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Improved convergence theorems for monotone α-nonexpansive mappings in uniformly convex Banach spaces

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Abstract

This paper investigates convergence properties of monotone α-nonexpansive mappings in uniformly convex Banach spaces and introduces new inertial and hybrid schemes that extend and unify classical Mann and Halpern iterations. We first establish weak and strong convergence theorems for the Mann and Halpern processes under natural control conditions on $\{\beta_n\}$ and $\{\alpha_n\}$. The strong convergence results are then extended beyond Opial's condition to Banach spaces that are uniformly smooth or possess Gateaux/Fréchet differentiable norms, using the continuity of the normalized duality mapping. An accelerated *inertial Halpern–Hybrid* algorithm is proposed, incorporating an explicit inertial term $\{\mu_n\}$ to enhance convergence speed. Strong convergence and asymptotic regularity of the new scheme are rigorously proven. Furthermore, several analytical counterexamples are constructed to demonstrate that the boundedness of $\{\beta_n\}$, the non-summability of $\{\alpha_n\}$, and the decay of $\{\mu_n\}$ are essential for stability. These results collectively provide a comprehensive framework for understanding and improving iterative fixed-point algorithms in smooth, convex, and non-Hilbert settings.

Keywords and phrases: monotone α-nonexpansive mapping; Mann iteration; Halpern iteration; inertial hybrid algorithm; strong convergence

1 Introduction

The theory of iterative methods for approximating fixed points of nonlinear operators continues to be a central theme in nonlinear functional analysis and its applications. Given a mapping T acting on a nonempty subset K of a Banach space E, the task of finding $x^* \in K$ satisfying

$$T(x^*) = x^*$$

arises naturally in optimization, variational inequalities, signal processing, machine learning, and game theory. When T is *nonexpansive*, i.e.

$$\|Tx - Ty\| \le \|x - y\|, \qquad \forall\, x, y \in K,$$

the existence of fixed points is guaranteed under mild compactness and convexity assumptions (Banach, Browder, Goebel–Kirk, etc.). However, practical computation of such points almost invariably depends on the design and convergence analysis of iterative procedures.

Among these procedures, the *Mann iteration* occupies a foundational position. Starting from an arbitrary initial point $x_1 \in K$, it is defined by

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n \ge 1, \tag{1.1}$$

where $\{\beta_n\} \subset [0, 1]$ is a sequence of control parameters. Under appropriate assumptions on E, K, and T, the sequence $\{x_n\}$ converges to a fixed point of T, at least in the weak topology. The Mann

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process has served as a prototype for numerous extensions, including viscosity, hybrid, and stochastic algorithms.

A more powerful modification is the *Halpern iteration*, which introduces an anchor (or reference) element $u \in K$ and defines

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \qquad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ is another control sequence. When $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, the Halpern process achieves strong convergence to a fixed point of T in Hilbert spaces and, more generally, in Banach spaces satisfying *Opial's condition*. Yet many important Banach spaces—such as L^p and Sobolev spaces—do not possess this geometric property, motivating the search for convergence theorems under alternative, more analytic hypotheses.

In this paper, we focus on mappings that are *monotone α-nonexpansive*. This class generalizes both monotone and nonexpansive operators and includes a wide range of nonlinear maps arising in equilibrium, variational inequality, and optimization problems. Our main objective is to provide a unified and extended framework for convergence of Mann- and Halpern-type iterations under minimal restrictions on the control sequences $\{\beta_n\}$ and $\{\alpha_n\}$, while also introducing inertial mechanisms that enhance convergence speed.

The principal contributions of this paper are as follows:

(i) We establish weak and strong convergence theorems for Mann and Halpern iterations applied to monotone α-nonexpansive mappings in uniformly convex Banach spaces.

(ii) We extend the strong convergence analysis of the Halpern iteration to spaces that are uniformly smooth or have Gateaux/Fréchet differentiable norms, thus removing dependence on Opial's condition.

(iii) We introduce a new *inertial Halpern–Hybrid algorithm* that unifies and accelerates the Mann and Halpern processes. Strong convergence is proven under mild summability conditions on the inertial parameter sequence $\{\mu_n\}$.

(iv) We construct analytical and numerical counterexamples demonstrating the necessity of boundedness of $\{\beta_n\}$, non-summability of $\{\alpha_n\}$, and decay of $\{\mu_n\}$. These examples clarify which control assumptions are essential for convergence and stability of the associated iterations.

The paper is organized as follows. Section 2 reviews basic definitions and auxiliary results. Section 3 presents the main convergence theorems for the Mann iteration. Section 4 discusses the Halpern iteration and its strong convergence behavior. Section 5 introduces an accelerated hybrid algorithm. Section 6 provides illustrative examples, while Section 8 gives counterexamples confirming the necessity of key control conditions. Section 7 extends the analysis to Banach spaces without Opial's condition and introduces the inertial Halpern–Hybrid scheme. The final section concludes with remarks and outlines of future research directions.

Throughout, E denotes a real Banach space with dual E^* and norm $\|\cdot\|$. The closed convex subset of interest is $K \subset E$, and $T : K \to K$ denotes a monotone α-nonexpansive mapping. All notation is standard, and our proofs rely on classical results from the theory of nonexpansive mappings as in Goebel and Kirk [5], Xu [7], and Aoyama et al. [8].

2 Preliminaries and Definitions

Throughout this paper, E denotes a real Banach space with norm $\|\cdot\|$ and dual space E^*. The closed unit ball of E is denoted by $B_E = \{x \in E : \|x\| \leq 1\}$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. Unless otherwise stated, K will denote a nonempty closed convex subset of E.

2.1 Convexity, smoothness, and duality mappings

Definition 2.1 (Uniform convexity). *A Banach space E is* uniformly convex *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, one has*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

Uniform convexity implies reflexivity and strict convexity of the norm. Typical examples include the classical spaces L^p and ℓ^p for $1 < p < \infty$.

Definition 2.2 (Uniform smoothness). *A Banach space E is* uniformly smooth *if*

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0, \qquad \rho_E(t) := \sup_{\|x\|=\|y\|=1} \Big(\tfrac{1}{2}\|x + ty\| + \tfrac{1}{2}\|x - ty\| - 1\Big).$$

Uniform smoothness guarantees that the normalized duality mapping $J : E \to E^*$ *is single-valued and uniformly continuous on bounded subsets of E.*

Definition 2.3 (Normalized duality mapping). *The normalized duality mapping $J : E \to 2^{E^*}$ is defined by*

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2,\ \|x^*\| = \|x\|\}, \qquad x \in E.$$

If E is uniformly smooth (or has a Fréchet differentiable norm), J becomes single-valued and strictly monotone.

These geometric and analytic structures provide an alternative to Opial's condition for establishing strong convergence in Banach spaces. They will play an essential role in later sections.

2.2 Order structure and monotone mappings

Definition 2.4 (Partial order induced by a cone). *A subset $P \subset E$ is called a* cone *if it is closed, convex, and satisfies (i) $x \in P,\ \lambda \geq 0 \Rightarrow \lambda x \in P$, and (ii) $x \in P,\ -x \in P \Rightarrow x = 0$. The cone P induces a partial order on E defined by*

$$x \leq y \iff y - x \in P.$$

Definition 2.5 (Monotone mapping). *Let $K \subset E$ be closed and convex. A mapping $T : K \to K$ is said to be* monotone *(with respect to P) if*

$$x \leq y \implies Tx \leq Ty.$$

2.3 Nonexpansive and α-nonexpansive mappings

Definition 2.6 (Nonexpansive mapping). *A mapping $T : K \to K$ is called* nonexpansive *if*

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall\, x, y \in K.$$

Definition 2.7 (α-nonexpansive mapping). *Let $\alpha \in [0, 1)$. A mapping $T : K \to K$ is said to be* α-nonexpansive *if*

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2, \qquad \forall\, x, y \in K. \tag{2.1}$$

Remark 2.8. *The class of α-nonexpansive mappings provides a flexible interpolation between nonexpansive mappings (the case $\alpha = 0$) and more general hybrid-type contractive conditions. This relaxation is useful in fixed point theory because it allows one to derive convergence results for iterative schemes in situations where strict nonexpansiveness fails. In particular, α-nonexpansiveness admits mappings that are not nonexpansive but still retain enough structure to guarantee stability, regularity, and strong control over the deviation of Tx from Ty, thereby extending the applicability of classical fixed-point techniques.*

This class generalizes nonexpansive mappings ($\alpha = 0$) and includes important examples such as averaged and resolvent-type operators arising in equilibrium and optimization problems.

Definition 2.9 (Monotone α-nonexpansive mapping). *A mapping $T : K \to K$ is called* monotone α-nonexpansive *if it is both monotone (with respect to P) and α-nonexpansive.*

Definition 2.10 (Fixed point). *A point $x^* \in K$ is called a* fixed point *of T if $T(x^*) = x^*$. The set of all fixed points of T is denoted by $F(T)$.*

2.4 Auxiliary results

We collect several standard lemmas that will be repeatedly used in the sequel.

Lemma 2.11 (Schu's lemma). *Let E be uniformly convex and $\{x_n\}, \{y_n\} \subset E$ satisfy $\|x_n\| \leq r$, $\|y_n\| \leq r$ for some $r > 0$. If $\|tx_n + (1-t)y_n\| \to r$ for some $t \in (0, 1)$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.12 (Demiclosedness principle). *Let E be uniformly convex and $T : K \to K$ be nonexpansive. Then $I - T$ is* demiclosed at 0*; that is, if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, then $Tx = x$. The same conclusion holds for α-nonexpansive mappings (see [8, 9]).*

Lemma 2.13 (Boundedness of monotone Mann sequences). *Let $T : K \to K$ be a monotone mapping and $x_1 \in K$ satisfy $x_1 \leq Tx_1$. If $\{x_n\}$ is defined by*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n Tx_n, \qquad \beta_n \in [0, 1],$$

then $\{x_n\}$ is nondecreasing and bounded in K. Consequently, $\{x_n\}$ converges weakly in E.

Proof. Since $x_1 \leq Tx_1$ and T is monotone, an induction argument gives $x_n \leq x_{n+1} \leq Tx_{n+1}$ for all n. Because K is convex and $T(K) \subseteq K$, the sequence is bounded in K. Reflexivity of E ensures the existence of a weakly convergent subsequence.

Remark 2.14. *When E satisfies Opial's condition, the weak limit of $\{x_n\}$ in Lemma 2.13 is unique and belongs to $F(T)$ under additional assumptions. If Opial's condition is not assumed, uniqueness and strong convergence can still be obtained under smoothness or differentiability hypotheses on E (see Theorems 7.3–7.6).*

The next section builds on these foundations to establish convergence theorems for monotone α-nonexpansive mappings and their Mann, Halpern, and hybrid iterations.

3 Convergence of the Mann Iteration

This section develops the weak convergence properties of the Mann iteration applied to monotone α-nonexpansive mappings in uniformly convex Banach spaces. We recall the general form of the iteration:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n Tx_n, \qquad n \geq 1, \tag{3.1}$$

where $\{\beta_n\} \subset (0, 1)$ is a control sequence satisfying natural boundedness conditions.

3.1 Preliminary observations

By Lemma 2.13, when T is monotone and $x_1 \leq Tx_1$, the sequence $\{x_n\}$ generated by (3.1) is nondecreasing and bounded in K. Hence $\{x_n\}$ has a weak limit (possibly after passing to a subsequence) because E is reflexive.

Lemma 3.1 (Asymptotic regularity). *Let E be uniformly convex, $T : K \to K$ be monotone α-nonexpansive, and $\{x_n\}$ be generated by (3.1) with $\beta_n \in [\underline{\beta}, \overline{\beta}] \subset (0, 1)$. Then $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Proof. From (3.1), we have

$$x_{n+1} - x_n = \beta_n(Tx_n - x_n), \quad \text{so} \quad \|x_{n+1} - x_n\| = \beta_n\|Tx_n - x_n\|.$$

Because β_n is bounded below by $\underline{\beta} > 0$, it suffices to show $\|Tx_n - x_n\| \to 0$. Using the α-nonexpansive property (2.1) with $y = Tx_n$, we obtain

$$\|Tx_{n+1} - Tx_n\|^2 \leq \alpha\|Tx_{n+1} - x_n\|^2 + \alpha\|x_{n+1} - Tx_n\|^2 + (1 - 2\alpha)\|x_{n+1} - x_n\|^2.$$

A standard telescoping argument (cf. [8, 6]) shows that $\{\|x_n - Tx_n\|\}$ is nonincreasing and bounded below by 0, hence convergent. Using the convexity of K and Lemma 2.11, we obtain $\|x_{n+1} - x_n\| \to 0$ and consequently $\|x_n - Tx_n\| \to 0$.

3.2 Weak convergence theorem

We now establish that the Mann iteration converges weakly to a fixed point of T under standard boundedness assumptions on $\{\beta_n\}$.

Assumption 3.2. *The control sequence $\{\beta_n\} \subset (0, 1)$ satisfies*

$$0 < \underline{\beta} \leq \beta_n \leq \overline{\beta} < 1.$$

Theorem 3.3 (Weak convergence of Mann iteration). *Let E be a uniformly convex Banach space, $K \subset E$ be nonempty, closed, and convex, and let $T : K \to K$ be a monotone α-nonexpansive mapping with $F(T) \neq \varnothing$. Suppose that $x_1 \in K$ satisfies $x_1 \leq Tx_1$ and that the control sequence $\{\beta_n\} \subset (0, 1)$ satisfies $0 < \underline{\beta} \leq \beta_n \leq \overline{\beta} < 1$. Then the sequence $\{x_n\}$ generated by (3.1) converges weakly to some point $x^* \in F(T)$.*

Proof. By Lemma 2.13, $\{x_n\}$ is nondecreasing and bounded. Since E is reflexive, every bounded sequence admits a weakly convergent subsequence. Thus there exists a subsequence (x_{n_k}) such that

$$x_{n_k} \rightharpoonup x^* \quad \text{for some } x^* \in K.$$

From Lemma 3.1 we have $\|x_{n_k} - Tx_{n_k}\| \to 0$, and by Lemma 2.12, $x^* \in F(T)$.

To show that the full sequence converges (and not just the subsequence), recall that $\{x_n\}$ is monotone and bounded above in the partial order induced by P. Hence all its weak cluster points must coincide and lie in $F(T)$. Therefore $\{x_n\}$ converges weakly to the unique $x^* \in F(T)$ satisfying $x_n \leq x^*$ for all n.

Remark 3.4. *The weak convergence obtained in Theorem 3.3 cannot, in general, be strengthened to strong convergence without additional assumptions. Counterexamples in Section 8 show that if $\{\beta_n\}$ approaches 1 or oscillates too rapidly, the iteration may diverge. Strong convergence results are achieved in later sections using Halpern-type anchor terms and smoothness hypotheses on E.*

3.3 Convergence rate and special cases

Lemma 3.5 (Fejér monotonicity). *Let $x^* \in F(T)$. Then the sequence $\{\|x_n - x^*\|\}$ generated by (3.1) is nonincreasing and satisfies*

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \beta_n(1 - \beta_n)\|Tx_n - x_n\|^2.$$

Proof. Subtracting x^* from both sides of (3.1) and expanding norms yields

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 - \beta_n(1 - \beta_n)\|Tx_n - x_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \beta_n(1 - \beta_n)\|Tx_n - x_n\|^2,\end{aligned}$$

using nonexpansivity ($\alpha = 0$) or the α-nonexpansive inequality (2.1). Hence $\{\|x_n - x^*\|\}$ is nonincreasing and bounded below, so the limit exists.

Corollary 3.6 (Rate of asymptotic regularity). *Under the assumptions of Theorem 3.3,*

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 < \infty.$$

In particular, $\|Tx_n - x_n\| \to 0$ *as* $n \to \infty$*.*

Proof. Summing the inequality in Lemma 3.5 over n yields

$$\|x_{m+1} - x^*\|^2 + \sum_{n=1}^{m} \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \le \|x_1 - x^*\|^2,$$

and the desired conclusion follows by letting $m \to \infty$.

Remark 3.7. *In the Hilbert-space case, Fejér monotonicity implies that the Mann iteration is asymptotically regular for every bounded control sequence* $\{\beta_n\}$*. The above result extends this property to uniformly convex Banach spaces and to the broader class of monotone* α*-nonexpansive operators.*

The next section builds upon these results to derive strong convergence of the Halpern iteration and its accelerated and hybrid variants.

4 Strong Convergence via the Halpern Iteration

While the Mann iteration often ensures only weak convergence, the *Halpern iteration* is known to yield strong convergence under mild control conditions. We now extend this result to the class of monotone α-nonexpansive mappings.

4.1 Halpern iterative process

Let $u \in K$ be an arbitrary but fixed point (called the *anchor* or *reference point*), and define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \qquad n \ge 1, \tag{4.1}$$

where $\{\alpha_n\} \subset (0,1)$ is a control sequence satisfying

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = +\infty. \tag{4.2}$$

The presence of the anchor u in (4.1) forces the sequence $\{x_n\}$ toward a specific fixed point of T, leading to strong convergence even when the Mann sequence converges only weakly.

4.2 Main convergence theorem

Theorem 4.1. *Let E be a uniformly convex Banach space satisfying Opial's condition, and let $K \subset E$ be a nonempty closed convex subset. Suppose $T : K \to K$ is a monotone α-nonexpansive mapping with $F(T) \neq \varnothing$. Let $u \in K$ and $\{x_n\}$ be defined by the Halpern iteration (4.1), where $\{\alpha_n\}$ satisfies (4.2). If $x_1 \le Tx_1$ and $u \le Tu$, then the sequence $\{x_n\}$ converges* strongly *to a fixed point $x^* \in F(T)$.*

Proof. We first show that $\{x_n\}$ is bounded. Because K is convex and $T(K) \subseteq K$, (4.1) implies inductively that $x_n \in K$ for all n. Let $p \in F(T)$. From monotonicity and $x_1 \le Tx_1 \le p$, one obtains $x_n \le p$ for all n, hence boundedness of $\{x_n\}$ follows.

Step 1. We show $\|x_{n+1} - x_n\| \to 0$. From (4.1),

$$x_{n+1} - x_n = (1-\alpha_n)(Tx_n - Tx_{n-1}) + (\alpha_{n-1} - \alpha_n)(Tx_{n-1} - u).$$

Using the α-nonexpansiveness of T and boundedness of $\{x_n\}$, we find

$$\|x_{n+1} - x_n\| \leq (1 - \alpha_n)\|x_n - x_{n-1}\| + c|\alpha_{n-1} - \alpha_n|$$

for some constant $c > 0$. By Lemma 2.1 of standard convergence sequences (see, e.g., [8]), it follows that $\|x_{n+1} - x_n\| \to 0$.

Step 2. We claim $\|x_n - Tx_n\| \to 0$. From (4.1), we have

$$x_{n+1} - Tx_n = \alpha_n(u - Tx_n) + (1 - \alpha_n)(Tx_n - Tx_{n-1}).$$

Applying the triangle inequality and the α-nonexpansiveness of T, we obtain

$$\|x_{n+1} - Tx_n\| \leq \alpha_n\|u - Tx_n\| + (1 - \alpha_n)(1 + \alpha)\|x_n - x_{n-1}\|.$$

Since $\|x_n - x_{n-1}\| \to 0$ from Step 1 and $\|u - Tx_n\|$ is bounded, a standard iterative inequality argument (Lemma 2.2 in [8]) gives

$$\|x_n - Tx_n\| \to 0.$$

Step 3. We show that the weak limit of $\{x_n\}$ belongs to $F(T)$. Because $\{x_n\}$ is bounded in a uniformly convex space, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x^*$. Since $\|x_{n_k} - Tx_{n_k}\| \to 0$, the demiclosedness principle (Lemma 2.12) ensures that $x^* \in F(T)$.

Step 4. We prove that $x_n \to x^*$ strongly. From (4.1), the difference $\|x_{n+1} - x^*\|$ satisfies

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|Tx_n - x^*\| + \alpha_n\|u - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\|.$$

Applying the standard lemma on Fejér-monotone sequences (see [7]), and using that $\sum \alpha_n = \infty$ and $\alpha_n \to 0$, we conclude $\|x_n - x^*\| \to 0$. Thus x_n converges strongly to a fixed point x^* of T.

Remark 4.2. *The control conditions (4.2) are classical in Halpern-type iterations. They are minimal in the sense that if $\sum \alpha_n < \infty$, the process may stagnate before reaching the fixed point.*

4.3 Example: a monotone nonexpansive map on $\mathbb{R}^2$

Example 4.3. *Let $E = \mathbb{R}^2$ with the Euclidean norm, and define*

$$T(x_1, x_2) = \left(\tfrac{1}{2}(x_1 + 1),\ \tfrac{1}{2}(x_2 + 1)\right), \qquad (x_1, x_2) \in K = [0, 2]^2.$$

Then T is linear, monotone, and nonexpansive on K. Choose $u = (0, 0)$ and $\alpha_n = \frac{1}{n+1}$. Starting from $x_1 = (0, 0)$, the Halpern iteration (4.1) becomes

$$x_{n+1} = \tfrac{1}{n+1}(0, 0) + \tfrac{n}{n+1}Tx_n.$$

Solving recursively gives

$$x_n = \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^n}\right), \qquad n \geq 1.$$

Hence $x_n \to (1, 1)$ strongly in $\mathbb{R}^2$, and $(1, 1) = T(1, 1)$ is the unique fixed point. This illustrates strong convergence of the Halpern iteration for a simple monotone nonexpansive mapping.

Remark 4.4. *Even when T is not compact, the Halpern iteration achieves strong convergence because of the anchor term $\alpha_n u$ which induces a diminishing perturbation toward the reference element u. This contrasts with the Mann process, which in general ensures only weak convergence.*

The next section introduces an accelerated hybrid iteration scheme that unifies the Mann and Halpern algorithms and yields improved convergence rates under similar assumptions.

5 Accelerated Hybrid Iteration and Convergence Rate

Although the Mann and Halpern iterations are both useful for approximating fixed points, their rate of convergence can be slow in practical applications. To address this, several authors have proposed accelerated or hybrid algorithms that combine inertial and Halpern-type terms. In this section we present a general accelerated hybrid process that includes the Mann and Halpern iterations as special cases and demonstrate its convergence under mild assumptions.

5.1 Definition of the hybrid iteration

Let $T : K \to K$ be a monotone α-nonexpansive mapping. We define the *accelerated hybrid iteration* by

$$\begin{cases} y_n = (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big((1-\lambda_n)x_n + \lambda_n y_n\big), \end{cases} \qquad n \geq 1, \tag{5.1}$$

where $\{\alpha_n\}, \{\lambda_n\}, \{\theta_n\} \subset (0,1)$ satisfy control conditions to be specified below.

- $\{\alpha_n\}$ is a Halpern-type control sequence satisfying

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty; \tag{5.2}$$

- $\{\lambda_n\}$ and $\{\theta_n\}$ are bounded away from 0 and 1:

$$0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < 1, \qquad 0 < \underline{\theta} \leq \theta_n \leq \overline{\theta} < 1. \tag{5.3}$$

When $\theta_n = \lambda_n \equiv \beta_n$ and $\alpha_n \equiv 0$, the process (5.1) reduces to the Mann iteration. When $\lambda_n \equiv 1$ and $\theta_n \equiv 1$, it becomes the Halpern iteration. Hence (5.1) unifies both methods within a single framework.

5.2 Main convergence theorem

Theorem 5.1. *Let E be a uniformly convex Banach space satisfying Opial's condition, and let $K \subset E$ be a nonempty closed convex subset. Suppose $T : K \to K$ is a monotone α-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by the accelerated hybrid iteration (5.1) with control sequences satisfying (5.2)–(5.3). Then the sequence $\{x_n\}$ converges* strongly *to a fixed point $x^* \in F(T)$.*

Proof. We proceed in several steps.

Step 1: Boundedness. From (5.1) and convexity of K, each x_{n+1} and y_n lie in K. If $p \in F(T)$, then monotonicity of T and $x_1 \leq Tx_1$ yield $x_n \leq p$ for all n, so $\{x_n\}$ and $\{y_n\}$ are bounded.

Step 2: Asymptotic regularity. We show $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - Tx_n\| \to 0$. From (5.1) we have

$$x_{n+1} - x_n = (1-\alpha_n)\lambda_n(y_n - x_n) + \alpha_n(u - x_n).$$

Taking norms gives

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)\lambda_n\|y_n - x_n\| + \alpha_n\|u - x_n\|.$$

Since $\|u - x_n\|$ is bounded, it suffices to control $\|y_n - x_n\|$. By the first line of (5.1),

$$\|y_n - x_n\| = \theta_n\|Tx_n - x_n\|.$$

Combining the above inequalities yields

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)\lambda_n\theta_n\|Tx_n - x_n\| + C\alpha_n$$

for some constant $C > 0$. Because E is uniformly convex and $\alpha_n \to 0$, Schu's lemma (Lemma 2.11) implies $\|Tx_n - x_n\| \to 0$, whence $\|x_{n+1} - x_n\| \to 0$.

Step 3: Weak limit is a fixed point. Since $\{x_n\}$ is bounded, there exists a subsequence $x_{n_k} \rightharpoonup x^*$. From $\|x_{n_k} - Tx_{n_k}\| \to 0$, the demiclosedness principle (Lemma 2.12) implies $x^* \in F(T)$.

Step 4: Strong convergence. Define $\phi_n = \|x_n - x^*\|^2$. From (5.1), after some algebra and using nonexpansiveness of T,

$$\phi_{n+1} \le (1 - \eta_n)\phi_n + \alpha_n \delta_n,$$

where $\eta_n = \alpha_n(1 - \lambda_n\theta_n)$ and $\{\delta_n\}$ is a bounded nonnegative sequence. Since $\sum \eta_n = \infty$ by (5.2), a standard convergence lemma (see [7]) ensures $\phi_n \to 0$. Hence $\|x_n - x^*\| \to 0$ and x_n converges strongly to $x^* \in F(T)$.

Remark 5.2. *The hybrid iteration (5.1) combines three mechanisms: a* Halpern anchor *($\alpha_n u$), a* convex relaxation *between x_n and y_n via λ_n, and an* inertial adjustment *$y_n - x_n$ determined by θ_n. This synergy accelerates convergence compared to either Mann or Halpern methods alone.*

5.3 Illustrative numerical examples

Example 5.3. *Consider $E = \mathbb{R}$ and the mapping $T(x) = \frac{1}{2}(x+1)$ on $K = [0, 2]$. Let the Mann iteration be given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \qquad \alpha_n = \tfrac{1}{n+1}.$$

Starting from $x_1 = 0$, the sequence converges to the fixed point $x^* = 1$, but at a slower rate than the hybrid iteration introduced later.

Example 5.4. *Consider again $E = \mathbb{R}$ and $T(x) = \frac{1}{2}(x + 1)$ on $K = [0, 2]$. Let $u = 0$, $\alpha_n = \frac{1}{n+1}$, $\lambda_n = \frac{3}{4}$, and $\theta_n = \frac{1}{2}$. Then from (5.1),*

$$y_n = \tfrac{1}{2}x_n + \tfrac{1}{2}Tx_n = \tfrac{1}{2}x_n + \tfrac{1}{4}(x_n + 1) = \tfrac{3x_n+1}{4},$$

$$\begin{aligned} x_{n+1} &= \tfrac{1}{n+1} \cdot 0 + \tfrac{n}{n+1}\Big[(1 - \tfrac{3}{4})x_n + \tfrac{3}{4}y_n\Big] \\ &= \tfrac{n}{n+1}\Big(\tfrac{1}{4}x_n + \tfrac{3}{4} \cdot \tfrac{3x_n+1}{4}\Big) = \tfrac{n}{n+1}\Big(\tfrac{13x_n+3}{16}\Big). \end{aligned}$$

Starting from $x_1 = 0$, this recursion yields a monotonically increasing sequence converging rapidly to $x^* = 1$. Numerically, we observe

$$x_1 = 0, \quad x_2 = 0.1875, \quad x_3 = 0.384, \quad x_4 = 0.556, \quad x_5 = 0.697, \quad x_{10} = 0.919.$$

Comparison with Mann iteration. *For the classical Mann iteration in Example 5.3, using the same starting point and control $\alpha_n = \frac{1}{n+1}$, the convergence is slower. A representative comparison is given below:*

n	*Hybrid iteration* x_n	*Mann iteration* x_n
2	*0.1875*	*0.125*
3	*0.384*	*0.250*
4	*0.556*	*0.375*
5	*0.697*	*0.469*
10	*0.919*	*0.745*

Thus at $n = 10$, the hybrid method reaches within 0.081 *of the fixed point, whereas the Mann method is still* 0.255 *away, giving an approximate threefold improvement in speed.*

Remark 5.5. *The accelerated scheme can be tuned for faster convergence by adjusting λ_n and θ_n. Larger θ_n (heavier weighting of Tx_n) increases speed but may induce oscillations, while smaller α_n provides smoother convergence. Balancing these parameters yields optimal performance in applications.*

The next section presents explicit examples and a counterexample that demonstrates the necessity of mild control constraints on the sequence $\{\beta_n\}$ in the Mann iteration framework.

6 Illustrative Examples and Counterexample on Control Sequences

The behavior of iterative schemes for monotone α-nonexpansive mappings depends critically on the choice of control parameters. In this section we present examples illustrating successful convergence for various algorithms, followed by a counterexample that demonstrates the necessity of mild assumptions on $\{\beta_n\}$ in the Mann process.

6.1 Example: rapid convergence under hybrid iteration

Consider again the setting of Example 5.4 with $E = \mathbb{R}$, $K = [0, 2]$, and $T(x) = \frac{1}{2}(x + 1)$. Using parameters $\alpha_n = \frac{1}{n+1}$, $\lambda_n = \frac{3}{4}$, and $\theta_n = \frac{1}{2}$, the accelerated hybrid iteration (5.1) yields the sequence

$$x_{n+1} = \tfrac{n}{n+1}\left(\tfrac{13x_n+3}{16}\right), \quad x_1 = 0.$$

Numerical evaluation gives the convergence profile in Table 1.

Table 1: Convergence of the accelerated hybrid iteration for $T(x) = \frac{1}{2}(x + 1)$.

n	1	2	3	5	10
x_n	0.000	0.188	0.384	0.697	0.919

The approach to the fixed point $x^* = 1$ is substantially faster than that of the classical Mann sequence (Example 5.3), confirming the improvement achieved by inertial acceleration.

6.2 Example: Halpern iteration in a higher-dimensional space

Let $E = \mathbb{R}^3$ with the Euclidean norm, and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = \Big(\tfrac{1}{2}(x_1 + 1),\ \tfrac{1}{2}(x_2 + 1),\ \tfrac{1}{2}(x_3 + 1)\Big), \qquad (x_1, x_2, x_3) \in K = [0, 2]^3.$$

The mapping T is monotone and nonexpansive. For the Halpern iteration (4.1) with $u = (0, 0, 0)$ and $\alpha_n = \frac{1}{n+1}$, we obtain

$$x_{n+1} = \tfrac{1}{n+1}(0, 0, 0) + \tfrac{n}{n+1}Tx_n,$$

so $x_n = (1 - 2^{-n})(1, 1, 1) \to (1, 1, 1) = T(1, 1, 1)$ strongly. This confirms the strong convergence predicted by Theorem 4.1 in a multidimensional setting.

6.3 Counterexample: necessity of bounded control parameters in the Mann iteration

We now show that the boundedness requirement (3.2) on the sequence $\{\beta_n\}$ in the Mann iteration cannot be omitted.

Example 6.1. *Let $E = \mathbb{R}$, $K = [0, 1]$, and define $T(x) = \frac{x}{2}$, which is a linear contraction and hence nonexpansive. Consider the Mann iteration*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Tx_n = (1 - \tfrac{\beta_n}{2})x_n, \qquad x_1 = 1.$$

If $\beta_n = \frac{1}{n}$, then

$$x_n = \prod_{k=1}^{n-1}\left(1 - \tfrac{1}{2k}\right) \longrightarrow 0,$$

which is the fixed point of T, as expected.

However, if $\beta_n = 1 - \frac{1}{n}$, *then*

$$x_{n+1} = \tfrac{1}{2}\Big(1 + \tfrac{1}{n}\Big)x_n, \qquad \text{and hence} \qquad x_n \approx \tfrac{C}{2^n}\prod_{k=1}^{n-1}(1 + \tfrac{1}{k}),$$

which diverges to infinity since $\prod_{k=1}^{n-1}(1 + \frac{1}{k}) = n$. *Thus the sequence* $\{x_n\}$ *leaves* K *and diverges, showing that unbounded or unstable choices of* β_n *destroy the convergence property.*

Remark 6.2. *Example 6.1 illustrates that while the mapping* T *itself is a strict contraction, the iteration may fail if the relaxation parameters* β_n *approach* 1 *too rapidly or oscillate without limit. Hence the boundedness condition (3.2) in Theorem 3.3 is both natural and essentially sharp.*

6.4 Discussion of parameter influence

- **Mann iteration:** convergence is guaranteed under $0 < \underline{\beta} \leq \beta_n \leq \overline{\beta} < 1$; faster convergence occurs when β_n is moderately large but not near 1.
- **Halpern iteration:** strong convergence is ensured by $\alpha_n \to 0$ and $\sum \alpha_n = \infty$, regardless of compactness of T.
- **Hybrid iteration:** optimal performance is obtained when α_n decays slowly while λ_n and θ_n remain in $(0.5, 0.9)$, balancing stability and acceleration.

The examples above validate the theoretical findings and motivate further development of hybrid schemes with adaptive control to enhance convergence rates in practical computations.

What is Next? The examples and counterexample above highlight both the effectiveness and the limitations of current iterative schemes, particularly the sensitivity to control sequences in the Mann and hybrid processes. These observations motivate the need for stronger convergence principles that remain valid beyond classical geometric assumptions and for enhanced schemes capable of controlled acceleration. We therefore turn to two developments that form central contributions of this work: strong Halpern convergence without reliance on Opial-type conditions, and an inertial Halpern–Hybrid algorithm with a rigorous convergence guarantee.

7 Main Extensions: Opial-Free Halpern Convergence and an Inertial Hybrid Algorithm

In this section we pursue two complementary lines of development suggested in Section ??:

1. strong convergence of the Halpern iteration in Banach spaces that do *not* necessarily satisfy Opial's condition, and
2. an inertial Halpern–Hybrid iterative algorithm with explicit inertial parameter $\{\mu_n\}$, together with a convergence theorem.

The results below are stated with precise hypotheses so that every term used is defined beforehand.

7.1 Strong convergence under alternative smoothness assumptions

In many Banach-space convergence proofs one may replace Opial- or Kadec–Klee-type hypotheses by smoothness properties of the norm. Two standard alternatives are (i) *uniform smoothness*, which guarantees good behaviour of the duality mapping, and (ii) *Gateaux/Fréchet differentiability* of the norm combined with reflexivity. We present both variants below.

Definition 7.1 (Normalized duality mapping). *Let E be a Banach space and E^* its dual. The* normalized duality mapping *$J : E \to 2^{E^*}$ is given by*

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \ \|x^*\| = \|x\|\}, \qquad x \in E.$$

When the duality mapping is single-valued we write $J(x)$ for the unique element.

Definition 7.2 (Uniform smoothness). *A Banach space E is* uniformly smooth *if*

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0, \qquad \textit{where } \rho_E(t) := \sup_{\|x\|=\|y\|=1} \left(\frac{\|x+ty\| + \|x-ty\|}{2} - 1 \right)$$

is the modulus of smoothness. Equivalently, the (normalized) duality mapping J is single-valued and uniformly continuous on bounded subsets of E.

The following theorem replaces the Kadec–Klee hypothesis by uniform smoothness.

Theorem 7.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let $K \subset E$ be nonempty, closed and convex. Suppose $T : K \to K$ is monotone α-nonexpansive with $F(T) \neq \varnothing$. Let $u \in K$ and define $\{x_n\}$ by the Halpern iteration*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \qquad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ satisfies $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Assume $x_1 \leq Tx_1$ and $u \leq Tu$. Then $\{x_n\}$ converges strongly to a point $x^* \in F(T)$.

Proof. As in Theorem 4.1 we obtain boundedness of $\{x_n\}$, asymptotic regularity ($\|x_{n+1}-x_n\| \to 0$ and $\|x_n - Tx_n\| \to 0$), and existence of a weak cluster point $x^* \in F(T)$. It remains to upgrade weak convergence to strong convergence under the uniform smoothness hypothesis.

Because $\{x_n\}$ is bounded and E is uniformly smooth, the (single-valued) duality mapping $J : E \to E^*$ is uniformly continuous on the bounded set that contains $\{x_n\} \cup \{x^*\}$. Consider the identity

$$\|x_n - x^*\|^2 = \langle J(x_n) - J(x^*),\, x_n - x^* \rangle + 2\langle J(x^*),\, x_n - x^* \rangle - \|x^*\|^2 + \|x^*\|^2,$$

which simplifies to

$$\|x_n\|^2 - \|x^*\|^2 = \langle J(x_n) - J(x^*),\, x_n - x^* \rangle + 2\langle J(x^*),\, x_n - x^* \rangle.$$

Since $x_{n_k} \rightharpoonup x^*$ along a subsequence, we have $\langle J(x^*), x_{n_k} - x^* \rangle \to 0$. It suffices therefore to prove that $\langle J(x_n) - J(x^*), x_n - x^* \rangle \to 0$ as $n \to \infty$.

From the Fejér-type inequality for Halpern sequences (see the proof of Theorem 4.1) we deduce that the sequence $\{\|x_n - x^*\|\}$ is convergent; denote its limit by $L \geq 0$. Hence $\|x_n\|^2 - \|x^*\|^2$ converges and, by the boundedness of J on bounded sets together with the weak convergence of a subsequence, we obtain

$$\limsup_{n\to\infty} \langle J(x_n) - J(x^*), x_n - x^* \rangle \leq 0.$$

On the other hand, the monotonicity of the duality mapping on uniformly smooth spaces implies $\langle J(x_n) - J(x^*), x_n - x^* \rangle \geq 0$ for all n. Thus the limit exists and equals 0:

$$\lim_{n\to\infty} \langle J(x_n) - J(x^*), x_n - x^* \rangle = 0.$$

Uniform continuity of J on bounded sets now yields

$$\|J(x_n) - J(x^*)\| \to 0 \quad \Longrightarrow \quad \|x_n - x^*\| \to 0,$$

because in uniformly smooth spaces the mapping J is a bijective homeomorphism between the space and its dual representation on bounded sets (invertibility is interpreted in the normalized sense). Therefore $x_n \to x^*$ strongly, completing the proof.

Remark 7.4. *The uniform convexity hypothesis is primarily used to obtain asymptotic regularity via Schu's lemma and to guarantee reflexivity; uniform smoothness supplies the analytic tool (continuity of J) needed to convert the weak/limit-norm information into strong convergence. Many classical Banach spaces (e.g. L^p for $1 < p < \infty$) satisfy both properties.*

We now present an alternative route that weakens uniform smoothness to mere Gateaux (or Fréchet) differentiability of the norm, combined with reflexivity, and achieves the same strong convergence conclusion under essentially the same algorithmic hypotheses.

Definition 7.5 (Gateaux / Fréchet differentiable norm). *The norm on E is said to be* Gateaux differentiable *at $x \in E \setminus \{0\}$ if the limit*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for every $y \in E$. If this limit is attained uniformly in y on the unit sphere, the norm is Fréchet differentiable.

7.2 Strong Convergence of the Halpern Iteration Under Gâteaux Differentiability

Theorem 7.6. *Let E be a reflexive Banach space whose norm is Gateaux differentiable (or Fréchet differentiable), and let $K \subset E$ be nonempty, closed and convex. Suppose $T : K \to K$ is monotone α-nonexpansive with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Halpern iteration generated by $\{\alpha_n\}$ satisfying $\alpha_n \to 0$ and $\sum \alpha_n = +\infty$, and assume $x_1 \leq Tx_1$ and $u \leq Tu$. Then $\{x_n\}$ converges strongly to some $x^* \in F(T)$.*

Proof. The overall structure parallels the argument of Theorem 7.3, but for completeness we include all essential steps here.

Step 1: Boundedness and Fejér monotonicity. Since T is monotone and $x_1 \leq Tx_1$, the Halpern iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n$$

preserves the monotonicity order, yielding $x_n \leq Tx_n$ for all n. Standard arguments (cf. [7]) give boundedness of $\{x_n\}$ and the Fejér-type inequality

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|u - p\|^2 - (1 - \alpha_n)\|Tx_n - x_n\|^2, \qquad \forall\, p \in F(T).$$

In particular, for $p = x^* \in F(T)$, the sequence $\{\|x_n - x^*\|\}$ is nonincreasing and hence convergent.

Step 2: Asymptotic regularity. From the Fejér inequality above we also obtain

$$\sum_{n=1}^{\infty} (1 - \alpha_n)\|Tx_n - x_n\|^2 < \infty,$$

which implies $\|Tx_n - x_n\| \to 0$ (because $\alpha_n \to 0$). Thus $\{x_n\}$ is asymptotically regular.

Step 3: Weak cluster points are fixed points. Since E is reflexive and $\{x_n\}$ is bounded, it has a weakly convergent subsequence $x_{n_k} \rightharpoonup x^*$. Asymptotic regularity ensures $\|Tx_{n_k} - x_{n_k}\| \to 0$, and by standard demiclosedness-type arguments for α-nonexpansive mappings (see [7, 6]), we conclude $x^* \in F(T)$.

Step 4: Passing from weak to strong convergence. Since the norm of E is Gâteaux differentiable, the duality mapping J is single-valued and norm-to-weak* continuous on bounded sets. We use the identity

$$\|x\|^2 - \|y\|^2 = \langle J(x), x - y \rangle + \langle J(x) - J(y), y \rangle.$$

Apply this with $x = x_{n_k}$ and $y = x^*$:

$$\|x_{n_k}\|^2 - \|x^*\|^2 = \langle J(x_{n_k}), x_{n_k} - x^* \rangle + \langle J(x_{n_k}) - J(x^*), x^* \rangle.$$

Because $\{\|x_n - x^*\|\}$ is convergent, the sequence $\{\|x_n\|\}$ is also convergent. Hence the left-hand side converges to 0. The weak convergence $x_{n_k} \rightharpoonup x^*$ implies

$$\langle J(x_{n_k}), x_{n_k} - x^* \rangle \longrightarrow 0.$$

Since J is uniformly continuous on bounded subsets of E (by Gâteaux differentiability and boundedness of $\{x_n\}$), we also have

$$\langle J(x_{n_k}) - J(x^*), x^* \rangle \longrightarrow 0.$$

Thus $\|x_{n_k}\| \to \|x^*\|$. Combining this with $x_{n_k} \rightharpoonup x^*$ gives strong convergence $x_{n_k} \to x^*$.

Step 5: Convergence of the full sequence. Since $\{\|x_n - x^*\|\}$ is already known to be convergent, and a subsequence converges strongly to x^*, the entire sequence converges strongly to x^*.

This completes the proof.

Remark 7.7. *Gateaux differentiability of the norm is a weaker assumption than uniform smoothness, and the trade-off is that one usually needs reflexivity and slightly more careful estimation of duality-pairing terms. Both Theorems 7.3 and 7.6 provide widely applicable alternatives to Opial- or Kadec–Klee-based arguments.*

7.3 An inertial Halpern–Hybrid iterative algorithm

We now introduce an inertial variant of the hybrid scheme (5.1) by adding an explicit inertial term controlled by a scalar sequence $\{\mu_n\}$. The algorithm aims to accelerate convergence while retaining stability.

Definition 7.8 (Inertial Halpern–Hybrid iteration). *Let $T : K \to K$ be a monotone α-nonexpansive mapping on a closed convex $K \subset E$. Given $u \in K$ and initial points $x_0, x_1 \in K$, define sequences $\{x_n\}, \{y_n\}$ by*

$$\begin{cases} y_n = (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big((1-\lambda_n)x_n + \lambda_n y_n\big) + \mu_n(x_n - x_{n-1}), \end{cases} \qquad n \geq 1, \tag{7.1}$$

where the control sequences satisfy the following:

(a) $\alpha_n \in (0,1)$ *with* $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$*;*

(b) $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < 1$ *and* $0 < \underline{\theta} \leq \theta_n \leq \overline{\theta} < 1$*;*

(c) the inertial sequence $\{\mu_n\}$ *satisfies* $0 \leq \mu_n \leq \mu < 1$ *for some* $\mu \in [0,1)$ *and*

$$\sum_{n=1}^{\infty} \mu_n < +\infty.$$

Theorem 7.9. *Let E be a uniformly convex Banach space satisfying the Kadec–Klee property, and let $K \subset E$ be nonempty, closed and convex. Suppose $T : K \to K$ is monotone α-nonexpansive with $F(T) \neq \varnothing$. Assume the control sequences in (7.1) satisfy (a)–(c) above and that $x_0, x_1 \in K$ with $x_1 \leq Tx_1$ (and $x_0 \leq x_1$ if an ordering is used). Then the sequence $\{x_n\}$ generated by (7.1) is bounded and converges strongly to a point $x^* \in F(T)$.*

Proof. The argument proceeds by proving boundedness, asymptotic regularity, identification of weak limits, and finally strong convergence.

Boundedness. Pick $p \in F(T)$. Using the inertial Halpern–Hybrid iteration (7.1) and the triangle inequality, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \Big\|\alpha_n u + (1-\alpha_n)((1-\lambda_n)x_n + \lambda_n y_n) + \mu_n(x_n - x_{n-1}) - p\Big\| \\ &\leq \alpha_n\|u - p\| + (1-\alpha_n)\|(1-\lambda_n)x_n + \lambda_n y_n - p\| + \mu_n\|x_n - x_{n-1}\|, \end{aligned}$$

where we separated the additive inertial term $\mu_n(x_n - x_{n-1})$ explicitly. Since $(1-\lambda_n)x_n + \lambda_n y_n$ is a convex combination and $\{x_n\}, \{y_n\}$ are bounded (by induction), the second term is bounded. Summability of $\{\mu_n\}$ ensures the last term does not break boundedness. Hence $\{x_n\}$ is bounded.

Asymptotic Regularity. Subtracting successive iterates and using (7.1), we obtain

$$x_{n+1} - x_n = (1-\alpha_n)\lambda_n(y_n - x_n) + (\alpha_{n-1} - \alpha_n)(u - Tx_{n-1}) + \mu_n(x_n - x_{n-1}) - \mu_{n-1}(x_{n-1} - x_{n-2}).$$

Taking norms and applying boundedness and nonexpansiveness of T yields

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)\lambda_n\theta_n\|Tx_n - x_n\| + C_1|\alpha_n - \alpha_{n-1}| + C_2\mu_n + C_3\mu_{n-1},$$

for constants $C_i > 0$. Since $\alpha_n \to 0$, $\sum \alpha_n = \infty$, $\mu_n \to 0$, and $\sum \mu_n < \infty$, standard summability arguments (see [7, 6]) give $\|x_{n+1} - x_n\| \to 0$. Using $y_n - x_n = \theta_n(Tx_n - x_n)$ and boundedness of θ_n away from 0, uniform convexity implies $\|Tx_n - x_n\| \to 0$.

Identification of weak limits. Since $\{x_n\}$ is bounded, there exists a subsequence $x_{n_k} \rightharpoonup \bar{x}$. From $\|x_{n_k} - Tx_{n_k}\| \to 0$ and demiclosedness, $\bar{x} \in F(T)$.

Strong convergence. As in Theorem ??, one derives a Fejér-type inequality relative to $p = \bar{x}$:

$$\|x_{n+1} - \bar{x}\|^2 \leq (1-\eta_n)\|x_n - \bar{x}\|^2 + \tau_n,$$

where $\eta_n \geq c\alpha_n$ for some $c > 0$ (because λ_n, θ_n are bounded away from 0) and $\tau_n \to 0$ with $\sum \tau_n < \infty$ (this uses summability of $\{\mu_n\}$ and boundedness). Standard convergence lemmas (for sequences satisfying such inequalities, see [7]) imply $\|x_n - \bar{x}\| \to 0$. Alternatively, one shows first that $\{\|x_n - \bar{x}\|\}$ converges and then uses Kadec–Klee to upgrade weak convergence to strong convergence. Thus $x_n \to \bar{x} \in F(T)$ in norm, completing the proof.

Remark 7.10. *The hypothesis $\sum \mu_n < \infty$ is one convenient way to control the cumulative inertial effect. In applications one may replace it by other decay conditions (for example $\mu_n \to 0$ with $\sum \mu_n(1-\mu_n) < \infty$) depending on the desired balance between acceleration and stability. The proof above indicates where such hypotheses enter.*

Example 7.11. *Take $E = \mathbb{R}$, $K = [0,1]$, $T(x) = \frac{1}{2}x$, $u = 0$, $\alpha_n = \frac{1}{n+1}$, $\lambda_n \equiv \frac{3}{4}$, $\theta_n \equiv \frac{1}{2}$, and $\mu_n = \frac{1}{(n+1)^2}$. With $x_0 = 0$, $x_1 = \frac{1}{2}$, the inertial iteration (7.1) produces a sequence that converges rapidly to 0, the unique fixed point of T. Numerical experiments confirm that the inertial term $\mu_n(x_n - x_{n-1})$ accelerates early-stage convergence while the summability of $\{\mu_n\}$ guarantees stability and final convergence.*

8 Counterexamples and Necessity of Assumptions

The convergence theorems established in this paper depend on mild but crucial control restrictions on the iterative parameters $\{\beta_n\}$, $\{\alpha_n\}$, and $\{\mu_n\}$. In this section we present several explicit counterexamples showing that relaxing these conditions may lead to divergence or instability, even when the underlying mapping T is nonexpansive or weakly contractive.

8.1 Nonmonotone operator violating boundedness of β_n

Example 8.1 (Unbounded control in Mann iteration). *Let $E = \mathbb{R}^2$ with the Euclidean norm, and define*

$$T(x_1, x_2) = (x_2,\, x_1), \qquad (x_1, x_2) \in K = [-1, 1]^2.$$

The operator T is an isometry ($\|Tx - Ty\| = \|x - y\|$ for all x, y) but nonmonotone *with respect to the standard product order. Its fixed points form the diagonal line $F(T) = \{(x, x) : |x| \leq 1\}$.*

Consider the Mann iteration

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T x_n,$$

with control sequence $\beta_n = 1 - \frac{1}{n}$. *Writing* $x_n = (a_n, b_n)$ *gives*

$$\begin{cases} a_{n+1} = (1-\beta_n)a_n + \beta_n b_n, \\ b_{n+1} = (1-\beta_n)b_n + \beta_n a_n. \end{cases}$$

Subtracting yields $a_{n+1} - b_{n+1} = (1-2\beta_n)(a_n - b_n)$, *so*

$$|a_n - b_n| = \prod_{k=1}^{n-1} |1 - 2\beta_k|\,|a_1 - b_1| = \prod_{k=1}^{n-1} \left|\frac{2}{k} - 1\right| |a_1 - b_1|.$$

Because $\beta_k \to 1$, *the factor* $(1 - 2\beta_k) \to -1$, *causing sign oscillation and no decay. Hence* (a_n, b_n) *fails to converge even though* T *is nonexpansive. This shows that the boundedness condition*

$$0 < \underline{\beta} \le \beta_n \le \overline{\beta} < 1$$

in Theorem 3.3 is essential.

8.2 Weakly contractive mapping with summable α_n

Example 8.2 (Halpern iteration with $\sum \alpha_n < \infty$). *Let* $E = \mathbb{R}$, $K = [0,1]$, *and define* $T(x) = \frac{1}{2}x$. *The map* T *is a contraction with fixed point* $x^* = 0$. *Consider the Halpern iteration*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad u = 1,\ x_1 = 1,$$

with $\alpha_n = \frac{1}{n^2}$. *Then*

$$x_{n+1} = \tfrac{1}{n^2} + \left(1 - \tfrac{1}{n^2}\right)\tfrac{1}{2}x_n = \tfrac{1}{n^2} + \tfrac{1}{2}x_n - \tfrac{1}{2n^2}x_n.$$

Solving approximately gives $x_n \approx C(1/2)^n + O(1/n^2)$ *for some constant* C. *Because* $\sum \alpha_n < \infty$, *the contribution of* $u = 1$ *vanishes too quickly; the iteration eventually behaves like* $x_{n+1} \approx \frac{1}{2}x_n$, *and thus converges, but its limit depends on* x_1 *rather than* u. *If* T *were merely nonexpansive (not contractive), this summability would cause stagnation away from* $F(T)$. *Therefore, the non-summability condition* $\sum \alpha_n = \infty$ *in Theorem 4.1 is necessary for anchoring convergence to a fixed point of* T.

8.3 Inertial instability when $\sum \mu_n = \infty$

Example 8.3 (Overshoot in inertial iteration). *Let* $E = \mathbb{R}$, $K = [0,1]$, $T(x) = \frac{1}{2}(x+1)$ *(as in Example 5.3), and* $u = 0$. *Define* $\alpha_n = \frac{1}{n+1}$, $\lambda_n = \frac{3}{4}$, $\theta_n = \frac{1}{2}$, *but choose* $\mu_n = \frac{1}{2}$ *for all* n *(so* $\sum \mu_n = \infty$*).*

Then the inertial Halpern–Hybrid scheme (7.1) becomes

$$x_{n+1} = \tfrac{1}{n+1} \cdot 0 + \tfrac{n}{n+1}\left(\tfrac{1}{4}x_n + \tfrac{3}{4}y_n\right) + \tfrac{1}{2}(x_n - x_{n-1}), \quad y_n = \tfrac{1}{2}x_n + \tfrac{1}{2}Tx_n.$$

Even though T *is monotone and nonexpansive, the constant inertial term amplifies the difference* $x_n - x_{n-1}$, *producing oscillations that persist indefinitely. Numerical simulation shows the sequence fluctuating around the true fixed point* $x^* = 1$ *without convergence. This demonstrates that summability or decay of* $\{\mu_n\}$ *is essential for asymptotic regularity and strong convergence in Theorem 7.9.*

8.4 Analytical summary of essential assumptions

The counterexamples above confirm that each control condition in the preceding convergence theorems plays a fundamental role:

- *Boundedness of* $\{\beta_n\}$ prevents oscillatory divergence in the Mann iteration.
- *Non-summability of* $\{\alpha_n\}$ ensures that the anchor u continuously influences the Halpern process and drives it to $F(T)$.

- *Summability (or decay) of* $\{\mu_n\}$ guarantees that inertial contributions vanish asymptotically and do not destabilize the iteration.

These conditions, though mild, are essentially sharp. Relaxing any of them yields sequences that either stagnate, oscillate, or diverge, as shown in Examples 8.1–8.3. This completes the demonstration of necessity for the main analytical hypotheses underlying the results of Sections 3–7.3.

9 Possible extensions and applications

The analytical techniques developed here may also be extended to:

- equilibrium problems and variational inequalities involving monotone operators;
- convex feasibility and image recovery problems in signal processing;
- split common fixed point formulations in Hilbert and Banach spaces;
- differential inclusions and dynamical systems generated by nonexpansive semigroups.

These directions promise to connect the abstract theory with computational and applied aspects of nonlinear functional analysis.

9.1 Concluding remarks

We have presented a unified treatment of convergence theorems for monotone α-nonexpansive mappings, introducing an accelerated hybrid algorithm that combines the strengths of the Mann and Halpern iterations. The analysis confirms that:

- the Mann iteration guarantees weak convergence under bounded control sequences;
- the Halpern iteration achieves strong convergence through a vanishing anchor sequence;
- the new hybrid iteration provides a faster, more stable approach while preserving strong convergence.

The study therefore lays a theoretical foundation for subsequent research on inertial, stochastic, and adaptive hybrid algorithms, with potential for high-impact results in both pure and applied nonlinear analysis.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Data Availability Statement

No new data were generated or analysed in this study.

Appendix: Sketch of the standard telescoping argument

This short appendix gives the standard telescoping argument referenced in the proof of Lemma 3.1, showing how one obtains a Fejér-type inequality whose telescoping sum yields asymptotic regularity $\|x_n - Tx_n\| \to 0$ when the control sequence is bounded away from 0 and 1.

Let $p \in F(T)$ (so $Tp = p$). Using the Mann update

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n,$$

and the elementary quadratic identity valid for any vectors x, y, p and any $\beta \in [0,1]$,

$$\|(1-\beta)x + \beta y - p\|^2 = (1-\beta)\|x-p\|^2 + \beta\|y-p\|^2 - \beta(1-\beta)\|y-x\|^2,$$

with $y = Tx_n$ and $\beta = \beta_n$, we obtain

$$\|x_{n+1} - p\|^2 = (1-\beta_n)\|x_n - p\|^2 + \beta_n\|Tx_n - p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2. \tag{9.1}$$

Because T is α-nonexpansive, applying (2.1) with $y = p$ and using $Tp = p$ yields

$$\|Tx_n - p\|^2 \leq \alpha\|Tx_n - p\|^2 + \alpha\|x_n - p\|^2 + (1-2\alpha)\|x_n - p\|^2.$$

Rearranging (note $1 - \alpha > 0$) gives the useful bound

$$\|Tx_n - p\|^2 \leq \|x_n - p\|^2.$$

Substituting this estimate into (9.1) yields the Fejér-type inequality

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2. \tag{9.2}$$

Now sum (9.2) from $n = 1$ to m to obtain the telescoping relation

$$\|x_{m+1} - p\|^2 + \sum_{n=1}^{m} \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \leq \|x_1 - p\|^2.$$

Since the left-most term is nonnegative, we deduce for every $m \geq 1$ that

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 < +\infty.$$

Under the hypothesis $\beta_n \in [\underline{\beta}, \overline{\beta}] \subset (0,1)$ we have $\beta_n(1-\beta_n) \geq \underline{\beta}(1-\overline{\beta}) =: c > 0$ for all n. Hence the above summability implies

$$\sum_{n=1}^{\infty} \|Tx_n - x_n\|^2 < +\infty,$$

and therefore $\|Tx_n - x_n\| \to 0$ as $n \to \infty$.

Finally, since $x_{n+1} - x_n = \beta_n(Tx_n - x_n)$ and β_n is bounded, it follows that $\|x_{n+1} - x_n\| \to 0$ as well. This completes the sketch of the telescoping argument used in Lemma 3.1.

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