ r^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{v-3} \\ g_{v-2} \\ g_{v-1} \\ g_{v} \end{pmatrix}
\end{aligned}
$$

$$P_2^{(3)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, P_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q_2^{(3)} = \begin{pmatrix} -\frac{7}{5} & -\frac{99}{160} & -\frac{9}{10} & \frac{149}{640} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{7}{135} & -\frac{83}{160} & -\frac{19}{30} & \frac{1831}{17280} \\ \frac{1}{45} & -\frac{41}{480} & \frac{47}{90} & \frac{3133}{5760} \end{pmatrix}$$

$$R_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{201}{640} \\ 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{1920} \\ 0 & 0 & 0 & -\frac{17}{5760} \end{pmatrix} \text{ and } \mathrm{S}_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{11}{288} \\ 0 & 0 & 0 & -\frac{3}{32} \end{pmatrix}$$

Therefore, the root condition of the first characteristics polynomial of the discrete schemes in (2.18) is determined by:

$$\begin{aligned} \Omega(\varsigma) &= \det(\varsigma P_2^{(3)} - P_1^{(3)}) \\ &= \left| \varsigma P_2^{(3)} - P_1^{(3)} \right| = 0 \end{aligned} \tag{3.3}$$

Now we have,

$$\Omega(\varsigma) = \left| \varsigma \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 & 0 & -\varsigma & 0 \\ \varsigma & 0 & -\varsigma & 0 \\ 0 & \varsigma & -\varsigma & 0 \\ 0 & 0 & -\varsigma & \varsigma \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right|$$

$$\Rightarrow \Omega(\varsigma) = \begin{pmatrix} 0 & 0 & -\varsigma & -1 \\ \varsigma & 0 & -\varsigma & 0 \\ 0 & \varsigma & -\varsigma & 0 \\ 0 & 0 & -\varsigma & \varsigma \end{pmatrix}$$

The following are obtained using Maple (2.17) software,
$\Omega(\varsigma) = -\varsigma^4 - \varsigma^3$, $\Rightarrow -\varsigma^4 - \varsigma^3 = 0$, $\Rightarrow \varsigma_1 = -1, \varsigma_2 = 0, \varsigma_3 = 0, \varsigma_4 = 0$. Since $|\varsigma_i| < 1,\ i = 1, 2, 3, 4$, the discrete schemes in (2.18) is zero stable.

3.4 Convergence

Since (2.12), (2.15) and (2.18) are consistent and zero stable, therefore they are convergent.

3.5 Region of Absolute Stability

The regions of absolute stability of the numerical methods for first order DDEs are considered. We considered finding the $P-$ and $Q-$stability by applying (2.12), (2.15) and (2.18) to the DDEs of this form:

$$\begin{aligned} &y'(t) = my(\mathrm{t}) \ + \ \mathrm{n}y(t-\tau), && t \geq t_0 \\ &y(t) = c(t), && t \leq t_0 \end{aligned} \tag{3.4}$$

where $c(t)$ is the initial function, a, b are complex coefficients, $\tau = ru,\ r \in \mathrm{Z}^+$, w is the step size and $u = \frac{\tau}{r}, r$ is a positive integer. Let $A_1 = rm$ and $A_2 = rn$, then from (2.12),
let

$$Y_{V+2} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-1} \\ y_v \end{pmatrix}, F_{V+2} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \end{pmatrix}, F_V = \begin{pmatrix} f_{v-1} \\ f_v \end{pmatrix}, G_{V+2} = \begin{pmatrix} g_{v+1} \\ g_{v+2} \end{pmatrix} \text{ and}$$

$$G_V = \begin{pmatrix} g_{v-1} \\ g_v \end{pmatrix}$$

Since

$$P_2^{(1)} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, P_1^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Q_2^{(1)} = \begin{pmatrix} -\frac{11}{12} & \frac{13}{48} \\ \frac{5}{12} & \frac{29}{48} \end{pmatrix}, R_2^{(1)} = \begin{pmatrix} 0 & -\frac{17}{48} \\ 0 & -\frac{1}{48} \end{pmatrix} \text{ and}$$

$$S_2^{(1)} = \begin{pmatrix} 0 & -\frac{1}{8} \\ 0 & -\frac{1}{8} \end{pmatrix}$$

we have,

$$P_2^{(1)} Y_{V+2} = P_1^{(1)} Y_{V+1} - r\left(\sum_{i=1}^{2}\left(Q_i^{(1)} F_{V+i} + rR_i^{(1)} G_{V+i} + rS_i^{(1)} G_{V+i}\right)\right) \tag{3.5}$$

Applying the same approach for (2.15), we have

$$Y_{V+3} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}, F_{V+3} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \end{pmatrix}, F_V = \begin{pmatrix} f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}, G_{V+3} = \begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \end{pmatrix} \text{ and}$$

$$G_V = \begin{pmatrix} g_{v-2} \\ g_{v-1} \\ g_v \end{pmatrix}$$

Since

$$P_2^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, P_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q_2^{(2)} = \begin{pmatrix} -\frac{7}{5} & -\frac{1}{5} & -\frac{11}{135} \\ -\frac{9}{20} & -\frac{29}{40} & \frac{83}{540} \\ -\frac{1}{20} & \frac{19}{40} & \frac{307}{540} \end{pmatrix}, R_2^{(2)} = \begin{pmatrix} 0 & 0 & -\frac{43}{135} \\ 0 & 0 & \frac{23}{1080} \\ 0 & 0 & \frac{7}{1080} \end{pmatrix}$$

$$\text{and } \mathrm{S}_2^{(2)} = \begin{pmatrix} 0 & 0 & \frac{2}{45} \\ 0 & 0 & -\frac{11}{180} \\ 0 & 0 & -\frac{19}{180} \end{pmatrix}$$

we have,

$$P_2^{(2)} Y_{V+2} = P_1^{(2)} Y_{V+1} - r\left(\sum_{i=1}^{2}\left(Q_i^{(2)} F_{V+i} + rR_i^{(2)} G_{V+i} + rS_i^{(2)} G_{V+i}\right)\right) \tag{3.6}$$

Applying the same approach for (2.18), we have:

$$Y_{V+4} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}, F_{V+4} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \end{pmatrix}, F_V = \begin{pmatrix} f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}, G_{V+4} = \begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \\ g_{v+4} \end{pmatrix}$$

$$\text{and } G_V = \begin{pmatrix} g_{v-3} \\ g_{v-2} \\ g_{v-1} \\ g_v \end{pmatrix}$$

Since

$$P_2^{(3)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, P_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q_2^{(3)} = \begin{pmatrix} -\frac{7}{5} & -\frac{99}{160} & -\frac{9}{10} & \frac{149}{640} \\ -\frac{17}{45} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{7}{135} & -\frac{83}{160} & -\frac{19}{30} & \frac{1831}{17280} \\ \frac{1}{45} & -\frac{41}{480} & \frac{47}{90} & \frac{3133}{5760} \end{pmatrix} R_2^{(3)} =$$

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{201}{640} \\ 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & -\frac{11}{1920} \\ 0 & 0 & 0 & -\frac{17}{5760} \end{pmatrix}, \text{ and } S_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{11}{288} \\ 0 & 0 & 0 & -\frac{3}{32} \end{pmatrix}$$

we have,

$$P_2^{(3)}Y_{V+2} = P_1^{(3)}Y_{V+1} - r\left(\sum_{i=1}^{2}\left(Q_i^{(3)}F_{V+i} + rR_i^{(3)}G_{V+i} + rS_i^{(3)}G_{V+i}\right)\right) \tag{3.7}$$

The polynomials of $P-$ and $Q-$stability are constructed by applying (3.5), (3.6) and (3.7) to (3.4) and (2.12), (2.15) and (2.18) to (3.4) as represented below:

$$\Omega^{(1)}(\varsigma) = \det\left[\begin{array}{l}\left(P_2^{(1)} - A_1\left(Q_2^{(1)} - R_2^{(1)} - S_2^{(1)}\right)\right)\varsigma^{2+u} \\ -\left(P_1^{(1)} - A_1\left(Q_1^{(1)} - R_1^{(1)} - S_1^{(1)}\right)\right)\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(1)} - R_i^{(1)} - S_i^{(1)}\right)\varsigma^i\right)\end{array}\right] \tag{3.8}$$

$$\Omega^{(2)}(\varsigma) = \det\left[\begin{array}{l}\left(P_2^{(2)} - A_1\left(Q_2^{(2)} - R_2^{(2)} - S_2^{(2)}\right)\right)\varsigma^{2+u} \\ -\left(P_1^{(2)} - A_1\left(Q_1^{(2)} - R_1^{(2)} - S_1^{(2)}\right)\right)\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(2)} - R_i^{(2)} - S_i^{(2)}\right)\varsigma^i\right)\end{array}\right] \tag{3.9}$$

$$\Omega^{(3)}(\varsigma) = \det\left[\begin{array}{l}\left(P_2^{(3)} - A_1\left(Q_2^{(3)} - R_2^{(3)} - S_2^{(3)}\right)\right)\varsigma^{2+u} \\ -\left(P_1^{(3)} - A_1\left(Q_1^{(3)} - R_1^{(3)} - S_1^{(3)}\right)\right)\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(3)} - R_i^{(3)} - S_i^{(3)}\right)\varsigma^i\right)\end{array}\right] \tag{3.10}$$

$$\Phi^{(1)}(\varsigma) = \det\left[P_2^{(1)}\varsigma^{2+u} - P_1^{(1)}\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(1)} - R_i^{(1)} - S_i^{(1)}\right)\varsigma^i\right)\right] \tag{3.11}$$

$$\Phi^{(2)}(\varsigma) = \det\left[P_2^{(2)}\varsigma^{2+u} - P_1^{(2)}\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(2)} - R_i^{(2)} - S_i^{(2)}\right)\varsigma^i\right)\right] \tag{3.12}$$

$$\Phi^{(3)}(\varsigma) = \det\left[P_2^{(3)}\varsigma^{2+u} - P_1^{(3)}\varsigma^{1+u} - A_2\left(\sum_{i=1}^{2}\left(Q_i^{(3)} - R_i^{(3)} - S_i^{(3)}\right)\varsigma^i\right)\right] \tag{3.13}$$

Making use of Maple 18 and MATLAB, the region of $P-$ and $Q-$ stability for (2.12), (2.15) and (2.18) are shown in Fig 1 to 6.
The $P-$stability regions in Figs 1 to 3 lie inside the open-ended region while the $Q-$stability regions in Figs 4 to 6 lie inside the enclosed region.

4 Numerical simulations/sensitivity analysis

In this section, some first-order delay differential equations shall be solved using (2.12), (2.15) and (2.18) of the discrete schemes been worked-out. The delay argument shall be evaluated using the idea of sequence derived by [13].

4.1 Implementation of Numerical Examples

Example 4.1.

$$\begin{aligned} y'(t) &= -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}),\ 0 \leq t \leq 3 \\ y(t) &= 1 + e^{-3t},\ t \leq 0 \end{aligned}$$

Exact solution $y(t) = 1 + e^{-3t}$

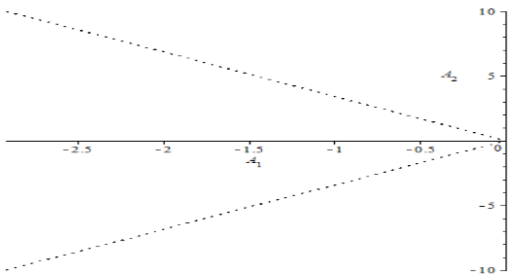

Figure 1: Region of $P-$stability (SDBAMM) in (2.12)

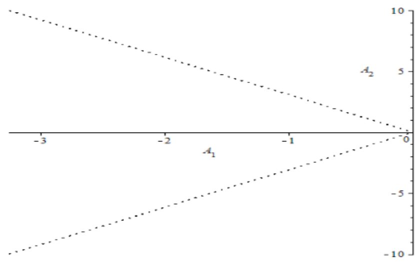

Figure 2: Region of $P-$ stability (SDBAMM) in (2.15)

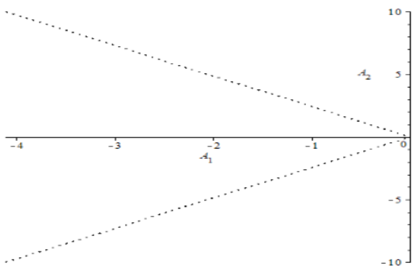

Figure 3: Region of $P-$ stability (SDBAMM) in (2.18)

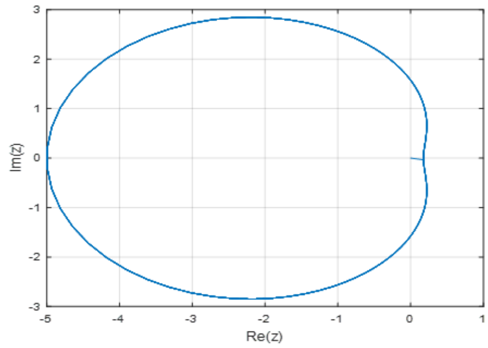

Figure 4: Region of $Q-$ stability (SDBAMM) in (2.12)

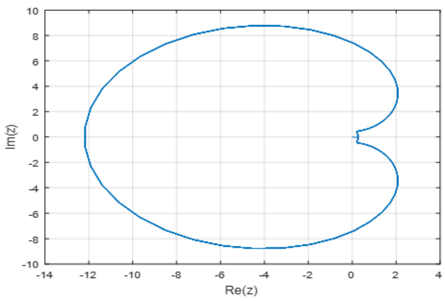

Figure 5: Region of $Q-$stability (SDBAMM) in (2.15)

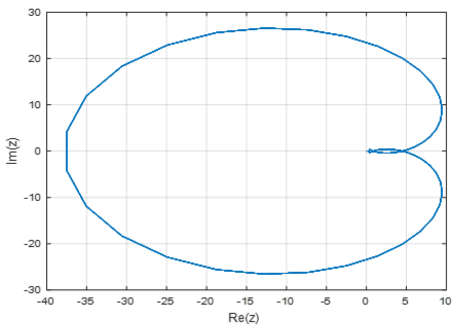

Figure 6: Region of $Q-$ stability (SDBAMM) in (2.18)

Example 4.2.

$$\begin{aligned} y'(t) &= -y(t-1+e^{-t}) + \sin(t-1+e^{-t}) + \cos(t),\ 0 \le t \le 3 \\ y(t) &= \sin(t),\ t \le 0 \end{aligned}$$

Exact Solution $y(t) = \sin(t)$

These examples were solved using the discrete schemes in (2.12), (2.15) and (2.18), and the solutions are presented below:

t	Exact Result	k= 2 Numerical Result	k= 3 Numerical Result	k= 4 Numerical Result
0.1	1.970445534	1.970437881	1.97044904	1.970443562
0.2	1.941764534	1.941788077	1.941758789	1.941766732
0.3	1.913931185	1.913918886	1.913952019	1.913926936
0.4	1.886920437	1.886945601	1.886920095	1.886939106
0.5	1.860707976	1.860695746	1.860704496	1.860703629
0.6	1.835270211	1.835294291	1.835287789	1.83527321
0.7	1.810584246	1.810572646	1.810584183	1.810579668
0.8	1.786627861	1.786650587	1.786624556	1.786645263
0.9	1.763379494	1.763368559	1.763395661	1.763375519
1.0	1.740818221	1.740839629	1.740818146	1.740820925
1.1	1.718923733	1.718913433	1.71892072	1.718919639
1.2	1.697676326	1.697696488	1.697691096	1.697691798
1.3	1.677056874	1.677047175	1.677056807	1.677053344
1.4	1.65704682	1.657065808	1.657044067	1.657049221
1.5	1.637628152	1.637619016	1.63764165	1.637624519
1.6	1.618783392	1.618801274	1.61878333	1.618797115
1.7	1.600495579	1.600486976	1.600493063	1.600492447
1.8	1.582748252	1.582765095	1.582760589	1.582750382
1.9	1.565525439	1.565517337	1.565525382	1.565522216
2.0	1.548811636	1.548827497	1.548809336	1.548823808
2.1	1.532591801	1.53258417	1.532603078	1.532589024
2.2	1.516851334	1.516866272	1.516851283	1.516853223
2.3	1.501576069	1.501568884	1.501573968	1.501573211
2.4	1.486752256	1.486766323	1.486762561	1.486763052
2.5	1.472366553	1.472359784	1.472366506	1.472364089
2.6	1.458406011	1.458419258	1.458404091	1.458407685
2.7	1.444858066	1.444851693	1.444867484	1.444855532
2.8	1.431710523	1.431723001	1.43171048	1.431720099
2.9	1.418951549	1.418945547	1.418949794	1.418949363
3.0	1.40656966	1.40658141	1.406578267	1.406571146

Table 1: Numerical Results of Example 4.1 for SDBAMM $k = 2, 3$ and 4

t	Exact Result	k= 2 Numerical Result	k= 3 Numerical Result	k=4 Numerical Result
0.1	0.009999833	0.010012581	0.009988966	0.010009576
0.2	0.019998667	0.019998667	0.019994091	0.02000444
0.3	0.0299955	0.03000849	0.029980057	0.030005243
0.4	0.039989334	0.039989334	0.039962721	0.039989334
0.5	0.049979169	0.049992396	0.049959023	0.049989263
0.6	0.059964006	0.059964006	0.059932691	0.059969988
0.7	0.069942847	0.069956306	0.06990007	0.069952942
0.8	0.079914694	0.079914694	0.079878552	0.079914694
0.9	0.089878549	0.089892235	0.089830946	0.089888979
1.0	0.099833417	0.099833417	0.09977407	0.099839597
1.1	0.109778301	0.109792207	0.109725753	0.109788731
1.2	0.119712207	0.119712207	0.119647916	0.119712207
1.3	0.129634143	0.129648265	0.129557837	0.129644892
1.4	0.139543115	0.139543115	0.139473765	0.139549484
1.5	0.149438132	0.149452464	0.149356768	0.149448881
1.6	0.159318207	0.159318207	0.159224568	0.159318207
1.7	0.169182349	0.169196885	0.169095817	0.1691934
1.8	0.179029573	0.179029573	0.178930766	0.179036122
1.9	0.188858895	0.188873629	0.188747564	0.188869946
2.0	0.198669331	0.198669331	0.19856525	0.198669331
2.1	0.2084599	0.208474826	0.208343295	0.208471235
2.2	0.218229623	0.218229623	0.218100257	0.21823634
2.3	0.227977524	0.227992637	0.227855546	0.227988858
2.4	0.237702626	0.237702626	0.237567886	0.237702627
2.5	0.247403959	0.247419253	0.247256231	0.24741556
2.6	0.257080552	0.257080552	0.256940343	0.257087427
2.7	0.266731437	0.266746904	0.266578239	0.266743038
2.8	0.276355649	0.276355649	0.276189248	0.276355649
2.9	0.285952225	0.285967861	0.285793468	0.285964073
3.0	0.295520207	0.295520207	0.295348246	0.295527228

Table 2: Numerical Results of Example 4.2 for SDBAMM $k = 2, 3$ and 4

5 Discussions and Conclusion

Here, the performances of the schemes derived in (2.12), (2.15) and (2.18), shall be applied in solving the two examples above by finding their absolute errors.

5.1 Analysis of Results

The analysis of results is obtained by evaluating absolute difference of exact results and numerical results. The results are summarized in the tables 3 to 4

t	k= 2 Error	k= 3 Error	k= 4 Error
0.1	7.65255E-06	3.50645E-06	1.97155E-06
0.2	2.35434E-05	5.74458E-06	2.19842E-06
0.3	1.22993E-05	2.08337E-05	4.24927E-06

0.4	2.51643E-05	3.41717E-07	1.86693E-05
0.5	1.22304E-05	3.48043E-06	4.34743E-06
0.6	2.40796E-05	1.75776E-05	2.99859E-06
0.7	1.16E-05	6.29702E-08	4.57797E-06
0.8	2.27259E-05	3.30507E-06	1.74019E-05
0.9	1.09353E-05	1.61667E-05	3.97534E-06
1.0	2.14083E-05	7.46817E-08	2.70432E-06
1.1	1.03004E-05	3.01343E-06	4.09443E-06
1.2	2.01619E-05	1.47699E-05	1.54719E-05
1.3	9.6995E-06	6.74982E-08	3.5305E-06
1.4	1.89882E-05	2.75282E-06	2.40118E-06
1.5	9.13562E-06	1.34984E-05	3.63262E-06
1.6	1.78822E-05	6.18061E-08	1.37232E-05
1.7	8.60281E-06	2.51581E-06	3.13181E-06
1.8	1.68426E-05	1.23366E-05	2.12963E-06
1.9	8.1017E-06	5.66995E-08	3.2227E-06
2.0	1.58609E-05	2.30009E-06	1.21719E-05
2.1	7.63101E-06	1.1277E-05	2.77701E-06
2.2	1.49375E-05	5.14917E-08	1.88851E-06
2.3	7.18507E-06	2.10107E-06	2.85807E-06
2.4	1.4067E-05	1.0305E-05	1.0796E-05
2.5	6.76874E-06	4.6741E-08	2.46374E-06
2.6	1.32467E-05	1.92031E-06	1.67369E-06
2.7	6.37322E-06	9.41778E-06	2.53422E-06
2.8	1.24776E-05	4.34291E-08	9.57557E-06
2.9	6.00225E-06	1.75525E-06	2.18625E-06
3.0	1.17503E-05	8.60726E-06	1.48626E-06

Table 3: Absolute Errors for SDBAMM $k = 2, 3$ and 4 using Example 4.1

t	k= 2 Error	k=3 Error	k=4 Error
0.1	1.27475E-05	1.08674E-05	9.7424E-06
0.2	6.66692E-12	4.57575E-06	5.77327E-06
0.3	1.29899E-05	1.54432E-05	9.7424E-06
0.4	1.33658E-11	2.66127E-05	6.63416E-12
0.5	1.32271E-05	2.01461E-05	1.00942E-05
0.6	1.05554E-11	3.13157E-05	5.98174E-06
0.7	1.3459E-05	4.27772E-05	1.00942E-05
0.8	1.08273E-11	3.61416E-05	1.91727E-11
0.9	1.36855E-05	4.76031E-05	1.04299E-05
1.0	1.31718E-11	5.93464E-05	6.18065E-06
1.1	1.39066E-05	5.25476E-05	1.04299E-05
1.2	1.10806E-11	6.42909E-05	1.10806E-11
1.3	1.4122E-05	7.63053E-05	1.07489E-05
1.4	4.42365E-11	6.93496E-05	6.36976E-06
1.5	1.43318E-05	8.13641E-05	1.07489E-05

1.6	1.4246E-11	9.36388E-05	8.5754E-11
1.7	1.45359E-05	8.65324E-05	1.10507E-05
1.8	2.58242E-11	9.88071E-05	6.54857E-06
1.9	1.47342E-05	0.000111331	1.10507E-05
2.0	4.93877E-12	0.00010408	1.04939E-10
2.1	1.49266E-05	0.000116604	1.13349E-05
2.2	8.08693E-11	0.000129366	6.71702E-06
2.3	1.5113E-05	0.000121978	1.13349E-05
2.4	2.71346E-11	0.00013474	7.28654E-11
2.5	1.52933E-05	0.000147729	1.16008E-05
2.6	7.84489E-12	0.000140209	6.87461E-06
2.7	1.54676E-05	0.000153197	1.16008E-05
2.8	6.41138E-11	0.000166401	3.58862E-11
2.9	1.56356E-05	0.000158757	1.18481E-05
3.0	1.6134E-10	0.00017196	7.02114E-06

Table 4: Absolute Errors of SDBAMM $k = 2, 3$ and 4 using Example 4.2

5.2 Conclusions

In conclusion, it is observed that SDBAMM schemes for step numbers $k = 2, 3$, and 4 are suitable for solving DDEs. The discrete schemes of (2.12), (2.15) and (2.18) obtained from the different continuous forms were proved to be convergent, $P-$stable and $Q-$stable. Also, it was discovered in tables 3 to 4 that the SDBAMM for $k = 4$ scheme performed better than the SDBAMM schemes for step numbers $k = 3$ and $k = 2$ when compared with the exact results respectively. It is recommended that the SDBAMM schemes of higher step numbers perform better than the SDBAMM schemes of lower step numbers. Further study should be carried out for step number $k = 5, 6, 7...$ on the formulation of discrete schemes of SDBAMM for solving DDEs without the introduction of interpolation formula in evaluating the delay argument.

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