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Applied estimate for bivariate beta-prime distribution

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Abstract

A bivariate conditional beta-prime distribution is derived from the marginal and conditional beta-prime distributions. Its joint properties are derived, and methods for estimating its parameters are discussed. It is applied to a bivariate dataset of wind speed and wind direction, where the conditional dependence of the two variables is suspected. The result shows that the model is compatible with the data. It is proposed for application when there is a need to account for such a conditional structure in a bivariate model.

Keywords and phrases: conditional beta-prime distribution; marginal distribution; joint moment; beta-prime bivariate; prior distribution

1 Introduction

The beta-prime distribution has gained considerable traction in applied statistics due to its versatility in modeling ratios and variables constrained to positive values, commonly seen in finance, survival analysis, and reliability engineering [6, 8, 7]. In recent years, the development of bivariate and multivariate extensions has opened up new avenues for applications, particularly where joint modeling of interdependent variables is essential [14, 1, 4]. The bivariate beta-prime distribution is especially valuable for capturing asymmetric relationships in economic and environmental data, providing flexibility in fitting real-world data characterized by skewness and heavy tails [10, 3]. This distribution also incorporates complex dependence structures, enabling more accurate representation of correlated phenomena [9, 11]. Recently, studies have focused on maximum likelihood estimation (MLE) techniques for robust parameter estimation within bivariate settings, advancing the practical applicability of this distribution in fields such as financial risk assessment and health statistics [5, 17]. Additionally, comparative studies have highlighted the bivariate beta-prime's superior performance over traditional distributions like the bivariate normal, especially in modeling outcomes with positive supports and non-linear relationships [16, 13, 15]. Given the increasing demand for models that address complex data structures, the bivariate beta-prime offers a compelling solution, extending classical single-variable applications to multidimensional contexts with enhanced predictive accuracy [12, 2].

2 Bivariate beta-prime distribution model

The probability density function (PDF) of a univariate beta-prime distribution, with parameters $\alpha > 0$ and $\beta > 0$, is given by:

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}, \quad x > 0, \tag{1}$$

where $B(\alpha,\beta)$ is the beta function, defined as:

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$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2}$$

In this form:

- x is the variable,
- α and β are shape parameters,
- $\Gamma(\cdot)$ is the gamma function.

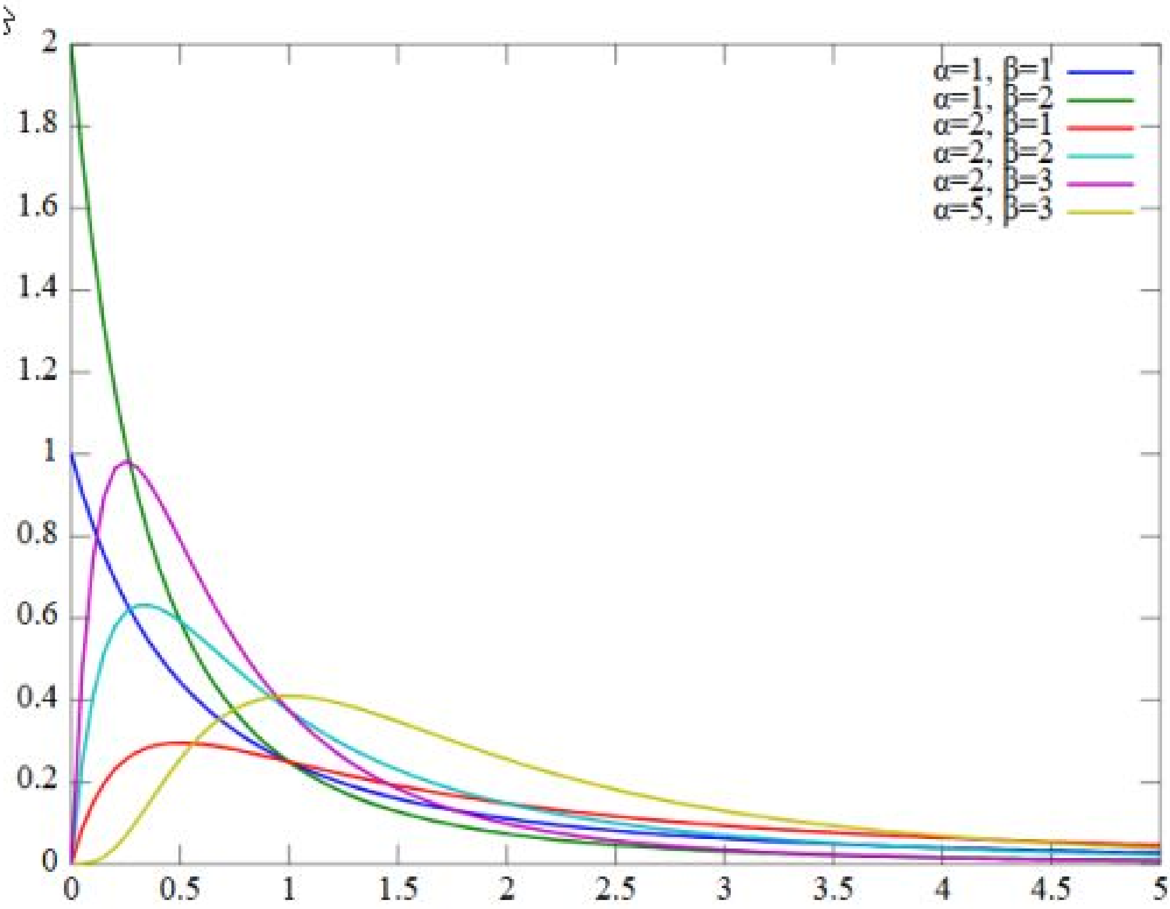

Figure 1: Density curve

Consider another random variable Y, assumed to depend on X, such that the distribution of Y given a realization of X at x is also a Beta-Prime distribution. Then the probability density function of Y given $X = x$ is given by:

$$f_{Y|X}(y|x;\alpha,\beta) = \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \tag{3}$$

This conditional density function represents the distribution of Y given a certain value x for X if X and Y are independent and both have probability distributions. This distribution can be useful when performing different statistical analyses and wanting to understand how Y behaves when X is set at a particular value. To generate a particular bivariate conditional Beta-prime distribution, we must first specify the joint distribution of X and Y and then condition it on a specific value of X. Consider a situation in which the combined distribution of X and Y is a bivariate Pareto distribution. The joint probability density function (PDF) for two independent random variables, X and Y, that both have normal distributions can be written as the product of their individual pdfs.

$$f(x,y) = f_{X,Y}(x,y) = f(x;\alpha,\beta)\cdot f_{Y|X}(y|x;\alpha,\beta) \tag{4}$$

The BCBPD is, therefore, given by:

$$f_{X,Y}(x,y) = \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)}\cdot\frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \tag{5}$$

2.1 Theorem

Theorem 1: The relation

$$f_{X,Y}(x,y) = \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)}\cdot\frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \tag{6}$$

This function is a true bivariate probability density function for $x, y \in (0,\infty)$.
Proof: Let

$$M = \int_0^\infty\int_0^\infty f_{X,Y}(x,y)\,dx\,dy. \tag{7}$$

then,

$$M = \int_0^\infty\int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)}\cdot\frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)}\,dx\,dy. \tag{8}$$

Separate the Integrals
Since the integrand is separable, we can rewrite the double integral as the product of two separate integrals:

$$M = \frac{1}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\left(\int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{1}\,dx\right)\left(\int_0^\infty \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{1}\,dy\right).$$

Thus, we need to solve the two integrals:

$$I_1 = \int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{1}\,dx,$$

and

$$I_2 = \int_0^\infty \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{1}\,dy.$$

Solve the First Integral I_1
The first integral is:

$$I_1 = \int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{1}\,dx.$$

This is a standard integral related to the Beta function. It is known that:

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx = \frac{B(\alpha,\beta)}{B(\alpha,\alpha+\beta)}.$$

For our case, $\alpha=\alpha_1$ and $\beta=\beta_1$, so:

$$I_1 = B(\alpha_1,\beta_1).$$

Solve the Second Integral I_2
The second integral is:

$$I_2 = \int_0^\infty \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{1}\, dy.$$

Using the same method as for I_1, we find:

$$I_2 = B(\alpha_2, \beta_2).$$

Now, we multiply the results from I_1 and I_2 and substitute them back into the expression for M:

$$M = \frac{1}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \cdot B(\alpha_1, \beta_1) \cdot B(\alpha_2, \beta_2).$$

The Beta functions cancel out, and we are left with:

$$M = 1.$$

Since $M = 1$, the relation (6) is a true bivariate probability density function.

2.2 Generalization

The multivariate beta-prime distribution generalizes the bivariate beta-prime distribution to d random variables. Let $X_1, X_2, \ldots, X_d$ be d random variables, where each X_i follows a beta-prime distribution with shape parameters α_i and β_i for each variable. The joint probability density function (PDF) of the multivariate beta-prime distribution can be written as:

$$f_{X_1,X_2,\ldots,X_d}(x_1, x_2, \ldots, x_d; \alpha_1, \beta_1, \ldots, \alpha_d, \beta_d, \rho) = \prod_{i=1}^{d} \frac{x_i^{\alpha_i-1}(1+x_i)^{-\alpha_i-\beta_i}}{B(\alpha_i, \beta_i)} \tag{9}$$

where:

- $f_{X_1,X_2,\ldots,X_d}(x_1, x_2, \ldots, x_d)$ is the joint PDF of the random variables $X_1, X_2, \ldots, X_d$,
- α_i and β_i are the shape and scale parameters for the i-th variable X_i,
- $B(\alpha_i, \beta_i)$ is the Beta function for each X_i,

3 Properties of beta-prime distribution

3.1 Marginal distribution of Y

Theorem 2 The marginal distribution of Y from the joint distribution of a bivariate beta-prime distribution, is obtained from:

$$f_{X,Y}(x, y) = \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1, \beta_1)} \cdot \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2, \beta_2)} \cdot c(F_X(x), F_Y(y); \rho),$$

where:

- $B(\alpha_i, \beta_i)$ is the Beta function for the marginals,
- $F_X(x)$ and $F_Y(y)$ are the cumulative distribution functions (CDFs) of X and Y,
- $c(F_X(x), F_Y(y); \rho)$ is a copula function with dependence parameter ρ.

The marginal distribution of Y, denoted $f_Y(y)$, is obtained by integrating out x from the joint distribution:

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx.$$

Substituting the expression for $f_{X,Y}(x,y)$:

$$f_Y(y) = \int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)} \cdot \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \cdot c(F_X(x), F_Y(y);\rho)\,dx.$$

Independent Case If X and Y are independent, then $c(F_X(x), F_Y(y);\rho) = 1$, so the marginal distribution simplifies to:

$$f_Y(y) = \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \int_0^\infty \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)}\,dx.$$

Since the integral over x of the beta-prime PDF integrates to 1, this reduces to:

$$f_Y(y) = \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)}.$$

Thus, the marginal distribution of Y is itself a univariate beta-prime distribution with parameters α_2 and β_2:

$$f_Y(y) = \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)}, \quad y > 0.$$

3.2 Posterior Distribution for the Bivariate Beta-Prime Model

Given observed data $\{(x_i,y_i)\}_{i=1}^n$ from a bivariate beta-prime distribution, we want to derive the posterior distribution for the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, \rho$.

1. Joint PDF of the bivariate beta-prime distribution

The joint probability density function (PDF) of a bivariate beta-prime distribution is given by:

$$f_{X,Y}(x,y;\alpha_1,\beta_1,\alpha_2,\beta_2,\rho) = \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)} \cdot \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \cdot c(F_X(x), F_Y(y);\rho), \quad (10)$$

where:

- $B(\alpha,\beta)$ is the beta function,
- $c(F_X(x), F_Y(y);\rho)$ is a copula density function that introduces dependence between X and Y through the parameter ρ.

2. Prior distributions for the parameters

We assume independent priors for each parameter $\alpha_1, \beta_1, \alpha_2, \beta_2$, and ρ, so the joint prior distribution is:

$$p(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho) = p(\alpha_1)\cdot p(\beta_1)\cdot p(\alpha_2)\cdot p(\beta_2)\cdot p(\rho). \quad (11)$$

3. Likelihood Function

Given the observed data $\{(x_i,y_i)\}_{i=1}^n$, the likelihood function is:

$$L(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho;\mathbf{X},\mathbf{Y}) = \prod_{i=1}^n f_{X,Y}(x_i,y_i;\alpha_1,\beta_1,\alpha_2,\beta_2,\rho). \quad (12)$$

Substituting for $f_{X,Y}(x,y;\alpha_1,\beta_1,\alpha_2,\beta_2,\rho)$, we have:

$$L(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho;\mathbf{X},\mathbf{Y}) = \prod_{i=1}^n \left(\frac{x_i^{\alpha_1-1}(1+x_i)^{-\alpha_1-\beta_1}}{B(\alpha_1,\beta_1)} \cdot \frac{y_i^{\alpha_2-1}(1+y_i)^{-\alpha_2-\beta_2}}{B(\alpha_2,\beta_2)} \right) \quad (13)$$

4. Posterior distribution using Bayes' theorem

According to Bayes' theorem, the posterior distribution for $\alpha_1, \beta_1, \alpha_2, \beta_2, \rho$ given $\mathbf{X}$ and $\mathbf{Y}$ is:

$$p(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho|\mathbf{X}, \mathbf{Y}) = \frac{L(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho; \mathbf{X}, \mathbf{Y}) \cdot p(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)}{p(\mathbf{X}, \mathbf{Y})}, \tag{14}$$

where

- $p(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ is the joint prior distribution,
- $p(\mathbf{X}, \mathbf{Y})$ is the marginal likelihood (or evidence), calculated as

$$p(\mathbf{X}, \mathbf{Y}) = \int\int\int\int\int L(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho; \mathbf{X}, \mathbf{Y}) \cdot p(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)\, d\alpha_1\, d\beta_1\, d\alpha_2\, d\beta_2\, d\rho. \tag{15}$$

Substituting $L(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho; \mathbf{X}, \mathbf{Y})$ and the prior, we get:

$$p(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho|\mathbf{X}, \mathbf{Y}) \propto \left(\prod_{i=1}^{n} f_{X,Y}(x_i, y_i; \alpha_1, \beta_1, \alpha_2, \beta_2, \rho)\right) \cdot p(\alpha_1) \cdot p(\beta_1) \cdot p(\alpha_2) \cdot p(\beta_2) \cdot p(\rho). \tag{16}$$

This is the unnormalized posterior distribution for the parameters of the bivariate beta-prime distribution.

4 Method of Parameter Estimation

The parameters of the BCBPD can be estimated using the maximum likelihood method of estimation. From the density in (6).

The joint probability density function of a bivariate beta-prime distribution is given by:

$$f_{X,Y}(x, y; \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{x^{\alpha_1-1}(1+x)^{-\alpha_1-\beta_1}}{B(\alpha_1, \beta_1)} \cdot \frac{y^{\alpha_2-1}(1+y)^{-\alpha_2-\beta_2}}{B(\alpha_2, \beta_2)},$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

1. Log-Likelihood Function

Given a dataset $\{(x_i, y_i)\}_{i=1}^{n}$, the log-likelihood function is:

$$\log L(\alpha_1, \beta_1, \alpha_2, \beta_2; \mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{n} \left[(\alpha_1 - 1)\log x_i - (\alpha_1 + \beta_1)\log(1 + x_i) - \log B(\alpha_1, \beta_1)\right]$$

$$+ \sum_{i=1}^{n} \left[(\alpha_2 - 1)\log y_i - (\alpha_2 + \beta_2)\log(1 + y_i) - \log B(\alpha_2, \beta_2)\right].$$

2. Partial derivatives

To find the MLEs, we set the partial derivatives with respect to α_1, β_1, α_2, and β_2 to zero: 1. Partial Derivative with Respect to α_1: The partial derivative of $\log L$ with respect to α_1 is:

$$\frac{\partial \log L}{\partial \alpha_1} = \sum_{i=1}^{n} \left(\log x_i - \log(1 + x_i) - \frac{\partial}{\partial \alpha_1} \log B(\alpha_1, \beta_1)\right)$$

Using the known result for the derivative of the log Beta function:

$$\frac{\partial}{\partial \alpha_1} \log B(\alpha_1, \beta_1) = \psi(\alpha_1) - \psi(\alpha_1 + \beta_1)$$

where $\psi(x)$ is the digamma function, we get:

$$\frac{\partial \log L}{\partial \alpha_1} = \sum_{i=1}^{n} \left(\log x_i - \log(1 + x_i) - (\psi(\alpha_1) - \psi(\alpha_1 + \beta_1)) \right).$$

Setting this equal to zero:

$$\sum_{i=1}^{n} \left(\log x_i - \log(1 + x_i) - (\psi(\alpha_1) - \psi(\alpha_1 + \beta_1)) \right) = 0.$$

2. Partial Derivative with Respect to β_1: The partial derivative with respect to β_1 is:

$$\frac{\partial \log L}{\partial \beta_1} = -\sum_{i=1}^{n} \log(1 + x_i) + \sum_{i=1}^{n} \left(\psi(\alpha_1) - \psi(\alpha_1 + \beta_1) \right).$$

Setting this equal to zero:

$$-\sum_{i=1}^{n} \log(1 + x_i) + \sum_{i=1}^{n} \left(\psi(\alpha_1) - \psi(\alpha_1 + \beta_1) \right) = 0.$$

3. Partial Derivative with Respect to α_2: The partial derivative with respect to α_2 is:

$$\frac{\partial \log L}{\partial \alpha_2} = \sum_{i=1}^{n} \left(\log y_i - \log(1 + y_i) - (\psi(\alpha_2) - \psi(\alpha_2 + \beta_2)) \right).$$

Setting this equal to zero:

$$\sum_{i=1}^{n} \left(\log y_i - \log(1 + y_i) - (\psi(\alpha_2) - \psi(\alpha_2 + \beta_2)) \right) = 0.$$

4. Partial Derivative with Respect to β_2: The partial derivative with respect to β_2 is:

$$\frac{\partial \log L}{\partial \beta_2} = -\sum_{i=1}^{n} \log(1 + y_i) + \sum_{i=1}^{n} \left(\psi(\alpha_2) - \psi(\alpha_2 + \beta_2) \right).$$

Setting this equal to zero:

$$-\sum_{i=1}^{n} \log(1 + y_i) + \sum_{i=1}^{n} \left(\psi(\alpha_2) - \psi(\alpha_2 + \beta_2) \right) = 0.$$

Thus, the system of equations to solve for the Maximum Likelihood Estimators (MLE) of the parameters α_1, β_1, α_2, and β_2 is:

$$\sum_{i=1}^{n} \left(\log x_i - \log(1 + x_i) - (\psi(\alpha_1) - \psi(\alpha_1 + \beta_1)) \right) = 0$$

$$-\sum_{i=1}^{n} \log(1 + x_i) + \sum_{i=1}^{n} \left(\psi(\alpha_1) - \psi(\alpha_1 + \beta_1) \right) = 0$$

$$\sum_{i=1}^{n} (\log y_i - \log(1 + y_i) - (\psi(\alpha_2) - \psi(\alpha_2 + \beta_2))) = 0$$

$$-\sum_{i=1}^{n} \log(1 + y_i) + \sum_{i=1}^{n} (\psi(\alpha_2) - \psi(\alpha_2 + \beta_2)) = 0$$

These equations must be solved numerically, as they are nonlinear. Techniques such as Newton-Raphson or optimization methods can be used for this purpose.

5 Real data application

The dataset under consideration consists of 40 observations of wind speed and wind direction, two key meteorological variables. Wind speed is measured in meters per second (m/s), and wind direction is expressed in radians. The unique characteristics of wind patterns make this dataset an ideal candidate for applying a bivariate conditional Beta-prime distribution, a model well-suited for circular data like wind direction. In this analysis, the bivariate model is used to explore the relationship between wind speed and wind direction. Specifically, we employ the Beta-prime distribution to model the conditional dependence of wind direction on wind speed. By fitting this distribution to the dataset, we can better understand how changes in wind speed affect the distribution of wind direction, offering insights into the underlying dynamics of wind patterns. The analysis includes visualizing the data through joint and 3D histograms, which further illustrate the interaction between these variables. dataset: 'Wind Speed': [11.21, 8.99, 12.49, 10.87, 10.10, 10.91, 6.95, 8.91, 13.21, 5.95, 6.36, 9.94, 11.71, 9.46, 9.58, 12.27, 9.03, 8.30, 8.96, 6.71, 11.77, 12.43, 8.76, 8.83, 12.88, 10.67, 12.11, 9.53, 11.18, 8.86, 8.89, 11.91, 8.76, 9.77, 11.59, 11.07, 7.30, 11.71, 9.43, 7.31], 'Wind Direction': [-0.035, 0.573, 2.067, -2.716, -1.003, 0.728, 0.547, -2.407, 0.155, 2.123, 0.637, 2.242, -2.535, 1.110, 1.369, -1.993, -1.891, -2.837, -2.688, -2.632, 2.352, -0.664, -1.578, -0.326, -0.695, 1.659, -0.907, -0.603, -2.775, -0.089, -1.777, 1.481, 1.736, -1.982, 1.372, -0.351, -0.066, 2.086, 0.019, -1.008]

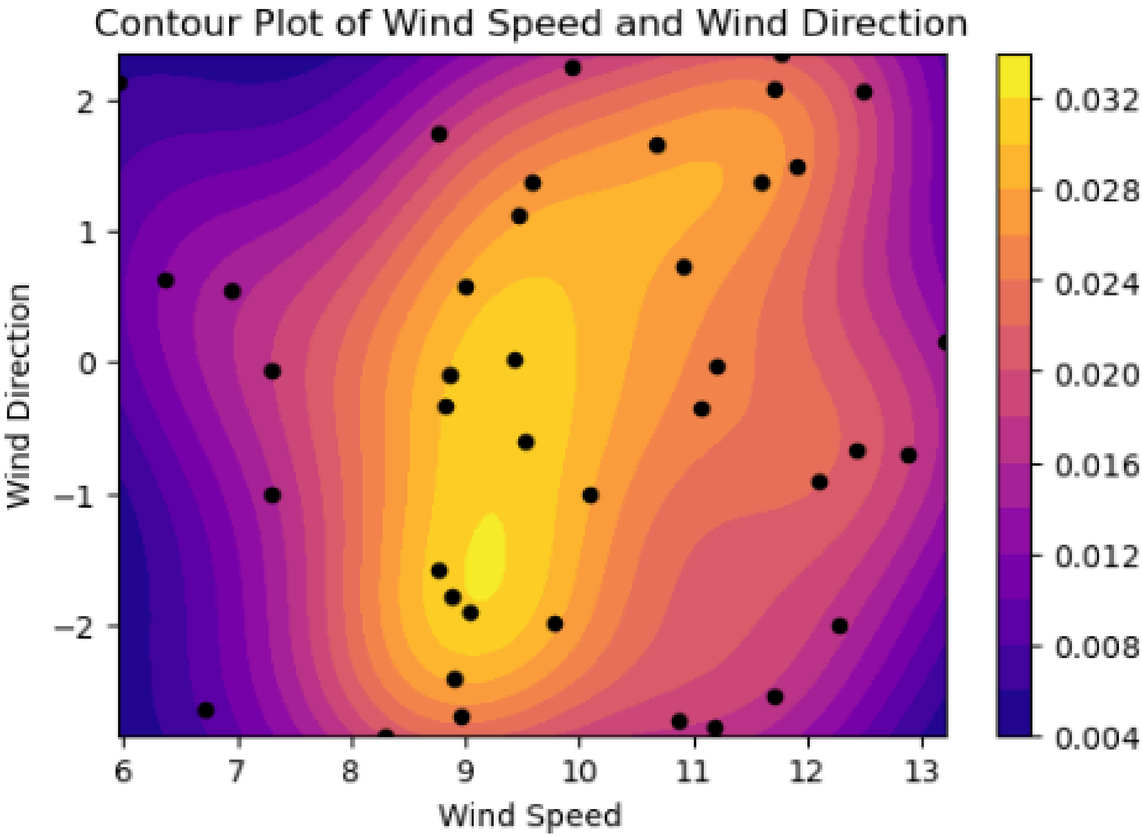

Figure 2: BCBPD contour plot of the wind speed and wind direction

The Kolmogorov-Smirnov (K-S) test returned a statistic of 0.101 and a p-value of 0.771. This indicates a small deviation between the observed data and the bivariate conditional Beta-prime distribution, suggesting that the empirical distribution of the wind speed and direction data closely aligns with the hypothesized model. The high p-value (0.771) suggests that there is no significant evidence to reject the null hypothesis, meaning that the data is consistent with being drawn from the bivariate conditional Beta-prime distribution. Thus, this distribution appears to be a suitable fit for the dataset. The 3D histogram is given below

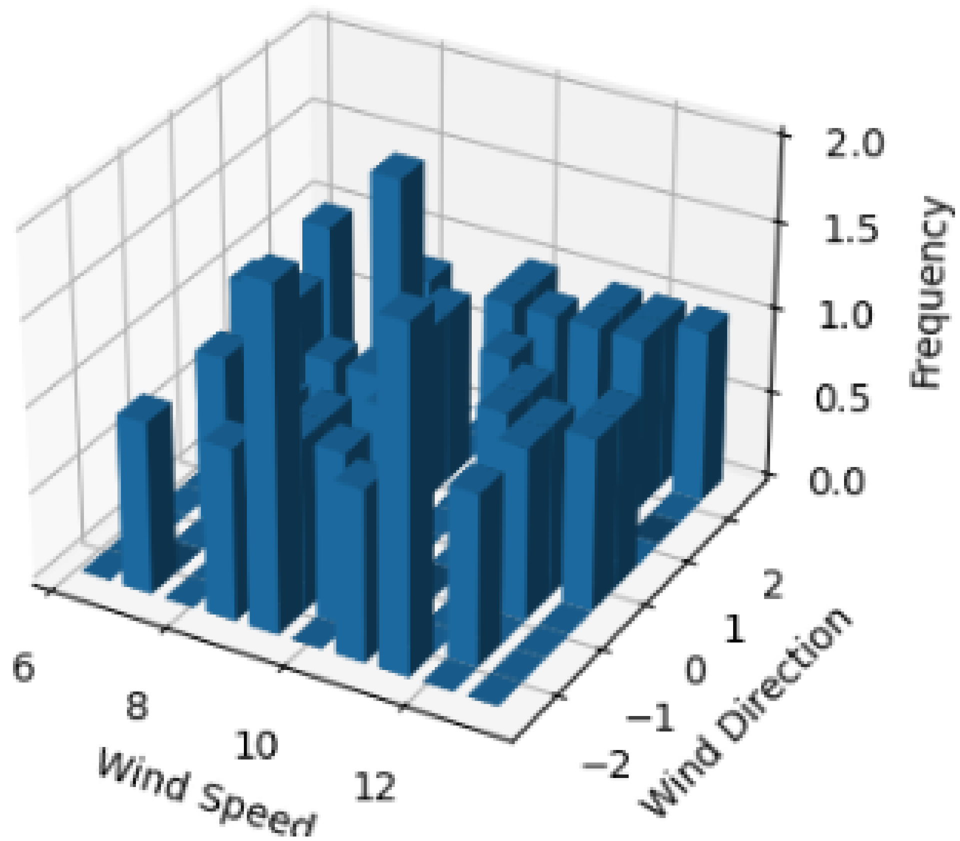

Figure 3: histogram plot of the wind speed and wind direction

6 Conclusion

The application of the bivariate conditional Beta-prime distribution to the dataset has provided valuable insights into the relationship between wind speed and wind direction. The density plots and visualizations illustrate the suitability of this distribution in modeling circular data, especially when the dependent variable exhibits a semicircular pattern. By fitting the beta-prime distribution, we were able to explore the marginal densities and conditional relationships, offering a robust framework for understanding the dynamics of wind behavior in meteorological studies. The approach demonstrated here underscores the flexibility and applicability of the beta-prime distribution in capturing the intricacies of circular data within a bivariate context.

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