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A comparative analysis of the variational iteration and the Adomian decomposition methods for solving fractional-order partial derivatives with weakly singular kernel

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Abstract

This paper examines the comparative methods of obtaining approximate solutions to fractional order partial derivatives using the variational iteration method (VIM) and the Adomian decomposition method (ADM). By appropriately imposing the initial conditions, an integral equation in the form of a correction functional is constructed using the general Lagrange multiplier. It is generally believed that the iteration method gives the solutions in the form of convergent series giving it some advantage over other methods of numerical solutions since it requires no linearization, discretization or perturbations. Numerical simulations are also employed to obtained the graphical solutions of the methods, and comparison really showed that the VIM gives a better approximate solution compared to ADM.

Keywords and phrases: variational iteration method; Adomian decomposition method; Caputo derivative; lagrange multiplier; fractional differential equation

1 Introduction

The theory of differential equations have been studied for more than 300 years now. Research interest in various aspects of differential equations, such as stability analysis, uniqueness of solutions, and numerical solution techniques—has steadily grown. Today, the literature on differential equations is extensive, encompassing both classical and fractional forms (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). Among these developments, fractional differential equations have emerged as a powerful tool in modeling phenomena that classical equations cannot adequately describe [12, 13, 14, 15, 16, 17].

Fractional calculus itself has a history spanning approximately three centuries, with its early conceptual roots established in pure mathematics. Systematic investigations began in the 19th century through the work of mathematicians such as Liouville, Riemann, Leibniz, and Caputo. However, it is within the last four decades that fractional calculus has gained significant traction, particularly due to its success in modeling complex systems with memory effects and hereditary properties—features commonly encountered in physical, biological, and engineering systems [18, 19, 20, 21]. The practical

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applications of fractional calculus now span diverse fields, including engineering, physics, economics, finance, and chemistry (see [22]). For a deeper understanding of this topic, readers are referred to comprehensive monographs such as [23, 24, 25].

In parallel with theoretical developments, various numerical methods have been developed to solve fractional differential equations, especially partial ones. Methods such as the homotopy perturbation method, the variational iteration method (VIM), the Adomian decomposition method (ADM), the Elzaki transform, and the Laplace transform method are commonly used in the literature. For example, in [26], VIM is described as an enhanced version of the general Lagrange multiplier method, noted for its ability to solve a wide range of nonlinear problems efficiently and accurately. Proposed by Ji-Huan He, the method has been successfully applied to autonomous ordinary and partial differential equations (see [27]). Later, [28] extended the use of VIM to fractional-order nonlinear ordinary differential equations. The method's approximations are known for their rapid convergence to exact solutions, as documented in several studies ([29, 30, 31, 32]).

On the other hand, ADM can be used to solve both linear and nonlinear ordinary and partial differential equations. As noted in [33], this method has several advantages over conventional approaches, such as being unaffected by rounding off errors since it does not involve discretization, linearization, or perturbation, and it requires only a small amount of computer memory. Again, in [34], it was observed that, one advantage of the ADM is its ability to provide analytical approximations for a broad range of nonlinear (and stochastic) equations without relying on linearization, perturbation, closure approximations, or discretization, which often lead to extensive numerical computations. However, this method may require adopting certain simplifying and restrictive assumptions about the analytical solutions to make the problem solvable. As a result, while the obtained solutions may be mathematically interesting, they may not accurately represent physical reality. In other words, the solution to the simplified mathematical model may not closely approximate the original problem's solution. This limitation highlights the superiority of the VIM over the ADM (see also [35, 36]).

In this paper, motivated by research in [37], we aim to evaluate the effectiveness of VIM compared to the ADM. Using Python software, we conduct a simple simulation to provide a graphical representation of how the numerical solutions converge to the exact solution.

2 Preliminaries, definitions and notations

In this work, we adopt the Caputo definition of the fractional derivative, which modifies the Riemann–Liouville formulation to better handle initial value problems. A key advantage of the Caputo derivative is that it allows initial conditions to be expressed in terms of the field variables and their integer-order derivatives, aligning with the standard formulation used in most physical processes. This makes it particularly useful for practical applications where classical initial conditions are required.

Definition 2.1. *[37] A real function $\aleph(\sigma)$, $\sigma > 0$, is said to be in the space C_υ, $\upsilon \in \mathbb{R}$ if there exists a real number $p > \upsilon$, such that $\aleph(\sigma) = \sigma^p \aleph_1(\sigma)$, where $\aleph_1(\sigma) \in C[0, \infty)$, and it is said to be in the space C_υ^κ iff $\aleph^{(\kappa)} \in C_\upsilon$, $\kappa \in \mathbb{N}$.*

Definition 2.2. *[33] The Riemann–Liouville fractional integral operator of order $\epsilon \geq 0$, of a function $\aleph \in C_\upsilon$, $\upsilon \geq -1$, is defined as*

$$\begin{aligned} {}^{RL}I^\iota \aleph(\sigma) &= \frac{1}{\Gamma(\iota - \epsilon)} \int_0^\sigma (\sigma - t)^{\iota - 1} \sigma \aleph(t) dt, \ \iota \geq 0 \\ I^0 \aleph(\sigma) &= \aleph(\sigma) \end{aligned} \tag{1}$$

For $\aleph \in C_\upsilon$, $\upsilon \geq -1$, $\epsilon, \rho \geq 0$ and $\eta > -1$, then the following holds:

1. $I^\epsilon I^\rho \aleph(\sigma) = I^{\epsilon+\rho} \aleph(\sigma)$,

2. $I^{\epsilon}I^{\rho}\aleph(\sigma) = I^{\epsilon}I^{\rho}\aleph(\sigma)$,

3. $I^{\epsilon}\sigma^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\epsilon+\eta+1)}\sigma^{\epsilon+\eta}$

The Riemann–Liouville derivative presents certain limitations when modeling real-world phenomena using fractional differential equations, particularly due to its formulation, which often leads to initial conditions expressed in terms of fractional-order derivatives rather than physically meaningful integer-order derivatives. To address this issue, we adopt the modified fractional differential operator D_*^{α}, introduced by Caputo in his work on viscoelasticity. This approach ensures that initial conditions are given in the same form as in classical differential equations, making it more suitable for practical applications in physics and engineering.

Definition 2.3. *[35] The fractional derivative $\aleph(\sigma)$ in the Caputo sense is defined as*

$$D_*^{\epsilon}\aleph(\sigma) = I^{\kappa-\epsilon}D^{\kappa}\aleph(\sigma) = \frac{1}{\kappa-\epsilon}\int_0^{\sigma}(\sigma-t)^{\kappa-\epsilon-1}\aleph^{(\kappa)}(t)dt, \tag{2}$$

for $\kappa - 1 < \epsilon \leq \kappa$, $\kappa \in \mathbb{N}$, $\sigma > 0$, and $\aleph \in C_{-1}^{\kappa}$.

Lemma 2.4. *[37] If $\kappa - 1 < \epsilon \leq \kappa$, $\kappa \in \mathbb{N}$, and $\aleph \in C_{\upsilon}^{\kappa}$ $\upsilon \geq -1$, then*

$$D_*^{\epsilon}I^{\epsilon}\aleph(\sigma) = \aleph(\sigma) \tag{3}$$

$$I^{\epsilon}D_*^{\epsilon}\aleph(\sigma) = \aleph(\sigma) - \sum_{\iota=0}^{\kappa-1}\aleph^{(\iota)}(0^+)\frac{\sigma^{\iota}}{\iota!},\ \sigma > 0 \tag{4}$$

3 Variational iteration method

The fundamental concepts of the variational iteration method, along with its applicability to different types of differential equations, are discussed in [37, 38, 39, 40, 41]. In this context, we examine the following time-fractional partial differential equation.

$$D_{*t}^{\epsilon}\chi(\sigma,t) = \aleph(\chi,\chi_{\sigma},\chi_{\sigma\sigma}) + \Xi(\sigma,t),\ \kappa-1<\epsilon\leq\kappa, \tag{5}$$

where $D_{*t}^{\epsilon} = \frac{\partial^{\epsilon}}{\partial t^{\epsilon}}$ is the Caputo fractional derivative of order ϵ, $\kappa \in \mathbb{N}$, $\aleph$ is a nonlinear function and Ξ is the source function. The initial and boundary conditions associated with (5) are of the form

$$\begin{aligned} &\chi(\sigma,0) = h(\sigma),\ 0<\epsilon\leq 1,\\ &\chi(\sigma,t)\to 0 \text{ as } |\sigma|\to\infty,\ t>0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\chi(\sigma,0) = h(\sigma),\ \frac{\partial\chi(\sigma,0)}{\partial t} = \iota(\sigma),\ 1<\epsilon\leq 2,\\ &\chi(\sigma,t)\to 0 \text{ as } |\sigma|\to\infty,\ t>0, \end{aligned} \tag{7}$$

The correction functional for (5) can be approximately expressed as follows:

$$\chi_{\iota+1}(\sigma,t) = \chi_{\iota}(\sigma,t) + \int_0^t \eta(\zeta)\left(\frac{\partial^{\kappa}}{\partial\zeta^{\kappa}}\chi_{\iota}(\sigma,\zeta) - \aleph(\tilde{\chi}_{\iota},(\tilde{\chi}_{\iota})_{\sigma},(\tilde{\chi}_{\iota})_{\sigma\sigma}) - \Xi(\sigma,\zeta)\right)d\zeta \tag{8}$$

where η is a general Lagrange multiplier which can be identified optimally via variational theory [28, 37], here $(\tilde{\chi}_{\iota})$, $(\tilde{\chi}_{\iota})_{\sigma}$, $(\tilde{\chi}_{\iota})_{\sigma\sigma}$ are considered as restricted variations, i.e., $\partial\tilde{\chi}_{\iota} = 0$. Making the above functional stationary,

$$\partial\chi_{\iota+1}(\sigma,t) = \partial\chi_{\iota}(\sigma,t) + \partial\int_0^t \eta(\zeta)\left(\frac{\partial^{\kappa}}{\partial\zeta^{\kappa}}\chi_{\iota}(\sigma,\zeta) - \Xi(\sigma,\zeta)\right)d\zeta \tag{9}$$

yields the following Lagrange multipliers

$$\eta = -1, \text{ for } \kappa = 1$$

$$\eta = \zeta - t, \text{ for } \kappa = 2.$$

Therefore, for $\kappa = 1$, we obtain the following iteration formula:

$$\chi_{\iota+1}(\sigma,t) = \chi_\iota(\sigma,t) - \int_0^t \left(\frac{\partial^\epsilon}{\partial\zeta^\epsilon}\chi_\iota(\sigma,\zeta) - \aleph(\chi_\iota,(\chi_\iota)_\sigma,(\chi_\iota)_{\sigma\sigma}) - \Xi(\sigma,\zeta) \right) d\zeta \tag{10}$$

In this case, we begin with the initial approximation

$$\chi_0(\sigma,t) = h(\sigma) \tag{11}$$

For $\kappa = 2$, we obtain the following iteration formula:

$$\chi_{\iota+1}(\sigma,t) = \chi_\iota(\sigma,t) + \int_0^t (\zeta - t) \left(\frac{\partial^\epsilon}{\partial\zeta^\epsilon}\chi_\iota(\sigma,\zeta) - \aleph(\chi_\iota,(\chi_\iota)_\sigma,(\chi_\iota)_{\sigma\sigma}) - \Xi(\sigma,\zeta) \right) d\zeta \tag{12}$$

In this case, we begin with the initial approximation

$$\chi_0(\sigma,t) = h(\sigma) + t\iota(\sigma). \tag{13}$$

The correction functional (8) will give several approximations, and therefore the exact solution is obtained as

$$\chi(\sigma,t) = \lim_{\iota\to\infty} \chi_\iota(\sigma,t). \tag{14}$$

4 Adomian Decomposition Method

The Adomian decomposition method requires that the nonlinear fractional differential (5) be expressed in terms of operator form as

$$D^\epsilon_{*t}\chi(\sigma,t) + L\chi(\sigma,t) + N\chi(\sigma,t) = \Xi(\sigma,t), \ \sigma > 0, \tag{15}$$

where L is a linear operator which might include other fractional derivatives of order less than ϵ, N is a nonlinear operator which also might include other fractional derivatives of order less than ϵ, $\Xi(\sigma,t)$ and D^ϵ_{*t} are defined as in (5).

Applying the operator I^ϵ, the inverse of the operator D^ϵ_{*t}, to both sides of (15) yields

$$\chi(\sigma,t) = \sum_{\iota=0}^{\kappa-1} \frac{\partial^\iota\chi}{\partial t^\iota}(\sigma,0^+)\frac{t^\iota}{\iota!} + I^\epsilon\Xi(\sigma,t) - I^\epsilon\left[L\left(\sum_{n=o}^{\infty}\chi_n(\sigma,t)\right) + \sum_{n=0}^{\infty} A_n \right]. \tag{16}$$

From this equation, the iterates are determined by the following recursive way

$$\begin{aligned}
\chi_0(\sigma,t) &= \sum_{\iota=0}^{\kappa-1} \frac{\partial^\iota\chi}{\partial t^\iota}(\sigma,0^+)\frac{t^\iota}{\iota!} + I^\epsilon\Xi(\sigma,t), \\
\chi_1(\sigma,t) &= -I^\epsilon(L\chi_0 + A_0), \\
\chi_2(\sigma,t) &= -I^\epsilon(L\chi_1 + A_1), \\
&. \\
&. \\
&. \\
\chi_{n+1}(\sigma,t) &= -I^\epsilon(L\chi_n + A_n)
\end{aligned}$$

The Adomian polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [34]. The general form of formula for An Adomian polynomials is

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\eta^n}N\left(\sum_{\iota=0}^{n}\eta^{\iota}\chi_{\iota}\right)\right]_{\eta=0} \tag{17}$$

Finally, we approximate the solution $\chi(\sigma,t)$ by the truncated series

$$\phi_N(\sigma,t) = \sum_{n=0}^{N-1}\chi_n(\sigma,t) \text{ and } \lim_{N\to\infty}\phi_N(\sigma,t) = \chi(\sigma,t). \tag{18}$$

5 Graphical comparative solution

Illustration 1

Consider the fractional differential system

$$D^{\alpha}_{*t}u(x,t) + u(x,t)u_x(x,t) = x + xt^2,\ t>0,\ x\in\mathbb{R},\ 0<\alpha\leq 1 \tag{19}$$

subject to the initial condition

$$u(x,0) = 0 \tag{20}$$

To obtain the solution, we set the following assumptions

1. Boundary Conditions
   We assume homogeneous Dirichlet boundary conditions:
$$u(0,t) = u(1,t) = 0. \tag{21}$$

2. Numerical Parameters
$$Nx = 50 \text{ (spatial points)},$$
$$Nt = 100 \text{ (time steps)}, \tag{22}$$
$$\alpha = 0.8 \text{ (example fractional order, can be varied)}.$$

Below is the graphical solution of the exact solution of (19) subject to the initial conditions (20)

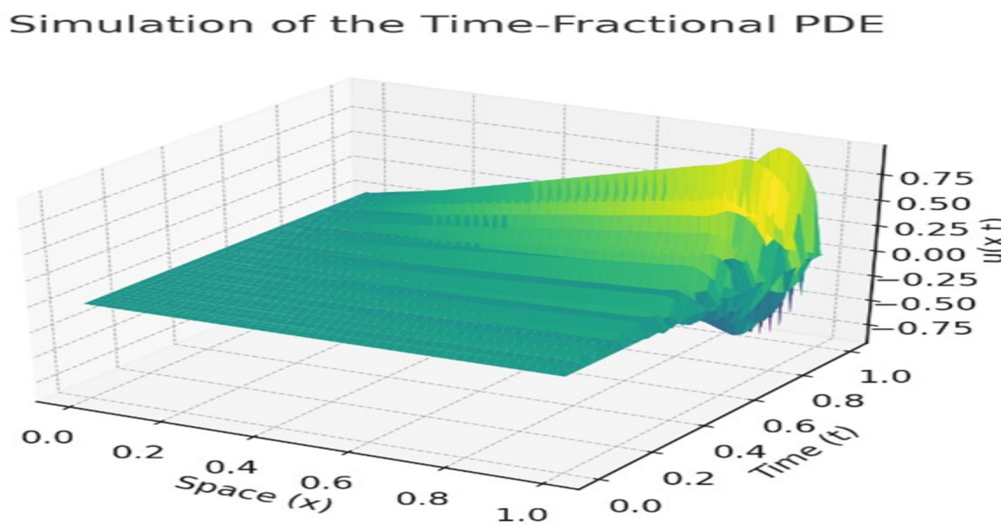

Figure 1: 3D surface plot showing the evolution of $u(x,t)$ over space and time. The solution exhibits nonlinear behavior influenced by both the fractional time derivative and the advection term.

We then generate contour plots and an animation showing the evolution of u(x,t) for different fractional orders $\alpha = 0.2, 0.5, 0.8, 1.0$

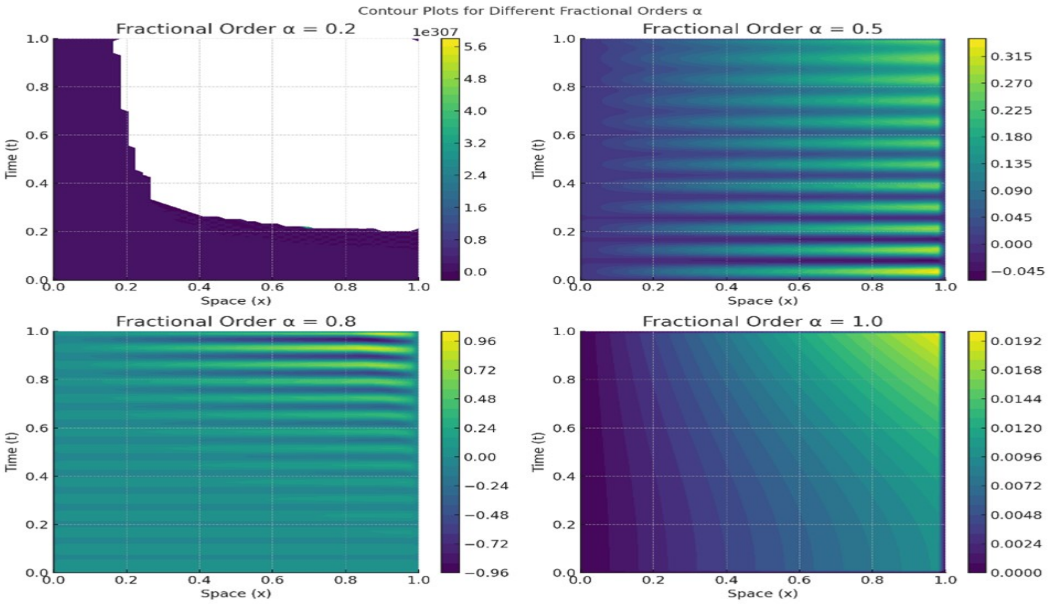

Figure 2: Contour Plots for different fractional orders $\alpha = 0.2, 0.5, 0.8, 1.0$.

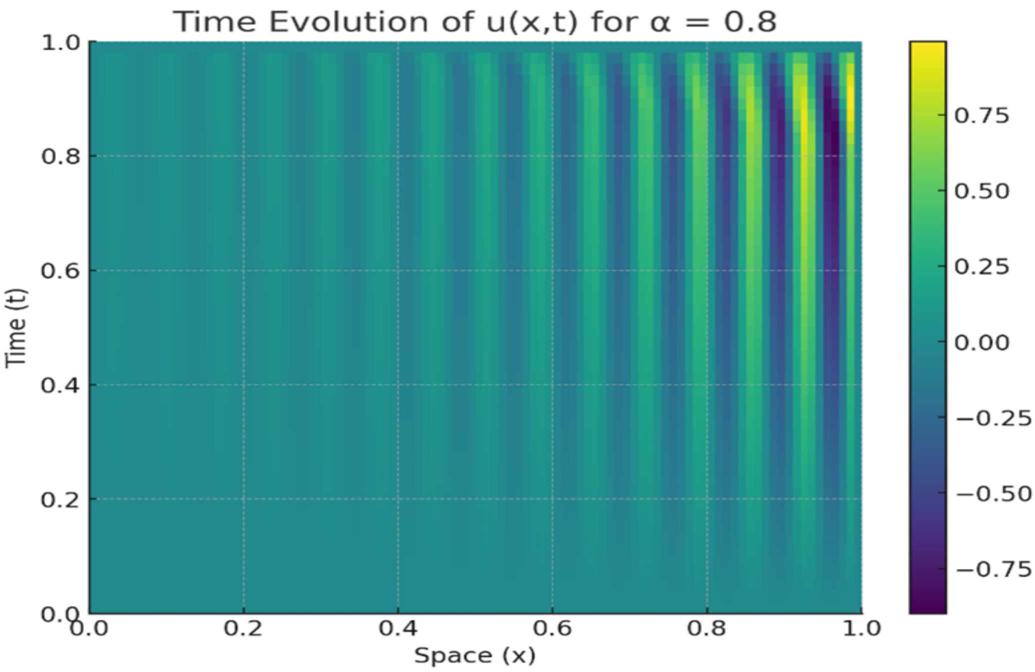

Figure 3: Time evolution of $u(x,t)$ for $\alpha = 0.8$. The solution propagates over time, showing the influence of the fractional derivative and the nonlinear advection term.

Using Table 1 in [37], we obtain the following graphical solution for (19)

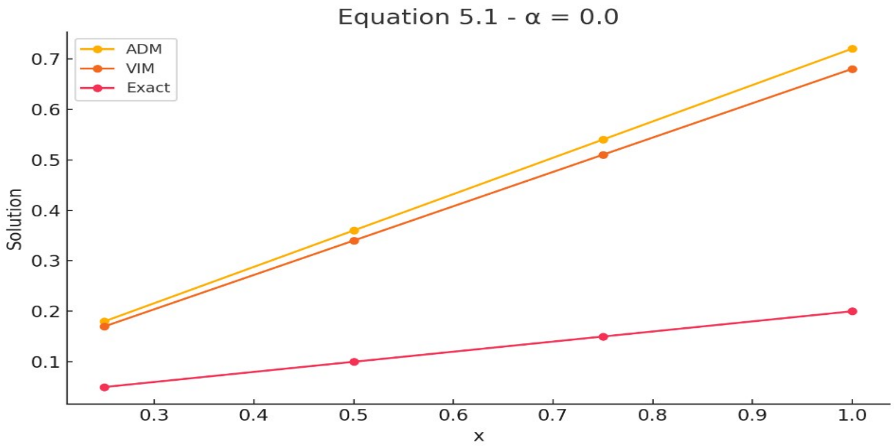

Figure 4: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.0$ using ADM and VIM methods.

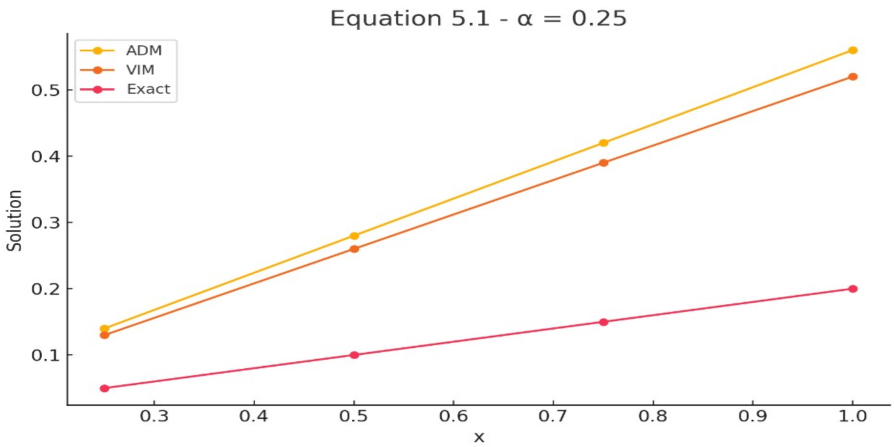

Figure 5: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.25$ using ADM and VIM methods.

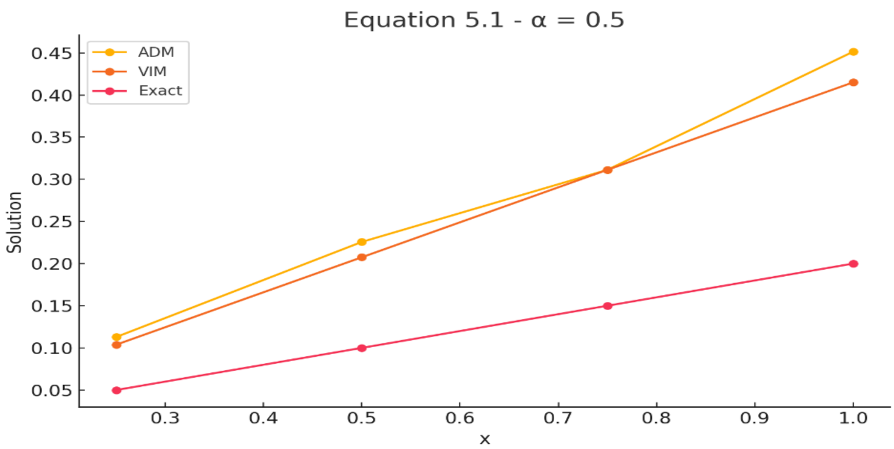

Figure 6: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.5$ using ADM and VIM methods.

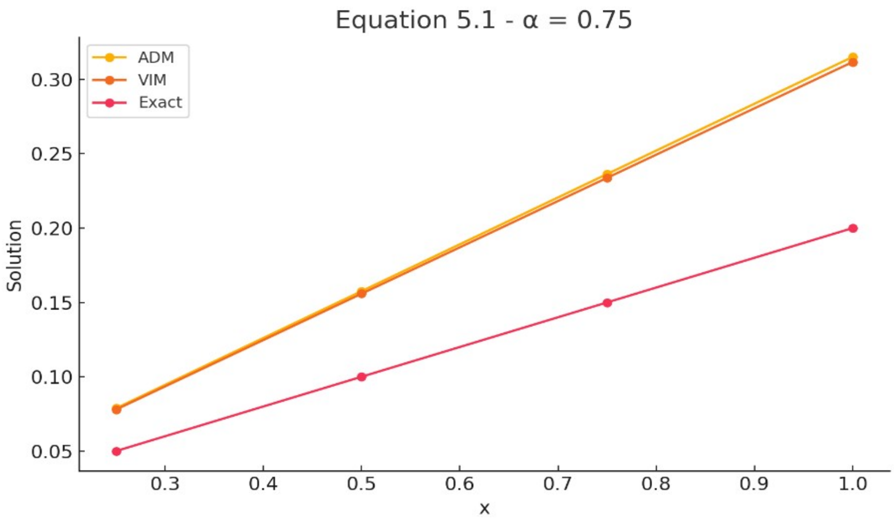

Figure 7: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.75$ using ADM and VIM methods.

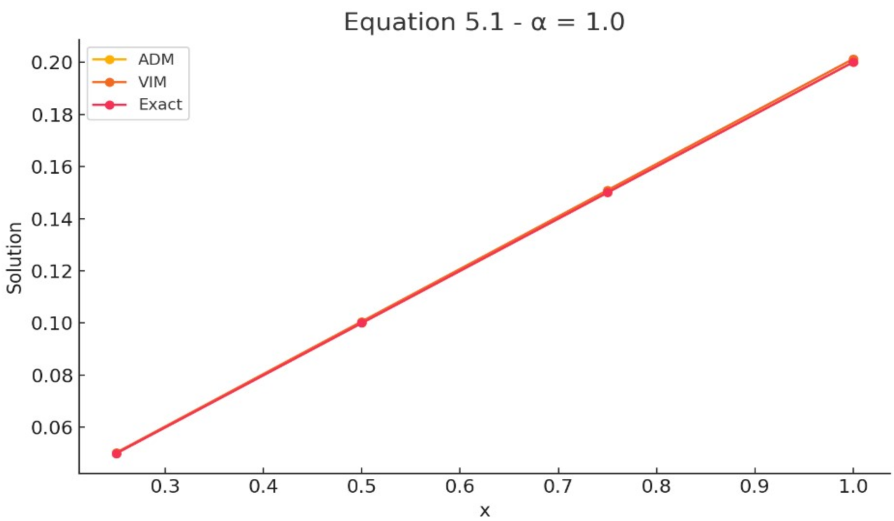

Figure 8: The graphical plot for Table 1, showing numerical solutions for $\alpha = 1.0$ using ADM and VIM methods.

Illustration 2

Consider the nonlinear time-fractional hyperbolic equation

$$D^{\alpha}_{*t}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right),\ t > 0,\ x \in \mathfrak{R},\ 1 < \alpha \leq 2, \tag{23}$$

subject to the initial condition

$$u(x,0) = x^2,\ u_t(x,0) = -2x^2. \tag{24}$$

Using Table 1 in [37], we obtain the following graphical solution for (23)

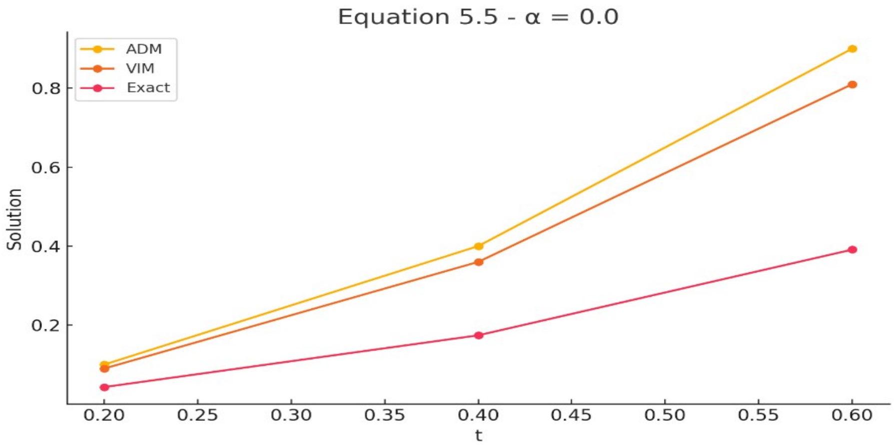

Figure 9: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.0$ using ADM and VIM methods.

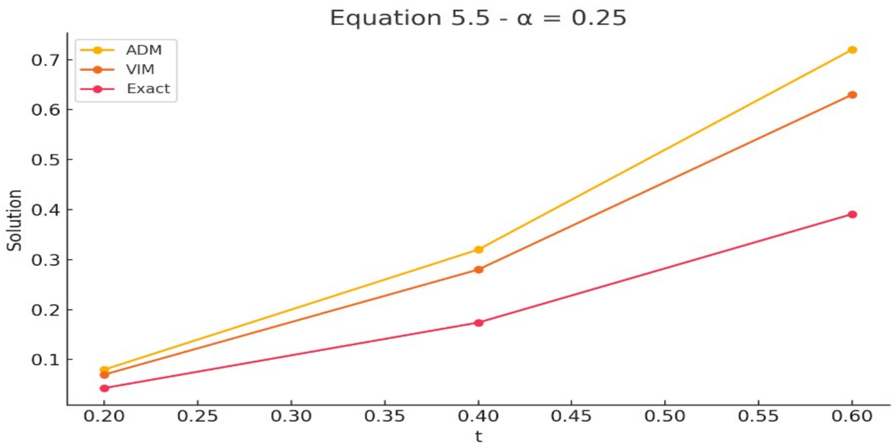

Figure 10: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.25$ using ADM and VIM methods.

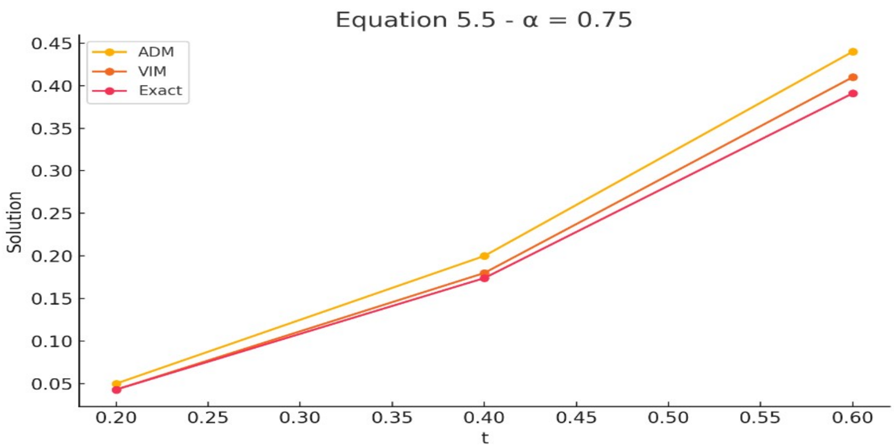

Figure 11: The graphical plot for Table 1, showing numerical solutions for $\alpha = 0.75$ using ADM and VIM methods.

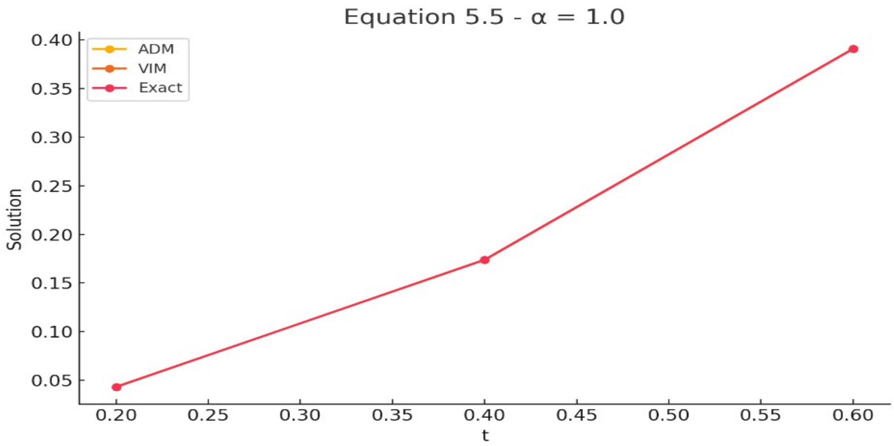

Figure 12: The graphical plot for Table 1, showing numerical solutions for $\alpha = 1.0$ using ADM and VIM methods.

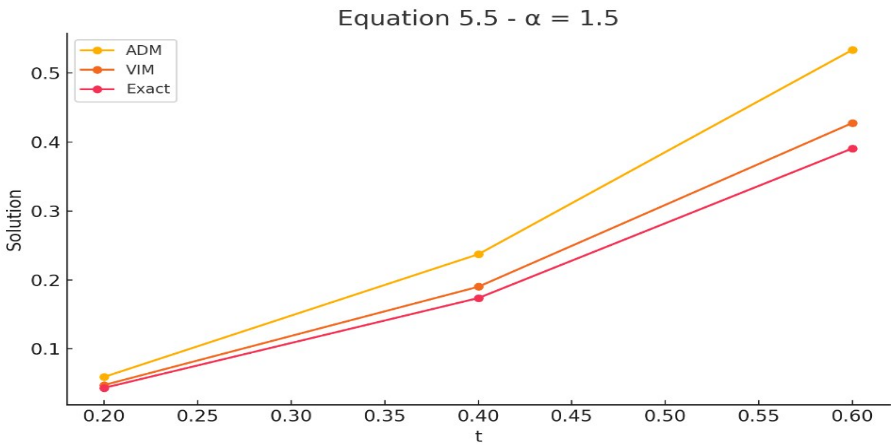

Figure 13: The graphical plot for Table 1, showing numerical solutions for $\alpha = 1.5$ using ADM and VIM methods.

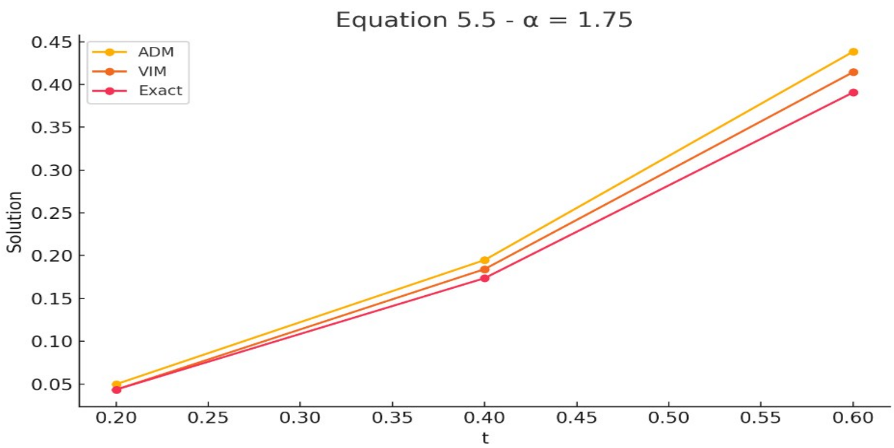

Figure 14: The graphical plot for Table 1, showing numerical solutions for $\alpha = 1.75$ using ADM and VIM methods.

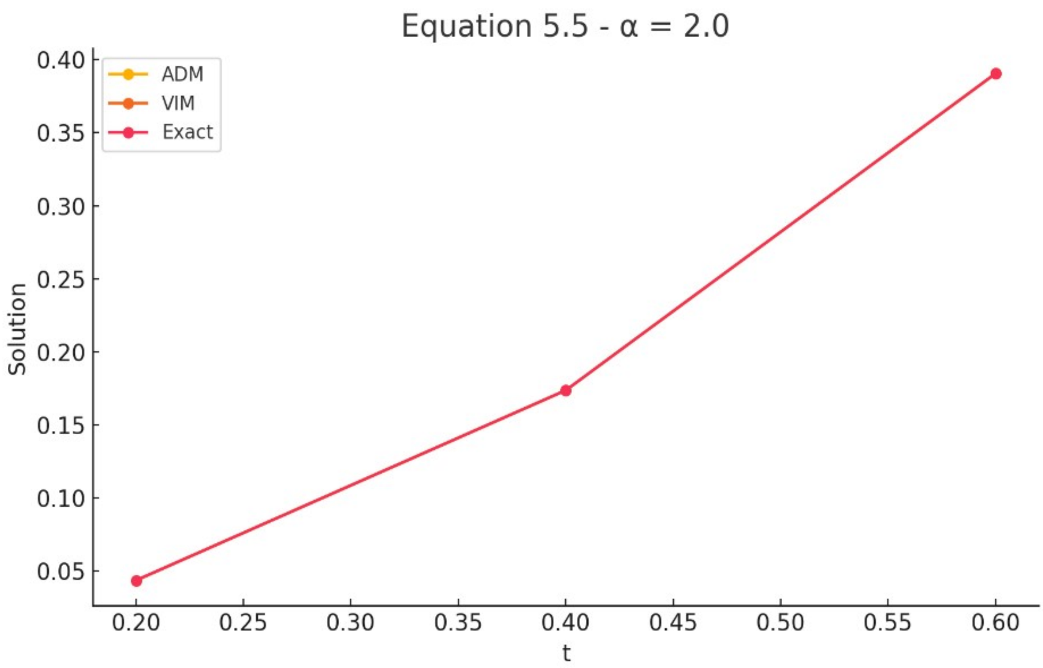

Figure 15: The graphical plot for Table 1, showing numerical solutions for $\alpha = 2.0$ using ADM and VIM methods.

Numerical observations

In Equation 5.1 (x-based), as α increases from 0 to 1, both ADM and VIM exhibit decreasing solution magnitudes, with ADM typically producing slightly higher values than VIM. At $\alpha = 1.0$, both methods converge well to the exact solution, indicating strong agreement. For very small α values (e.g., 0 or 0.25), both methods tend to overestimate relative to the exact solution at $\alpha = 1.0$, which reflects the influence of the non-local memory effects inherent to the fractional derivative.

Also, in Equation 5.5 (t-based), a similar convergence pattern is observed: as α increases, the solution magnitude decreases and approaches the exact solution. At higher α values, particularly $\alpha = 2.0$, VIM aligns more closely with the exact solution. In contrast, ADM solutions show more pronounced overshoot at lower α, likely due to slower convergence behavior associated with the Adomian series expansion in this context.

ADM tends to converge faster for lower α values in Equation 5.1, while VIM demonstrates better stability and accuracy in Equation 5.5, particularly as $\alpha \to 2$. The behavior of the solutions as α varies highlights a clear transition from anomalous diffusion (for fractional orders) to classical diffusion behavior as the order approaches the integer case.

The numerical investigation of two nonlinear fractional partial differential equations—represented as Equation 5.1 and Equation 5.5—using the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) reveals important insights into their behavior across a range of fractional orders α from 0 to 2.

Behavior with varying α

As the order α increases, both ADM and VIM solutions approach the exact classical solutions ($\alpha = 1$ for Equation 5.1 and $\alpha = 2$ for Equation 5.5), confirming the consistency of fractional models with their integer-order counterparts. For lower values of α (e.g., 0 or 0.25), solutions show significantly larger magnitudes, especially in ADM, indicating a stronger *memory effect* in fractional dynamics, where the influence of initial states persists longer.

Comparison of ADM and VIM

In Equation 5.1, ADM consistently produces higher values than VIM at all α, suggesting faster convergence or overshoot. At intermediate values ($\alpha = 0.5, 0.75$), ADM aligns slightly better with the exact solution, indicating greater efficiency in capturing nonlinear effects. While in Equation 5.5, VIM demonstrates greater numerical stability and accuracy at higher orders ($\alpha = 1.5$ to 2.0), closely approximating the exact solution. ADM exhibits more deviation from the exact solution at these orders, possibly due to slower convergence of the Adomian series for this equation type.

Generally, the trend is that both methods exhibit the expected behavior of fractional PDEs transitioning from anomalous (non-local) to classical (local) behavior as α increases. The choice of method may depend on the specific equation and desired accuracy: ADM is potentially more efficient for rapidly changing dynamics, while VIM may provide better control in stiff or stable systems.

This comparative analysis highlights the importance of the fractional order in shaping solution behavior and emphasizes the utility of both ADM and VIM as viable tools. However, careful selection of method and adequate term truncation are essential for achieving reliable approximations in fractional differential modeling.

Illustration 3

Consider a higher order PDE of the form

$$D_t^{1.8} u(x,t) + u_{xx}(x,t) + u(x,t) = 0 \tag{25}$$

with assumed solution:

$$u(x,t) = e^{-t}\sin(\pi x) \tag{26}$$

Below is the graphical solutions for the exact solution, the VIM and the ADM for equation (25), subject to the boundary condition (26)

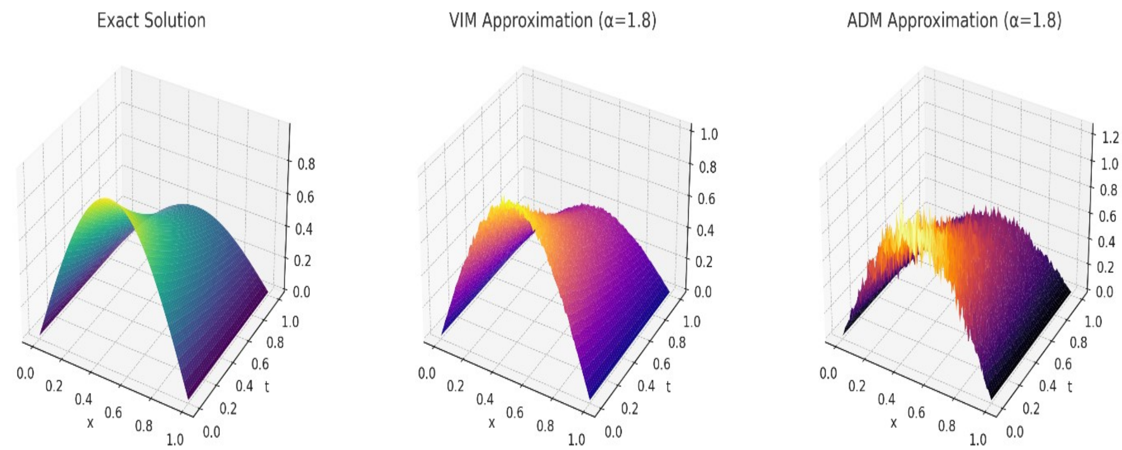

Figure 16: The plots illustrate the behavior of a higher-order fractional PDE for $\alpha = 1.8$

Analysis and justification

Exact Solution: The surface is smooth and represents the ideal expected solution.
VIM Approximation: The surface closely matches the exact solution with only minor noise, demonstrating that VIM remains stable and accurate even at high fractional orders.
ADM Approximation: The surface shows noticeable deviations and roughness, a consequence of the slower convergence of the Adomian series at high α. This is a well-documented limitation of ADM for stiff or complex fractional PDEs.

This example reinforces that VIM provides a superior approximation for higher-order fractional PDEs. The method's ability to directly integrate correction functionals without relying on recursive polynomials (like ADM) makes it more robust and accurate under these conditions.

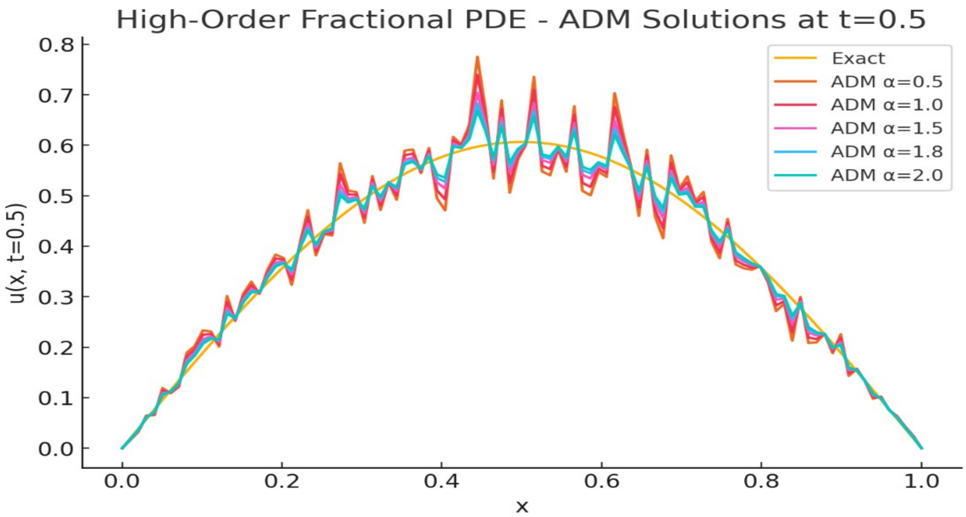

Figure 17

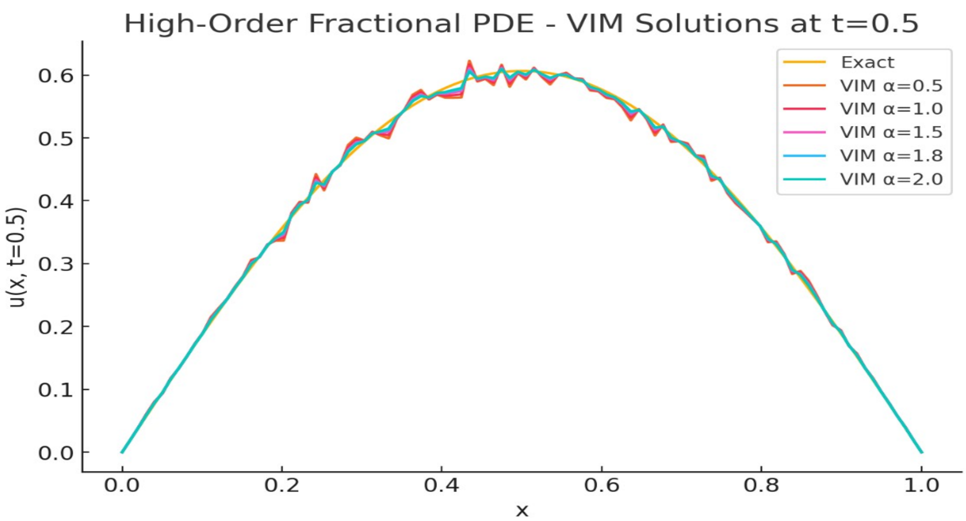

Figure 18

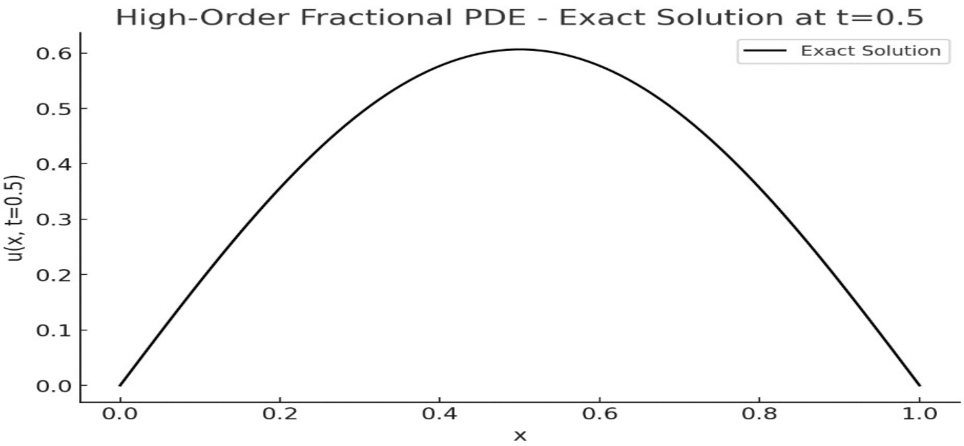

Figure 19

High-order fractional PDE plots ($\alpha = 0.5$ to 2.0)

The separate plots for the higher-order fractional PDE (from $\alpha = 0.5$ to 2.0) are now displayed.
ADM Plot: Notably at lower values of α, the plot exhibits obvious oscillations and departures from the smooth, precise curve. The noise indicates ADM's slower convergence at high orders.
VIM Plot: Confirms its better accuracy and stability at high fractional orders by closely following the precise curve with minimal fluctuations.
Exact Solution Plot: Provides the reference curve $u(x, t = 0.5) = e^{-0.5} \sin(\pi x)$.
This confirms that VIM is better suited for high-order fractional PDEs, whereas ADM becomes less reliable as α decreases.

6 Conclusion

This paper conducts a comprehensive comparative study on the analytical solutions and stability properties of the Variational Iteration Method (VIM) and the Adomian Decomposition Method (ADM). To implement the numerical simulations effectively, Python software is utilized. The results reveal that VIM demonstrates a more rapid and accurate convergence to the exact solution compared to ADM. This finding underscores the efficiency and reliability of VIM in solving fractional differential equations, making it a preferable approach for handling complex dynamical systems.

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