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The Singly Implicit Runge-Kutta Methods for nonlinear singular initial value problems

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Abstract

The use of collocation process for constructing numerical methods for solving ordinary differential equations has been attractive for nonlinear singular initial value problems. In this paper, a class of singly implicit Runge-Kutta methods has been developed which can efficiently solve nonlinear singular initial value problems in ordinary differential equations. The methods are derived based on collocation at the polynomial nodes which produce dense output within the integration interval. Preliminary numerical calculations using the new methods clearly show improved performance, efficiency and effectiveness compared to some methods with strong algebraic stability properties. In order to capture the system behavior for long period, graphic surface curves of phase plots were generated, which show very strange and new behaviors. The obtained surface phase plots are portions of the phase space of the systems, often depicting real world observed facts.

Keywords and phrases: block hybrid method; collocation method; continuous scheme; singular initial value problem; stiff oscillatory system

1 Introduction

Among nonlinear systems, numerous academic disciplines and fields, including natural science, engineering, social science, psychology and economics, have conducted substantial research in recent years on the analysis and use of chaotic phenomena [1]. Collocation methods have been proved beyond reasonable doubt to be good candidates with A-stable regions of absolute stability for the application to chaotic occurrences. This stability property is very important for solving stiff problems and problems with highly oscillating solutions [2,3]. In addition, due to the success of the enormous roles which various collocation methods have been playing especially in the integration of differential equations [4,5,6] we are very much motivated to employ the class of collocation methods for the treatment of singular systems with oscillatory solutions. In particular, the class is very efficient in dealing with stiff, highly oscillatory problems and dynamical systems with non linearity which are of great importance in the application fields, such as in science, engineering and technology. Therefore, the research work in this paper is motivated by the desire to develop singly implicit Runge-Kutta methods which are more accurate in dealing with both linear and nonlinear singular initial value problems with oscillatory solutions of the form

$$y'(x) = f(x, y(x)), \qquad [x_0, b], \tag{1}$$

with initial condition

$$y(x_0) \ = \ y_0.$$

The numerical integration of (1) involves difficulties because the solution often features a narrow initial

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boundary layer region of rapid change, as well as so-called stiffness in the outer region where $x > 0$ [7]. By using a combination of numerical techniques, that is, single-step and multistep collocations, we can obtain an effective solution method. The solution of the equation in (1) can be describe by the following interpolation polynomial

$$y(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + ... + \phi_{p-1}x^{p-1} = \sum_{i=0}^{p-1} \phi_i x^i. \tag{2}$$

We set the sum $p = t + s$ to be able to determine $\{\phi_i\}$ uniquely, where $t > 0$ denotes the number of interpolation points used and $s > 0$ are the distinct collocation points chosen from the interval $\bar{x}_j \in [x_n, x_{n+1}]$. Our main aim is to utilize not only the interpolation points $\{x_{n+i}\}$ but also several collocation points on the interpolation interval and to fit $y(x)$ and $y'(x)$ in the sense of [8]. Now to determine the parameters $\{\phi_i\}, \;\; i = 0, 1, 2, .., p-1$ we impose the following conditions,

$$y(x_{n+i}) = y_{n+i}, \qquad (i = 0, 1, 2, ..., t-1), \tag{3}$$

$$y'(c_{n+j}) = y'_{n+j}, \qquad (j = 0, 1, 2, ..., s-1), \tag{4}$$

Here c_{n+j} are the collocation points distributed on the step-points array, y_{n+i} are the interpolation data of $y(x)$ on $\{x_{n+i}\}$, and $y'(x)$ are the collocation data of $y(x)$ on $\{c_{n+j}\}$. In fact, equations (3) and (4) can be expressed in a matrix form as follows

$$M\phi = V, \tag{5}$$

where the p-square matrix M, p-vectors ϕ and V are defined as follows

$$M = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^t & \cdots & x_n^{t+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^t & \cdots & x_n^{t+s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^t & \cdots & x_{n+t-1}^{t+s-1} \\ 0 & 1 & 2x_0 & \cdots & tx_1^{t-1} & \cdots & D'x_1^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_s & \cdots & tx_s^{t-1} & \cdots & D'x_s^{p-1} \end{pmatrix}, \tag{6}$$

$$\phi = (\phi_0, \phi_1, \phi_2, ..., \phi_{p-1})^T, \;\; V = (y_n, y_{n+u}, ..., y_{n+t-1}, y'_n, y'_{n+u}, ..., y'_{n+s-1})^T$$

where D' in (6) represent the first derivative corresponding to the differentiation with respect to x. Similar to the Vandermonde matrix M in (5) is non-singular as proved explicitly by Akinola and Akoh in [9]. A closed form solution for the system in (5) is presented which has been obtained by considering the inverse of the matrix M, that is,

$$\phi = M^{-1}V \quad where \quad M^{-1} = U. \tag{7}$$

Re-arranging equations (5)-(7) to have the multistep collocation formula of the form

$$y(x) = \sum_{j=0}^{t-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \upsilon_j(x) f(\bar{x}_j, y(\bar{x}_j)), \tag{8}$$

where $\phi_j(x)$ and $\upsilon_j(x)$ are continuous coefficients of the method and are also assumed polynomials of the form

$$\phi_j(x) = \sum_{i=0}^{t+s-1} \phi_{j,i+1} x^i, \quad j \in \{0, 1, 2, ..., t-1\}, \tag{9}$$

$$h\upsilon_j(x) = h \sum_{i=0}^{t+s-1} \upsilon_{j,i+1} x^i, \quad j = 0, 1, 2, ..., s-1, \tag{10}$$

The step size h in (8) can be a constant or varied in the numerical integration process. The basis function $\{x^i\}$ in (9) and (10) is defined as

$$H_i(x) = \sum_{i=0}^{p-1} x^i.$$

Here y and f are smooth real N-dimensional vector functions. The numerical constant coefficients $\phi_{j,i+1}$ and $\upsilon_{j,i+1}$ in (9) and (10) are to be determined. They are selected so that accurate approximations of well-behaved problems are obtained efficiently. In fact, the above constant coefficients can be obtained from the component of the matrix M^{-1}. The choice $U = M^{-1}$ leads to the determination of the numerical constant coefficients $\phi_{j,i+1}$ and $\upsilon_{j,i+1}$ in (9) and (10). Actual evaluation of the matrices M and U are carried out with a computer algebra system, for example Maple.
The method in (8) can be more efficient than the standard conventional Runge-Kutta methods since the number of function evaluations is fewer. It is convenient to write the coefficients of the defining method in (8) evaluated at some points in the block matrix form as

$$Y = e \bigotimes y_n + h(A \bigotimes I_N)F(Y),$$

$$y_{n+1} = y_n + h(b^T \bigotimes I_N)F(Y), \tag{11}$$

where $A = a_{ij}$ denotes the $(t+s) \times (t+s)$ real matrix, $b^T = b_i$ and $c = c_i$ are real vectors of dimension $(t+s)$ and $c_i \in [0, 1], \ i = 1, 2, ..., t+s-1$. I is the identity matrix of size equal to the differential equation system to be solved and N is the dimension of the system. Also $\bigotimes$ is the Kronecker product of two matrices and *e* is the $s \times I$ vector of units. For simplicity, we rewrite the method in (11) as follows

$$Y = y_n + hAF(Y),$$

$$y_{n+1} = y_n + hb^T F(Y). \tag{12}$$

Butcher's array provides the following description of the singly implicit Runge-Kutta collocation methods compactly as follows,

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{13}$$

If matrix A in the Butcher's array is a lower triangular matrix with a zero main diagonal, explicit Runge-Kutta method is obtained. This approach is called a diagonally implicit method if matrix A is a lower triangular matrix with non-zero members on the major diagonal. Because the internal stage depends on the current stage as well as on previous stages, the method is also known as the implicit Runge-Kutta method. The implicit Runge-Kutta technique evaluates the stage values using Newton's iteration. The implicit methods have an algebraic nicety that the explicit methods lack because the set of implicit methods under a sufficiently natural operation is homomorphic to a particular group. Conversely, the explicit methods subset is merely a semigroup [10].

Theorem 1.1: *Let y(x) in (8)be differentiable of any order required and satisfies the interpolation and collocation conditions defined explicitly in (3) and (4). Then*

$$y(x) \;=\; V^T(M^{-1})^T(H_i(x))^T \tag{14}$$

is the proposed continuous scheme, where M and U are as defined in (5) and (7) respectively and for the existence of the square matrix U in (7), M in (5) is assumed to be a non-singular matrix.

Proof: The proof is similar to the approach given in Yakubu et al.[11,12].

2 Development of the Solution Methods

2.1 Singly Implicit Runge-Kutta Method of Order Four

In this segment we consider some specific methods of the singly implicit Runge-Kutta methods with high order of accuracy. In order to make the methods competitive with respect to their counterparts in the Runge-Kutta family(see, for example [13, 14, 15, 16, 17]) the collocation parameters are carefully chosen in such a way that the methods will have smaller error constant with A-stable regions of absolute stability. This is because among the Runge-Kutta methods, those with smaller error constants, A-stable or L-stable are preferred when integrating differential equations. Hence, the $(M-1)$ collocation points called the off-steps or intra-step points, are introduced to improve the accuracy of the methods. They are denoted by u and v. These values are also specifically chosen to ensure that the methods can accurately approximate linear and nonlinear singular problems and to maintain zero stability (see, [18]). Following [12] we obtain the first method of order four as follows,

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{7-\sqrt{7}}{14} & \frac{55+2\sqrt{7}}{504} & \frac{98+7\sqrt{7}}{504} & \frac{98-47\sqrt{7}}{504} & \frac{1+2\sqrt{7}}{504} \\
\frac{7+\sqrt{7}}{14} & \frac{55-2\sqrt{7}}{504} & \frac{98+47\sqrt{7}}{504} & \frac{98-7\sqrt{7}}{504} & \frac{1-2\sqrt{7}}{504} \\
1 & \frac{1}{9} & \frac{7}{18} & \frac{7}{18} & \frac{1}{9} \\
\hline
 & \frac{14}{126} & \frac{49}{126} & \frac{49}{126} & \frac{14}{126}
\end{array} \tag{15}$$

2.2 Singly Implicit Runge-Kutta Method of Order Five

In the second method, we also choose the collocation points carefully to have a better method with the mentioned requirements above in the first subsection. Hence in addition to u and v above, we select w to have three interior collocation points that will give a better method as follows,

$$\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7-\sqrt{7}}{14} & \frac{322595-90290\sqrt{7}}{88200} & \frac{-902090-258755\sqrt{7}}{88200} & \frac{1752240-558480\sqrt{7}}{88200} & \frac{-1225490-415205\sqrt{7}}{88200} & \frac{96845-31490\sqrt{7}}{88200} \\
\frac{1}{2} & \frac{27643}{10080} & \frac{-79576-6615\sqrt{7}}{10080} & \frac{129696}{10080} & \frac{-79576+6615\sqrt{7}}{10080} & \frac{6853}{10080} \\
\frac{7+\sqrt{7}}{14} & \frac{64519+18058\sqrt{7}}{17920} & \frac{-245098-83041\sqrt{7}}{17920} & \frac{350448+111696\sqrt{7}}{17920} & \frac{-180418-51751\sqrt{7}}{17920} & \frac{19369+6298\sqrt{7}}{17920} \\
1 & \frac{12236}{630} & \frac{-51107-2940\sqrt{7}}{630} & \frac{85512}{630} & \frac{-51107+2940\sqrt{7}}{630} & \frac{5096}{630} \\
\hline
 & \frac{12236}{630} & \frac{-51107-2940\sqrt{7}}{630} & \frac{85512}{630} & \frac{-51107+2940\sqrt{7}}{630} & \frac{5096}{630}
\end{array} \tag{16}$$

2.3 Singly Implicit Runge-Kutta Method of Order Five

Following a similar choice as in method (16) above but varying the value of w we obtain the below method as,

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7-\sqrt{7}}{14} & \frac{164353-44866\sqrt{7}}{52920} & \frac{1801926-709695\sqrt{7}}{52920} & \frac{-1869056+595712\sqrt{7}}{52920} & \frac{-216482+201887\sqrt{7}}{52920} & \frac{145719-46818\sqrt{7}}{52920} \\
\frac{3}{4} & \frac{120827}{17920} & \frac{664048-376859\sqrt{7}}{17920} & -\frac{1558144}{17920} & \frac{664048+376859\sqrt{7}}{17920} & \frac{122661}{17920} \\
\frac{7+\sqrt{7}}{14} & \frac{164353+44866\sqrt{7}}{52920} & \frac{-216482-201887\sqrt{7}}{52920} & \frac{-1869056+595712\sqrt{7}}{52920} & \frac{1801926+709695\sqrt{7}}{52920} & \frac{145719+46818\sqrt{7}}{52920} \\
1 & \frac{4226}{270} & \frac{27979-15512\sqrt{7}}{270} & -\frac{65152}{270} & \frac{27979+15512\sqrt{7}}{270} & \frac{5238}{270} \\
\hline
 & \frac{4226}{270} & \frac{27979-15512\sqrt{7}}{270} & -\frac{65152}{270} & \frac{27979+15512\sqrt{7}}{270} & \frac{5238}{270}
\end{array}
\tag{17}
$$

We can apply the derived methods in two ways as block methods or as Singly Implicit Runge-Kutta collocation methods.

3 Analysis of the Properties of the SIRKMs

3.1 Order of the Singly Implicit RK Methods

Among the fundamental properties of the singly implicit Runge-Kutta methods that were derived in the previous section, are order, zero-stability, consistency, order of convergence, and stability characteristics which we will calculate in this section. In order to determine the above properties of the new methods, we first compute the order of method (17) in subsection (2.3) above. In accordance with Lambert [19, 20], we obtain the matrices of coefficients as follows:

$$
\phi_0 = -\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \phi_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \phi_w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \phi_v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
$$

$$
\upsilon_0 = \begin{bmatrix} \frac{164353-44866\sqrt{7}}{52920} \\ \frac{120827}{17920} \\ \frac{164353+44866\sqrt{7}}{52920} \\ \frac{4226}{270} \end{bmatrix}, \quad \upsilon_u = \begin{bmatrix} \frac{1801926-709695\sqrt{7}}{52920} \\ \frac{664048-376859\sqrt{7}}{17920} \\ \frac{-216482-201887\sqrt{7}}{52920} \\ \frac{27979-15512\sqrt{7}}{270} \end{bmatrix}, \quad \upsilon_w = \begin{bmatrix} \frac{-1869056+595712\sqrt{7}}{52920} \\ -\frac{1558144}{17920} \\ -\frac{1869056-595712\sqrt{7}}{52920} \\ -\frac{65152}{270} \end{bmatrix},
$$

$$
\upsilon_v = \begin{bmatrix} \frac{-216482+201887\sqrt{7}}{52920} \\ \frac{664048+376859\sqrt{7}}{17920} \\ \frac{1801926+709695\sqrt{7}}{52920} \\ \frac{27979+15512\sqrt{7}}{270} \end{bmatrix}, \quad \upsilon_1 = \begin{bmatrix} \frac{145719-46818\sqrt{7}}{52920} \\ \frac{122661}{17920} \\ \frac{145719+46818\sqrt{7}}{52920} \\ \frac{5238}{270} \end{bmatrix}.
$$

To obtain the order, we substitute the above matrices of coefficients into the following,

$$C_0 = \phi_0 + \phi_u + \phi_w + \phi_v + \phi_1 = 0,$$

$$C_1 = [u\phi_u + w\phi_w + v\phi_v + \phi_1] - (\upsilon_0 + \upsilon_u + \upsilon_w + \upsilon_v + \upsilon_1) = 0,$$

$$C_2 = \frac{1}{2}[(u)^2\phi_u + (w)^2\phi_w + (v)^2\phi_v + \phi_1] - ((u)\upsilon_u + (w)\upsilon_w + +(v)\upsilon_v + \upsilon_1) = 0,$$

$$C_3 = \frac{1}{3}[(u)^3\phi_u + (w)^3\phi_w + (v)^3\phi_v + \phi_1] - \left((u)^2\upsilon_u + (w)^2\upsilon_w + (v)^2\upsilon_v + \upsilon_1\right) = 0,$$

$$\vdots \qquad\qquad \vdots,$$

$$C_6 = \frac{1}{6}[(u)^6\phi_u + (w)^6\phi_w + (v)^6\phi_v + \phi_1] - \left((u)^5\upsilon_u + (w)^5\upsilon_w + (v)^5\upsilon_v + \upsilon_1\right) \neq 0,$$

and after routine algebraic simplifications we have,

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \; but \; C_6 \neq 0,$$

which indicates that the highest order of the newly derived methods is five (5) with error constants and orders displayed in Table 1.

3.2 Zero-stability Analysis of the SIRK Methods

Since zero stability guarantees that the method will yield significant findings even when the initial conditions are just approximate, it is an essential feature of new numerical techniques. A non-zero-stable numerical method may yield results that are not exactly the same as the original solution or that grow unbounded during computation. In certain circumstances, the method may be considered unreliable and may require modification in order to produce more precise results. Zero-stability provides an analytical method for the stability of a numerical approach by efficiently treating it as a difference scheme. The resulting difference scheme's is equal to a numerical method's zero-stability. Expressing the singly implicit Runge-Kutta method in a matrix form gives the equation as follows

$$U^{(1)}Y_\mu = U^{(0)}Y_{\mu-1} + h\left(V^{(0)}F_{\mu-1} + V^{(1)}F_\mu\right), \tag{18}$$

where

$$Y_\mu = (y_{n+u}, y_{n+w}, ..., y_{n+v}, y_{n+1})^T, \;\; Y_{\mu-1} = (y_{n-u}, y_{n-w}, ..., y_{n-v}, y_n)^T,$$

$$F_\mu = (f_{n+u}, f_{n+w}, ..., f_{n+v}, f_{n+1})^T, \;\; F_{\mu-1} = (f_{n-u}, f_{n-w}, ..., f_{n-v}, f_n)^T,$$

and the matrices $U^{(1)}$,$U^{(0)}$,$V^{(1)}$,and $V^{(0)}$ in this case are a maximum of 4 by 4 matrices, with their corresponding entries given by the coefficients of the method in (17). The matrices for the method are given as below where $U^{(1)}$ is the matrix identity and of dimension 4.

$$U^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad U^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$V^{(1)} = \begin{bmatrix} \frac{1801926-709695\sqrt{7}}{52920} & \frac{-1869056+595712\sqrt{7}}{52920} & \frac{-216482+201887\sqrt{7}}{52920} & \frac{145719-46818\sqrt{7}}{52920} \\ \frac{664048-376859\sqrt{7}}{17920} & \frac{-1558144}{17920} & \frac{664048+376859\sqrt{7}}{17920} & \frac{122661}{17920} \\ \frac{-216482-201887\sqrt{7}}{52920} & \frac{-1869056-595712\sqrt{7}}{52920} & \frac{1801926+709695\sqrt{7}}{52920} & \frac{145719+46818\sqrt{7}}{52920} \\ \frac{27979-15512\sqrt{7}}{270} & \frac{-65152}{270} & \frac{27979+15512\sqrt{7}}{270} & \frac{5238}{270} \end{bmatrix},$$

$$V^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{164353-44866\sqrt{7}}{52920} \\ 0 & 0 & 0 & \frac{120827}{17920} \\ 0 & 0 & 0 & \frac{164353+44866\sqrt{7}}{52920} \\ 0 & 0 & 0 & \frac{4226}{270} \end{bmatrix}.$$

We note that the zero-stability of a method is just a limit as h tends to zero. Therefore, as $h \to 0$, the method in (17) tends to the difference system

$$U^{(1)} Y_\mu - U^{(0)} Y_{\mu-1} \;=\; 0,$$

with the first characteristic polynomial $\rho(R)$ of the form

$$\rho(R) = det\left(RU^{(1)} - U^{(0)}\right). \tag{20}$$

Substituting values of $U^{(1)}$ and $U^{(0)}$ from (19) into equation(20) we have

$$\rho(R) = \left(R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right). \tag{21}$$

On simplification we have,

$$\rho(R) = R^3(R-1) = 0, \quad R = (0,0,0,1). \tag{22}$$

Since $R = (0,0,0,1)$, method (17) is zero stable.
Similarly, reformulating methods (15) and (16) in block form using the matrix difference equation of the form in (18) we can show that (15) and (16) are zero-stable.

Definition 3.1:[13] *(Zero-stable) A family of methods in (8) are said to be zero-stable if the roots condition*

$$\rho(R) = det\left[\sum_{i=0}^{k} A^i \lambda^{k-1}\right] = 0,$$

satisfies the inequality $|\lambda_j| \leq 1,\; j = 1, 2, 3, ..., k$. *But when the inequality reduces to* $|\lambda_j| = 1$, *then the multiplicity should not be greater than two.*

Based on the definition in 3.1, the newly derived singly implicit Runge-Kutta methods are zero-stable.

From our calculations the orders and error constants for the constructed singly implicit Runge-Kutta methods are presented in Table 1. It is clear from the Table that the methods are of high-order and hence more accurate than some of their counterparts in existence.

Table 1: Orders and Error Constants of the SIRK Methods

Method	Order	Error constant
Order four method (15)	$p = 4$	$C_5 = \ -1.2500 \times 10^{-2}$
Order five method (16)	$p = 5$	$C_6 = \ 6.3750 \times 10^{-2}$
Order five method (17)	$p = 5$	$C_6 = 3.4138 \times 10^{-2}$

Definition 3.2:[14](Consistency) *The family of methods in (8) are consistent if the order p of the methods satisfy the inequality* $p \geq 1$,
(i) $\rho(1) = 0$,
and
(ii) $\rho'(1) = \sigma(1)$, *where* $\rho(z)$ and $\sigma(z)$*are the* 1^{st} *and* 2^{nd} *characteristic polynomials which are assumed to have no common factor.*

From Table 1 and definition 3.1, we can attest that the singly implicit Runge-Kutta methods derived are consistent.

Definition 3.3:[19](A(α)-stable) *A family of methods in (8) are $A(\alpha)$-stable* with the inequality $0 < \alpha \leq \pi/2$ *and the angular sector*

$$S_\alpha = \{z \in C : |arg(-z)| < \alpha, z \neq 0\},$$

is contained in the stability domain $\Re$.
When the interval of absolute stability consist of the negative real axis, the method is said to be A_0-stable.

Definition 3.4:[21] *The family of methods given in (8) are said to be irreducible if the continuous coefficients* $\phi_j(x)$,*and* $\upsilon_j(x)$ *have no common factor.*

Since the singly implicit Runge-Kutta methods are consistent, zero-stable, by Henrici[22] they are convergent.

3.3 Order of Convergence of the SIRK Methods

In this section we investigate the order of convergence of the newly derived sinlgly implicit Runge-Kutta methods (SIRKMs).

Definition 3.5: [19] *A function* $f(x, y)$ *satisfies the Lipschitz condition on the domain* $D \subseteq R^2$ *if there exists a constant* $L > 0$ *such that*

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*|,$$

for all $(x, y), (x, y^*) \in D$.
Definition 3.6:*A linear multistep method (LMM) is said to be convergence if for all initial value problem subject to the above Lipschitz condition*

$$\lim_{h \to 0} y_n = y^*(x_n),$$

for all solutions $\{y_n\}$ *of the method*(see, [19] page 22).

Now we demonstrate the convergence using a member y_{n+1} of a block from method (15) by assuming that the exact solution of y_{n+1} is y^*_{n+1} which is written precisely as

$$y^*_{n+1} = y^*_n + \frac{14hf_n}{126} + \frac{49hf_{n+u}}{126} + \frac{49hf_{n+v}}{126} + \frac{14hf_{n+1}}{126} - \frac{12}{960}h^6 y^{*(6)}(\varsigma_n) + R_7,$$

where R_7 is the remainder term. Subtract y_{n+1} from y^*_{n+1} to get

$$\begin{aligned} y^*_{n+1} - y_{n+1} = y^*_n + \left[\tfrac{14hf_n}{126} + \tfrac{49hf_{n+u}}{126} + \tfrac{49hf_{n+v}}{126} + \tfrac{14hf_{n+1}}{126} \right] \\ -y_n + \left[\tfrac{14hf_n}{126} + \tfrac{49hf_{n+u}}{126} + \tfrac{49hf_{n+v}}{126} + \tfrac{14hf_{n+1}}{126} \right] \\ -\frac{12}{960}h^6 y^{*(6)}(\varsigma_n) + R_7. \end{aligned}$$

Expanding the above equation to have

$$\begin{aligned} & y^*_{n+1} - y_{n+1} = y^*_n - y_n \\ & \quad + \frac{14h}{126}[f(x_n, y^*_n) - f(x_n, y_n)] + \frac{49h}{126}[f(x_{n+u}, y^*_{n+u}) - f(x_{n+u}, y_{n+u})] \\ & + \frac{49h}{126}[f(x_{n+v}, y^*_{n+v}) - f(x_{n+v}, y_{n+v})] + \frac{14h}{126}[f(x_{n+1}, y^*_{n+1}) - f(x_{n+1}, y_{n+1})] \\ & \quad - \frac{12h^6}{960} y^{*(6)}(\varsigma_n) + R_7. \end{aligned}$$

Let $d_n = y^*_n - y_n,\ d_{n+u} = y^*_{n+u} - y_{n+u}, ..., d_{n+1} = y^*_{n+1} - y_{n+1}$ in line with two references [23,24]. Taking the absolute values on both left and right hand sides, and imposing the triangular inequality using the Lipschitz condition defined in (3.5) above yields

$$\begin{aligned} |d_{n+1}| \leq \left(1 + \tfrac{14h}{126}L\right)|d_n| + \tfrac{49h}{126}L|d_{n+u}| + \tfrac{49h}{126}L|d_{n+v}| + \tfrac{14h}{126}L|d_{n+1}| \\ - \frac{12h^6}{960} y^{*(6)}(\varsigma_n) + R_7. \end{aligned}$$

This simplifies to

$$\begin{aligned} \left(1 - \tfrac{14h}{126}L\right)|d_{n+1}| \leq \left(1 + \tfrac{14h}{126}L\right)|d_n| + \tfrac{49h}{126}L|d_{n+u}| + \tfrac{49h}{126}L|d_{n+v}| \\ - \tfrac{12h^6}{960} y^{*(6)}(\varsigma_n) + R_7. \end{aligned} \tag{23}$$

As $h \to 0$ with the exception of $|d_n|$ every other term on the right hand side of (23) tend to zero while on the left hand side we are left with $|d_{n+1}|$ and by the definition of convergence, we have

$$\lim_{h \to 0} y_{n+1} = y^*_{n+1}.$$

Thus $|d_{n+1}| \leq |d_n|$ and $y^*_{n+1} - y^*_n \leq y_{n+1} - y_n$ holds. The convergence of y_{n+1} is established following the works of [23, 24]. In the same manner, it can be shown that the other members of the block hybrid method $[y_{n+u}, y_{n+v}]$ converge. Hence, we have shown that the new singly implicit Runge-Kutta methods are convergent, with the remainder term R_7 as,

$$R_{p+2} = C_{p+2}h^{p+2}y^{*(p+2)}(\varsigma_n) = O(h^7),$$

which shows that the highest order of convergence of the new methods is 7.

3.4 Regions of Absolute Stability (RAS) of the SIRKMs

To study the linear stability properties of the singly implicit Runge-Kutta methods we reformulate them as general linear methods (see, [10]). Hence, we use the notations introduced by [13] where a general linear method is represented by a partitioned $(s+r)\times(s+r)$ matrix (containing A,U,B,V),

$$\begin{bmatrix} Y^{[n]} \\ y^{[n-1]} \end{bmatrix} = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] \begin{bmatrix} hf(Y^{[n]}) \\ y^{[n]} \end{bmatrix}, \quad n = 1, 2, \cdots, N, \tag{24}$$

where

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix}, \; y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}, \; f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \vdots \\ f(Y_s^{[n]}) \end{bmatrix}, \; y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix},$$

and r denotes quantities as output from each step and input to the next step and s denotes stage values used in the computation of the steps $Y_1, Y_2, \cdots, Y_s$. The coefficients of these matrices indicate the relationship between the various numerical quantities that arise in the computation of the stability regions. The elements of the matrices A, U, B and V are substituted into the stability matrix. To simplify the calculation of the region of absolute stability we swap out the matrices V and U for C and D respectively. In the sense of [25] we apply (24) to the linear test equation $y' = \lambda y, \; x \geq 0$ and $\lambda \in$ C,(where C is a complex number) which leads to the recurrence relation

$$y^{[n+1]} = M(z)y^{[n]}, \; n = 1, 2, \cdots, N-1, \; Z = \lambda h,$$

where the stability matrix $M(z)$ is defined by

$$M(z) = C + zB(I - zA)^{-1}D.$$

For the stability regions, we adopt the approach of Akinola and Akoh [9] where the singly implicit Runge-Kutta methods are expressed as,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_{n+u} \\ \vdots \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} y_{n-u} \\ \vdots \\ y_{n} \end{pmatrix}$$

$$+h \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} f_{n+u} \\ \vdots \\ f_{n+1} \end{pmatrix} + h \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} f_{n-u} \\ \vdots \\ f_{n} \end{pmatrix}$$

with

$$AY_n = BY_{M-1} + hCF_M + hDF_{M-1}.$$

Using method (16) to show the calculation of region of absolute stability following Akinola and Akoh [9] we have,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+u} \\ y_{n+w} \\ y_{n+v} \\ y_{n+1} \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-u} \\ y_{n-w} \\ y_{n-v} \\ y_n \end{bmatrix},$$

$$C = \begin{bmatrix} -\frac{902090+258755\sqrt{7}}{88200} & \frac{1752240-558480\sqrt{7}}{88200} & -\frac{1225490-415205\sqrt{7}}{88200} & \frac{96845-31490\sqrt{7}}{88200} \\ -\frac{79576+6615\sqrt{7}}{10080} & \frac{129696}{10080} & -\frac{79576-6615\sqrt{7}}{10080} & \frac{6853}{10080} \\ -\frac{245098+83041\sqrt{7}}{17640} & \frac{350448+111696\sqrt{7}}{17640} & -\frac{180418+51751\sqrt{7}}{17640} & \frac{19369+6298\sqrt{7}}{17640} \\ -\frac{51107+2940\sqrt{7}}{630} & \frac{85512}{630} & -\frac{51107-2940\sqrt{7}}{630} & \frac{5096}{630} \end{bmatrix} \begin{bmatrix} f_{n+u} \\ f_{n+w} \\ f_{n+v} \\ f_{n+1} \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{322595-90290\sqrt{7}}{88200} \\ 0 & 0 & 0 & 0 & \frac{27643}{10080} \\ 0 & 0 & 0 & 0 & \frac{64519+18058\sqrt{7}}{17640} \\ 0 & 0 & 0 & 0 & \frac{12236}{630} \end{bmatrix} \begin{bmatrix} f_{n-u} \\ f_{n-w} \\ f_{n-v} \\ f_n \end{bmatrix}.$$

Substituting the matrices (A, B, C, D)into the characteristic equation we have

$$\rho(\eta, z) = det(\eta I - M(z)) \;=\; det[r(A - Cz) - (B - Dz)]. \tag{25}$$

The determinant produces the stability polynomial of the method. Going by the stability polynomial of the method, the region of absolute stabilty $\Re$ of the method is given by

$$\Re = x \in C : \rho(\eta, z) = 1 \rightarrow |\eta| \leq 1.$$

Computing the stability functions give the stability polynomials of the methods, which are plotted to produce the required graphs of the regions of absolute stability of the methods (MDs) as shown in Figures 1, 2 and 3 respectively.

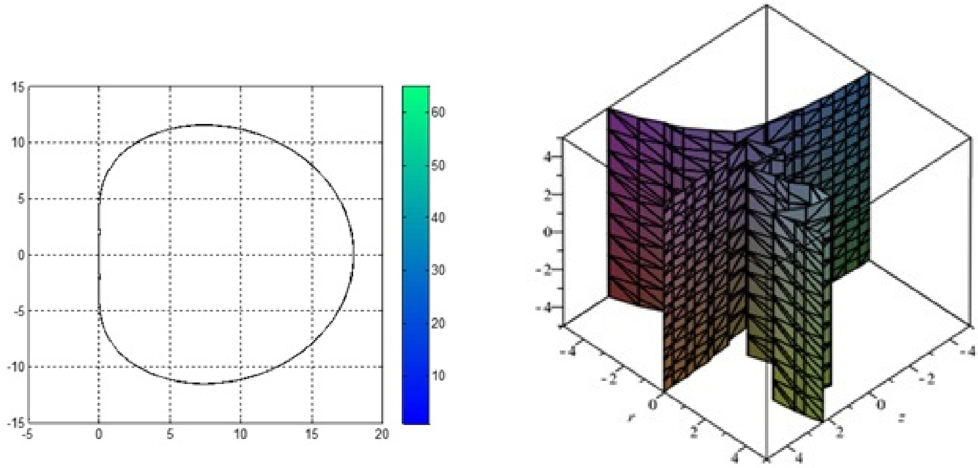


Figure 1: RAS and the Graphic Surface of SIRK Method(15)

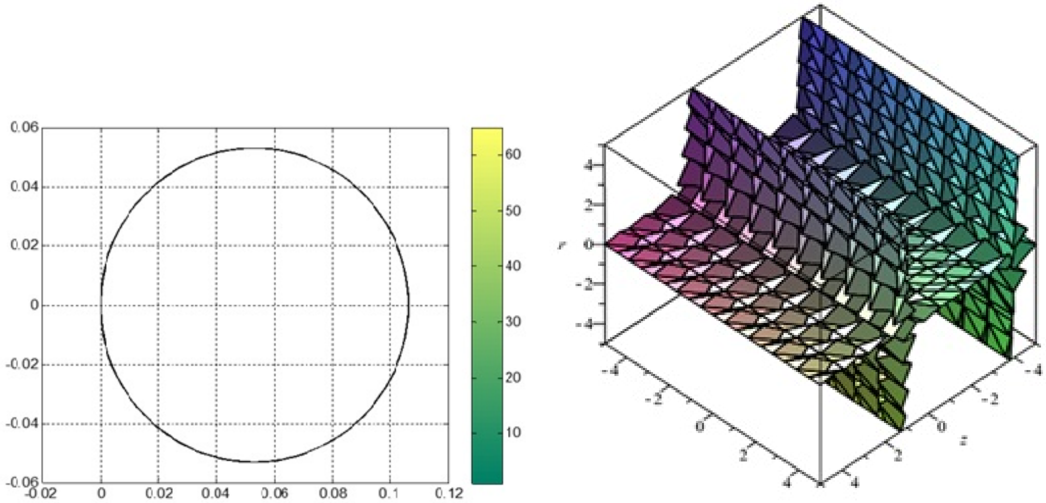


Figure 2: RAS and the Graphic Surface of SIRK Method (16)

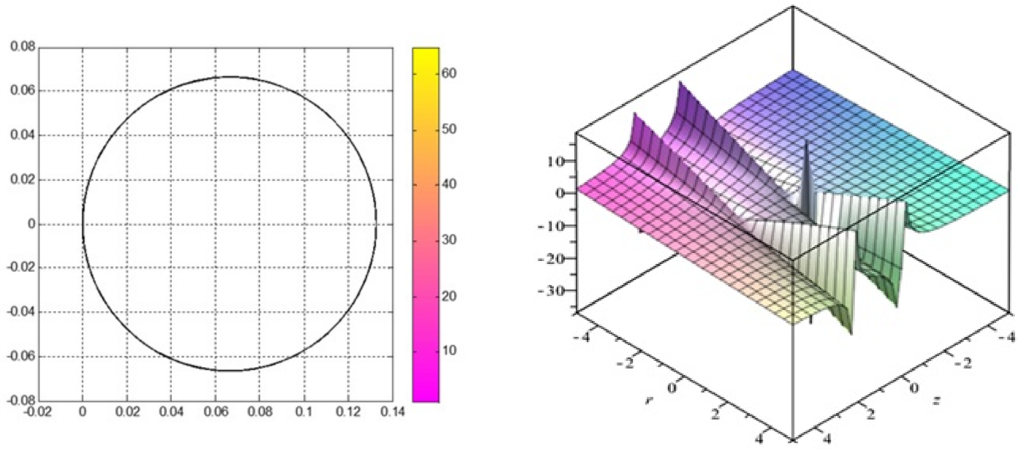


Figure 3: RAS and the Graphic Surface of SIRK Method (17)

Remark 1.1: *In the stable singly implicit Runge-Kutta methods the graphs of RAS of the new methods are drawn. The RAS of the singly implicit Runge-Kutta methods are A-stable, since the regions consist of the complex plane outside the enclosed Figures.*
Further, there are always some distinguished features observed among the three methods derived. These features can be visualized graphically with the help of different shapes given to their plotted regions as well as in their applications to solve differential equations. Taking into account the region of absolute stability, for example, is very crucial for a method with a nonlinear nature. Today the most effective method for the development and analysis of numerical schemes has been taken over by the new concept of graphic surface plot. Some of the most well-known works by [26, 27] introduced theoretical elements of the graphic surface that are now considered to be the most powerful approach for the formulation and analysis of numerical schemes.

4 Preliminary Results

The numerical solution must, of course, be piecewise continuous across the integration interval for the majority of differential equations, as opposed to solutions provided at certain discrete points. The necessity to depict the true representation of their natural or physical interpretation of what they represent in natural phenomena is the reason for this study. Instead of using discrete variable approaches, it is usually far more advantageous to assess the function at some points in the past that would not have been a grid point in the integration. We have used MatLab, a computer algebra program, to write some computer programs for the new derived methods' implementation. We computed these solution and compared the results with the exact solutions side by side in Tables and the solution curves in Figures. Five test problems are used to evaluate the singly implicit Runge-Kutta methods' performance. In the presentation and explanation of the results, we indicate the function evaluations by *nfe* and the exact solutions by *Ext*.

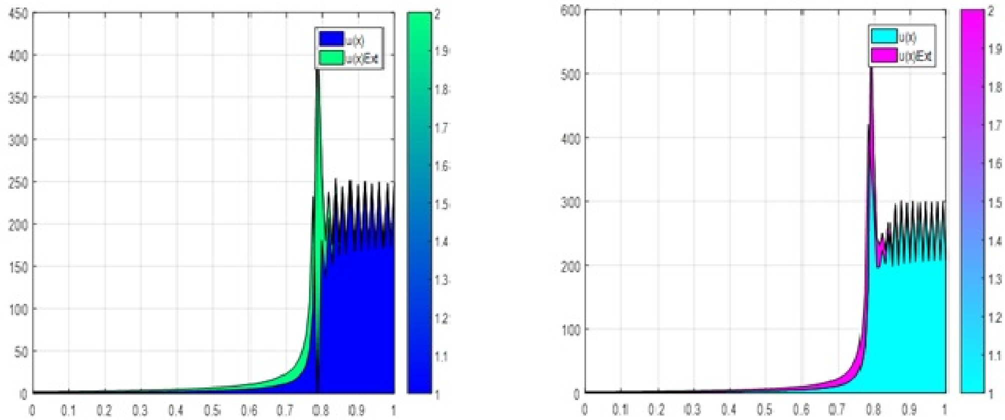
Example 1: A non-linear singular IVP [27, 28]: The initial layer or inner solution of this stiff problem satisfies the initial condition, and the outer solution matches the inner one. Numerous methods, including adaptive meshes, exponential-fitting, and concepts based on the theory of matching asymptotic expansions, have been quantitatively applied to this kind of problem, which is given by

$$u'(x) = 1 + u(x)^2, \quad u(0) = 1,$$

with theoretical solution $u(x) = tan(x + \pi/4)$ which has a simple pole when x is equal to $\pi/4$.
Here we applied the method in Yakubu et al. [29] and method (15) and the results are shown in Table 2 below, while the graphic surface curves are displayed in Figure 4.

Table 2: Absolute errors in the numerical integration of example 1

x	u	Yakubu et al.[29]$\lvert u(x) - u_n(x)\rvert$	Method(15)$\lvert u(x) - u_n(x)\rvert$
2	u	$2.220446049250313 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
50	u	$8.881784197001252 \times 10^{-16}$	0
150	u	$1.776356839400251 \times 10^{-15}$	$2.220446049250313 \times 10^{-16}$
250	u	$2.442490654175344 \times 10^{-15}$	$4.440892098500626 \times 10^{-16}$
500	u	$6.439293542825908 \times 10^{-15}$	$4.440892098500626 \times 10^{-16}$



(a) (b)

Figure 4: Graphic surface curves for example 1 with nfe =500

Example 2: A non-linear singular IVP [30]: This example was conveniently selected to demonstrate the robustness of the proposed methods.

$$u'(x) = \frac{u(x)}{3} + \frac{u(x)^2}{6}, \quad 0 \leq x \leq 1, \qquad u(0) = 2,$$

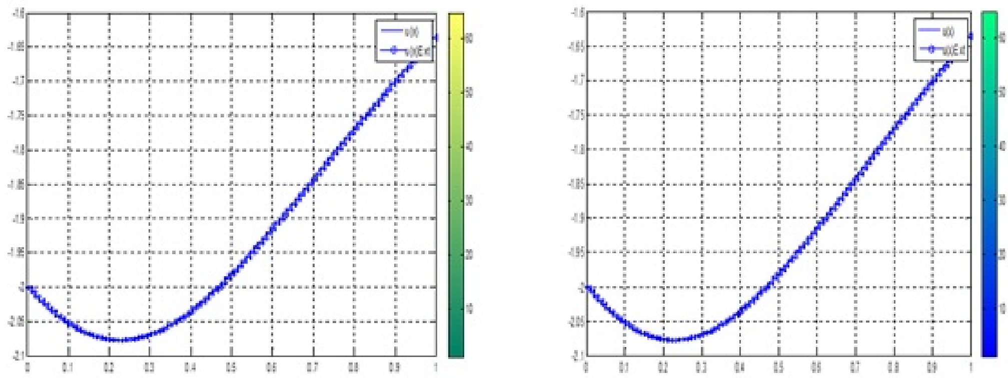
with theoretical solution

$$u(x) = -\frac{2}{(4x - 4 + 5e^{-x})^{1/3}}$$

Upon application of the SIRK method on the example above we obtain the results, which are depicted in Table 3 below, while the graphic solution curves are displayed in Figure 5. As depicted the two methods are vying over superiority of performance.

Table 3: Absolute errors in the numerical integration of example 2

x	u	Kwami et al.[31] $\lvert u(x) - u_n(x)\rvert$	Method(16) $\lvert u(x) - u_n(x)\rvert$
5	u	$1.110223024625157 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
50	u	$1.110223024625157 \times 10^{-16}$	$8.881784197001252 \times 10^{-16}$
150	u	$5.551115123125783 \times 10^{-16}$	$4.440892098500626 \times 10^{-16}$
250	u	$9.992007221626409 \times 10^{-16}$	$1.776356839400251 \times 10^{-15}$
500	u	$1.998401444325282 \times 10^{-15}$	$3.108624468950438 \times 10^{-15}$



(a) (b)

Figure 5: Solution curves for example 2 with nfe =500

Example 3: A non-linear Riccati singular IVP [32]: This example was selected to demonstrate the performance of the proposed methods.

$$u'(x) = 4 - 4u(x) + u(x)^2, \quad 0 \leq x \leq 1, \qquad u(0) = 2,$$

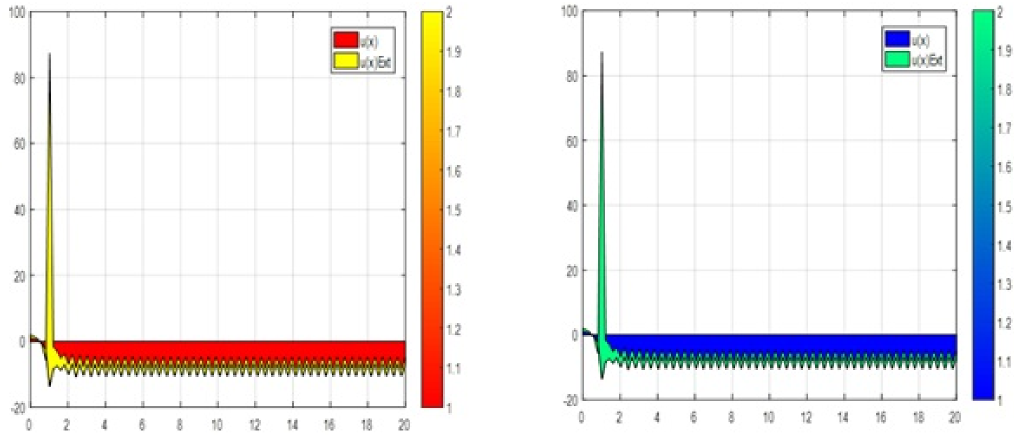
with exact solution

$$u(x) = \frac{2x-1}{x-1},$$

and has a pole at $x = 1$. This is a challenging problem for numerical method, especially those with limited regions of absolute stability. As seen in Table 4 the two methods are competing over accuracy of results. We observed that the two methods have no difficulties in dealing with the solution at the pole, when $x = 1$. We also observed that the bigger the number of steps is so it is the corresponding error. The graphic surface curves plots are shown in Figure 6.

Table 4: Absolute errors in the numerical integration of example 3

x	u	Yakubu et al.[29]$\lvert u(x) - u_n(x)\rvert$	Method(17)$\lvert u(x) - u_n(x)\rvert$
5	u	$8.104628079763643 \times 10^{-15}$	$1.110223024625157 \times 10^{-16}$
50	u	$1.317834730230061 \times 10^{-13}$	$1.110223024625157 \times 10^{-16}$
150	u	$4.031219802413943 \times 10^{-13}$	$5.551115123125783 \times 10^{-16}$
250	u	$6.780132011385831 \times 10^{-13}$	$9.992007221626409 \times 10^{-16}$
500	u	$1.379230063491832 \times 10^{-12}$	$1.998401444325282 \times 10^{-15}$



(a) (b)

Figure 6: Graphic surface curves for example 3 with nfe =500

Example 4: Linear mildly stiff differential equation: This example is a mildly system of stiff initial value problem in ordinary differential equation.

$$u_1'(x) = -8u_1(x) + 7u_2(x), \quad u_1(0) = 1,$$
$$u_2'(x) = 42u_1(x) - 43u_2(x), \quad u_2(0) = 8,$$

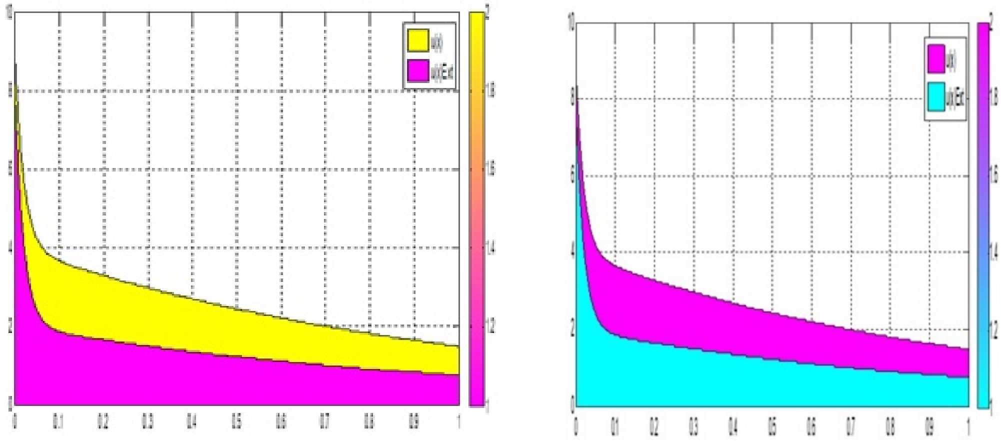
with exact solution given by

$$u_1(x) = 2e^{-x} - e^{-50x},$$
$$u_2(x) = 2e^{-x} + 6e^{-50x}.$$

The problem shows that to solve stiff equations the stability of a good method should impose no limitation on the step size, and hence it requires a large unbounded stability region. The results obtained from the experiment are given in Table 5 and the graphic surface curves are displayed in Figure 7, which shows that the singly implicit methods are good methods for stiff systems.

Table 5: Absolute errors in the numerical integration of example 4

x	u_i	Yakubu et al.[29] $\lvert u(x) - u_n(x)\rvert$	Method(15) $\lvert u(x) - u_n(x)\rvert$
5	u_1	$1.409945471486651 \times 10^{-08}$	$2.688619993307384 \times 10^{-09}$
	u_2	$8.459672873328827 \times 10^{-08}$	$1.613171907166588 \times 10^{-08}$
50	u_1	$1.901500734646788 \times 10^{-09}$	$3.625963973519220 \times 10^{-10}$
	u_2	$1.140901373375414 \times 10^{-08}$	$2.175580160468371 \times 10^{-09}$
250	u_1	$4.662936703425658 \times 10^{-15}$	$4.440892098500626 \times 10^{-16}$
	u_2	$4.662936703425658 \times 10^{-15}$	$6.661338147750939 \times 10^{-16}$
500	u_1	$5.773159728050814 \times 10^{-15}$	$7.771561172376096 \times 10^{-16}$
	u_2	$5.773159728050814 \times 10^{-15}$	$6.661338147750939 \times 10^{-16}$



(a) (b)

Figure 7: Graphic surface curves for example 4 with nfe =500

Example 5: Nonlinear highly stiff problem(The Kaps Problem): A highly stiff differential equation is a challenging problem for most numerical integrators. The method in (15) was applied to solve the problem and the obtained results are shown in the Table below

$$u_1'(x) = -1002u_1(x) + 1000u_2(x)^2, \quad u_1(0) = 1,$$

$$u_2'(x) = u_1(x) - u_2(x)(1 + u_2(x)), \quad u_2(0) = 1,$$

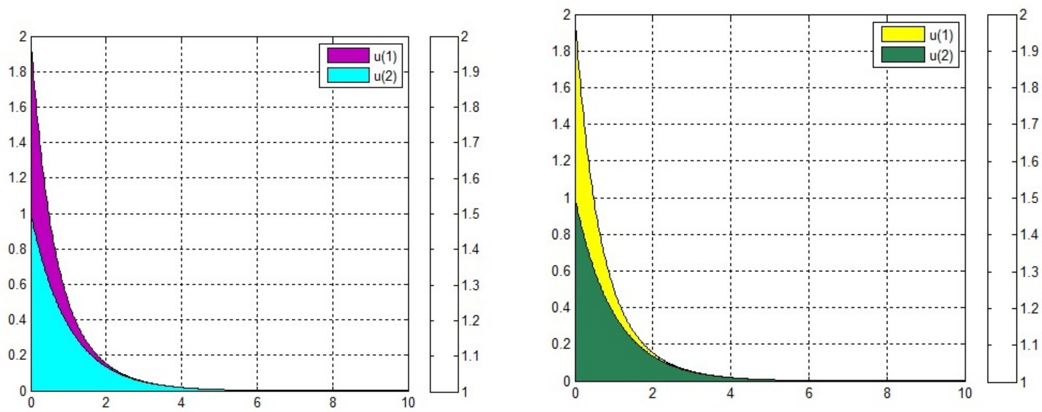
with exact solution given by

$$u_1(x) = e^{-2x},$$
$$u_2(x) = e^{-x}.$$

For this problem, numerical results are obtained with step-size $h = 0.1$ and are shown in Table 6. It is clear from the numerical evidences that the proposed singly implicit Runge-Kutta method outperformed the method in [31] which is considered for comparison in terms of accuracy and computational cost. In Figure 8, we have displayed the plotted graphic surface curves. The problem has been integrated on the interval [0, 10] and the results are presented in the Table of values below.

Table 6: Absolute errors in the numerical integration of example 5

x	u_i	Kwami et al.[31]$\|u(x) - u_n(x)\|$	Method(15)$\|u(x) - u_n(x)\|$
5	u_1	$7.195191100068943 \times 10^{-04}$	$6.088642135428923 \times 10^{-04}$
	u_2	$4.261624617640791 \times 10^{-07}$	$6.769571064824120 \times 10^{-07}$
50	u_1	$2.103378373718295 \times 10^{-05}$	$1.934210145547455 \times 10^{-05}$
	u_2	$6.392282789202675 \times 10^{-07}$	$8.495431147448329 \times 10^{-08}$
150	u_1	$7.081422117357319 \times 10^{-09}$	$6.386446382107380 \times 10^{-09}$
	u_2	$3.863915408030411 \times 10^{-08}$	$1.387533509963901 \times 10^{-09}$
500	u_1	$4.941950518946409 \times 10^{-21}$	$4.174928811030460 \times 10^{-21}$
	u_2	$1.071921137934739 \times 10^{-13}$	$1.112152981337260 \times 10^{-15}$



(a) (b)

Figure 8: Graphic surface curves for example 5 with nfe =500

5 Concluding Remarks

Three numerical techniques for approximating linear and nonlinear singular initial value problems are presented in this paper. Specifically, singly implicit Runge-Kutta procedures of orders 4 and 5 were developed by combining a multistep collocation with a one-step collocation process. Analysis of these methods indicates that they exhibit convergence when certain stability and consistency conditions are satisfied. Ultimately, we demonstrate a computational validation of the convergence order of the methods outlined in the Table of values and the graphic surface curves presented in the previous section in Figures, while also supporting our findings with several test problems. In a future publication, we will explore implicit Runge-Kutta techniques capable of attaining high-order convergence, including an analysis of their symmetry characteristics. We also aim to expand the use of related Runge-Kutta methods to address modeled differential equation challenges that arise in various fields, such as cardiac electrophysiology and populations exhibiting tumor-immune interactions.

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References

[1] Yakubu, D.G.(2018). Accurate multistep multi-derivative collocation methods applied to chaotic systems, J. Mod. Meth.Numer. Math., 9(1-2), 1-15.

[2] Petzold, L.R.,Jay, L.O., and Yen, J.(1997). Numerical solution of highly oscillatory ordinary differential equations, Acta Numer., 61, 437-483.

[3] Petzold, L.(1981). An efficient numerical method for highly oscillatory ordinary differential equations, SIAM J. Numer. Anal., 18(3), 455-479.

[4] Costabile, F. and Napoli, A.(2004). Stability of Chebychev collocation methods,Comput. Math. Appl., 47, 659- 666.

[5] Costabile, F. and Napoli, A.(2011). A class of collocation methods for numerical integration of initial value problems, Comput. Math. Appl., 62, 3221- 3235.

[6] Ascher, U. and Bader, G.(1986). Stability of collocation at Gaussian points, SIAM J. Numer. Anal., 23(2), 412- 422.

[7] Zhong-Mei, G. U., Nefedov, N.N., and Malley, R.E. Jr (1989). On singular singularly perturbed initial value problems, SIAM. J. Appl. Math., 49, 1- 25.

[8] Mitsui, T. and Yakubu D.G (2011). Two-step Family of "Look-ahead" Linear Multistep Method for ODEs, Sci.& Engin. Rev., 52,181- 188.

[9] Akinola, R.O. and Akoh, A.S.(2023). A seventh-order computational algorithm for the solution of stiff systems of differential equations, J. Nig. Math. Soc., 42(3), 215 – 242.

[10] Butcher, J. C.(2008). Numerical Methods for Ordinary Differential Equations, Second Edition,John Wiley and Sons, LTD, New York.

[11] Yakubu, D.G., Abba, I. B., and Markus, S.(2015). Two-step second derivative high-order methods with two off-step points for stiff systems in ordinary differential equations, J. Afrika Math., 26(5-6), 1081-1093.

[12] Yakubu, D.G., Kumleng G. M., and Markus, S.(2017).Second derivative Runge-Kutta collocation methods based on Lobatto nodes for stiff systems, J. Mod Meth. Numer Math., 8(1-2), 118-138.

[13] Burrage, K. and Butcher, J.C.(1979). Stability criteria for implicit Runge-Kutta method, SIAM J. Numer. Anal., 16, 46- 57.

[14] Vigo-Aguiar, J. and Ramos, H.(2007). A family of A-stable Runge-Kutta collocation methods of high order for initial value problems, IMA J. Numer. Anal., 27(4), 798-817.

[15] Gonzalez-Pinto, S. Gonzalez-Concepcion, and C. Montijino, J.I., (1994). Iterative scheme for Gauss methods, J. Comput. Math. Applic., 27(7), 67 - 81.

[16] Gonzalez-Pinto, S., Hernandez-Abreu, D., and Montijino, J.I.(2010). An efficient family of strongly A-stable Runge-Kutta collocation methods for stiff system and DAEs Part I: Stability and order results, J. Comput. Applic. Math., 234, 1105 - 1116.

[17] Yakubu D.G., Manjak N. H., Buba S.S., and Maksha A.I.(2011). A family of uniformly accurate order Runge-Kutta collocation methods, J. Comput. Appl. Math., 30(2), 315-330.

[18] Fatunla S.O.(1991). Block methods for second order ODEs, Intern. J.Comput. Math., 41: 55-63.

[19] Lambert, J.D.(1973).Computational Methods in Ordinary Differential Equations, John Wiley and Sons, Ltd.

[20] Lambert, J.D.(1991). Numerical Methods for Ordinary Differential Equations, The initial Value Problem, John Wiley and Sons, Ltd, New York.

[21] Higinio, R. and Patricio, M.F.(2014). Some new implicit two-step multiderivative methods for solving special second-order IVP's, Appl. Math. Comput., 239, 227–241.

[22] Henrici, P.(1962). Discrete Variable Methods in Ordinary Differential Equations, John Wiley and Sons Ltd, New York.

[23] Ismail, N.I.N., Majid, Z.A., and Senu, N.(2020). Solving neutral delay differential equation of Pantograph type, Malaysian Journal of Mathematical Sciences, 14S, 107-121.

[24] Akinola, R. O., Nchini, A., and Agbanwu, D.O.(2024). An accurate fifth-order A-stable method for solving stiff pharmacokinetic models, J. Nig. Math. Soc., accepted.

[25] Dahlquist, G.(1963). A special stability problem for linear multistep methods, BIT J. Numer. Math., 3, 27- 43.

[26] Ramos, H., Qureshi, S., and Soomro, A.(2021). Adaptive step-size approach for Simpson's-type block methods with time efficiency and order stars, Computational and Applied Mathematics, 40, 219.

[27] Qureshi, S., Soomro, A., and Evren Hıncal, E.(2021). A new family of A- acceptable nonlinear methods with fixed and variable stepsize approach, Compt. Math. Meth., 3(6), e1213.

[28] Ramos, J. I.(2005). Linearization techniques for singularly-perturbed initial-value problems of ordinary differential equations, Appl. Math.Comput., 163, 1143-1163.

[29] Yakubu, D. G.,Onumanyi, P., and Chollom, J. P.(2004). A new family of general linear methods based on the block Adams-Moulton multistep methods, Science Forum J. Pure and Applied Sciences, 7(1), 99-106.

[30] Shateyi, S.(2023). On the application of the block hybrid methods to solve linear and non-linear first order differential equations, Axioms, 12(189), 12020189.

[31] Kwami, A.M., Kumleng, G. M., Kolo, A. M., and Yakubu, D.G.(2016). Block hybrid multistep methods for the numerical integration of stiff systems of ordinary differential equations arising from chemical reactions, Abacus, Journal of the Mathematical Association of Nigeria, 42(2), 134-164.

[32] Ramos, H.(2007). A non-standard explicit integration scheme for initial-value problems, Appl. Math. Comput., 189, 710-718.