International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 8 Issue 1, 2025

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

Jump-Diffusion model with currency uncertainty: A real option approach

I. Adinya*

Abstract

This paper analyzes the investment decision under currency uncertainty using a real options framework where the project value follow a jump-diffusion process, while the investment cost a geometric Brownian motion (GBM). We derive the Hamilton-Jacobi-Bellman (HJB) equation and optimal investment threshold explicitly. The model is compared with a purely GBM-based approach to highlight the effect of jump risk. The results show that higher currency ratio and jump intensity greatly impact the investment threshold, encouraging earlier investment. Sensitivity analyses are conducted on key parameters highlighting their impact. Our findings provide insights into investment timing for industries and firms operating in volatile international markets, where currency fluctuations are ever present.

Keywords and phrases: Currency Uncertainty; Optimal stopping; jump-diffusion model; real options; financial modeling.

1 Introduction

Uncertainty is a critical issue in investment, particularly for firms operating in international markets where currency fluctuations and economic shocks introduce significant risks. Since the pioneering work of [1] and [2] in option pricing, financial models of investment under uncertainty have evolved significantly. A major extension is the real options framework which provides a powerful tool for modeling investment decisions under uncertainty, allowing firms to account for the value of waiting before committing capital to a project.

Investment decisions under uncertainty have been extensively modeled using real options theory ([3], [4], [5]). These models highlight how firms benefit from postponing investment in volatile environments until conditions become more favorable. Early works assumed a constant investment cost, with the project value following a geometric Brownian motion (GBM) ([5], [6]). Recent advancements have explored stochastic investment costs ([7], [8], [9]), incorporated stochastic volatility ([10]) and regime-switching models ([11]), allowing for more flexible representations of uncertainty. However, most real options models assume continuous dynamics, ignoring the possibility of sudden jumps due to macroeconomic shocks, policy changes, financial crises or uncertaintain observed in exchange rate markets.

The inclusion of jump processes in financial modeling originated with [2], who introduced Poisson-driven jumps to account for discontinuities in asset prices. Since then, jump-diffusion models have been widely applied in option pricing ([13], [14]) and interest rate modeling ([15]). However, their application to real options theory and investment under exchange rate risk remains limited. Recent studies ([16]) suggest that firms exposed to currency fluctuations tend to delay investment longer than firms facing only diffusion-based exchange rate risks. This delay results from the heightened risk of sudden devaluation, which can dramatically alter the value of future revenues and investment costs.

Real options models traditionally assume deterministic investment costs, but empirical evidence suggests that investment costs are rarely fixed over time. Recent works have extended this framework

*Department of Mathematics, University of Ibadan, Ibadan, Nigeria; E-mail: iniadinya@gmail.com

to incorporate stochastic capital expenditures ([9]). However, these models often neglect the combined effects of currency uncertainty and jump risks, despite strong empirical evidence that foreign exchange markets exhibit discontinuous price movements ([2]). Standard exchange rate models rely on log-normal diffusion processes, yet empirical studies show that currency markets frequently experience abrupt jumps, requiring more advanced modeling techniques ([16]). While some research has examined the effect of exchange rate fluctuations on capital investment, many firms remain highly sensitive to sudden currency shocks that standard diffusion models fail to capture. Studies, like [16], investigate the role of currency jumps in investment timing but do not explicitly derive optimal investment thresholds under stochastic capital costs.

Although the literature on real options is extensive, two key gaps remain unaddressed. First, most models fail to incorporate jumps that arise from macroeconomic events. Second, existing research often neglects the interaction between stochastic investment costs and currency uncertainty, which is crucial for firms engaging in international capital projects. By ignoring these factors, prior models may underestimate the risks faced by multinational corporations and foreign investors.

To address these gaps, this paper develops a real options model that incorporates jump-diffusion dynamics and a novel currency uncertainty ratio to quantify the effect of exchange rate fluctuations on investment thresholds. The theoretical framework extends existing real options models by deriving explicit solutions for optimal investment thresholds under jump dynamics. The model further examines how jump intensity and jump size affect investment strategies, providing a more comprehensive understanding of the decision-making process under currency uncertainty. This study also highlights how capital cost fluctuations interact with foreign exchange risk to influence investment timing.

The remainder of this paper is structured as follows. Section 2 presents the model formulation and in Section 3 the optimal stopping stategy is derived. Section 4 provides numerical simulations and sensitivity analyses and Section 5 concludes

2 Methodology

This section introduces the mathematical tools and concepts required for the formulation of the optimal investment problem under currency uncertainty. We assume that all stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions of right-continuity and completeness.

Definition 2.1 (Geometric Brownian Motion (GBM)). *A stochastic process X_t is said to follow a Geometric Brownian Motion (GBM) if it satisfies the stochastic differential equation (SDE):*

$$dX_t = \mu_X X_t dt + \sigma_X X_t dB_t, \tag{1}$$

where μ_X is the drift rate, σ_X is the volatility, and B_t is a standard Brownian motion.

Remark 2.2. GBM is widely used in financial modeling as it ensures that the process remains strictly positive, which is particularly useful for modeling asset prices and investment values.

Definition 2.3 (Jump-Diffusion Process). *A jump-diffusion process for a state variable X_t is governed by the stochastic differential equation (SDE):*

$$dX_t = \mu_X X_t dt + \sigma_X X_t dB_t + X_{t^-} dJ_t, \tag{2}$$

where: μ_X is the drift term (expected rate of return or growth); σ_X is the volatility; B_t is a standard Brownian motion and J_t is a jump process modeled by a compound Poisson process:

$$dJ_t = \sum_{i=1}^{N_t} (Y_i - 1). \tag{3}$$

Here: N_t is a Poisson process with intensity λ_X, representing the expected number of jumps per unit time; Y_i are i.i.d. jump sizes from an independent distribution $F_Y(y)$; and $Y_i - 1$ represents the relative magnitude of jumps.

The jump-diffusion model allows for both continuous variability and discontinuous jumps, making it more realistic for capturing real-world exchange rate shocks compared to pure GBM models. The jump component introduces sudden movements at random times, capturing macroeconomic shocks, policy changes, or financial crises.

Remark 2.4 (Special Cases). If $\lambda_X = 0$, the jump-diffusion model reduces to the standard geometric Brownian motion (GBM). If $\sigma_X = 0$, the model describes a pure jump process.

Definition 2.5 (Stopping Time). *A random variable τ is a stopping time with respect to $\{\mathcal{F}_t\}$ if for every $t \geq 0$,*

$$\{\tau \leq t\} \in \mathcal{F}_t. \tag{4}$$

In investment decision-making, the problem of determining the right time to invest is naturally framed as an optimal stopping problem. The decision-maker seeks to maximize the expected value of the investment opportunity by choosing an optimal stopping time τ^*.

Definition 2.6 (Value Function). *The value function associated with an optimal stopping problem is given by:*

$$F(V, I) = \sup_{\tau \geq 0} \mathbb{E}\left[e^{-r\tau}\mathcal{G}(V_\tau, I_\tau)\right], \tag{5}$$

where V_t and I_t represent the project value and investment cost, respectively, and $\mathcal{G}(V, I)$ is the reward function.

Remark 2.7. The optimal stopping time τ^* is chosen to maximize the expected discounted payoff, meaning that the firm waits until the investment is most favorable given the stochastic nature of the project value and investment cost.

2.1 Currency uncertainty ratio

To explicitly quantify exchange rate fluctuations, we introduce the currency uncertainty ratio D_t, defined as:

Definition 2.8 (Currency Uncertainty Ratio). *Let E_t represent the exchange rate at time t, defined as the amount of domestic currency per unit of foreign currency. The **currency uncertainty ratio** is given by:*

$$D_t = \frac{E_{t0}}{E_t}, \tag{6}$$

where E_{t0} is the initial exchange rate. This ratio adjusts future cash flows to account for exchange rate fluctuations.

Remark 2.9. When the exchange rate E_t increases (domestic currency depreciates), the currency uncertainty ratio D_t decreases, reducing the effective value of investment payoffs in domestic currency terms.

This ratio captures the impact of exchange rate fluctuations on project valuation. The project value, adjusted for currency uncertainty, is then given by:

$$V_t = D_t C_f, \tag{7}$$

where C_f is the foreign-currency-denominated cash flow.

2.2 Model formulation

In this section, we consider a jump diffusion process for the project value and GBM for the investment cost.
The dynamics of V and I involve two standard Brownian motions $W_v(t)$ and $W_I(t)$. We also introduce a non-Gaussian jump term represented by a Poisson processes $(N(t))_{t\geq 0}$ with rate λ respectively, that count the jumps, with jump size Z_i, $i = 1, 2, \ldots$, which are a sequence of independent and identically distributed (i.i.d.) non-negative random variables. The jump terms describe major changes in the market that imply sudden increase or decrease in the investment costs. We assume that the processes $W_v(t)$ and $W_I(t)$ are stochastically independent of the jump descriptor $N(t)$ and $Z_i, i = 1, 2, \ldots Z(t)$ and defined on the same probability space. We model the processes $V(t)$ as a jump-diffusion whose dynamics is given by

$$\frac{dV(t)}{V(t^-)} = \mu_v dt + \sigma_v dW_v(t) + d\left(\sum_{i=1}^{N(t)} Z_i - 1\right), \tag{8}$$

where $V(t^-) = \lim_{\substack{u\to t\\ u<t}} V(u)$ is the left-hand limit and represents the value of $V(t)$ just before a jump at time t and $I(t)$ as GBM:

$$\frac{dI(t)}{I(t^-)} = \mu_I dt + \sigma_I dW_I(t), \tag{9}$$

μ_v, μ_I are drift terms and σ_v, σ_I are volatilities.
We solve the stochastic differential equation (8), using Ito's Lemma for jump processes (Cont and Tankov (2004)), to obtain

$$V(t) \quad = V(0)\exp\left(\left(\mu_v - \tfrac{\sigma_v^2}{2}\right)t + \sigma_v W_v(t) + \textstyle\sum_{i=1}^{N(t)} Y_i\right) \tag{10}$$

Similarly, the dynamics of the investment cost I is as follows:

$$\begin{cases} \dfrac{dI(t)}{I(t^-)} & = (\mu_I)\, dt + \sigma_I dW_I(t), \\ I(0) > 0 \end{cases} \tag{11}$$

and

$$I(t) \quad = I(0)\exp\left(\left(r_I - \tfrac{\sigma_I^2}{2}\right)t + \sigma_I W_I(t)\right) \tag{12}$$

3 Results and discussions

In this section, we formulate the optimal stopping problem for investment under currency uncertainty where the project value follows a jump-diffusion process and the investment cost follows a geometric Brownian motion. This follows from the corresponding Hamilton-Jacobi-Bellman (HJB) equation, and we then solve for the optimal investment threshold.

3.1 Formulation of the optimal stopping problem

The firm seeks to determine the optimal timing for investment to maximize its expected discounted profits. The decision to invest is modeled as an optimal stopping problem, where the firm chooses the stopping time τ^* that maximizes the expected present value of investment payoffs. Following the optimal stopping problem formulation, we seek to find $F(v, i)$ and $\tau^* \in \mathcal{T}$ such that

$$F(v, i) = \sup_{\tau\geq 0} \mathcal{P}_J^{\tau}(v, i) = \mathcal{P}_J^{\tau^*}(v, i), \tag{13}$$

where

$$\mathcal{P}_J(V,I) = \mathbb{E}\left[e^{-r\tau}\mathcal{G}_J\left(V(\tau), I(\tau)\right)\right], \tag{14}$$

The value function, $F(v,i)$, is the solution to the optimal stopping problem, representing the optimal expected payoff given the current state ([19]). $\mathcal{P}_J(V,I)$ (the performance criterion) is the objective function that the decision-maker seeks to optimize. It is typically expressed as an expected value of some payoff function $\mathcal{G}_J$.
The value function represents the expected discounted payoff when an investor optimally times their investment. The value function is defined as:

$$\mathcal{P}_J(V,I) = \mathbb{E}\left[e^{-r\tau}\mathcal{G}_J\left(V(\tau), I(\tau)\right)\right], \tag{15}$$

where $\mathcal{G}_J\left(V,I\right)$ is the investment payoff function. The firm will optimally invest at the first hitting time:

$$\tau^* = \inf\{t \geq 0 : (V_t, I_t) \in \mathcal{D}\}, \tag{16}$$

where $\mathcal{D}$ is the stopping region in which it becomes optimal to invest. The continuation region, where investment is postponed, is denoted by $\mathcal{C}$.

Remark 3.1. The problem involves two sources of uncertainty: (i) the jump-diffusion process for the project value V_t, and (ii) the geometric Brownian motion for the investment cost I_t. This makes the optimal investment decision depend on both continuous and discontinuous sources of risk.

Next we derive the payoff, $\mathcal{G}_J$, from the performance criterion.
The function $\mathcal{G}_J$ represents the firm's expected discounted net cash flow upon investment. It is obtained by considering the project's future cash flows, adjusted for currency uncertainty and jump risk. Upon investment, the firm receives cash flows from the project, denoted as V_t and incurs an investment cost I_t. The objective is to compute the expected present value of the project's future profits.
We define the project's cash flows as proportional to its value $D_t V_T$, where D_t is the currency uncertainty ratio, representing how fluctuations in foreign exchange rates affect the value of cash flows.

We now consider the expected value of the firm, which is given by the discounted expectation of future profits.

Lemma 3.2. *The expected discounted net payoff is given by*

$$\mathcal{P}_J(V,I) = \mathbb{E}\left[e^{-r\tau}\mathcal{G}_J\left(V(\tau), I(\tau)\right)\right] \tag{17}$$

where

$$\mathcal{G}_J\left(v,i\right) = \frac{V_\tau D}{r - \mu_V - \zeta\lambda} - \frac{I_\tau}{r - \mu_I} \tag{18}$$

Proof. The performance criterion, $\mathcal{P}_J(V,I)$, which indicates the value of the option when the firm executes it at time τ is given by

$$\mathcal{P}_J(V,I) = \mathbb{E}\left[\int_\tau^\infty e^{-rt}\left(V_t D - I_t\right)dt \,|\mathcal{F}_\tau\right].$$

By change of variable, $s = t - \tau$, we have $s = 0$ when $t = \tau$. Thus

$$\mathcal{P}_J(V,I) = \mathbb{E}\left[\int_0^\infty e^{-rs}\left(V_{s+\tau} D - I_{s+\tau}\right)ds \,|\mathcal{F}_\tau\right].$$

By the tower property of conditional expectation and strong Markov property and using standard results for expectations of jump-diffusion processes (see [2], [14]), we get:

$$\begin{aligned}\mathcal{P}_J(V,I) &= \mathbb{E}\left[\mathbb{E}\left(\int_0^\infty e^{-rs}\left(V_{s+\tau} D - I_{s+\tau}\right)ds \mid V_\tau, I_\tau\right)|\mathcal{F}_\tau\right] \\ &= \mathbb{E}\left[\mathcal{G}_J\left(V(\tau), I(\tau)\right)|\mathcal{F}_\tau\right]\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_J\left(V(\tau), I(\tau)\right) &= \mathbb{E}\left(\int_0^\infty e^{-rs}\left(V_{s+\tau}D - I_{s+\tau}\right)ds \mid V_\tau, I_\tau\right) \\
&= \int_\tau^\infty e^{-rt}\left[D\mathbb{E}\left(V_{t+\tau} \mid V_\tau\right) - \mathbb{E}\left(I_{t+\tau} \mid I_\tau\right)\right]dt \\
&= \frac{V_\tau D}{r - \mu_V - \zeta\lambda} - \frac{I_\tau}{r - \mu_I}
\end{aligned}$$

3.2 Hamilton-Jacobi-Bellman (HJB) Equation

In this section we derive the HJB equation associated with the optimal stopping problem for investment decisions under a jump-diffusion process for project value and a geometric Brownian motion (GBM) for investment cost. In the continuation region $\mathcal{C}$, where the firm has not yet invested, the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$rF(V, I) = \mathcal{L}F(V, I), \tag{14}$$

where $\mathcal{L}$ is the infinitesimal generator of the joint process (V_t, I_t), given by:

$$\mathcal{L}F(V, I) = \mu_V V\frac{\partial F}{\partial V} + \mu_I I\frac{\partial F}{\partial I} + \frac{1}{2}\sigma_V^2 V^2\frac{\partial^2 F}{\partial V^2} + \frac{1}{2}\sigma_I^2 I^2\frac{\partial^2 F}{\partial I^2} + \lambda\mathbb{E}\left[F(ZV, I) - F(V, I)\right]. \tag{15}$$

Thus, the HJB is given by:

$$\mu_V V\frac{\partial F}{\partial V} + \mu_I I\frac{\partial F}{\partial I} + \frac{1}{2}\sigma_V^2 V^2\frac{\partial^2 F}{\partial V^2} + \frac{1}{2}\sigma_I^2 I^2\frac{\partial^2 F}{\partial I^2} + \lambda\mathbb{E}\left[F(VZ, I) - F(V, I)\right] - rF(V, I) = 0 \tag{16}$$

Remark 3.3. The inclusion of the jump term makes the HJB equation a partial integro-differential equation (PIDE), which is more complex than traditional optimal stopping problems under GBM.

To simplify the problem and reduce the HJB equation, we introduce the ratio $y = V/I$ which is the project value-to-cost ratio and assume:

$$F(V, I) = If(y). \tag{17}$$

Using this transformation, we apply the chain rule to compute derivatives and substitute them into the HJB equation to obtain a reduced one-dimensional equation in terms of y.

We assume that the value function can be rewritten as:

$$F(V, I) = If(y).$$

Then

$$\frac{\partial F}{\partial V} = \frac{\partial}{\partial V}\left(If(y)\right).$$

Since $I = V/y$,

$$\frac{\partial F}{\partial V} = If'(y)\cdot\frac{1}{I} = f'(y),$$

and

$$\frac{\partial^2 F}{\partial V^2} = \frac{\partial}{\partial V}f'(y) = \frac{1}{I}f''(y).$$

Similarly for I,

$$\frac{\partial F}{\partial I} = f(y) - yf'(y),$$

and

$$\frac{\partial^2 F}{\partial I^2} = \frac{1}{I^2}\left(y^2 f''(y) - 2yf'(y)\right).$$

Substituting into the HJB Equation and simplifying, we obtain:

$$\frac{1}{2}\sigma_y^2 y^2 f''(y) + \mu_y y f'(y) + \lambda \mathbb{E}[f(Zy)] - (r - \mu_I - \lambda) f(y) = 0. \tag{18}$$

where:

$$\sigma_y^2 = \sigma_V^2 + \sigma_I^2, \quad \mu_y = \mu_V - \mu_I.$$

To derive explicit expressions for the investment threshold and optimal value function we assume

$$f(y) = A_1 y^{\beta_1}. \tag{19}$$

Now substituting $f(y) = A_1 y^{\beta_1}$, $f'(y) = A_1\beta_1 y^{\beta_1 - 1}$ and $f''(y) = A_1\beta_1(\beta_1 - 1)y^{\beta_1 - 2}$. into (18) we obtain a characteristic equation:

$$\frac{1}{2}\beta_1(\beta_1 - 1)(\sigma_V^2 + \sigma_I^2) + \beta_1(\mu_V - \mu_I - \sigma_I^2) - (r - \mu_I) + \lambda(\mathbb{E}[Z^{\beta_1}] - 1) = 0.$$

The characteristic equation is derived from a second-order differential equation. It has two roots:say β_1 (positive root) and β_2 (negative root). To ensure economic feasibility and a finite expected value of the investment option, we require the exponent β_1 to be positive. Simplifying further, we have the reduced HJB equation:

$$\frac{1}{2}(\sigma_V^2 + \sigma_I^2)\beta_1^2 + \left(\mu_V - \mu_I - \sigma_I^2 - \frac{1}{2}(\sigma_V^2 + \sigma_I^2)\right)\beta_1 + \lambda\mathbb{E}[Z^{\beta_1}] - \lambda - r + \mu_I = 0. \tag{20}$$

Solving for β:

$$\beta_{1,2} = \frac{-\left(\mu_y - \sigma_I^2 - \frac{1}{2}\sigma_y^2\right) \pm \sqrt{\left(\mu_y - \sigma_I^2 - \frac{1}{2}\sigma_y^2\right)^2 - 2\sigma_y^2(r - \mu_I - \lambda(\mathbb{E}[Z^\beta] - 1))}}{\sigma_y^2}. \tag{21}$$

The general solution is:

$$f(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2}, \tag{22}$$

but y^{β_2} is unbounded as $y \to \infty$, we set $A_2 = 0$. Thus

$$f(y) = A_1 y^{\beta_1} \tag{23}$$

and the value function is:

$$F(V, I) = A_1 V^{\beta_1} I^{-\beta_1}, \quad \text{if } y < y^*. \tag{24}$$

Remark 3.4. The optimal investment threshold is defined as the critical ratio $y^* = V/I$ at which the firm exercises its option to invest. To ensure a well-posed optimal stopping problem, we impose boundary conditions at y^*. The value function must satisfy the following free boundary conditions at y^*:
At the boundary y^*, the value function must match the investment payoff (value matching condition):

$$F(y^*, I) = \mathcal{G}(y^*, I). \tag{25}$$

For optimality, the value function must be differentiable at y^* (smooth pasting condition):

$$\left.\frac{\partial F}{\partial V}\right|_{y^*} = \left.\frac{\partial \mathcal{G}}{\partial V}\right|_{y^*}. \tag{26}$$

These conditions ensure that the investment value function is continuous and differentiable at the optimal stopping threshold. The value matching condition ensures that the firm is indifferent between continuing and stopping at y^*, while the smooth pasting condition ensures that the marginal benefit of waiting equals the marginal cost. These conditions allow us to solve for the investment threshold y^* and the constant A_1 in the value function.

Combined with the boundary conditions, (25) and (26), we can derive the optimal investment strategy and the value of firm. The next proposition summarizes the results as follows:

Proposition 3.5. *The investment value function is given by:*

$$F(V,I) = \begin{cases} A_1 V^{\beta_1} I^{-\beta_1}, & \text{if } \frac{V}{I} < y^*, \\ \frac{VD}{r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1)} - \frac{I}{r-\mu_I}, & \text{if } \frac{V}{I} \geq y^*. \end{cases} \tag{27}$$

where β_1 *satisfies* (21),

$$A_1 = \frac{1}{(r-\mu_I)(\beta_1-1)} \cdot \left(\frac{D(\beta_1-1)}{\beta_1(r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1))} \right)^{\beta_1}.$$

and

the optimal investment threshold is given as:

$$y^* = \frac{\beta_1}{(r-\mu_I)} \cdot \frac{(r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1))}{D(\beta_1-1)}. \tag{28}$$

Proof. Standard formulation for real options analysis (see [4]) shows that the value of the option to invest, denoted by F, is equal to

$$F(V) = AV^{\beta_1}.$$

In our case, since both the investment cost and project value are stochastic, we have

$$F(V,I) = A\left(\frac{V}{I}\right)^{\beta_1}.$$

This is derived from solving the HJB equation, (16). The value-matching condition at $y = y^*$ gives:

$$A_1(y^*)^{\beta_1} = \frac{Dy^*}{r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1)} - \frac{1}{r-\mu_I}. \tag{29}$$

The smooth-pasting condition ensures differentiability:

$$\beta_1 A_1 (y^*)^{\beta_1-1} = \frac{D}{r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1)}. \tag{30}$$

Solving for A_1 and y^*, we obtain:

$$y^* = \frac{\beta_1}{(r-\mu_I)} \cdot \frac{(r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1))}{D(\beta_1-1)}, \tag{31}$$

$$A_1 = \frac{1}{(r-\mu_I)(\beta_1-1)} \cdot \left(\frac{D(\beta_1-1)}{\beta_1(r-\mu_V-\lambda(\mathbb{E}[Z_1^\beta]-1))} \right)^{\beta_1}. \tag{32}$$

Remark 3.6. The expectation $\mathbb{E}[Z^\beta]$ plays a crucial role in determining the behavior of investment decisions in the presence of jumps. However, its form is highly sensitive to the assumed distribution of the jump sizes Z. Different jump models lead to distinct functional forms for $\mathbb{E}[Z^\beta]$, affecting the resulting equilibrium conditions and optimal strategies.

In Kou's model ([13]) double-exponential framework, where jumps follow a distribution with densities decaying at different exponential rates for positive and negative jumps, we can explicitly compute $\mathbb{E}[Z^{\beta_1}]$, which helps us solve for β_1 in the characteristic equation. The model assumes that the log of the jump size follows a double-exponential distribution. This means:

$$Z \sim e^J, \quad J \sim \text{Double Exponential}(\eta_1, \eta_2, p).$$

Let p_1 and q_1 be the probability of the upward and downward jumps respectively, such that $p_1, q_1 \geq 0$ and $p_1 + q_1 = 1$. Using the moment generating function of the double-exponential distribution, then, ζ, which is the expected relative jump size ([20]), is given by,

$$\zeta = \mathbb{E}\left[Z_i - 1\right] = \frac{p_1\eta_1}{\eta_1 - 1} + \frac{q_1\eta_2}{\eta_2 + 1} - 1. \tag{33}$$

The logarithms of jump sizes $Y_i = \ln(Z_i)$ form a sequence of independent, identically distributed (i.i.d.) random variables whose density function has the asymmetric double exponential distribution of the form ([20]):

$$f_Y(y) = p_1\eta_1 e^{-\eta_1 y} 1_{y\geq 0} + q_1\eta_2 e^{\eta_2 y} 1_{y<0},$$

which ensures that the expected value of processes are finite) and $\eta_2 > 0$.
Using the moment generating function of the double-exponential distribution, the expectation is:

$$\mathbb{E}[Z^{\beta_1}] = p\frac{\eta_1}{\eta_1 - \beta_1} + (1-p)\frac{\eta_2}{\eta_2 + \beta_1}, \quad \text{for} \quad \beta_1 < \eta_1 \quad \text{and} \quad \beta_1 > -\eta_2.$$

Therefore (20) becomes

$$\frac{1}{2}(\sigma_V^2 + \sigma_I^2)\beta_1^2 + \left(\mu_V - \mu_I - \sigma_I^2 - \frac{1}{2}(\sigma_V^2 + \sigma_I^2)\right)\beta_1 + \lambda\left(p\frac{\eta_1}{\eta_1 - \beta_1} + q\frac{\eta_2}{\eta_2 + \beta_1}\right) - \lambda - r + \mu_I = 0.$$

Thus,

$$y^* = \frac{\beta_1}{r - \mu_I}\left(\frac{r - \mu_V - \lambda\left(\frac{p\eta_1}{\eta_1 - \beta_1} + \frac{q\eta_2}{\eta_2 + \beta_1} - 1\right)}{D(\beta_1 - 1)}\right),$$

and

$$A_1 = \frac{1}{(r - \mu_I)(\beta_1 - 1)}\left(\frac{D(\beta_1 - 1)}{\beta_1\left(r - \mu_V - \lambda\left(\frac{p\eta_1}{\eta_1 - \beta_1} + \frac{q\eta_2}{\eta_2 + \beta_1} - 1\right)\right)}\right)^{\beta_1}.$$

In [?], the logarithm of the jump sizes follows a normal distribution:

$$\ln Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2).$$

In this case, the expectation takes an exponential quadratic form:

$$\mathbb{E}[Z^\beta] = e^{\beta\mu_Z + \frac{1}{2}\beta^2\sigma_Z^2}.$$

Note, also that more flexible jump models, such as Lévy processes or tempered stable distributions, yield different moment expressions, often requiring numerical evaluation. In each case, the structure of $\mathbb{E}[Z^\beta]$ critically influences the solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation and the investment decision rules.

4 Sensitivity analysis and discussions

Investment decisions under uncertainty require careful numerical analysis to understand how different factors influence optimal investment timing. This section provides a computational study of the optimal investment threshold y^* and value function $F(V, I)$ under the jump-diffusion model. We conduct a sensitivity analysis to evaluate the impact of volatility, jump intensity, and currency risk. Comparisons between jump-diffusion and traditional geometric Brownian motion (GBM) models are also discussed. We present in table 1 a summary of the key variables used in this study.

Table 1: Description of parameters used.

parameter	Description
r	Risk-free discount rate
D	Currency ratio
μ_V	Drift of project value V_t
μ_I	Drift of investment cost I_t
σ_V	Volatility of V_t
σ_I	Volatility of I_t
D	Currency uncertainty ratio
λ	Jump intensity
p	Probability of upward jump
$1-p$	Probability of downward jump
η_1	Decay rate of upward jumps
η_2	Decay rate of downward jumps
$\mathbb{E}[Z]$	Expected jump size
y^*	Optimal investment threshold
A_1	Constant in value function
β_1	Exponent in value function

The valuation of investment opportunities under a Jump-Diffusion model is highly sensitive to several economic and stochastic parameters. Each plays a distinct role in shaping the value function $F(V, I)$ and the corresponding investment threshold y^* which is the value-to-cost ratio. A higher μ_V increases the expected future value of the project, raises the dominant root β_1, and thus amplifies the investment value. It lowers the threshold y^*, encouraging earlier investment. An increase in μ_I raises the expected growth in cost, reducing the incentive to delay. This increases $F(V, I)$ and also reduces y^*, leading to earlier investment action. Higher volatility in either value or cost increases the option value of waiting. This delays investment by raising the threshold y^*, especially when uncertainty is substantial. A higher discount rate diminishes the present value of future payoffs, reducing the value function and raising the threshold y^*, thereby postponing investment. This is a key parament in this study.

Greater jump frequency can increase or decrease $F(V, I)$ depending on the nature of the jumps. If upward jumps dominate, investment value rises and y^* falls; if downward jumps dominate, the opposite occurs. The probability of upward jumps (p) and the distribution of jump magnitudes (η_1, η_2) significantly affect the shape of $\mathbb{E}[Z^\beta]$, and hence the characteristic roots β. A fatter right tail (lower η_1) increases investment value, while a fatter left tail (lower η_2) introduces downside risk and lowers value. Among all parameters, μ_V, λ, and the jump distribution parameters (p, η_1, η_2) exhibit the most pronounced influence on the investment opportunity's value. The choice of jump model not only alters the solution structure of the HJB equation but also deeply affects economic intuition, timing, and risk profiles of the investment strategy.

In line with previous literature on investment under uncertainty with jumps (see, e.g., [20], [4], [7]), the sensitivity analysis in this section 4 adopts a set of parameters that reflect a typical investment en-

vironment subject to both diffusion and jump risks. The following values were used unless otherwise specified: drift of project value $\mu_V = 0.1$, drift of investment cost $\mu_I = 0$, volatility of project value $\sigma_V = 0.2$, volatility of investment cost $\sigma_I = 0.1$, a discount rate $r = 0.05$, time horizon $T = 1$ year, and 252 trading steps. . The jump intensity was set to $\lambda = 1$, with a 60% probability of upward jumps ($p = 0.6$) and symmetric exponential tails with $\eta_1 = \eta_2 = 5$.
In some figures, these parameters were slightly adjusted to highlight sensitivity effects and demonstrate how changes in jump characteristics impact the valuation of investment opportunities.

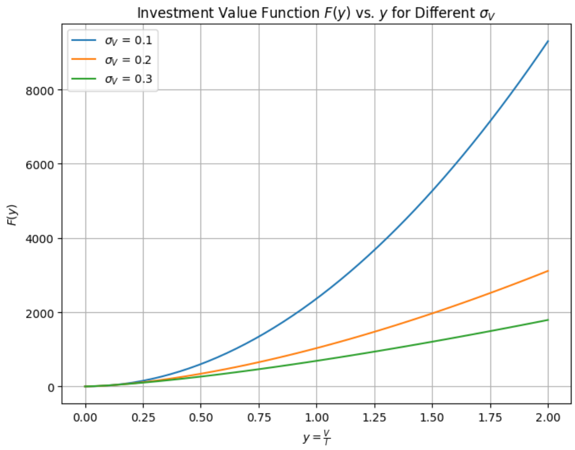

Figure 1: Value of Investment opportunity $F(y)$ for $\sigma = 0.1, 0.2, 0.3$.

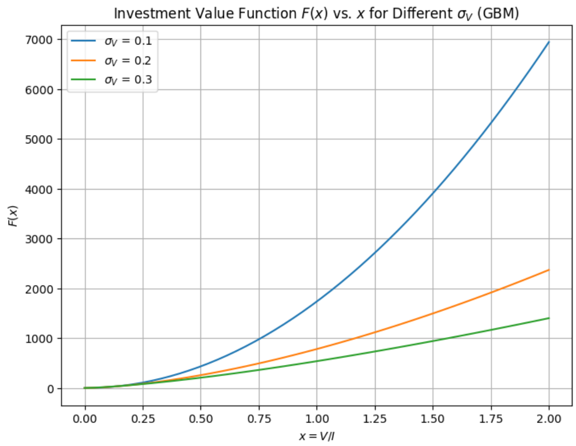

Figure 2: Same as figure 1 for GBM

We placed the plots of both cases; Jump difusion (JD hereafer) and GBM for the project value. Figures 1 and 2 plot F as a function of V/I in both casees. - The plot shows that for higher D, F is higher for all y, reflecting better investment value Note that in the JD case, there is increase in F.

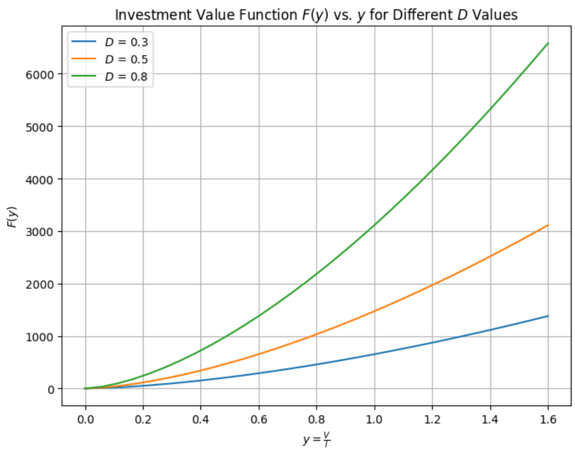

Figure 3: Value of Investment opportunity $F(y)$ for $D = 0.3, 0.5, 0.8$.

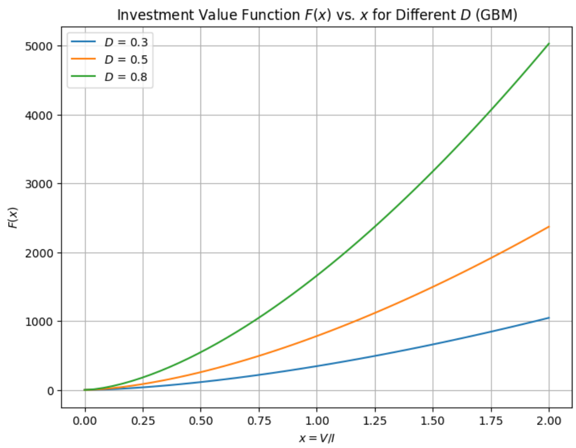

Figure 4: Same as figure 3 for GBM

Figures 3 and 4 evaluates how currency uncertainty impacts investment valuation. Notice that F increases when D increases. Thus, greater currency fluctuation increases the value of the investment opportunity in bothe cases but more for JB. We can also observe how project volatility affects investment thresholds in figures 5 and 6. Investment is very sensitive to volatility in project values, hence firms delay investment under greater volatility. Higher σ_V increases the value of waiting because uncertainty makes investment timing more important.

Another interesting observation is seen in Figures 7 and 8. Notice that an increase in D results a decrease in the critical value. This highlights the impact of currency fluctuations on investment

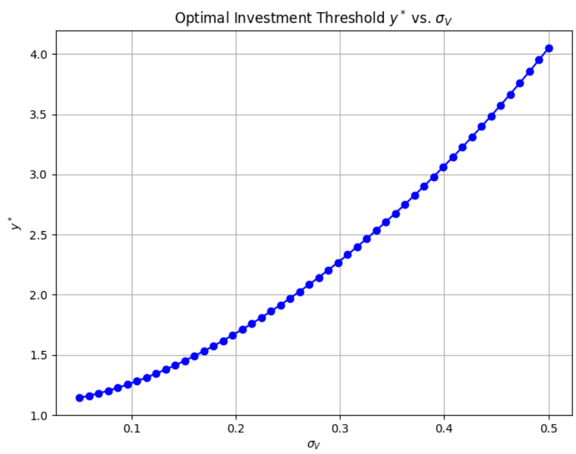

Figure 5: Critical value as a function of volatility,

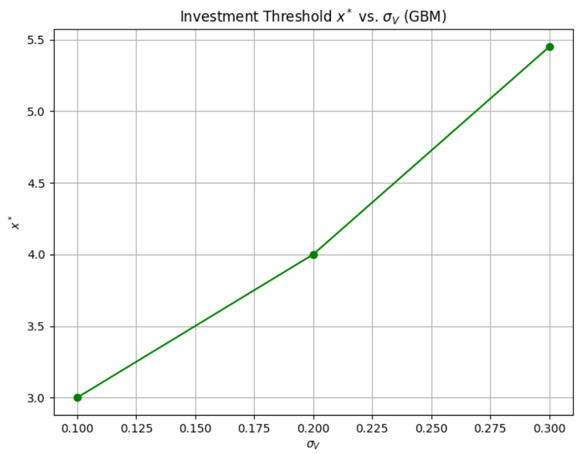

Figure 6: Same as figure 5 for GBM

decisions. A higher D lowers y^*, encouraging earlier investment and investment becomes optimal sooner.

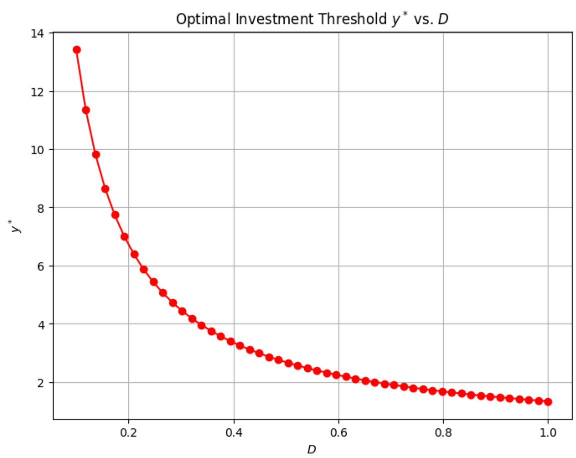

Figure 7: Critical value as a function of currency ratio

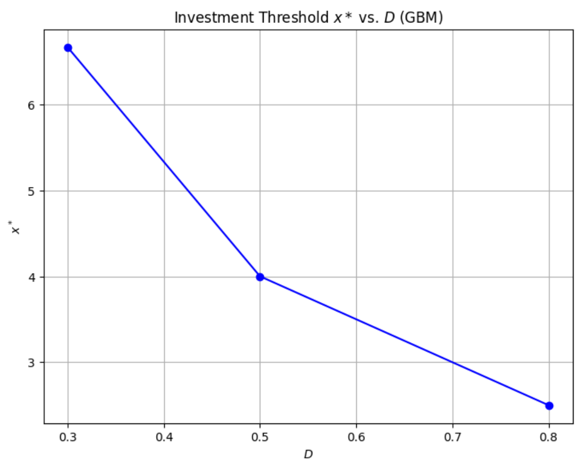

Figure 8: Same as figure 7 for GBM

Remark 4.1. The presence of jumps significantly alters investment timing compared to purely diffusive models. Real-world firms should consider currency hedging strategies when making long-term investment decisions.

5 Conclusion

In international finance, multinational corporations (MNCs) and firms engaged in cross-border investments face not only volatility risk but also extreme movements in exchange rates. Ignoring these fluctuations can lead to miscalculations in investment timing and valuation. By integrating a currency uncertainty ratio, we provide a more refined measure of exchange rate risk and its effect on optimal investment strategies. This paper extends the real options model by incorporating jump-diffusion dynamics in project valuation with currency ratio. The results highlight the role of jumps in accelerating or delaying investment, depending on their frequency and magnitude. Compared to a GBM-only model, jump diffusion impacts by making investment decisions more sensitive to jump intensity and expected jump size. The choice of jump distribution directly affects the investment opportunity valuation, optimal thresholds, and risk-adjusted return calculations.

Our findings highlight the importance of uncertainty in investment decisions. When investment costs and project values fluctuate, firms should adjust their investment thresholds accordingly. The model introduces additional risk considerations that can significantly affect investment timing. Policymakers and corporate managers must consider these factors in financial planning and capital budgeting strategies.
Future research could explore heterogeneous jump distributions or time-dependent jump intensities to better capture sector-specific investment behaviors. Additionally, empirical validation using industry data could refine our theoretical insights.

References

[1] Black, F. and Scholes, M. 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81: 637–659.

[2] Merton, R. C., 1973. The theory of rational option pricing. Bell Journal of Economics and Management Science 4 (Spring), 141-183.

[3] Myers, S. C., 1977, Determinants of corporate borrowing. Journal of Financial Economics, 5: 147–175.

[4] Dixit, A. K., and Pindyck, R. S. (1994). Investment under uncertainty. Princeton University Press, UK.

[5] McDonald, R., and Siegel, D. (1986). The value of waiting to invest. The Quarterly Journal of Economics, 101(4), 707-727.

[6] Pindyck, Robert S. (1991). Irreversibility, Uncertainty, and Investment. Journal of Economic Literature, Vol. 29 , pp. 1110-1152.

[7] Adinya, I. and Ekhaguere, G. O. S. (2022). Real options and investment timing in stochastic environments. Journal of Mathematical Finance, 12(4), 567-589.

[8] Adinya, I (2021). The Divestment Value of a Build Operate and Transfer Project in a Levy Market. Transactions of the Nigerian Association of Mathematical Physics. 16, 291 - 304.

[9] Miao, J., and Wang, N. (2007). Investment, consumption, and hedging under incomplete markets. Journal of Economic Dynamics and Control, 31(11), 3648-3680.

[10] Ting, S. H. M., Ewald, C. O., and Wang, W. K. (2013). On the investment–uncertainty relationship in a real option model with stochastic volatility. Mathematical Social Sciences, 66(1), 22-32.

[11] Bun, N. A. (2023). A Real Options Analysis of Regime Switching and Lead Time on the Investment Decision of a Monopoly (Doctoral dissertation, Tilburg University).

[12] Merton, R.C. 1976. Option Pricing When Underlying Stocks are Discontinuous. Journal of Financial Economics 3:125 - 144.

[13] Kou, S. G. (2002). A jump-diffusion model for option pricing. Management Science, 48(8), 1086-1101.

[14] Cont, R., and Tankov, P. (2004). Financial modelling with jump processes. Chapman and Hall/CRC.

[15] Wu, L. (2024). Interest rate modeling: Theory and practice. Chapman and Hall/CRC.

[16] Kozlov, R. (2023). The Effect of Interest Rate Changes on Consumption: An Age-Structured Approach. Economies, 11(1), 23.

[17] Grenadier, S. R., and Malenko, A. (2010). Real options and competitive strategy. Journal of Financial Economics, 97(2), 240-257

[18] Oliveira, C. A. D. (2014). Investment and exchange rate uncertainty under different regimes. Estudos Econômicos (São Paulo), 44, 553-577.

[19] Oksendal, B. and Sulem, A. (2005) Applied Stochastic Control of Jump Diffusions. Springer, New York.

[20] Kou, S. G., and Wang, H. (2004). Option Pricing under a Double Exponential Jump-Diffusion Model. Management Science, 50(9), 1178-1192.