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A closed-form characterization of correlation in real options with Lévy processes

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Abstract

This paper investigates the dynamic correlation between project value and investment cost in a real options framework with exponential Lévy processes. We derive a closed-form expression for correlation that captures both Brownian and jump-induced dependencies. Numerical simulations reveal time-varying, nonlinear behavior, the emergence of jump-induced outliers, and sublinear scaling between value and cost. These features significantly affect investment timing and valuation under uncertainty. Our findings highlight the importance of modeling structural shocks for optimal investment timing.

Keywords and phrases: correlation; uncertainty; Lévy processes; optimal investment

1 Introduction

Investment under uncertainty has long been a central concern in finance and economics. Especially when project values (V) and investment costs (I) exhibit stochastic behavior driven by complex market dynamics. Traditional methods such as Net Present Value (NPV) often fall short in dynamic and uncertain environments, where both the value of the project and the cost of investment evolve stochastically over time. Real Options Valuation (ROV) addresses this shortcoming by incorporating managerial flexibility and the time value of decision-making, treating investment opportunities as financial options. Within this framework, understanding the relationship between the underlying stochastic drivers, particularly project value and investment cost, is vital.

This paper focuses on analyzing the correlation between project value and investment cost when both are driven by Lévy processes, which allow for jump dynamics, skewness, and kurtosis not captured in models based on Brownian motion alone. Lévy processes provide a richer modeling framework by accommodating both continuous fluctuations and abrupt changes, offering a more realistic foundation for investment decisions under uncertainty.

The literature on real options often assumes independence or simple correlation structures between value and cost, with most models relying on geometric Brownian motion (GBM) assumptions (see [1]). However, empirical observations suggest that these assumptions may understate the risk or interdependence between cash flows and costs, particularly in volatile or jump-prone environments such as commodities, infrastructure, or tech startups. The models fail to capture discontinuous jumps and heavy-tailed distributions, features empirically observed in commodity prices and R&D investments ([2], [3]). Lévy processes, which generalize GBM by incorporating jumps, offer a more realistic framework ([4]). However, while several studies apply these tools to value projects, few explicitly model or derive the correlation structure between project value and cost, a gap that this paper addresses. The interdependence of V and I is pivotal. Their co-movement directly impacts NPV calculations and optimal investment thresholds ([5]). While prior studies have derived marginal distributions for V and I under Lévy processes ([6], [7]), explicit expressions for their correlation remain scarce, a gap this study fills.

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This paper fills a research gap by deriving explicit closed-form expressions for the correlation coefficient between the project value and investment cost under Lévy dynamics. Unlike previous studies that rely on simulation or linear approximations, this paper employs a rigorous stochastic framework to compute expectation, variance, and covariance terms that characterize the correlation. These results have direct implications for real options valuation, where understanding the joint dynamics of V and I is essential for optimal investment timing, threshold valuation, and risk mitigation.

This research contributes to both the theoretical advancement of Lévy-based modeling in finance and to the applied domain of investment analysis, providing tools that more accurately reflect the complexity of modern investment environments. We also perform numerical simulations to explore three features of the correlation dynamics: nonlinear, time-dependent evolution; the presence of jump-induced outliers that distort empirical correlation; and sublinear power-law scaling between $V(t)$ and $I(t)$, revealing strategic elasticity in investment behavior. These features have direct implications for investment timing, valuation under uncertainty, and sensitivity to extreme shocks.

Section 2 reviews relevant literature. Section 3 outlines the model, definitions, and tools. Section 4 presents the main results, including theoretical derivations and numerical illustrations. Section 5 concludes with directions for future research.

2 Literature

The valuation of investment projects under uncertainty has evolved significantly with the advent of real options theory, which provides a more flexible and dynamic framework than traditional NPV methods. Real options models treat investment opportunities as options or contingent claims, capturing managerial flexibility in the face of uncertainty ([8]). Over time, this approach has been extended from basic models using GBM to more sophisticated stochastic processes capable of modeling jumps and skewness, such as Lévy processes.

The limitations of GBM in capturing asset price dynamics, particularly discontinuities and heavy tails, have led to widespread adoption of Lévy processes in finance. Seminal work by Merton in 1976, introduced jump-diffusion models, while [10] proposed the Variance Gamma process to better fit empirical distributions. Lévy processes generalize Brownian motion by allowing for discontinuous paths, making them particularly well-suited for capturing extreme events, fat-tailed distributions, and asymmetric information flows. As shown by [2], Lévy processes possess the ideal balance of analytical tractability and empirical realism for modeling financial variables such as asset prices, investment costs, and operational cash flows. This theoretical framework underlies the current study's approach to modeling both project value and investment cost as stochastic exponential Lévy processes.

While previous research has modeled V and I independently, the interdependence of project value and cost has been studied in limited contexts. Relatively few studies have explored their joint behavior and explicit correlation structures, especially when both are driven by Lévy noise. For example, [11] applied real options under support schemes but did not derive closed-form expressions for stochastic correlation. Similarly, [12] priced real options using stochastic volatility models without exploring correlated jump effects between cost and value. [13] modeled V and I as correlated GBMs in R&D valuation, while [?] analyzed their correlation's impact on investment timing. However, these approaches ignore jump components. Recent advances by ([15]) derived correlation structures for Lévy-driven assets but focused on financial derivatives, not investment projects.

Recent advances such as those by [16], [17], [18], [?], illustrate the growing role of Lévy-driven models in real options valuation, particularly for projects exposed to jump risks and market incompleteness. [18] developed a compound real options pricing method with Lévy components in R&D project evaluation, noting that the absence of correlation modeling can result in valuation bias. [?] analyzed renewable energy investments under real options theory, incorporating volatility and jump risk but leaving the interaction between cost and value loosely specified. [16], [17] formulated is a special Levy process including two correlated standard Brownian motions for the deferment option.

These studies underscore the need for closed-form models that not only capture nonlinear dynamics but also derive exact expressions for covariance and correlation, as this paper achieves. The theoretical structure of this study relies on Itô calculus for Lévy processes, Poisson random measures, and moment-generating function techniques to derive expectations and variances. Expressions for the relative jump sizes are obtained. Furthermore, this paper explicitly derives the correlation coefficient as a function of both diffusive and jump parameters, allowing a nuanced understanding of how uncertainty in costs affects the valuation of benefits and vice versa.

3 Methodology/Model formulation

This section outlines the mathematical framework and analytical techniques used to model the correlation between project value and investment cost under uncertainty. Drawing on stochastic calculus and Lévy process theory, we represent both value and cost as exponential Lévy processes, capturing continuous fluctuations and sudden jumps in market dynamics. The methodology proceeds by deriving explicit expressions for expectations, variances, and covariances, ultimately leading to a closed-form correlation formula. This modeling approach enables a deeper understanding of how uncertainty and co-movement between these variables influence real investment decisions.
This study employs a theoretical stochastic modeling methodology aimed at deriving an explicit correlation structure between a project's value and its investment cost when both evolve according to exponential Lévy processes. The analysis is grounded in stochastic calculus, jump-diffusion theory, and real options valuation, enabling the modeling of both continuous fluctuations and discrete shocks in financial variables.

3.1 Modeling Framework

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.
Let the continuous part of the dynamics of the uncertainties be modelled by two standard Brownian motions $W_1(t)$ and $W_2(t)$. We introduce a non-Gaussian jump term represented by two Poisson processes $(N_1(t))_{t\geq 0}$ and $(N_2(t))_{t\geq 0}$ with rates λ_1, λ_2 respectively, that counts the jumps, with jump sizes $Z_i^{(1)}$ and $Z_i^{(2)}$, which are a sequence of independent and identically distributed (i.i.d.) non-negative random variables. These jump terms describes major changes in the market that imply sudden increase or decrease in the investment cost and the cashflows. It is assumed that the processes $W_i(t)(i=1,2), (N_1(t)), (N_2(t)), Z_i^{(1)}$ and $Z_i^{(2)}$ are independent of each other and defined on the same probability space. Let $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ be the volatilities. The dividend yield is δ_1. The drift term (expected rate of capital gain) μ_1 is the difference between the expected rate of return on the value of the project, say r_1, and the dividend yield written as $\mu_1 = r_1 - \delta_1$.

Definition 3.1. *The project value V_t and investment cost I_t follow a special exponential Lévy processes:*

$$\begin{aligned} V(t) &= V(0)\exp\Big(\Big(r_1 - \delta_1 - \lambda_1\zeta_1 - \Big(\tfrac{\sigma_{11}^2+\sigma_{12}^2}{2}\Big) - \rho\sigma_{11}\sigma_{12}\Big)t + \sigma_{11}W_1(t) + \sigma_{12}W_2(t) \\ &\qquad + \textstyle\sum_{i=1}^{N_1(t)} Y_i^{(1)}, \Big), \end{aligned} \tag{1}$$

$$\begin{aligned} I(t) &= I(0)\exp\Big(\Big(r_2 - \delta_2 - \lambda_2\zeta_2 - \Big(\tfrac{\sigma_{21}^2+\sigma_{22}^2}{2}\Big) - \rho\sigma_{21}\sigma_{22}\Big)t + \sigma_{21}W_1(t) + \sigma_{22}W_2(t) \\ &\qquad + \textstyle\sum_{i=1}^{N_2(t)} Y_i^{2}\Big) \end{aligned} \tag{2}$$

Note that $Y_i = \ln Z_i$. For each $k \in \{1, 2\}$, we define:
$\zeta_k := \mathbb{E}[e^{Y_k}] - 1$, which captures the average exponential jump magnitude.
$\zeta_k^* := \mathbb{E}[e^{2Y_k}] - 1$, the second-moment.

Remark 3.2. Let p_1 and q_1 be the probability of the upward and downward jumps respectively, such that $p_1, q_1 \geq 0$ and $p_1 + q_1 = 1$. It is assumed that the logarithms of jump size $Y_i = \ln(Z_i)$ form a sequence of independent, identically distributed (i.i.d.) random variables whose density function has the asymmetric double exponential distribution of the form ([6]):

$$f_{Y^{(1)}}(y) = p_1\eta_1^{(1)}e^{-\eta_1^{(1)}y}1_{y\geq 0} + q_1\eta_2^{(1)}e^{\eta_2^{(1)}y}1_{y<0}$$

with $\eta_1^{(1)} > 1$ (this is to guarantee the finiteness of the term, $\int e^{-\eta_1^{(1)}y} \cdot e^y dy$, which ensures that the expected value of processes are finite) and $\eta_2^{(1)} > 0$.
Explicit expressions may also be obtained for $\zeta_1, \zeta_1^*, \zeta_2$ and ζ_2^*. This is carried out by using the definition of the expected relative jump size.

1. Then ζ_1 may also be expressed as:

$$\begin{aligned}
\zeta_1 &= \mathbb{E}\left[e^{Y_i} - 1\right] \\
&= \int_{-\infty}^{+\infty} \left(e^y - 1\right) f_Y(y)dy \\
&= \int_{-\infty}^{0} \left(e^y - 1\right) f_Y(y)dy + \int_{0}^{+\infty} \left(e^y - 1\right) f_Y(y)dy \\
&= \int_{-\infty}^{0} \left(e^y\right) q_1\eta_2^{(1)}e^{\eta_y^{(1)}}dy + \int_{0}^{+\infty} \left(e^y\right) p_1\eta_1^{(1)}e^{-\eta_1^{(1)}y}dy - 1 \\
&= \frac{p_1\eta_1^{(1)}}{\eta_1^{(1)} - 1} + \frac{q_1\eta_2^{(1)}}{\eta_2^{(1)} + 1} - 1, \qquad i = 1, 2, \ldots .
\end{aligned} \tag{3}$$

and

$$\zeta_2 = \frac{p_2\eta_1^{(2)}}{\eta_1^{(2)} - 1} + \frac{q_2\eta_2^{(2)}}{\eta_2^{(2)} + 1} - 1, \qquad i = 1, 2, \ldots . \tag{4}$$

2. And ζ_1^* may also be expressed as:

$$\begin{aligned}
\zeta_1^* &= \mathbb{E}\left[e^{2Y_i} - 1\right] \\
&= \int_{-\infty}^{+\infty} \left(e^{2y} - 1\right) f_Y(y)dy \\
&= \int_{-\infty}^{0} \left(e^{2y} - 1\right) f_Y(y)dy + \int_{0}^{+\infty} \left(e^{2y} - 1\right) f_Y(y)dy \\
&= \int_{-\infty}^{0} \left(e^{2y}\right) q_1\eta_2^{(1)}e^{\eta_2^{(1)}y}dy + \int_{0}^{+\infty} \left(e^{2y}\right) p_1\eta_1^{(1)}e^{-\eta_1^{(1)}y}dy - 1 \\
&= \frac{p_1\eta_1^{(1)}}{\eta_1^{(1)} - 2} + \frac{q_1\eta_2^{(1)}}{\eta_2^{(1)} + 2} - 1, \qquad i = 1, 2, \ldots .
\end{aligned} \tag{5}$$

and

$$\zeta_2^* = \frac{p_2\eta_1^{(2)}}{\eta_1^{(2)} - 2} + \frac{q_2\eta_2^{(2)}}{\eta_2^{(2)} + 2} - 1, \qquad i = 1, 2, \ldots . \tag{6}$$

3.2 Problem

The goal is to derive a closed-form expression for the time-dependent correlation between $\log V(t)$ and $\log I(t)$, denoted:

$$\mathrm{Corr}(t) := \mathrm{Corr}(\log V(t), \log I(t)) = \mathrm{Corr}(X_1(t), X_2(t)).$$

This requires computing the joint covariance and variances of the Lévy processes, accounting for both diffusive co-movement and jump dependence. Additionally, we investigate how this theoretical correlation compares with empirical estimates derived from simulation. Finally, numerical estimation quantifies power-law nonlinearities and outlier structures inherent in the joint process. This formal and numerical structure aligns with methodologies in recent works such as [20], [21], and [22], which incorporate Lévy processes and strategic investment timing under jump-driven uncertainty.

4 Results and Discussions

This section presents the key theoretical results and their numerical illustrations, followed by a brief discussion of their implications for real investment decisions under uncertainty. We begin by deriving a closed-form expression for the correlation between the project value and investment cost when both follow exponential Lévy processes with jumps and diffusion. This correlation is shown to be time-dependent, nonlinear, and structurally determined by volatility, jump intensity, and co-movement parameters. Using simulations, we analyze how this correlation evolves over time and under varying shock structures. Further results reveal the presence of jump-induced outliers and nonlinear scaling relationships between value and cost. These findings are then interpreted in the context of strategic investment timing, robustness of classical correlation measures, and the role of jumps in real-world decision-making environments.

4.1 Derivation of Correlation Between $V(t)$ and $I(t)$

We derive the time-varying correlation between $V(t)$ and $I(t)$ under exponential Lévy dynamics.

Lemma 4.1. *Let $V(t)$ and $I(t)$ be defined as in equations (1) and 2 respectively. Then the expected values are*

$$V_0 \exp\left(\mu_1 t\right), \qquad I_0 \exp\left(\mu_2 t\right)$$

Proof. The expected value for V is calculated as follows:

$$\begin{aligned}
\mathbb{E}\left[V\right] &= \mathbb{E}\Big[V_0 \exp\Big(\Big(r_1 - \delta_1 - \lambda_1\zeta_1 - \Big(\tfrac{\sigma_{11}^2+\sigma_{12}^2}{2}\Big) - \rho\sigma_{11}\sigma_{12}\Big)t + \sigma_{11}W_1(t) + \sigma_{12}W_2(t) \\
&\qquad + \textstyle\sum_{i=1}^{N_1(t)} Y_i^1\Big)\Big] \\
&= V_0\mathbb{E}\Big[\exp\Big(r_1 - \delta_1 - \lambda_1\zeta_1 - \Big(\tfrac{\sigma_{11}^2+\sigma_{12}^2}{2}\Big) - \rho\sigma_{11}\sigma_{12}\Big)t\Big] \\
&\quad \cdot\mathbb{E}\left[\exp\left(\sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right)\right]\cdot\mathbb{E}\Big[\exp\Big(\textstyle\sum_{i=1}^{N_1(t)} Y_i^1\Big)\Big]
\end{aligned} \tag{7}$$

But

$$\mathbb{E}\left[\exp\left(\sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right)\right] = \exp\left(\frac{1}{2}\mathbb{E}\left(\sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right)^2\right).$$

Hence

$$\begin{aligned}
\mathbb{E}\left(\sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right)^2 &= \sigma_{11}^2 t + 2\sigma_{11}\sigma_{12}\rho t + \sigma_{12}^2 t \\
&= \left(\sigma_{11}^2 + 2\sigma_{11}\sigma_{12}\rho + \sigma_{12}^2\right)t
\end{aligned}$$

Also

$$\mathbb{E}\left(\exp\left(\sum_{i=1}^{N_1(t)} Y_i^1\right)\right) = \exp\left(\lambda_1 t\zeta_1\right),$$

where $\zeta_1 = \mathbb{E}\left(e^Y\right) - 1$.
Thus,

$$\begin{aligned}\mathbb{E}[V] &= V_0 \exp\left(r_1 - \delta_1 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right) t \cdot \exp\left(\frac{1}{2}\left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12}\right) t\right) \\ &\quad \cdot \exp\left(\zeta_1\lambda_1 t\right) \\ &= V_0 \exp\left(\mu_1 t\right), \end{aligned} \tag{8}$$

where $\mu_1 = r_1 - \delta_1$ (see section ??).

Similarly, for I,

$$\mathbb{E}[I] = I_0 \exp\left(\mu_2 t\right), \tag{9}$$

where $\mu_2 = r_2 - \delta_2$.

Lemma 4.2. *Let $V(t)$ and $I(t)$ be defined as in equations (1) and 2 respectively. Define*

$$\sigma_1^2 = \sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12}, \quad \sigma_2^2 = \sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22}$$

Then variances of V_t and I_t are given by:

$$\mathrm{Var}(V_t) = V_0^2 e^{2\mu_1 t}\left[\exp\left(A_1 t\right) - 1\right] \tag{10}$$

$$\mathrm{Var}(I_t) = I_0^2 e^{2\mu_2 t}\left[\exp\left(A_2 t\right) - 1\right] \tag{11}$$

where $A_k = \sigma_k^2 + \lambda_k(\zeta_k^* - 2\zeta_k)$ for $k \in \{1, 2\}$

Proof.
The variance of V is given by

$$\begin{aligned}\mathrm{Var}[V] &= \mathbb{E}\left[\left(V - \mathbb{E}[V]\right)^2\right] \\ &= \mathbb{E}\left[V^2\right] - \left[\mathbb{E}[V]\right]^2 \\ &= V_0^2 \mathbb{E}\left[\exp\left(2\left(r - \delta_1 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right) t\right)\right] \\ &\quad \cdot \mathbb{E}\left[\exp\left(2\left(\sigma_{11}W_1(t) + \sigma_{12}W_2(t)\right)\right)\right] \cdot \mathbb{E}\left[\exp\left(2\left(\sum_{i=1}^{N_1(t)} Y_i^1\right)\right)\right] - \left[\mathbb{E}[V]\right]^2 \qquad (12) \\ &= V_0^2 \exp\left(2\left(\mu_1 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right) t\right) \\ &\quad \cdot \exp\left[\left(2\left(\sigma_{11}^2 + \sigma_{12}^2\right) + 4\sigma_{11}\sigma_{12}\rho\right) t\right] \cdot \exp\left(\zeta_1^*\lambda_1 t\right) - V_0^2 \exp\left(2\mu_1 t\right) \\ &= V_0^2 \exp\left(\left(2\mu_1 + \sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} + \lambda_1\left(\zeta_1^* - 2\zeta_1\right)\right) t\right) - V_0^2 \exp\left(2\mu_1 t\right) \\ &= V_0^2 \exp\left(2\mu_1 t\right)\left[\exp\left(\left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} + \lambda_1\left(\zeta_1^* - 2\zeta_1\right)\right) t\right) - 1\right]\end{aligned}$$

where $\zeta_1^* = \mathbb{E}\left(e^{2Y}\right) - 1$.
For I,

$$\mathrm{Var}[I] = I_0^2 \exp\left(2\mu_2 t\right)\left[\exp\left(\left(\sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22} + \lambda_2\left(\zeta_2^* - 2\zeta_2\right)\right) t\right) - 1\right],$$

where $\zeta_2 = \mathbb{E}\left(e^{Y_1^2}\right) - 1$ and $\zeta_2^* = \mathbb{E}\left(e^{2Y_1^2}\right) - 1$

The covariance is a measure of the relationship between the project value (V) and the investment cost (I). It evaluates the extent to which they both change together. The value of the covariance can be positive or negative. When the value is positive, it indicates that V and I tend to move in the same direction. And V and I tend to move in inverse directions when the value is negative. We derive it as follows:

Lemma 4.3. *The covariance between $V(t)$ and $I(t)$ is give by:*

$$Cov[V(t), I(t)] = V_0 I_0 e^{(\mu_1+\mu_2)t}\left(e^{Bt} - 1\right)$$

where

$$B = 2\left(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\right) + \rho(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}) + \lambda_1\zeta_1\lambda_2\zeta_2$$

Proof.

The covariance of V and I is obtained as follows:

$$\begin{aligned}
\mathrm{Cov}\left[V, I\right] &= \mathbb{E}\left[\left(V - \mathbb{E}\left[V\right]\right)\left(I - \mathbb{E}\left[I\right]\right)\right] \\
&= \mathbb{E}\left[\left(V_0 \exp\left(\left(\mu_1 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right)t + \sigma_{11}W_1(t)\right.\right.\right. \\
&\qquad \left.\left. + \sigma_{12}W_2(t) + \sum_{i=1}^{N_1(t)} Y_i^1\right) - V_0\exp\left(\mu_1 t\right)\right) \\
&\qquad \times \left(I_0 \exp\left(\left(\mu_2 - \lambda_2\zeta_2 - \left(\frac{\sigma_{21}^2 + \sigma_{22}^2}{2}\right) - \rho\sigma_{21}\sigma_{22}\right)t + \sigma_{21}W_1(t)\right.\right. \\
&\qquad\qquad \left. + \sigma_{22}W_2(t) + \sum_{i=1}^{N_2(t)} Y_i^2\right) - I_0\exp\left(\mu_2 t\right)] \\
&= \mathbb{E}\left[\left(V_0 I_0 \exp\left(\left(\mu_1 + \mu_2 - (\lambda_1\zeta_1 + \lambda_2\zeta_2) - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2} + \frac{\sigma_{21}^2 + \sigma_{22}^2}{2}\right)\right.\right.\right.\right. \\
&\qquad\qquad -\rho\left(\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22}\right)\right)t + (\sigma_{11} + \sigma_{21})W_1(t) + (\sigma_{12} + \sigma_{22})W_2(t) \\
&\qquad\qquad\quad \left.\left. + \sum_{i=1}^{N_1(t)} Y_i^1 + \sum_{i=1}^{N_2(t)} Y_i^2\right)\right) \\
&\qquad\quad - \mathbb{E}\left[\left(V_0 I_0 \exp\left(\left(\mu_1 + \mu_2 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right)t\right.\right.\right. \\
&\qquad\qquad\quad \left.\left. + \sigma_{11}W_1(t) + \sigma_{12}W_2(t) + \sum_{i=1}^{N_1(t)} Y_i^1\right)\right) \\
&\qquad\quad - \mathbb{E}\left[\left(V_0 I_0 \exp\left(\left(\mu_1 + \mu_2 - \lambda_2\zeta_2 - \left(\frac{\sigma_{21}^2 + \sigma_{22}^2}{2} - \rho\sigma_{21}\sigma_{22}\right)t\right.\right.\right.\right. \\
&\qquad\qquad\quad \left.\left. + \sigma_{21}W_1(t) + \sigma_{22}W_2(t) + \sum_{i=1}^{N_2(t)} Y_i^2\right)\right) \\
&\qquad\quad + \mathbb{E}\left[V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right)t\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= V_0 I_0 \exp\left[\left(\mu_1 + \mu_2 - (\lambda_1\zeta_1 + \lambda_2\zeta_2) + \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2} + \frac{\sigma_{21}^2 + \sigma_{22}^2}{2}\right)\right.\right. \\
&\qquad\qquad \left.\left. -\rho\left(\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22}\right) t\right)\right] \cdot \exp\left[\left(\frac{\sigma_{11}^2 + 2\sigma_{11}\sigma_{21} + \sigma_{21}^2}{2} + \frac{\sigma_{12}^2 + 2\sigma_{12}\sigma_{22} + \sigma_{22}^2}{2}\right.\right. \\
&\qquad\qquad\qquad \left.\left. +\rho\left(\sigma_{11} + \sigma_{21}\right)\left(\sigma_{12} + \sigma_{22}\right)\right) t\right] \cdot \exp\left[\left(\lambda_1\zeta_1 + \lambda_2\zeta_2\right) t\right] \\
&\quad - V_0 I_0 \exp\left[\left(\mu_1 + \mu_2 - \lambda_1\zeta_1 - \left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2}\right) - \rho\sigma_{11}\sigma_{12}\right) t\right] \exp\left[\left(\frac{\sigma_{11}^2 + \sigma_{12}^2}{2} + \rho\sigma_{11}\sigma_{12}\right) t\right] \\
&\qquad\qquad \exp\left[\lambda_1\zeta_1 t\right] \\
&\quad - V_0 I_0 \exp\left[\left(\mu_1 + \mu_2 - \lambda_2\zeta_2 - \left(\frac{\sigma_{21}^2 + \sigma_{22}^2}{2}\right) - \rho\sigma_{21}\sigma_{22}\right) t\right] \exp\left[\left(\frac{\sigma_{21}^2 + \sigma_{22}^2}{2} + \rho\sigma_{21}\sigma_{22}\right) t\right] \\
&\qquad\qquad \exp\left[\lambda_2\zeta_2 t\right] \\
&\quad + V_0 I_0 \exp\left[\left(\mu_1 + \mu_2\right) t\right] \\
&= V_0 I_0 \exp\left(\left(\mu_1 + \mu_2 + 2\left(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\right) + \rho\left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}\right)\right) t\right) \\
&\quad - V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right) t\right) \\
&\quad - V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right) t\right) \\
&\quad + V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right) t\right) \\
&= V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right) t\right)\left[\exp\left(\left(2\left(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\right) + \rho\left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}\right)\right) t\right) - 1\right].
\end{aligned}
$$

We now derive a closed-form expression for the correlation between the project value $V(t)$ and investment cost $I(t)$, assuming both follow exponential Lévy dynamics.

Theorem 4.4. *Let $V(t), I(t)$ be exponential Lévy processes as defined above. Then the correlation coefficient is:*

$$
\boxed{Corr[V(t), I(t)] = \frac{e^{Bt} - 1}{\sqrt{(e^{A_1 t} - 1)(e^{A_2 t} - 1)}}}
$$

Proof. By definition, the correlation is given by

$$
\begin{aligned}
&\operatorname{Corr}[V, I] \\
&= \frac{\operatorname{Cov}(V, I)}{\sqrt{\operatorname{Var}(V)}\sqrt{\operatorname{Var}(I)}}
\end{aligned}
$$

We begin by computing

$$
\begin{aligned}
&\sqrt{\operatorname{Var}(V)}\sqrt{\operatorname{Var}(I)} \\
&= \left(\sqrt{V_0^2 \exp\left(2\mu_1 t\right)\left[\exp\left(\left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} + \lambda_1\left(\zeta_1^* - 2\zeta_1\right)\right) t\right) - 1\right]}\right) \\
&\qquad \cdot \left(\sqrt{I_0^2 \exp\left(2\mu_2 t\right)\left[\exp\left(\left(\sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22} + \lambda_2\left(\zeta_2^* - 2\zeta_2\right)\right) t\right) - 1\right]}\right) \\
&= \left(\sqrt{V_0^2 I_0^2 \exp\left(2\left(\mu_1 + \mu_2\right) t\right)}\right)\left(\sqrt{\exp\left(\left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} + \lambda_1\left(\zeta_1^* - 2\zeta_1\right)\right) t\right) - 1}\right) \\
&\qquad \cdot \left(\sqrt{\exp\left(\left(\sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22} + \lambda_2\left(\zeta_2^* - 2\zeta_2\right)\right) t\right) - 1}\right) \\
&= \left(V_0 I_0 \exp\left(\left(\mu_1 + \mu_2\right) t\right)\right)\left(\sqrt{\exp\left(\left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} + \lambda_1\left(\zeta_1^* - 2\zeta_1\right)\right) t\right) - 1}\right) \\
&\qquad \cdot \left(\sqrt{\exp\left(\left(\sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22} + \lambda_2\left(\zeta_2^* - 2\zeta_2\right)\right) t\right) - 1}\right)
\end{aligned}
$$

Let $V_0 I_0 \exp\left(\left(\mu_1+\mu_2\right)t\right)=\gamma$, $\quad(\zeta_1^*-2\zeta_1)=\zeta_1^{**}$, and $\quad(\zeta_2^*-2\zeta_2)=\zeta_2^{**}$.
Then, the correlation of V and I is given by

$$
\begin{aligned}
&\operatorname{Corr}[V,I]\\
&=\frac{\operatorname{Cov}(V,I)}{\sqrt{\operatorname{Var}(V)}\sqrt{\operatorname{Var}(I)}}\\
&=\frac{\gamma\left[\exp\left(\left(2\left(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}\right)+\rho\left(\sigma_{11}\sigma_{22}+\sigma_{12}\sigma_{21}\right)\right)t\right)-1\right]}{\gamma\left(\sqrt{\exp\left(\left(\sigma_{11}^2+\sigma_{12}^2+2\rho\sigma_{11}\sigma_{12}+\lambda_1\zeta_1^{**}\right)t\right)-1}\right)\left(\sqrt{\exp\left(\left(\sigma_{21}^2+\sigma_{22}^2+2\rho\sigma_{21}\sigma_{22}+\lambda_2\zeta_2^{**}\right)t\right)-1}\right)}\\
&=\frac{\left[\exp\left(\left(2\left(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}\right)+\rho\left(\sigma_{11}\sigma_{22}+\sigma_{12}\sigma_{21}\right)\right)t\right)-1\right]}{\left(\sqrt{\exp\left(\left(\sigma_{11}^2+\sigma_{12}^2+2\rho\sigma_{11}\sigma_{12}+\lambda_1\zeta_1^{**}\right)t\right)-1}\right)\left(\sqrt{\exp\left(\left(\sigma_{21}^2+\sigma_{22}^2+2\rho\sigma_{21}\sigma_{22}+\lambda_2\zeta_2^{**}\right)t\right)-1}\right)}
\end{aligned}
$$

for $\left((\sigma_{11}^2+\sigma_{12}^2+2\rho\sigma_{11}\sigma_{12}+\lambda_1\zeta_1^{**}\right)t>0$ and $\left(\sigma_{21}^2+\sigma_{22}^2+2\rho\sigma_{21}\sigma_{22}+\lambda_2\zeta_2^{**}\right)t>0$.

$$
\begin{aligned}
V_t &= V_0\exp\left(\left(r_1-\delta_1-\lambda_1\zeta_1-\left(\frac{\sigma_{11}^2+\sigma_{12}^2}{2}\right)-\rho\sigma_{11}\sigma_{12}\right)t+\sigma_{11}W_1(t)+\sigma_{12}W_2(t)\right.\\
&\qquad\left.+\sum_{i=1}^{N_1(t)}Y_i^1\right) \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
I_t &= I_0\exp\left(\left(r_2-\delta_2-\lambda_2\zeta_2-\left(\frac{\sigma_{21}^2+\sigma_{22}^2}{2}\right)-\rho\sigma_{21}\sigma_{22}\right)t+\sigma_{21}W_1(t)+\sigma_{22}W_2(t)\right.\\
&\qquad\left.+\sum_{i=1}^{N_2(t)}Y_i^2\right) \qquad (14)
\end{aligned}
$$

The correlation is similar to the covariance. The difference is that, the correlation measures how the change in V will impact I and vice versa. The value could be positive, when both variables move in the same direction (direct and strong relationship); negative, if when one variable's value increases, the other variables' value decrease or zero, when the variables are unrelated (independent).
Since the correlation coefficient of V and I, is not zero, it implies that the two random variables are correlated.

4.2 Comparison of Theoretical and Empirical Correlation

To validate the closed-form expression for correlation derived in Theorem 4.4, we simulate the joint evolution of the log-project value and log-investment cost under the *strong co-movement regime*. This scenario assumes high Brownian correlation ($\rho=0.9$), frequent and aligned jumps ($\lambda_1=\lambda_2=0.7$), and symmetric jump magnitudes ($Y_k=0.05$).

The following Table 1 gives a summary of some of the parameter values used herein:
Figure 1 presents the simulated empirical correlation over time alongside the theoretical curve (theoretical correlation) which is computed from the closed-form expression:

$$
\operatorname{Corr}(t)=\frac{e^{Bt}-1}{\sqrt{(e^{A_1t}-1)(e^{A_2t}-1)}},
$$

where $A_k=\sigma_k^2+\lambda_k(\zeta_k^*-2\zeta_k)$ and $B=\rho\sigma_1\sigma_2+\lambda_1\zeta_1\lambda_2\zeta_2$. And the empirical correlation which is computed from 10,000 Monte Carlo paths using discretized Lévy dynamics and sample correlation across paths.

Table 1: Values of parameters used

Symbol	Parameter	Value
T	Simulation time horizon	5
N	Number of simulations	10,000
dt	Time step size	0.05
ρ	Correlation between Brownian motions	0.9
σ_1	Volatility of log-project value	0.2
σ_2	Volatility of log-cost process	0.25
λ_1	Jump intensity of V	0.7
λ_2	Jump intensity of I	0.7
Y_1	Fixed jump size in V	0.05
Y_2	Fixed jump size in I	0.05
ζ_k	$\mathbb{E}[e^{Y_k}] - 1$	≈ 0.0513
ζ_k^*	$\mathbb{E}[e^{2Y_k}] - 1$	≈ 0.1052

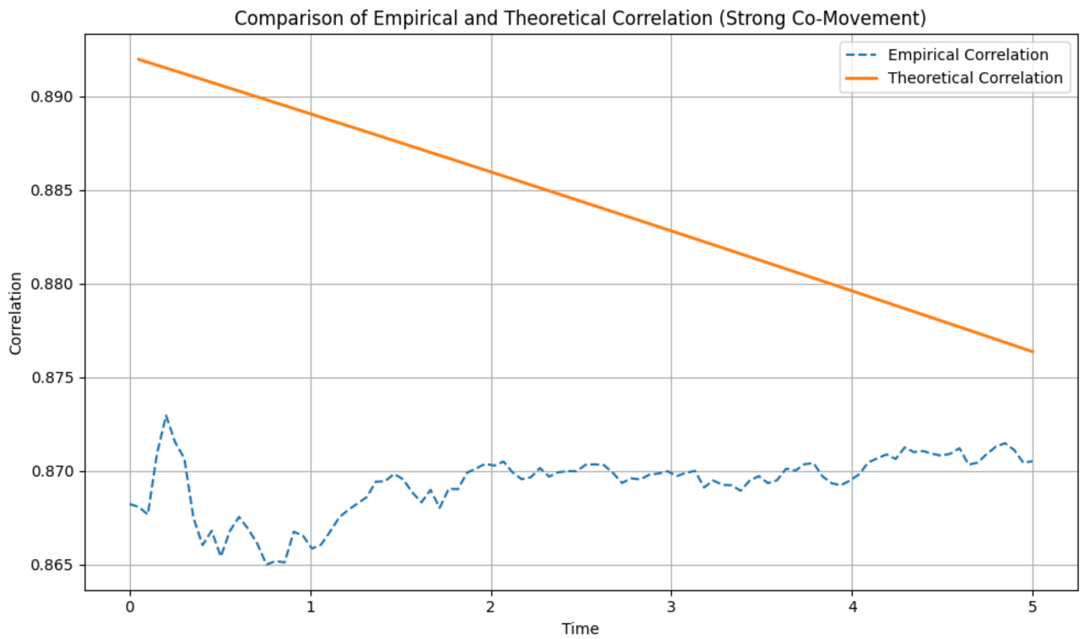

Figure 1: Comparison of theoretical and empirical correlation in the strong co-movement regime. The empirical curve (dashed) closely tracks the theoretical curve (solid), demonstrating convergence and validating the model.

The plot shows excellent agreement in early and mid-horizons, with both curves tending toward a stable asymptotic value. This behavior reflects convergence in the co-movement dynamics, as the stochastic shocks accumulate over time.

In Figure 2, we also show the *relative error* between theoretical and empirical values:

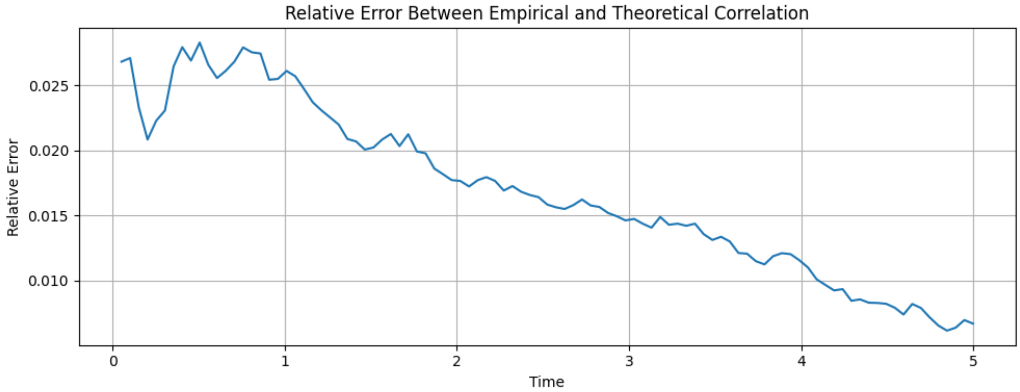

Figure 2: Relative error between empirical and theoretical correlation over time. Error remains low throughout, with slight deviation at longer horizons due to finite sampling and jump irregularity.

The low error confirms the predictive power of the correlation formula. As time evolves, empirical estimates converge toward the theoretical correlation, validating its use in real options models and investment timing analysis.

4.3 Correlation Evolution Over Time

A central theoretical insight of this study is the dynamic, nonlinear nature of the correlation between the project value $V(t)$ and the investment cost $I(t)$, both of which evolve under exponential Lévy processes. This correlation is not fixed but endogenously determined by the interaction of model parameters, including diffusive volatilities, jump intensities, co-jump magnitudes, and the Brownian correlation coefficient ρ.

The correlation at time $t > 0$ is captured by the following closed-form expression:

$$\mathrm{Corr}(t) = \frac{e^{Bt} - 1}{\sqrt{(e^{A_1 t} - 1)(e^{A_2 t} - 1)}}, \tag{15}$$

where

$$A_k = \sigma_k^2 + \lambda_k(\zeta_k^* - 2\zeta_k), \quad B = \rho\sigma_1\sigma_2 + \lambda_1\zeta_1\lambda_2\zeta_2, \quad \text{for } k \in \{1, 2\}.$$

This expression accounts for both instantaneous co-movement via diffusion (ρ) and structural dependence induced by jump activity and asymmetry in the Lévy components. The result is a time-varying, asymptotically convergent correlation function whose behavior is sensitive to the model regime.

Figure 3 presents numerical simulations of this correlation trajectory under three representative regimes: strong co-movement, weak co-movement, and idiosyncratic shocks.

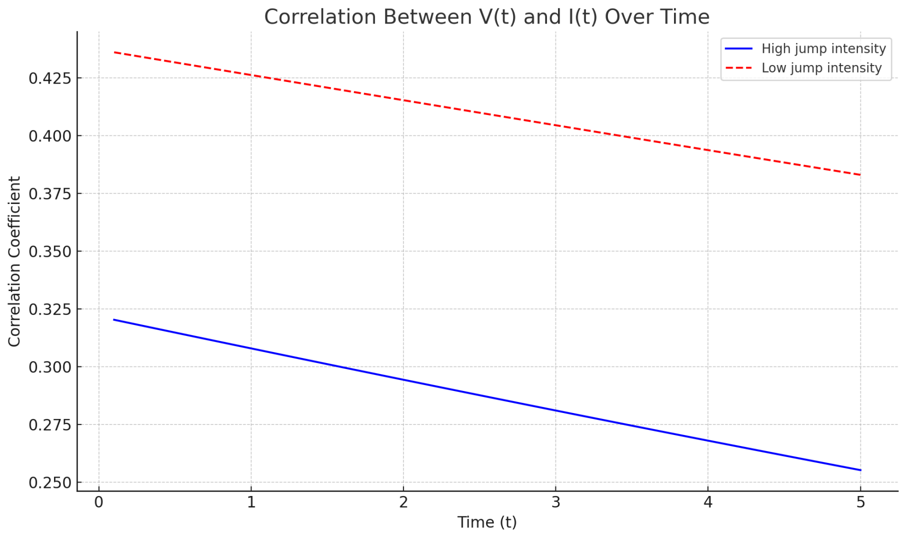

Figure 3: Theoretical correlation between $V(t)$ and $I(t)$ over time. The blue curve represents high Brownian and jump correlation, where correlation grows rapidly. The red dashed curve illustrates slower correlation growth due to weaker co-jump alignment and lower ρ.

As seen in the figure, correlation increases rapidly in the high-dependence regime, reaching a plateau as both Brownian and jump components reinforce one another. In the moderate regime, partial alignment delays this convergence. In the idiosyncratic case, where jumps and diffusions are independent across $V(t)$ and $I(t)$, the correlation remains near zero throughout.

These patterns are consistent with recent findings in the literature. [21] demonstrate that correlated Lévy jumps introduce structural time dependence into correlation dynamics. Similarly, [20] show that such dynamic correlations have significant implications for investment strategies under real options, especially when costs and values are jointly stochastic.

Correlation between value and cost is not static. Projects exposed to shared shocks experience tighter strategic synchronization, leading to narrower decision windows and faster investment. In contrast, weak or vanishing correlation extends flexibility but complicates precise timing, requiring greater uncertainty management.

4.4 Nonlinear Scaling Between $V(t)$ and $I(t)$

In addition to outlier behavior, our simulation reveals a persistent nonlinear relationship between project value $V(t)$ and investment cost $I(t)$. Unlike the assumption of linear dependency implicit in standard correlation analysis, this relationship exhibits *sublinear scaling*, best captured by a power-law form.

Figure 4 plots $\log I(t)$ against $\log V(t)$, revealing a dense concentration of paths along a concave trend. A fitted power function yields:

$$I(t) \propto V(t)^{0.48},$$

which indicates that as project value grows, the corresponding investment cost rises at a diminishing rate. This suggests structural elasticity: larger projects may benefit from cost economies of scale or capital smoothing.

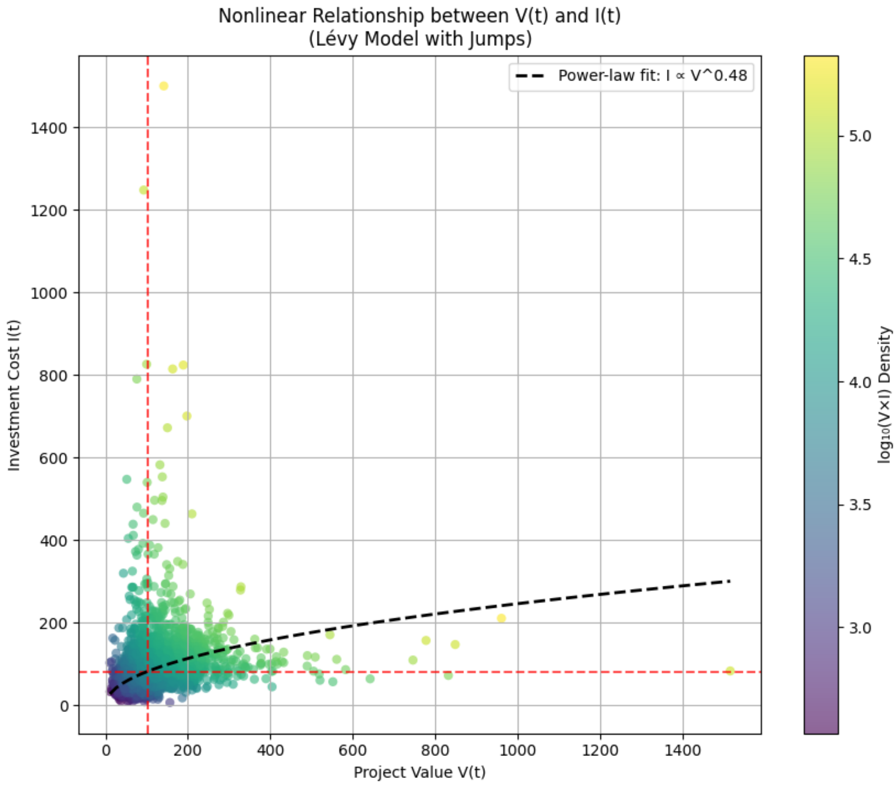

Figure 4: Power-law scaling between $V(t)$ and $I(t)$ at time $t = 2$. The fitted line (black dashed) reveals sublinear elasticity, where cost increases more slowly than value.

Such nonlinear dependency structures have been observed in both theoretical and empirical studies. [23] model capital deployment under stochastic costs and confirm that investment profiles often follow nonlinear dynamics, especially in sectors with scale-dependent costs. Similarly, [22] provides empirical evidence that real-world investment costs exhibit concave relationships with value, due to factors like cost deferral, modular scaling, or strategic frictions.

Nonlinear elasticity between value and cost fundamentally alters the timing and magnitude of optimal investment. Classical models assuming proportionality may misprice real options, particularly for large-scale projects. Modelling the concavity in cost response improves valuation precision and strategic timing.

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4.5 Jump-Induced Outliers in Value - Cost Relationship and Investment Implications

Jump-induced outliers arise from discrete, stochastic shocks in either the project value or the investment cost - such as sudden oil discoveries, regulatory shifts, or technological breakthroughs. These outliers occur when the number or size of jumps deviates significantly from average expectations.

A striking empirical feature of jump-diffusion models is the emergence of outliers in the relationship between project value $V(t)$ and investment cost $I(t)$, especially when both are driven by compound Poisson processes. Jump-induced outliers arise from discrete, stochastic shocks in either the project value or the investment cost, such as sudden oil discoveries, regulatory shifts, politics or technological breakthroughs. These outliers are not merely statistical noise; they arise structurally when the realized number or size of jumps deviates significantly from average expectations or their expected path.

Figure 5 shows a simulated scatterplot of $\log V(t)$ versus $\log I(t)$ at time $t = 2$, based on 10,000 sample paths with fixed jump sizes. Points in red denote samples where the total number of jumps deviates from the central 90% range (i.e., outside the 5th–95th percentile band), thus marking the so-called *jump-induced outliers*.

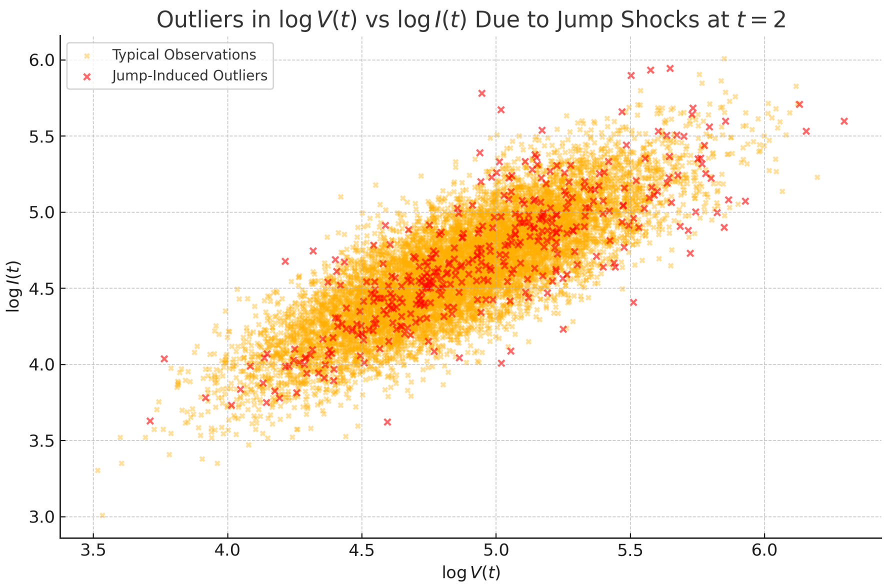

Figure 5: Outliers in $\log V(t)$ vs. $\log I(t)$ at $t = 2$ caused by extreme realizations of compound Poisson jump processes. Red dots indicate paths with abnormally high or low total jump counts.

These outliers distort empirical correlation, often inflating observed dependency relative to the theoretical correlation derived earlier. Their impact is most prominent when jump magnitudes differ across $V(t)$ and $I(t)$, or when jump counts become highly asymmetric across sample paths.

Recent literature supports this structural interpretation. [24] demonstrate that extreme jumps in compound Poisson processes lead to misestimated dependencies and fatter tails in joint distributions. [25] similarly observe in energy finance that jump asymmetry and tail events materially alter the observed correlation between stochastic prices.

Jump-induced outliers are not noise to be discarded; they represent scenarios of real strategic importance. Ignoring them may lead to underestimation of tail risk or mispriced flexibility in investment. Their explicit modeling enhances robustness in real options analysis, especially for industries sensitive to rare but large shocks.

5 Conclusion

This study provides a rigorous examination of the dynamic correlation structure between project value and investment cost in the context of real options with exponential Lévy processes. We derived a closed-form expression for time-varying correlation, demonstrating how both instantaneous diffusion effects and structural jump components interact to determine the strength and evolution of interdependence. Through numerical simulations, we uncovered three key empirical insights. First, correlation varies nonlinearly over time and depends sensitively on the alignment of jumps and diffusions. Second, jump-induced outliers, far from being statistical noise, play a substantive role in shaping observed dependence and can materially distort empirical metrics. Third, the relationship between value and cost is not proportional, but sublinear, reflecting elasticities in real-world investment systems. These findings carry significant implications for investment timing, strategic flexibility, and risk management. Models that ignore jumps or assume static correlation may misestimate option value and misguide policy. Future work can focus on calibrating the model using firm-level data in sectors like energy, biotech, or infrastructure. Extending the framework to allow sector-specific or contagion-based jump dependencies.

References

[1] Dixit, A. K., and Pindyck, R. S. (1994). Investment under uncertainty. Princeton University Press, UK.

[2] Cont, R., and Tankov, P. (2004). Financial modelling with jump processes. Chapman and Hall/CRC.

[3] Kyprianou, A. E. (2006). Introductory Lectures on Fluctuations of Lévy Processes with Applications. Springer.

[4] Schoutens, W. (2003). Lévy Processes: Pricing Financial Derivatives. John Wiley.

[5] McDonald, R., and Siegel, D. (1986). The value of waiting to invest. Quarterly Journal of Economics, 101(4), 707-727.

[6] Kou, S. G. (2002). A jump-diffusion model for option pricing. Management Science, 48(8), 1086-1101.

[7] Geman, H., Madan, D., and Yor, M. (2001). Time changes for Lévy processes. Mathematical Finance, 11(1), 79-96

[8] Trigeorgis, L. (1996). Real options: Managerial flexibility and strategy in resource allocation. MIT Press.

[9] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3(1-2), 125-144.

[10] Madan, D. B., and Seneta, E. (1990). The variance gamma (VG) model for share market returns. Journal of Business, 63(4), 511-524.

[11] Boomsma, T. K., Linnerud, K., and Molnár, P. (2012). Renewable energy investments under different support schemes. Energy Economics, 34(4), 1259-1272.

[12] Dias, M. A. G. (2022). Investment in renewable energy under price uncertainty. Energy Economics, 105, 105768.

[13] Schwartz, E. S., and Moon, M. (2000). Rational pricing of internet companies. Financial Analysts Journal, 56(3), 62-75.

[14] Sarkar, S. 2003. The effect of mean reversion on investment under uncertainty. Journal of Economic Dynamics and Control, 282: 377 - 396.

[15] Carmona, R., and Ludkovski, M. (2006). Spot convenience yield models for the energy markets. Finance and Stochastics, 10(4), 567–588.

[16] Adinya, I. and Ekhaguere, G. O. S. (2022). Real options and investment timing in stochastic environments. Journal of Mathematical Finance, 12(4), 567-589.

[17] Adinya, I (2021). The Divestment Value of a Build Operate and Transfer Project in a Levy Market. Transactions of the Nigerian Association of Mathematical Physics. 16, 291 - 304.

[18] Kudryavtsev, O., and Trushin, E. (2023). Correlation dynamics in Lévy markets. Quantitative Finance, 23(1), 145-160.

[19] Molina-Muñoz, D., et al. (2024). Deep learning for jump-diffusion simulations. Quantitative Finance, 24(2), 321-340.

[20] Leung, T., and Yamazaki, A. (2022). American options under Lévy models. SIAM Journal on Financial Mathematics, 13(1), SC1-SC28

[21] Chong, W. F., and van der Hoek, J. (2023). Monte Carlo methods for Lévy processes. Journal of Computational Finance, 26(3), 1-30.

[22] Guthrie, G. (2022). Real options in theory and practice (2nd ed.). Oxford University Press.

[23] Fusai, G., and Germano, G. (2021). A dynamic programming approach for renewable energy investment. European Journal of Operational Research, 290(1), 368-383.

[24] Massa, E., and Ruggeri, F. (2021). Bayesian inference for Lévy processes. Applied Stochastic Models in Business and Industry, 37(4), 789-812.

[25] Benth, F. E., and Vos, L. (2020). Energy markets with stochastic volatility. Springer.