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Factorization in the semigroup generated by the idempotents of the semigroup of full contractions of a finite chain

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Abstract

Let $[n] = \{1, 2, \ldots, n\}$ be a finite chain and let $\mathcal{CT}_n$ be the semigroup of full contractions on $[n]$. It was shown in [3] that the semigroup generated by idempotents is a regular semigroup (denoted by $\langle E(\mathcal{CT}_n)\rangle$). In this paper, we investigate factorization properties in this semigroup.

Keywords and phrases: full contractions maps; regular element; idempotents; rank properties

1 Introduction

Denote $[n] = \{1, 2, \ldots, n\}$ to be a finite chain and let $\mathcal{T}_n$ denote the semigroup of full transformations of $[n]$. A transformation $\alpha \in \mathcal{T}_n$ is said to be *order preserving* (resp., *order reversing*) if (for all $x, y \in [n]$) $x \leq y$ implies $x\alpha \leq y\alpha$ (resp., $x\alpha \geq y\alpha$); *order decreasing* if (for all $x \in [n]$) $x\alpha \leq x$; an *isometry* (i. e., *distance preserving*) if (for all $x, y \in [n]$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in [n]$) $|x\alpha - y\alpha| \leq |x - y|$. Let $\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : (for\ all\ x, y \in [n])\ |x\alpha - y\alpha| \leq |x - y|\}$ be the semigroup of full contractions on $[n]$, as such $\mathcal{CT}_n$ is a subsemigroup of $\mathcal{T}_n$. Certain algebraic and combinatorial properties of this semigroup (i. e., $\mathcal{CT}_n$) and some of its subsemigroups have been investigated, for example see [1, 2, 3].

Let $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$ denote the *subsemigroup of all order preserving or order reversing* and *subsemigroup of all order preserving full contractions of* $[n]$, respectively. These subsemigroups are both known to be non-regular and their Green's relations as well as starred Green's relations have been characterized in [3].

Let S be a semigroup and U be a subset of S, then U is said to *generate* S if every element in S is factorizable into product of elements in U, and moreover, $|U|$ is said to be the *rank* of S (denoted by $Rank(S)$) if

$$|U| = \min\{|A| : A \subseteq S \text{ and } \langle A\rangle = S\}.$$

The notation $\langle U\rangle = S$ means that U generates the semigroup S. The rank properties of several semigroups of transformations were investigated; for example, see [4, 5, 6, 7, 8, 9, 10]. Recently, Zubairu and Aliyu in [11] proved that the ranks of the subsemigroups $Reg(\mathcal{ORCT}_n)$ and $E(\mathcal{ORCT}_n)$ are 4 and 3, respectively (where $Reg(\mathcal{ORCT}_n)$ denotes the collection of all regular elements of $\mathcal{ORCT}_n$ and $E(\mathcal{ORCT}_n)$ denotes the collection of all idempotent elements of $\mathcal{ORCT}_n$; $Reg(\mathcal{ORCT}_n)$ and $E(\mathcal{ORCT}_n)$ were shown to be subsemigroups in [3]). As noted in [11], there are several subsemigroups of full contractions whose ranks are yet to be known, including the rank of the semigroup $\mathcal{CT}_n$.

It was shown in [3] that, contrary to $E(\mathcal{ORCT}_n)$, $E(\mathcal{CT}_n)$ does not form a subsemigroup. However, it has been shown that the semigroup $\langle E(\mathcal{CT}_n)\rangle$ generated by the idempotents in $\mathcal{CT}_n$ is a regular subsemigroup. Thus, the need to investigate the kind of elements these idempotents generate. This will pave way to understand the rank properties of the monoid $\mathcal{CT}_n$). For basic concepts in semigroup theory, we refer the reader to [12, 13, 14].

Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix} \in \mathcal{T}_n \ \ (1 \leq p \leq n), \tag{1}$$

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then the *height* of α is defined and denoted as $h(\alpha) = |\text{Im } \alpha| = p$, so also, $x_i\alpha^{-1} = A_i$ $(1 \leq i \leq p)$ are equivalence classes under the relation $\ker \alpha = \{(x, y) \in [n] \times [n] : x\alpha = y\alpha\}$. Furthermore, we denote the partition of $[n]$ by the relation $\ker \alpha$ $(A_1, \ldots, A_p)$ by $\textbf{Ker } \alpha$ and also, $\text{fix}(\alpha) = |\{x \in [n] : x\alpha = x\}|$. A subset T_α of $[n]$ is said to be a *transversal* of the partition $\textbf{Ker } \alpha$ if $|T_\alpha| = p$, and $|A_i \cap T_\alpha| = 1$ $(1 \leq i \leq p)$. A transversal T_α is said to be *convex* if for all $x, y \in T_\alpha$ with $x \leq y$ and if $x < z < y$ $(z \in [n])$, then $z \in T_\alpha$. A partition $\textbf{Ker } \alpha$ is said to be a *convex partition* if it has a convex transversal. For an arbitrary $x \in \mathbb{R}$, denote $\lfloor x \rfloor$ to be the greatest inter less or equal to x. An element a in a semigroup S is said to be an *idempotent* if $a^2 = a$. We shall denote $E(S)$ to be the set of idempotent elements of S. We shall also denote $J_r = \{\alpha \in \langle E(\mathcal{CT}_n)\rangle : |\text{Im } \alpha| = r\}$, to be the collection of elements in $\langle E(\mathcal{CT}_n)\rangle$, whose height is exactly r. It is well known that $\langle E(\mathcal{CT}_n)\rangle = J_1 \cup \ldots \cup J_n$.

We further introduce the following concept, which will be useful in our subsequent discussions. In the chain $[n]$, consider the first r elements, i.e., $1, \ldots, r$, and call them a *period* of length r. Then use the remaining $n - r$ elements to form a *zigzag* (or a *jagged*) structure of equal length r until the $n - r$ elements are exhausted. This formation is unique (since it consists of zigzags of equal periods). If the number of periods of equal length (say r) in the formed zigzag is m, we refer to such a shape as an m-fold shape. This number can easily be computed from the relation $\lfloor \frac{n-1}{r-1} \rfloor$. Thus, every n-chain has $\lfloor \frac{n-1}{r-1} \rfloor$ folds with a period length of r. For the purposes of illustration, let $n = 11$ and $r = 3$. We can form this zigzag in the following manners: set-wise $\{\{1, 5, 9\}, \{2, 4, 6, 8, 10\}, \{3, 7, 11\}\}$ or in counting manners $1, 2, 3-$, $3, 4, 5-$, $5, 6, 7-$, $7, 8, 9-$, and $9, 10, 11$, which is obviously a 5-fold. We now introduce the following definition. A convex partition $\textbf{Ker } \alpha$ of $[n]$ of height r is said to be an $\lfloor \frac{n-1}{r-1} \rfloor$-*fold partition* if it has $\lfloor \frac{n-1}{r-1} \rfloor$ number of convex transversal. The above paragraph proves the following lemma:

Lemma 1.1. *The chain $[n]$ can be decomposed into zigzag of $\lfloor \frac{n-1}{r-1} \rfloor$-folds for every period $r \leq n$ and hence there exists $\alpha \in J_r$ such that $\textbf{Ker } \alpha$ is an $\lfloor \frac{n-1}{r-1} \rfloor$-folds for every height r .*

Proof. In the chain $[n]$, we consider the first r elements, namely $1, \ldots, r$, and refer to this as a *period* of length r. We then use the remaining $n - r$ elements to construct a *zigzag* (or *jagged*) structure, also of length r, until we exhaust the $n - r$ elements. This configuration is unique, as it consists of zigzags formed from equal periods. If the number of equal-height periods (with height r) in the constructed zigzag is m, we denote this shape as an m-fold shape. This number can be readily computed using the formula $\lfloor \frac{n-1}{r-1} \rfloor$. Therefore, every n-chain contains $\lfloor \frac{n-1}{r-1} \rfloor$ folds, each with a period height of r, as required.

To formulate a zigzag from the chain $[n]$, the initial point does not always have to be point 1; it can be another point, say $i > 1$. As such, we provide the following definition: A zigzag of period r is said to be *regular* if its initial point is 1, and it is *irregular* if its initial point is any $i > 1$. In this case, it will have a total of $\lfloor \frac{n-i}{r-1} \rfloor$ periods, each of length r. The example in the last paragraph before the above lemma is a regular zigzag. However, the zigzag $1, 2, 3-$, $3, 4, 5, 6-$, $6, 7, 8, 9-$, $9, 10, 11$, with period $r = 4$ and $n = 11$, has a total of 2 periods, each of length 4 (i.e., a 2-fold), namely the periods $3, 4, 5, 6$ and $6, 7, 8, 9$. Notice that the lengths of the other periods are less than 4, and that the initial point of the first period of length 4 is 3, i.e., $i = 3$. It is easy to verify using the formula $\lfloor \frac{n-i}{r-1} \rfloor$ that there are 2 periods, each of length 4 (i.e., a 2-fold). Now we give the following definition: An element $\alpha \in \langle E(\mathcal{CT}_n)\rangle$ is said to be $\lfloor \frac{n-i}{r-1} \rfloor$-fold if $\textbf{Ker } \alpha$ has a total of $\lfloor \frac{n-i}{r-1} \rfloor$ periods, each of length r, for any initial point $i \geq 1$.

Before we proceed, let's describe the Green's equivalences on the semigroup $\langle E(\mathcal{CT}_n)\rangle$. Notice that the semigroup $\langle E(\mathcal{CT}_n)\rangle$ is a regular subsemigroup of the full transformation semigroup $\mathcal{T}_n$. Therefore, by [12], Prop. 2.4.2, it inherits the Green's $\mathcal{L}$ and $\mathcal{R}$ characterizations of the semigroup $\mathcal{T}_n$, but not necessarily the $\mathcal{D}$ equivalence. Thus, we have the following lemma.

Lemma 1.2. *Let $\alpha, \beta \in \langle E(\mathcal{CT}_n)\rangle$ then*

i $\alpha\mathcal{R}\beta$ *if and only if* $\text{Im } \alpha = \text{Im } \beta$;

ii $\alpha\mathcal{L}\beta$ *if and only if* $\ker \alpha = \ker \beta$.

2 The elements in $\langle E(\mathcal{CT}_n)\rangle$ of height 2

It is worth noting that elements in $\mathcal{CT}_n$ of height 2 are regular elements (see, [3]). Moreover, they are all idempotent generated as demonstrated in the lemma below:

Lemma 2.1. $J_2 = \{\alpha \in \mathcal{CT}_n : |\text{Im }\alpha| = 2\} = \langle E(\{\alpha \in \mathcal{CT}_n : |\text{Im }\alpha| = 2\})\rangle$. *In other words, every element of height 2 is generated by idempotents of height 2 in $\mathcal{CT}_n$.*

Proof. Let $\alpha = \begin{pmatrix} A_1 & A_2 \\ x_1 & x_2 \end{pmatrix} \in \mathcal{CT}_n$. Denote $T_\alpha = \{t_1, t_2\}$ as a convex transversal of $\mathbf{Ker}\,\alpha$, which exists since α is regular. Now, let $\{B_1, B_2\}$ be a partition of $[n]$ such that $x_1, t_1 \in B_1$ and $x_2, t_2 \in B_2$. Thus, the maps defined as $\gamma_1 = \begin{pmatrix} A_1 & A_2 \\ t_1 & t_2 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} B_1 & B_2 \\ x_1 & x_2 \end{pmatrix}$ are obviously idempotents in $\mathcal{CT}_n$. Moreover, it is easy to see that $\alpha = \gamma_1\gamma_2$, as required.

3 The elements in $\langle E(\mathcal{CT}_n)\rangle$ of height $n-1$

We begin this section by describing the idempotents of height $n-1$ in $\mathcal{CT}_n$ in the following lemma.

Lemma 3.1. *If $\epsilon \in E(J_{n-1})$ then ϵ is in one of the following forms:*

$$\epsilon = \begin{pmatrix} 1 & 2 & \ldots & n-2 & \{n-1, n\} \\ 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix}, \ \epsilon = \begin{pmatrix} 1 & 2 & \ldots & \{n-2, n\} & n-1 \\ 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix},$$

$$\epsilon = \begin{pmatrix} \{1,2\} & 3 & 4 & \ldots & n \\ 2 & 3 & 4 & \ldots & n \end{pmatrix}, \ \epsilon = \begin{pmatrix} 2 & \{1,3\} & 4 & \ldots & n \\ 2 & 3 & 4 & \ldots & n \end{pmatrix}.$$

Proof. Notice that ϵ is an idempotent of height $n-1$. It follows, by definition, that Fix $(\epsilon) =$ Im $\epsilon = \{1, \ldots, n-1\}$ or Fix $(\epsilon) =$ Im $\epsilon = \{2, \ldots, n\}$. In the former case, the remaining element n has 2 degrees of freedom to form a kernel class that admits the image set; similarly, in the latter case, the element 1 has 2 degrees of freedom to form a kernel class that admits the image set. This implies that either $\mathbf{Ker}\,\epsilon = \{\{1\}, \{2\}, \ldots, \{n-1, n\}\}$, $\mathbf{Ker}\,\epsilon = \{\{1\}, \{2\}, \ldots, \{n-2, n\}, \{n\}\}$, $\mathbf{Ker}\,\epsilon = \{\{1,2\}, \{3\}, \ldots, \{n\}\}$, or $\mathbf{Ker}\,\epsilon = \{\{2\}, \{1,3\}, \{4\}, \ldots, \{n\}\}$. The result now follows easily.

Denote the idempotents in $E(J_{n-1})$ as:

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \ldots & n-2 & \{n-1, n\} \\ 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix}, \ \epsilon_2 = \begin{pmatrix} 1 & 2 & \ldots & \{n-2, n\} & n-1 \\ 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix},$$

$$\epsilon_3 = \begin{pmatrix} \{1,2\} & 3 & 4 & \ldots & n \\ 2 & 3 & 4 & \ldots & n \end{pmatrix} \text{ or } \epsilon_4 = \begin{pmatrix} 2 & \{1,3\} & 4 & \ldots & n \\ 2 & 3 & 4 & \ldots & n \end{pmatrix}.$$

Then, consider the following multiplication table:

$\circ$	ϵ_1	ϵ_2	ϵ_3	ϵ_4
ϵ_1	ϵ_1	ϵ_1	ϵ_5^*	ϵ_6^*
ϵ_2	ϵ_2	ϵ_2	ϵ_7^*	ϵ_8^*
ϵ_3	ϵ_5^*	ϵ_7^*	ϵ_3	ϵ_3
ϵ_4	ϵ_6^*	ϵ_8^*	ϵ_4	ϵ_4

,

where

$$\epsilon_5^* = \begin{pmatrix} \{1,2\} & 3 & \ldots & n-2 & \{n-1, n\} \\ 2 & 3 & \ldots & n-2 & n-1 \end{pmatrix}, \ \epsilon_6^* = \begin{pmatrix} 2 & \{1,3\} & 4 & \ldots & n-2 & \{n-1, n\} \\ 2 & 3 & 4 & \ldots & n-2 & n-1 \end{pmatrix},$$

$$\epsilon_7^* = \begin{pmatrix} \{1,2\} & 3 & \dots & \{n-2,n\} & n-1 \\ 2 & 3 & \dots & n-2 & n-1 \end{pmatrix} \text{ and } \epsilon_8^* = \begin{pmatrix} 2 & \{1,3\} & \dots & \{n-2,n\} & n-1 \\ 2 & 3 & \dots & n-2 & n-1 \end{pmatrix}$$

are all idempotents of height $n-2$.

It is now clear from the above table that the idempotents in $E(J_{n-1})$ generate idempotents in J_{n-2}. These generated idempotents are all 1-folds for $n \geq 5$. Thus, we have the following lemmas.

Lemma 3.2. *For $n \geq 4$, every idempotent in $E(J_{n-1})$ is a 1-fold.*

Lemma 3.3. *For $n \geq 5$, $E(J_{n-1})$ generates 1-fold idempotents in J_{n-2}.*

3.1 1-fold elements in $\langle E(\mathcal{CT}_n)\rangle$ of height r

In this section, we investigate the structural properties of a 1-fold element in $\langle E(\mathcal{CT}_n)\rangle$ of height r. We begin with the following lemma.

Lemma 3.4. *Let $\epsilon \in E(J_r)$ be such that $\mathrm{Im}\ \epsilon = \{i, i+1, \dots, i+r-1\}$ ($n \geq 4$). If $|\{1, 2, \dots, i-1\}| < r-1$ and $|\{i+r, i+r+1, \dots, n\}| < r-1$, then ϵ is a 1-fold idempotent. Hence there does not exists $\epsilon' \in E(J_r)$ with $\mathrm{Im}\ \epsilon = \{i, i+1, \dots, i+r-1\}$ such that ϵ' is $\lfloor\frac{n-i}{r-1}\rfloor$-folds with $\lfloor\frac{n-i}{r-1}\rfloor \geq 2$.*

Proof. Fix $i, i+1, \dots, i+r-1$ and distribute the remaining elements $\{i+r, i+r+1, \dots, n\}$ to the blocks $\{i\}, \{i+1\}, \dots, \{i+r-1\}$ maximally, so that each element is placed in at most one block in a descending order, starting with the block $\{i+r-1\}$ down to $\{i+j\}$ for some $j \geq 1$. This maximum distribution can be done as follows: place $i+r$ into $\{i+r-2\}$, place $i+r+1$ into $\{i+r-3\}, \dots$, place n into $\{i+j\}$. Since $(n-i)-(r-1) < r-1$, we have $j \geq 1$ (note that this is not the only way to distribute the elements so that ϵ is a contraction; however, to maximize the number of periods of order r, we distribute in a one-to-one fashion). Similarly, we distribute the elements in $\{1, 2, \dots, i\}$ to the blocks, starting by placing $i-1$ into the block $\{i+1\}$, $i-2$ into the block $\{i+2\}, \dots$, and 1 into the block $\{i+k\}$. Since $i-1 < r-1$, we conclude that $k < r-1$. Notice that this is a zigzag of 1 period of order r and the other periods are of order less than r. It follows that ϵ is a $1-$fold idempotent. Hence there does not exists $\epsilon' \in E(J_r)$ with $\mathrm{Im}\ \epsilon = \{i, i+1, \dots, i+r-1\}$ such that ϵ' is at least a $2-$fold idempotent. The proof is now complete.

It is worth noting that if $\mathrm{Im}\ \epsilon = \{1, 2, \dots, r\}$ or $\mathrm{Im}\ \epsilon = \{2, 3, \dots, r+1\}$ then the number of periods depends on the height r. For $\frac{n+2}{2} \leq r \leq n$ ($n \geq 2$), it is easy to verify that $\lfloor\frac{n-i}{r-1}\rfloor = 1$, and for $1 \leq r < \frac{n+2}{2}$, $\lfloor\frac{n-i}{r-1}\rfloor \geq 1$. Thus, to have an $\lfloor\frac{n-i}{r-1}\rfloor-$fold of greater than 1, the length of the period must be in the range $2 \leq r < \frac{n+2}{2}$ (since obviously, if $r = 1$ then the number of the period is also 1 for any n). Let $J_r^* = \{\alpha \in J_r : \mathrm{Im}\ \alpha = \{1, 2, \dots, r\} \text{ or } \mathrm{Im}\ \alpha = \{2, 3, \dots, r+1\}\}$. Therefore, we have the following lemma.

Lemma 3.5. *If $\frac{n+2}{2} \leq r \leq n$ ($n \geq 5$). Then $E(J_r^*)$ generates $E(J_r^*) \cup E(J_{r-1}^*)$ and every element in $E(J_{r-1}^*)$ is a 1-fold idempotent.*

Proof. Let $\frac{n+2}{2} < r \leq n$ (where $n \geq 5$). It is easy to show that $n - r < r$ for every $\frac{n+2}{2} < r \leq n$. Thus, by Lemma 3.4, every element in $E(J_r^*)$ is a 1-fold. Now, let $\epsilon_1, \epsilon_2 \in E(J_r^*)$ such that either

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & A_i & \cdots & A_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \text{ or } \epsilon_1 = \begin{pmatrix} 2 & \{1,3\} & 4 & \cdots & i & A_{i+1} & \cdots & A_{r+1} \\ 2 & 3 & 4 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix}$$

or

$$\epsilon_1 = \begin{pmatrix} \{1,2\} & 3 & 4 & \cdots & i & A_{i+1} & \cdots & A_{r+1} \\ 2 & 3 & 4 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

and also

$$\epsilon_2 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & B_i & \cdots & B_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \text{ or } \epsilon_2 = \begin{pmatrix} 2 & \{1,3\} & 4 & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & 4 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix}$$

or

$$\epsilon_2 = \begin{pmatrix} \{1,2\} & 3 & 4 & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & 4 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

where $i \geq 3$. The blocks A_j and B_j are stationary for all j, i.e., $j \in A_j$ and $j \in B_j$.

We now have different cases to consider. If

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & A_i & \cdots & A_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & B_i & \cdots & B_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix},$$

then it is easy to see that $\epsilon_1\epsilon_2 = \epsilon_1 \in E(J_r^*)$ (note that, in this case, $F(\epsilon_1\epsilon_2) = F(\epsilon_1)$).

Now, if

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & A_i & \cdots & A_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

then

$$\epsilon_1\epsilon_2 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i-1 & A_i & \cdots & A_r \\ 2 & 3 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \in E(J_{r-1}^*).$$

Note that $|F(\epsilon_1\epsilon_2)| = |\{2,\ldots,r\}| = r-1$, i.e., $|\text{im } \epsilon_1\epsilon_2| = r-1$. Since $\frac{n+2}{2} < r$, we have $\frac{n+2}{2} \leq r-1$, and thus $\epsilon_1\epsilon_2$ is a 1-fold idempotent, as required.

Next, if

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & A_i & \cdots & A_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} \{1,2\} & 3 & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

then

$$\epsilon_1\epsilon_2 = \begin{pmatrix} \{1,2\} & 3 & \cdots & i-1 & A_i & \cdots & A_r \\ 2 & 3 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \in E(J_{r-1}^*).$$

It is not difficult to see that $\epsilon_1\epsilon_2$ is a 1-fold idempotent.

If

$$\epsilon_1 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i & A_{i+1} & \cdots & A_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} 1 & 2 & \cdots & i-1 & B_i & \cdots & B_r \\ 1 & 2 & \cdots & i-1 & i & \cdots & r \end{pmatrix},$$

then

$$\epsilon_1\epsilon_2 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i-1 & A_i & \cdots & A_r \cup A_{r+1} \\ 2 & 3 & \cdots & i-1 & i & \cdots & r \end{pmatrix} \in E(J_{r-1}^*).$$

Now, if

$$\epsilon_1 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i & A_{i+1} & \cdots & A_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

then $\epsilon_1\epsilon_2 = \epsilon_1 \in E(J_r^*)$.

Finally, if

$$\epsilon_1 = \begin{pmatrix} 2 & \{1,3\} & \cdots & i & A_{i+1} & \cdots & A_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} \{1,2\} & 3 & \cdots & i & B_{i+1} & \cdots & B_{r+1} \\ 2 & 3 & \cdots & i & i+1 & \cdots & r+1 \end{pmatrix},$$

then notice that $F(\epsilon_1\epsilon_2) = F(\epsilon_1)$ and $\epsilon_1\epsilon_2 = \epsilon_1 \in E(J_r^*)$. The proof is complete.

We now note the following:

Remark 3.6. As a consequence of the above lemma, it follows that idempotents with Im $\alpha = \{1,2,\ldots,r\}$ or Im $\alpha = \{2,3,\ldots,r+1\}$ of height $r \geq \frac{n+2}{2}$ does not generate an $(i \geq 2)-$fold idempotents of lower height. Moreover, elements in $\langle E(\mathcal{CT}_n)\rangle$ with Im $\alpha = \{1,2,\ldots,r\}$ or Im $\alpha = \{2,3,\ldots,r+1\}$ of height $\frac{n+2}{2} \leq r \leq n$ $(n \geq 5)$ are all idempotents.

Now let $K_r = \{\epsilon \in \langle E(\mathcal{CT}_n)\rangle : |\mathrm{Im}\ \epsilon| \leq r\}$. We have the following corollary.

Corollary 3.7. *If $n \geq 5$ and $\frac{n+2}{2} \leq r \leq n$. Then $\langle J_{n-1}\rangle = K_r$.*

It is now clear that there exist i-fold idempotents ($i \geq 2$) whenever $r \leq \lfloor \frac{n+2}{2} \rfloor$. For if $r \leq \lfloor \frac{n+2}{2} \rfloor$, then $n - r \leq \frac{n}{2}$, and consequently, there is a tendency to have a zigzag with i periods ($i \geq 2$), each of order r. Thus, we have the following lemma.

Lemma 3.8. *Let $\epsilon \in E(J_r)$ be such that $\mathrm{Im}\ \epsilon = \{i, i+1, \ldots, i+r-1\}$. If $|\{1, 2, \ldots, i\}| \geq r-1$ or $|\{i+r, i+r+1, \ldots, n\}| \geq r-1$. Then there exists $\epsilon' \in E(J_r)$ such that ϵ' is an $\lfloor \frac{n-i}{r-1} \rfloor$−folds with $\lfloor \frac{n-i}{r-1} \rfloor \geq 2$.*

Proof. Let $\mathrm{Im}\ \epsilon = \{i, i+1, \ldots, i+r-1\}$ and $|[n] \setminus \mathrm{Im}\ \epsilon| \geq r-1$. Thus $[n] \setminus \mathrm{Im}\ \epsilon = \{i+r, i+r+1, \ldots, n\}$. Now fix $i, i+1, \ldots, i+r-1$. Since $|[n] \setminus \mathrm{Im}\ \epsilon| \geq r-1$ then we can form a zigzag with the elements $i+r, i+r+1, \ldots, n$ of $\lfloor \frac{n-i}{r-1} \rfloor$− periods each of order r. Notice that if $|\{i+r, i+r+1, \ldots, n\}| = r-1$ then we can form the following zigzag $\{i, n\}, \{i+1, n-1\}, \ldots, \{i+r-2, i+r\}, \{i+r-1\}$, which is obviously a 2 period zigzag and thus

$$\epsilon' = \begin{pmatrix} \{i, n\} & \{i+1, n-1\} & \cdots & \{i+r-2, i+r\} & \{i+r-1\} \\ i & i+1 & \ldots & i+r-2 & i+r-1 \end{pmatrix}$$

is a 2-folds idempotent. It is now obvious that if $|\{j+r, j+r+1, \ldots, n\}| \geq r-1$ then we can get more number of periods from the formulation of the zigzag, as such the number of periods is greater than or equal to 2, i. e., $\lfloor \frac{n-i}{r-1} \rfloor \geq 2$. We can as well prove in a similar way for the case when $|\{i+r, i+r+1, \ldots, n\}| \geq r-1$.

Remark 3.9.
(a) For a subset $\{i, \ldots, r+i-1\}$ of $[n]$ of order r, if $|\{1, 2, \ldots, i\}| \geq r-1$ or $|\{i+r, i+r+1, \ldots, n\}| \geq r-1$, then we can find an element $\alpha \in E(\mathcal{CT}_n)$ whose $\mathrm{Im}\ \alpha = \{i, \ldots, r+i-1\}$ and **Ker** α is an r−fold with $r \geq 2$;

(b) For a subset $\{i, \ldots, r+i-1\}$ of $[n]$ of order r, if both $|\{1, 2, \ldots, i\}| \geq r-1$ and $|\{i+r, i+r+1, \ldots, n\}| \geq r-1$, then we can find an element $\alpha \in E(\mathcal{CT}_n)$ whose $\mathrm{Im}\ \alpha = \{i, \ldots, r+i-1\}$ and **Ker** α is an r−fold with $r \geq 3$.

Lemma 3.10. *For every $r < \frac{n+2}{2}$ there exists $\epsilon \in E(J_r)$ such that ϵ is an $\lfloor \frac{n-i}{r-1} \rfloor$−fold idempotent with $\lfloor \frac{n-i}{r-1} \rfloor \geq 2$.*

Theorem 3.11. *Every regular element $\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ x+1 & x+2 & \ldots & x+r \end{pmatrix} \in \langle E(J_r)\rangle$ can be express as product of two idempotents in $E(J_r)$.*

Proof. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ x+1 & x+2 & \ldots & x+r \end{pmatrix} \in \langle E(J_r)\rangle$. Denote $T_\alpha = \{a+1, \ldots, a+r\}$ to be a convex transversal of **Ker** α with either $a+i \in A_i$ or $a+r-i+1 \in A_i$ ($1 \leq i \leq r$). Notice that $\mathrm{Im}\ \alpha = \{x+1, \ldots, x+r\}$ is also convex. Thus either $x = a$(or $x+i = a+r-i+1$) or $x \neq a$. If $x = a$(or $x+i = a+r-i+1$) the result follows easily, since α will be an idempotent. Now suppose $x \neq a$. There are two cases to consider i. e., either $a+i \in A_i$ or $a+r-i+1 \in A_i$ ($1 \leq i \leq r$).

If $a+i \in A_i$. Then form a 2-fold zigzag as: $\{1, 2, \ldots, a+1, x+r\}$, $\{a+2, x+r-1\}$, $\{a+3, x+r-2\}$, …, $\{x+1, a+r, \ldots, n\}$. Now let

$$\epsilon_1 = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a+1 & a+2 & \ldots & a+r \end{pmatrix} \in E(J_r)$$

and let

$$\epsilon_2 = \begin{pmatrix} \{1, 2, \ldots, a+1, x+r\} & \{a+2, x+r-1\} & \ldots & \{x+1, a+r, \ldots, n\} \\ x+r & x+r-1 & \ldots & x+1 \end{pmatrix} \in E(J_r).$$

It is now easy to see that $\epsilon_1\epsilon_2 = \alpha$.

Now if $a + r - i + 1 \in A_i$ $(1 \leq i \leq r)$. Now form a 2-fold zigzag as: $\{1, 2, \ldots, x + 1, a + r\}$, $\{x + 2, a + r - 1\}$, $\{x + 3, a + r - 2\}$, …, $\{a + 1, x + r, \ldots, n\}$ and let

$$\epsilon_1 = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a+r & a+r-1 & \ldots & a+1 \end{pmatrix} \in E(J_r)$$

and let

$$\epsilon_2 = \begin{pmatrix} \{1, 2, \ldots, x+1, a+r\} & \{x+2, a+r-1\} & \ldots & \{a+1, x+r, \ldots, n\} \\ x+1 & x+2 & \ldots & x+r \end{pmatrix} \in E(J_r).$$

Then it is now easy to see that $\epsilon_1\epsilon_2 = \alpha$.

Remark 3.12. The regular (non-idempotent) elements in $\langle E(\mathcal{CT}_n)\rangle$ are generated by two idempotents with one of them is at least 2−fold idempotent.

4 Discussions and conclusion

The research focuses on the structure and properties of the semigroup of full contractions $\mathcal{CT}_n$ on a finite chain $[n] = \{1, 2, \ldots, n\}$. The key points include:

- The identification of idempotent generated regular subsemigroup denoted as $\langle E(\mathcal{CT}_n)\rangle$.
- An examination of various properties associated with idempotents, including their heights and the conditions under which they can generate other idempotents.
- A detailed exploration of zigzag structures within chains, which illuminate the periodic nature of the elements involved and aid in understanding idempotent behavior.
- Discussion of Green's relations and the characterization of regular versus non-regular elements within the semigroup.

The investigation of the idempotent structure within the semigroup of full contractions $\mathcal{CT}_n$ reveals significant relationships between idempotents and their heights, providing insight into the underlying algebraic framework. The findings emphasize that:

- Idempotents play a crucial role in defining the structure of $\mathcal{CT}_n$ and contribute significantly to the overall behavior of the semigroup.
- The characterization of idempotent-generated elements leads to a clearer understanding for intricate interactions among elements.
- Future research should focus on exploring the ranks of various subsemigroups within $\mathcal{CT}_n$, as well as deepening the understanding of idempotent properties and their practical applications in combinatorial and algebraic contexts.

These insights pave the way for further advancements in understanding the rank of the whole structure $\mathcal{CT}_n$. We wrap up the paper with the following question.
QUESTION: What are the ranks of the semigroups $\mathcal{CT}_n$ and $\langle E(\mathcal{CT}_n)\rangle$?

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