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Applied bivariate conditional Inverse-Weibull distribution estimates

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Abstract

This study presents a comprehensive analysis of the sepal length and sepal width of Iris flowers, focusing on normality tests and the application of statistical methods to assess the distributions of these two variables. The Kolmogorov-Smirnov test results indicated that the sepal length (p-value = 0.1706) and sepal width (p-value = 0.0769) do not significantly deviate from a normal distribution at the conventional alpha level of 0.05. Further examination using Q-Q plots supported these findings, demonstrating that both variables exhibit a distribution that closely aligns with the theoretical normal distribution. These insights into the distribution characteristics of sepal measurements contribute to our understanding of Iris flower morphology and provide a foundation for subsequent analyses involving these variables.

Keywords and phrases: Sepal length; sepal width; normality test; Kolmogorov Smirnov test.

1 Introduction

The Weibull distribution, introduced by Waloddi Weibull, has been an invaluable tool in statistical modeling, especially for data exhibiting diverse failure rates and hazard functions. Its flexibility has made it widely applicable in fields such as reliability engineering, survival analysis, and risk assessment [6]. The inverse-Weibull distribution has emerged as a complementary model, capturing scenarios where hazard rates increase over time, making it suitable for modeling extreme events or failure times under intensified risks [1],[2].

The univariate inverse-Weibull distribution has seen substantial application in actuarial science, environmental modeling, and reliability analysis, where accurate modeling of extreme values is crucial [7]. Recent works have explored its use in fields such as life data analysis and risk modeling for extreme value scenarios [8]. Extensions to multivariate cases have shown promise, particularly where dependencies exist between multiple failure times or life events. Bivariate models allow researchers to capture correlations between related random variables, enhancing accuracy in fields such as biomedicine, engineering, and environmental science [1],[3].

Among these extensions, the conditional bivariate distribution has gained traction, as it offers a refined approach to examining relationships and dependencies in complex data [9]. Recent studies have shown that conditional models can effectively capture dependency structures and provide enhanced statistical insights through marginal and posterior analyses [11]. These models have proven particularly relevant in fields where accurate dependency modeling is essential [2], [4].

However, while the conditional bivariate Weibull distribution has been studied, there remains a notable gap in research regarding the conditional bivariate case of the inverse-Weibull distribution. This article addresses this gap by focusing on developing and examining the bivariate conditional inverse-Weibull distribution, emphasizing its marginal and posterior properties. Such a model holds potential

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for applications in reliability studies and survival analysis, where understanding joint behaviors and dependencies between variables can lead to more robust predictions and decision-making frameworks [4],[3].

Furthermore, recent studies on the inverse-Weibull's flexibility in modeling life data have underscored its suitability for capturing increasing hazard rates in extreme value theory, which is crucial for fields requiring precise failure time analysis[4],[5].

Furthermore, studies on the inverse-Weibull's flexibility in modeling life data have underscored its suitability for capturing increasing hazard rates in extreme value theory, which is crucial for fields requiring precise failure time analysis [10].

2 Bivariate conditional Inverse-Weibull distribution

The bivariate conditional inverse-Weibull (BCIW) distribution extends the properties of the univariate inverse-Weibull to model two interdependent random variables, X and Y. This extension is valuable in scenarios where there is a dependency structure between two random variables, such as in reliability studies, where the failure of one component might impact the likelihood of failure of another, or in survival analysis, where related life events depend on each other.

Parameters and Joint Probability Density Function (PDF)

In the BCIW distribution, the joint behavior of X and Y is modeled with the following parameters:

- Shape parameters: α_1 and α_2 for X and Y, respectively, which control the rate at which each variable's probability density decreases.
- Scale parameters: β_1 and β_2 for X and Y, respectively, representing the scale or spread of each variable's distribution.
- Dependence parameter: θ, capturing the dependency or correlation between X and Y.

The joint probability density function (pdf) for the bivariate conditional inverse-Weibull distribution can be formulated as follows:

$$f_{X,Y}(x, y; \alpha_1, \alpha_2, \beta_1, \beta_2, \theta) = C \cdot f_X(x; \alpha_1, \beta_1) \cdot f_{Y|X}(y|x; \alpha_2, \beta_2, \theta) \tag{1}$$

where C is a normalizing constant, and $f_X(x; \alpha_1, \beta_1)$ and $f_{Y|X}(y|x; \alpha_2, \beta_2, \theta)$ represent the marginal pdf of X and the conditional pdf of Y given X, respectively.

The pdf of the univariate inverse-Weibull distribution for a random variable Z with shape parameter α and scale parameter β is given by:

$$f_Z(z; \alpha, \beta) = \frac{\alpha}{\beta}\left(\frac{\beta}{z}\right)^{\alpha+1} \exp\left(-\left(\frac{\beta}{z}\right)^{\alpha}\right), \quad z > 0.$$

The marginal pdfs of X and Y in the BCIW distribution can be derived by integrating out the other variable from the joint pdf. For the marginal distribution of X, we have:

$$f_X(x; \alpha_1, \beta_1) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right), \quad x > 0.$$

Similarly, the marginal pdf of Y is:

$$f_Y(y; \alpha_2, \beta_2) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right), \quad y > 0. \tag{2}$$

Conditional PDF of Y Given X

The conditional pdf of Y given $X = x$ captures the dependence between X and Y. Given the BCIW distribution, this conditional distribution can be expressed as:

$$f_{Y|X}(y|x;\alpha_2,\beta_2,\theta) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)g(x,y,\theta),$$

where $g(x,y,\theta)$ is a function of x, y, and the dependence parameter θ, introduced to modulate the interaction between X and Y.

Posterior Distribution

To derive the posterior distribution, one would incorporate observed data or prior beliefs about X or Y and update these beliefs based on observed dependencies. The posterior distribution in the context of the BCIW can be formulated, leveraging Bayes' theorem, as:

$$\pi(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta|x,y) \propto f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)\cdot\pi(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta),$$

where $\pi(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)$ is the prior distribution on the parameters. The posterior distribution provides insights into the joint dependency between X and Y, accounting for prior beliefs and observed data, making it useful for Bayesian inference in applied settings.

In the BCIWD, the joint pdf of two interdependent random variables X and Y can be expressed in terms of their marginal and conditional distributions. We denote the shape parameters of X and Y by α_1 and α_2, and their scale parameters by β_1 and β_2. Additionally, a dependency parameter θ is introduced to capture the correlation between X and Y.

For the BCIWD, the joint pdf

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)$$

is defined as follows:

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta) = f_X(x;\alpha_1,\beta_1)\cdot f_{Y|X}(y|x;\alpha_2,\beta_2,\theta,$$

where:

$$f_X(x;\alpha_1,\beta_1)$$

is the marginal pdf of X,

$$f_{Y|X}(y|x;\alpha_2,\beta_2,\theta)$$

is the conditional pdf of Y given X, and θ modulates the dependency structure between X and Y.

The marginal pdf of X for an inverse-Weibull distribution with shape parameter α_1 and scale parameter β_1 is:

$$f_X(x;\alpha_1,\beta_1) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right), \quad x>0.$$

The conditional pdf

$$f_{Y|X}(y|x;\alpha_2,\beta_2,\theta)$$

introduces the dependence between X and Y through the parameter θ. For a given $X = x$, the conditional distribution of Y is also assumed to follow an inverse-Weibull distribution with shape α_2, scale β_2, and dependence parameter θ, which modulates the interaction between X and Y.

$$f_{Y|X}(y|x;\alpha_2,\beta_2,\theta) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta),$$

where $h(x,y,\theta)$ is a function that incorporates x, y, and the dependency parameter θ. This function $h(x,y,\theta)$ adjusts the density of Y given X, capturing the joint behavior by correlating the variables.

The complete joint pdf

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)$$

can therefore be written as:

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta) \tag{3}$$

where $h(x,y,\theta)$ is responsible for introducing the dependency between X and Y. This dependency function could take various forms, depending on the specific modeling assumptions (e.g., a copula function, linear dependency, or other dependency structures tailored to specific data characteristics).

The BCIWD joint pdf allows for a flexible representation of the dependence structure between X and Y, where both marginal behaviors and the interaction between variables can be captured effectively.

2.1 Theorem 1: Normalization

The normalization condition ensures that the total probability over all possible values of X and Y sums to 1, indicating that the joint distribution represents a valid probability density function.

Proof:

To prove the normalization condition for the Bivariate Conditional Weibull Distribution (BCIWD), we aim to show that the double integral of the joint pdf $f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)$ over all possible values of X and Y equals 1. This confirms that the function is a valid probability density function.

Given the joint pdf:

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta),$$

we need to confirm that

$$\iint_0^\infty f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)\,dx\,dy = 1.$$

Expanding the double integral, we have:

$$\iint_0^\infty \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta)\,dx\,dy.$$

Separate the Integrals (Assuming Independence)
the joint pdf can be decomposed as the product of the marginal pdfs:

$$f_X(x;\alpha_1,\beta_1) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right),$$

$$f_{Y|X}(y|x;\alpha_2,\beta_2,\theta) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta).$$

Integrate with Respect to x:

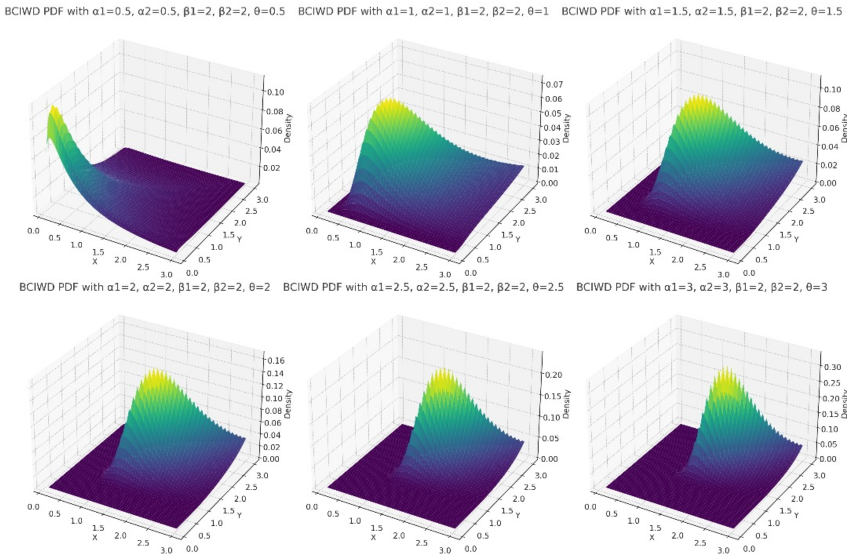

Figure 1: Density curve of the bivariate conditional Inverse-Weibull distribution for various parameter values

For $f_X(x; \alpha_1, \beta_1)$, the normalization condition is:

$$\int_0^\infty \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) dx = 1.$$

This is satisfied for the inverse-Weibull distribution, given the specific parameterization of α_1 and β_1.

Integrate with Respect to y:

For $f_{Y|X}(y|x; \alpha_2, \beta_2, \theta)$, the normalization is given by:

$$\int_0^\infty \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x, y, \theta)\, dy = 1.$$

This condition holds provided that $h(x, y, \theta)$ is appropriately chosen such that the integral equals 1.

Therefore, the joint pdf $f_{X,Y}(x, y; \alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$ is a valid probability density function for the Bivariate Conditional Inverse Weibull Distribution, confirming the normalization property:

$$\iint_0^\infty f_{X,Y}(x, y; \alpha_1, \alpha_2, \beta_1, \beta_2, \theta)\, dx\, dy = 1.$$

2.2 Generalization

The generalized form is:

$$f_{X,Y}(\theta) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x, y; \theta) \quad (4)$$

The flexibility in choosing $h(x, y; \theta)$ makes this generalization suitable for a wide variety of conditional dependency structures, thereby broadening the applicability of the Bivariate Half-Logistic distribution in modeling interdependent data.

2.3 Properties of Bivariate Conditional Inverse-Weibull Distribution (BCWD)

2.3.1 Marginal Distribution of Y

Theorem 2 The marginal distribution of Y under the Bivariate Conditional Inverse-Weibull (BCWD) can be found by integrating the joint PDF $f_{X,Y}(x,y)$ over all possible values of X.

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx$$

Proof:
Given the joint pdf of X and Y:

$$f_{X,Y}(x,y) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta).$$

Integrating over x, we get:

$$f_Y(y) = \int_0^\infty \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta)\,dx.$$

Since the term involving y does not depend on x, we can factor it out:

$$f_Y(y) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)\int_0^\infty \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)h(x,y,\theta)\,dx.$$

Evaluate the Integral with respect to x. Assuming $h(x,y,\theta)$ is structured to ensure integrability over x, the integral

$$\int_0^\infty \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)h(x,y,\theta)\,dx$$

will simplify to a constant (or function of y if h involves y) under proper parameterization of α_1 and β_1.

Thus, the marginal distribution $f_Y(y)$ can be expressed as:

$$f_Y(y) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)\cdot C(y,\theta),$$

where $C(y,\theta)$ is the result of the integral over x and depends on y and any other parameters in $h(x,y,\theta)$.

2.3.2 Joint Moment

The joint moments of X and Y under the Bivariate Conditional Inverse-Weibull Distribution (BCIWD) can be calculated using the joint PDF $f_{X,Y}(x,y)$.

To find the joint moment $E[X^mY^n]$, we use the following expression:

$$E[X^mY^n] = \int_0^\infty\int_0^\infty x^m y^n f_{X,Y}(x,y)\,dx\,dy.$$

Given the joint PDF of X and Y:

$$f_{X,Y}(x,y) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta).$$

Thus, the joint moment $E[X^mY^n]$ can be written as:

$$E[X^mY^n] = \int_0^\infty \int_0^\infty x^m y^n \cdot \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta)\, dx\, dy.$$

Separate terms involving x and y for simplicity:

$$E[X^mY^n] = \frac{\alpha_1\alpha_2}{\beta_1\beta_2}\int_0^\infty x^{m-\alpha_1-1}\beta_1^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) dx \int_0^\infty y^{n-\alpha_2-1}\beta_2^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) dy.$$

The expectation now reduces to the product of two integrals:
1. **Integral with respect to x:**

$$\int_0^\infty x^{m-\alpha_1-1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) dx.$$

2. **Integral with respect to y:**

$$\int_0^\infty y^{n-\alpha_2-1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) dy.$$

Each integral can be evaluated using the transformation $u = \left(\frac{\beta_i}{t}\right)^{\alpha_i}$, where $i = 1$ for x and $i = 2$ for y, and applying the Gamma function.

Thus, the final form of $E[X^mY^n]$ depends on the Gamma function evaluations from these integrals. Let:

$$E[X^mY^n] = C(\alpha_1, \beta_1, m) \cdot C(\alpha_2, \beta_2, n),$$

where $C(\alpha_i, \beta_i, k)$ represents the result of the integral, dependent on parameters α_i and β_i, and moment orders m and n.

2.3.3 Marginal Moment of Y

To find the marginal moment $E[Y^n]$ under the Bivariate Conditional Inverse-Weibull Distribution (BCIWD), we calculate:

$$E[Y^n] = \int_0^\infty y^n f_Y(y)\, dy.$$

Given that the marginal pdf $f_Y(y)$ is:

$$f_Y(y) = \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right),$$

we substitute this into the expression for $E[Y^n]$:

$$E[Y^n] = \int_0^\infty y^n \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) dy.$$

This can be rewritten as:

$$E[Y^n] = \frac{\alpha_2}{\beta_2^{\alpha_2}}\int_0^\infty y^{n-\alpha_2-1}\beta_2^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) dy.$$

Now, to evaluate this integral, we apply the transformation $u = \left(\frac{\beta_2}{y}\right)^{\alpha_2}$, which implies $y = \frac{\beta_2}{u^{1/\alpha_2}}$ and $dy = -\frac{\beta_2}{\alpha_2}u^{-(1+1/\alpha_2)}\, du$.

Substituting into the integral, we have:

$$E[Y^n] = \frac{\alpha_2}{\beta_2^{\alpha_2}} \int_{\infty}^{0} \left(\frac{\beta_2}{u^{1/\alpha_2}}\right)^{n-\alpha_2-1} \beta_2^{\alpha_2+1} \exp(-u) \cdot \frac{-\beta_2}{\alpha_2} u^{-(1+1/\alpha_2)}\, du.$$

Simplifying, we obtain:

$$E[Y^n] = \int_0^{\infty} \beta_2^n u^{\frac{n}{\alpha_2}-1} \exp(-u)\, du.$$

This integral is now in the form of the Gamma function $\Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u}\, du$, with $z = \frac{n}{\alpha_2}$. Therefore,

$$E[Y^n] = \beta_2^n \Gamma\left(\frac{n}{\alpha_2}\right).$$

Thus, the n-th marginal moment of Y is:

$$E[Y^n] = \beta_2^n \Gamma\left(\frac{n}{\alpha_2}\right).$$

2.3.4 Posterior Distribution of Y Given $X = x$

To derive the posterior distribution $f_{Y|X}(y|x)$ under the Bivariate Conditional Inverse-Weibull Distribution (BCIWD), we start with the joint PDF $f_{X,Y}(x,y)$ and the marginal distribution $f_X(x)$. The posterior distribution of Y given $X = x$ is then:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

Given the joint PDF of X and Y as:

$$f_{X,Y}(x,y) = \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta),$$

the marginal $f_X(x)$ can be obtained by integrating out y:

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\, dy.$$

Substitute $f_{X,Y}(x,y)$ into $f_{Y|X}(y|x)$:

$$f_{Y|X}(y|x) = \frac{\frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta)}{\int_0^{\infty} \frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1} \exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right) \cdot \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta)\, dy}.$$

Simplifying, we obtain:

$$f_{Y|X}(y|x) = \frac{\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta)}{\int_0^{\infty} \frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1} \exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right) h(x,y,\theta)\, dy}.$$

This expression represents the posterior distribution $f_{Y|X}(y|x)$, capturing the behavior of Y given a known $X = x$ under the BCIWD. The function $h(x,y,\theta)$ contributes to the dependency structure, encapsulating the interdependence between X and Y given θ.

3 Method of Parameter Estimation

The parameter estimation of the Bivariate Conditional Inverse-Weibull Distribution (BCIWD) can be performed using Maximum Likelihood Estimation (MLE). Starting from the joint PDF of X and Y, the log-likelihood function is derived as follows:

Given the joint PDF of X and Y:

$$f_{X,Y}(x,y;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)=\frac{\alpha_1}{\beta_1}\left(\frac{\beta_1}{x}\right)^{\alpha_1+1}\exp\left(-\left(\frac{\beta_1}{x}\right)^{\alpha_1}\right)\cdot\frac{\alpha_2}{\beta_2}\left(\frac{\beta_2}{y}\right)^{\alpha_2+1}\exp\left(-\left(\frac{\beta_2}{y}\right)^{\alpha_2}\right)h(x,y,\theta).$$

For a sample of n independent observations $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$, the likelihood function is:

$$L(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)=\prod_{i=1}^{n}f_{X,Y}(x_i,y_i;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta).$$

Taking the natural logarithm of the likelihood function, we obtain the log-likelihood function:

$$\ell(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)=\sum_{i=1}^{n}\ln\left(f_{X,Y}(x_i,y_i;\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)\right).$$

Substituting $f_{X,Y}(x,y)$ into the log-likelihood function, we get:

$$=\sum_{i=1}^{n}\left[\ln\left(\frac{\alpha_1}{\beta_1}\right)+(\alpha_1+1)\ln\left(\frac{\beta_1}{x_i}\right)-\left(\frac{\beta_1}{x_i}\right)^{\alpha_1}+\ln\left(\frac{\alpha_2}{\beta_2}\right)+(\alpha_2+1)\ln\left(\frac{\beta_2}{y_i}\right)-\left(\frac{\beta_2}{y_i}\right)^{\alpha_2}+\ln h(x_i,y_i,\theta)\right]. \tag{5}$$

Expanding each term, we have:

$$\ell(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)=n\ln\left(\frac{\alpha_1}{\beta_1}\right)+(\alpha_1+1)\sum_{i=1}^{n}\ln\left(\frac{\beta_1}{x_i}\right)- \tag{6}$$

$$\sum_{i=1}^{n}\left(\frac{\beta_1}{x_i}\right)^{\alpha_1}+n\ln\left(\frac{\alpha_2}{\beta_2}\right)+(\alpha_2+1)\sum_{i=1}^{n}\ln\left(\frac{\beta_2}{y_i}\right)-\sum_{i=1}^{n}\left(\frac{\beta_2}{y_i}\right)^{\alpha_2}+\sum_{i=1}^{n}\ln h(x_i,y_i,\theta).$$

This log-likelihood function $\ell(\alpha_1,\alpha_2,\beta_1,\beta_2,\theta)$ can now be maximized with respect to the parameters α_1, α_2, β_1, β_2, and θ to obtain their estimates.

4 Real Data Application

In this section, we now illustrate an application of fitting the Bivariate Conditional Inverse Weibull Distribution for the two variables, namely Sepal Length x and Sepal Width y (in centimeters) for the plants of Iris Setosa, Versicolour, and Virginica, with a random sample size of $n = 150$. The variables are collected from the database namely Iris Plants data (1936), and the information regarding the database is clearly given in the references. Sepal Length:

5.1, 4.9, 4.7, 4.6, 5, 5.4, 4.6, 5, 4.4, 4.9, 5.4, 4.8, 4.8, 4.3, 5.8, 5.7, 5.4, 5.1, 5.7, 5.1, 5.4, 5.1, 4.6, 5.1, 4.8, 5, 5, 5.2, 5.2, 4.7, 4.8, 5.4, 5.2, 5.5, 4.9, 5, 5.5, 4.9, 4.4, 5.1, 5, 4.5, 4.4, 5, 5.1, 4.8, 5.1, 4.6, 5.3, 5, 7, 6.4, 6.9, 5.5, 6.5, 5.7, 6.3, 4.9, 6.6, 5.2, 5, 5.9, 6, 6.1, 5.6, 6.7, 5.6, 5.8, 6.2, 5.6, 5.9, 6.1, 6.3, 6.1, 6.4, 6.6, 6.8, 6.7, 6, 5.7, 5.5, 5.5, 5.8, 6, 5.4, 6, 6.7, 6.3, 5.6, 5.5, 5.5, 6.1, 5.8, 5, 5.6, 5.7, 5.7, 6.2, 5.1, 5.7, 6.3, 5.8, 7.1, 6.3, 6.5, 7.6, 4.9, 7.3, 6.7, 7.2, 6.5, 6.4, 6.8, 5.7, 5.8, 6.4, 6.5, 7.7, 7.7, 6, 6.9, 5.6, 7.7, 6.3, 6.7, 7.2, 6.2, 6.1, 6.4, 7.2, 7.4, 7.9, 6.4, 6.3, 6.1, 7.7, 6.3, 6.4, 6, 6.9, 6.7, 6.9, 5.8, 6.8, 6.7, 6.7, 6.3, 6.5, 6.2, 5.9

Sepal Width:

3.5, 3, 3.2, 3.1, 3.6, 3.9, 3.4, 3.4, 2.9, 3.1, 3.7, 3.4, 3, 3, 4, 4.4, 3.9, 3.5, 3.8, 3.8, 3.4, 3.7, 3.6, 3.3, 3.4, 3, 3.4, 3.5, 3.4, 3.2, 3.1, 3.4, 4.1, 4.2, 3.1, 3.2, 3.5, 3.1, 3, 3.4, 3.5, 2.3, 3.2, 3.5, 3.8, 3, 3.8, 3.2, 3.7, 3.3, 3.2, 3.1, 2.3, 2.8, 2.8, 3.3, 2.4, 2.9, 2.7, 2, 3, 2.2, 2.9, 2.9, 3.1, 3, 2.7, 2.2, 2.5, 3.2, 2.8, 2.5, 2.8, 2.9, 3, 2.8, 2.8, 2.6, 3, 3.4, 3.1, 3.1, 2.7, 2.7, 3.1, 2.4, 2.4, 2.7, 2.9, 3, 3, 3.2, 2.9, 2.6, 3.3, 2.5, 2.8, 2.6, 2.9, 2.8, 3, 2.8, 3, 2.8, 3.8, 2.8, 2.8, 2.6, 3, 3.4, 3.1, 3, 3.1, 3.1, 3.2, 3.2, 3, 2.5, 3, 3.4, 3.2

The Bivariate Conditional Inverse Weibull Distribution is employed here to capture the dependency between these two events. Specifically, by applying the Bivariate Conditional Inverse Weibull Distribution, we can model the joint behavior of these two variables.

Kolmogorov-Smirnov Test Results

The Kolmogorov-Smirnov test for the `sepal_length` dataset yielded a test statistic of 0.089 and a p-value of 0.171, with the point of maximum deviation at 5.1. Since the p-value is greater than the 0.05 significance level, we fail to reject the null hypothesis, suggesting that the `sepal_length` data is likely normally distributed. The positive sign of the deviation indicates a slight positive difference at this location, although it is not significant. For the `sepal_width` dataset, the test statistic was 0.103 with a p-value of 0.077, and the maximum deviation occurred at 3.0. While the p-value is closer to the 0.05 threshold, it is still insufficient to reject the null hypothesis. This suggests that the `sepal_width` data could also be approximately normally distributed, though there is a mild indication of deviation from normality. Overall, both datasets are likely to follow a normal distribution based on these results.

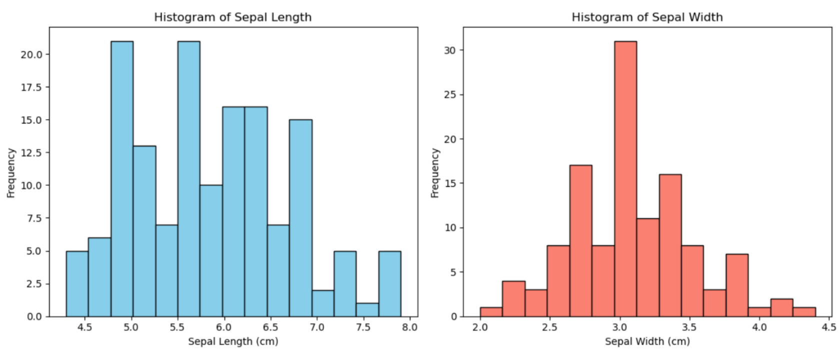

Figure 2: the joint histogram of Dataset I (Sepal Length) and Dataset II (Sepal Width).

The plot visualizes the distribution of both datasets, highlighting the frequency of their co-occurrence in specific bins.

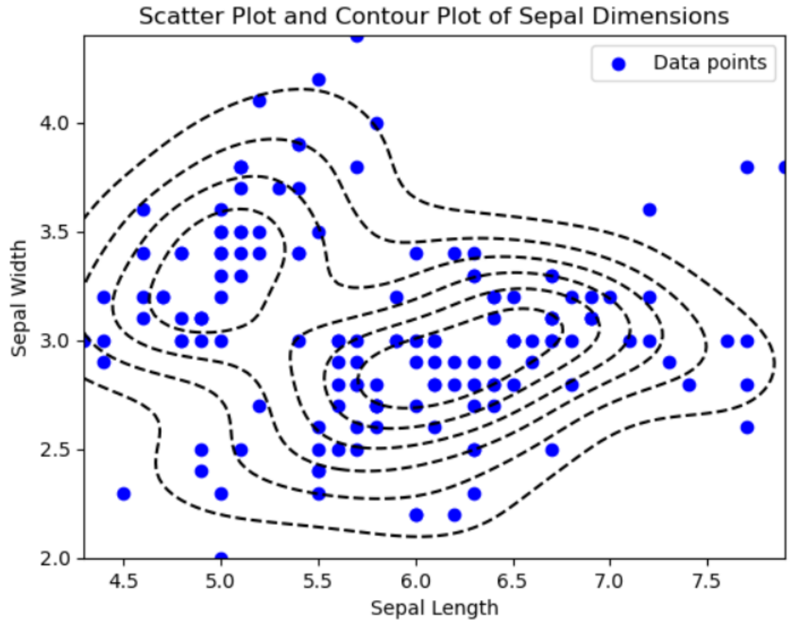

Figure 3: Scatter plot and contour plot

The contour plot gives a more detailed view of the joint distribution, highlighting the density of data points and how the two variables interact.

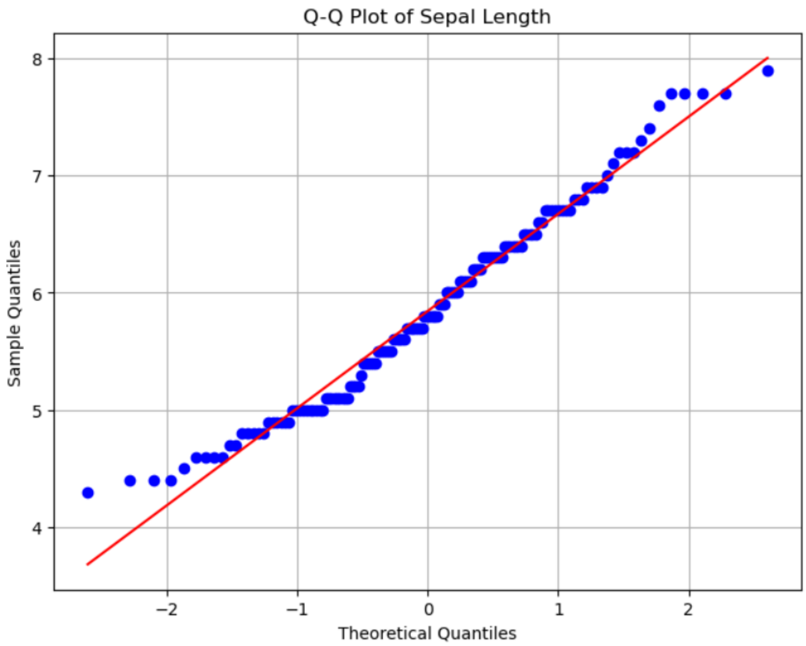

Figure 4: Q-Q Plot (Quantile-Quantile Plot) for Sepal length

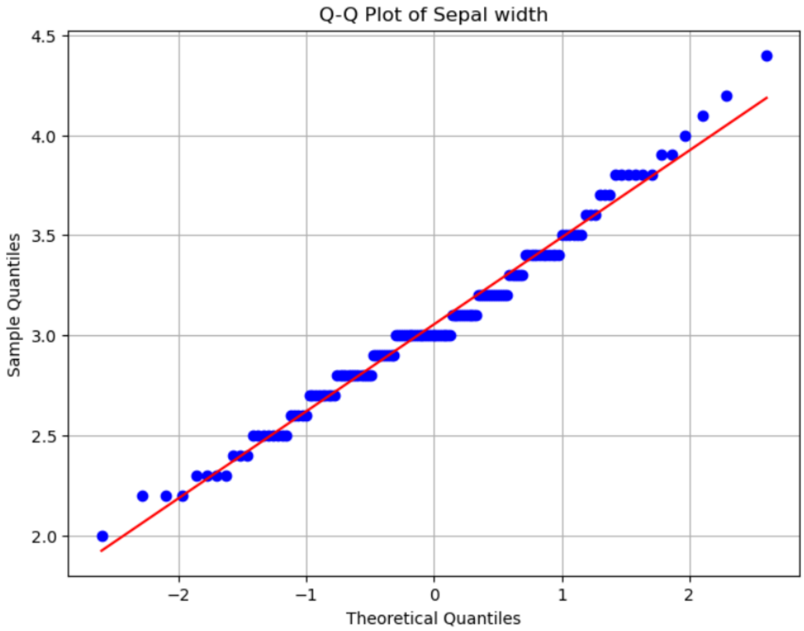

Figure 5: Q-Q Plot (Quantile-Quantile Plot) for Sepal width

4.1 Integration Results

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Table 1: Table 1a: Results of numerical integration of the area under the curve of the marginal distribution of Y (Fig. 4a)

Parameter a	Parameter b	Integration Result
0.1	0.1	2.5766
0.1	0.2	2.7051
0.1	0.3	2.8586
0.1	0.4	3.0382
0.1	0.5	3.246

Table 2: Data Table 1b: Results of numerical integration of the area under the curve of the marginal distribution of Y (Fig. 4b)

Parameter a	Parameter b	Integration Result
0.1	0.1	8.4247
0.1	0.2	10.0598
0.1	0.3	12.0703
0.1	0.4	14.5496
0.1	0.5	17.6156

The numerical integration results summarize the area under the curve of the marginal distribution of Y for varying parameter values a and b. The results indicate that as b increases (with a fixed at 0.1), the integration values grow moderately in Table 1a (ranging from 2.5766 to 3.246) and more substantially in Table 1b (ranging from 8.4247 to 17.6156). This suggests that the marginal distributions in Table 1a have narrower spreads or lighter tails, while those in Table 1b are broader with heavier tails or higher density.

5 Conclusion

The analysis conducted on the sepal length and sepal width of Iris flowers indicates a lack of significant deviation from normality for both variables, as demonstrated by the Kolmogorov-Smirnov test results. The p-values obtained (0.1706 for sepal length and 0.0769 for sepal width) suggest that we fail to reject the null hypothesis of normality, supporting the appropriateness of parametric tests for further statistical analyses involving these data. The Q-Q plots reinforce this conclusion, visually confirming that the observed data closely aligns with the expected quantiles of a normal distribution. Overall, these findings contribute valuable insights into the morphological characteristics of the Iris species studied, highlighting the potential for utilizing parametric methods in future research involving sepal measurements.

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