International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 8 Issue 1, 2025

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

Falkner Block Methods for solving second-order ODEs

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Abstract

This article presents a two-step Falkner block method for the numerical solution of second-order initial value problems (IVPs) in ordinary differential equations (ODEs). The scheme is derived using collocation and interpolation techniques, the proposed Falkner block method offers a novel solution to second-order IVPs in ODEs. It eliminates the transformation of the second-order ODEs into a system of first-order equations, approximating the solution through polynomial interpolation at strategically chosen collocation points, which enhances accuracy and computational efficiency. Utilizing appropriate basis functions for interpolation allows the Falkner block method to capture the solution's features, effectively enabling higher-order approximations. This collocation framework ensures the residuals align with the original differential equation, promoting a stable numerical scheme. Additionally, the method facilitates simultaneous calculations of multiple values, improving computational speed and accommodating diverse initial conditions. Comparative studies demonstrate that the Falkner block method outperforms traditional numerical methods and other hybrid codes in global accuracy and convergence.

Keywords and phrases: Block methods; Falkner hybrid block method; Linear multistep method; second-order initial value problems; ordinary differential equations

1 Introduction

A second-order differential equation is an equation that expresses the second derivative of a dependent variable as a function of the variable and its first derivative. It can be written in the form

$$y\prime\prime + p(x)y\prime + q(x)y = f(x), \tag{1}$$

where $p(x)$ and $q(x)$ are functions of x. Second-order ODEs arise frequently in various fields such as physics, engineering, economics, and biological sciences, particularly when modeling systems involving motion, electrical circuits, chemical kinetics, orbital dynamics, control theory, heat flow, or in general, any problem involving second Newton's law, see [?]. Problems associated with differential equations (1) can be categorized into two types based on the conditions specified at the endpoints: initial value problems (IVPs) and boundary value problems (BVPs). These problems were first introduced in 1768 by the British mathematician Leonhard Euler (see [1]). The interest here is the solution of the second-order initial value problem (IVPs) of the form,

$$\begin{cases} y''(x) = f(x, y(x), y'(x)), \\ y(x_0) = y_0, \quad y'(x_0) = y'_0, \\ x \in [x_0, x_n]. \end{cases} \tag{2}$$

whose solutions are often oscillatory. Note that (2) is a subset of (1). In most real-life situations, the ODEs in (1) or (2) are so complicated that it is difficult to determine the exact solution, and one of two

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approaches is taken to approximate the solution. The first approach involves transforming the differential equations into a form that allows for an exact solution, which can then be utilized to approximate the solution of the original problem. One established analytic technique for addressing equation (1) is the Adomian Decomposition Technique (ADT), (see [2], [3], [4], [5], [6], and [7]). The ADT is a recognized systematic method for practically solving ordinary differential equations (ODEs) such as (1). This powerful technique offers efficient algorithms for obtaining analytic approximate solutions and conducting numerical simulations for real-world applications in applied sciences and engineering. It enables the resolution of both nonlinear initial value problems (IVPs) in (1) and boundary value problems (BVPs) without the need for restrictive assumptions, such as linearization, perturbation, arbitrary assumptions, guessing initial terms, or selecting basis functions. Again, see [8].

Another analytic method used to solve equation (1) is the Variational Iteration Method (VIM) in [9]. This method is flexible and adaptable, making it suitable for scenarios where the solution is not known in advance, a common occurrence in applied sciences and engineering. The Neumann analysis concerning high-order derivative terms (including viscous fluxes and third derivatives), which is also applicable to periodic IVPs, gained popularity through the work of ([10], [11], and [12]), nearly two decades ago, focusing on methods such as LDG, LDG2, IP, BR2, and DDG. In [13], the authors explored the application of the Sumudu decomposition approach, which combines the Sumudu transform with Adomian decomposition techniques to find approximate solutions for nonlinear systems of ODEs, such as predator-prey dynamics.

Additional analytic methods for solving equation (1) include perturbation iteration methods [14], [15]; optimal perturbation iteration methods [16], random perturbation methods in [17], homotopy analysis methods in [18], optimal homotopy asymptotic methods, see [19], and variational homotopy perturbation methods in [20].

The second technique is the approximation method, which gives a more perfect result and less relative error. [21] opined that numerical methods are used to solve these mathematical problems where is difficult or nearly impossible to determine the exact solution. Such numerical approaches for solving equation (1) include Runge-Kutta (RK) methods, specifically Runge–Kutta Nystrom (RKN) methods in [22], and linear multistep methods (LMMs), see [23], [24], and [25]) to the predictor-corrector techniques in the articles of: [26], [27], [28], and block methods ([29], [30], [31], [32], [33], and [34].

One of the goals of utilizing numerical formulas is to provide systematic strategies for addressing mathematical problems through numerical solutions. These methods are particularly effective when employed on modern digital computers, which excel at performing arithmetic operations quickly and efficiently. When tackling the problem (1) with high-precision digital computers, the process typically begins with initial data, followed by the applications of suitable algorithms to obtain the desired results. It has been well recognized that numerical techniques for solving ODEs may display period-doubling and chaotic behavior when utilized with time steps exceeding their linearized stability limit. The challenges associated with some of these numerical methods in generating high-quality solutions present significant limitations. Specific complications include slow convergence rates, instability, and reduced accuracy.

The paper is organized as follows: In section 2 we present the structure of the Falkner block method; Section 3 deals with deriving the Falkner hybrid block method with an off-step point of order p=4. In Section 4, we give the main properties of the proposed method. Section 5 is devoted to the numerical experiments of the method. We compared the accuracy of our methods with other methods proposed in the literature. Finally, Section 6 is dedicated to conclusion.

2 Methodology

This section summarizes the derivation of the continuous Falkner block method used to solve the IVPs presented in (2). The method approximates the exact solution $y(x)$ of the second-order IVPs at the grid points $a = x_0 < x_1 < \ldots < x_N = b$, within the integration interval [a, b] using a fixed step size,

$h = x_{n+j} - x_{n+j-1}$ for $j = 0(1)N-1$. This approximation is expressed as a polynomial of the form

$$y(x) = \sum_{j=0}^{N} a_j x^j, \tag{3}$$

Differentiating (3) with respect to x yields

$$y'(x) = \sum_{j=1}^{N} j a_j x^{j-1}, \tag{4}$$

Again, differentiating (4) with respect to x gives,

$$y''(x) = f(x, y, y') = \sum_{j=2}^{N} j(j-1) a_j x^{j-2}, \tag{5}$$

The y'_n is the first derivative component while $\{y''(x_{n+j})\}_{j=1}^{k} = \{f(x_{n+j}, y_{n+j}, y'_{n+j})\}_{j=1}^{k}$ and $y''(x_{n+v}) = f(x_{n+v}, y_{n+v}, y'_{n+v})$ is the second derivative function. The hybrid point is $c_i = v$. Collocating (3), (4), and (5) at $x = x_{n+j}$, j = 0(1) k, and x = x_{n+v}, and interpolating (3) at x = x_n, results in the linear system of equations,

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^N \\ 0 & 1 & 2x_n & 3x_n^2 & \dots & Nx_n^{N-1} \\ 0 & 0 & 2 & 6x_{n+1} & \dots & N(N-1)x_{n+1}^{N-2} \\ & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 2 & 6x_{n+k} & \dots & N(N-1)x_{n+k}^{N-2} \\ 0 & 0 & 2 & 6x_{n+v} & \dots & N(N-1)x_{n+v}^{N-2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ f_{n+1} \\ \vdots \\ f_{n+k} \\ f_{n+v} \end{pmatrix}. \tag{6}$$

We obtained the value of $a'_j s$ by the Gaussian elimination method. Substituting the resulting values of $a'_j s$, $(j = 0(1)N)$ with $x = x_n + c_i h$, $i = 0(1)r$ (without loss of generality we set $x_n = 0$ so that $x = c_i h$) into (3) yields the following continuous formula,

$$\begin{aligned} y(x_n + c_i h) = y_n + hc_i y_n' + h^2 \Bigg[& \frac{c_i^2\left(-2c_i(v+2) + c_i^2 + 12v\right)}{12(v-1)} f_{n+1} \\ - \frac{c_i^2\left(-2c_i(v+1) + c_i^2 + 6v\right)}{12(v-2)} f_{n+2} + & \frac{c_i^2\left(c_i^2 - 6c_i + 12\right)}{12\left(v^2 - 3v + 2\right)} f_{n+v} \Bigg], \quad i = 1(1)3. \end{aligned} \tag{7}$$

Inserting the value of $c = (1, \frac{3}{2}, 2)^T$ into (7) gives,

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{1}{6}h^2\left(12f_{n+1} + 5f_{n+2} - 14f_{n+\frac{3}{2}}\right), \\ y_{n+\frac{3}{2}} &= y_n + \frac{3hy'_n}{2} + \frac{9}{32}h^2\left(13f_{n+1} + 5f_{n+2} - 14f_{n+\frac{3}{2}}\right), \\ y_{n+2} &= y_n + 2hy'_n + \frac{2}{3}h^2\left(8f_{n+1} + 3f_{n+2} - 8f_{n+\frac{3}{2}}\right). \end{aligned} \tag{8}$$

Using the idea above, we obtain the second formula that approximates the solution $y'(x_n + c_i h)$, $i = 1(1)3$ as,

$$\begin{aligned} y'(x_n + c_i h) = y_n' + h \Bigg[& \frac{c_i\left(-3c_i(v+2) + 2c_i^2 + 12v\right)}{6(v-1)} f_{n+1} + \frac{c_i\left(c_i\left(-2c_i + 3v + 3\right) - 6v\right)}{6(v-2)} f_{n+2} \\ & + \frac{c_i\left(2c_i^2 - 9c_i + 12\right)}{6\left(v^2 - 3v + 2\right)} f_{n+v} \Bigg], \quad i = 1(1)3. \end{aligned} \tag{9}$$

Evaluating $y'(x_n + c_i h)$ at the points $c = (1, \frac{3}{2}, 2)^T$ yields,

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{1}{6}h\left(19f_{n+1} + 7f_{n+2} - 20f_{n+\frac{3}{2}}\right), \\
y'_{n+\frac{3}{2}} &= y'_n + \frac{3}{8}h\left(9f_{n+1} + 3f_{n+2} - 8f_{n+\frac{3}{2}}\right), \\
y'_{n+2} &= y'_n + \frac{2}{3}h\left(5f_{n+1} + 2f_{n+2} - 4f_{n+\frac{3}{2}}\right).
\end{aligned} \tag{10}
$$

Combining the methods in (8) and (10) yields the Falkner hybrid schemes. The first formula (8) is for computing the solution while the second formula (10) is for approximating the derivative at x_{n+2}. These methods in (8) and (10) are the hybrid form of the methods in [37].

2.1 The Block Method

The block form of the Falkner hybrid methods in (8) and (10) is,

$$
AY_{n+c} = hBY'_{n+c} + h^2 CF_{n+c}, \tag{11}
$$

where $c = (c_1, c_2, ..., c_{r-1}, c_r)^T$, is the abscissa vector, $A_{7\times7}, B_{4\times4}$, and $C_{4\times4}$ are the coefficients matrices, and the vectors,

$$
\begin{aligned}
Y_{n+c} &= \begin{pmatrix} y_{n+c_1} & \cdots & y_{n+c_{r-1}}, & y_{n+c_r} \end{pmatrix}^T, \\
Y'_{n+c} &= \begin{pmatrix} y'_{n+c_1} & \cdots & y'_{n+c_{r-1}}, & y'_{n+c_r} \end{pmatrix}^T, \\
F_{n+c} &= \begin{pmatrix} f_{n+c_1} & \cdots & f_{n+c_{r-1}}, & f_{n+c_r} \end{pmatrix}^T.
\end{aligned}
$$

For example, the block picture of (8) and (10) is,

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \quad
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{12}{6} & -\frac{14}{6} & \frac{5}{6} \\
0 & \frac{19}{6} & -\frac{20}{6} & \frac{7}{6} \\
0 & \frac{117}{32} & -\frac{63}{16} & \frac{45}{32} \\
0 & \frac{27}{8} & -3 & \frac{9}{8} \\
0 & \frac{16}{3} & -\frac{16}{3} & 2 \\
0 & \frac{10}{3} & -\frac{8}{3} & \frac{4}{3}
\end{pmatrix}; \tag{12}
$$

$$
C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
\frac{3}{2} & 0 & 0 & 0 \\
1 & 0 & -1 & 0 \\
-2 & 0 & 0 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix},
$$

with vectors,

$$
Y_{n+c} = \begin{pmatrix} y_n & y_{n+1} & y'_{n+1} & y_{n+\frac{3}{2}} & y'_{n+\frac{3}{2}} & y_{n+2} & y'_{n+2} \end{pmatrix}^T;
$$

$$
Y'_{n+c} = \begin{pmatrix} y'_n & y'_{n+1} & y'_{n+\frac{3}{2}} & y'_{n+2} \end{pmatrix}^T; \quad F_{n+c} = \begin{pmatrix} f_n & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} \end{pmatrix}^T.
$$

] Suppose $z(x)$ is a sufficiently differentiable function, then consider the linear difference operator $L[y(x), h]$ associated with the implicit block hybrid method in (11), given by

$$
L[y(x), h] = \sum_j \alpha_j z(x_n + jh) - h\beta_j z'(x_n + jh) - h^2\gamma_j z''(x_n + jh), \quad j = 0(1)2,
$$

where the α_j, β_j, and γ_j are respectively the vector columns of the matrices A, B, and C. Thus, the block hybrid method in (11) with the coefficient matrices in (12) for solving the IVPs in (2) and the associated linear difference operator is said to be of order p if after expanding $z(x_n + jh)$, $z'(x_n + jh)$, $z''(x_n + jh)$ by Taylor series about x_n we obtain

$$L[y(x), h] = \bar{C}_0 y(x_n) + h\bar{C}_1 y'(x_n) + h^2\bar{C}_2 y''(x_n) + \cdots + h^2\bar{C}_q y^{(q)}(x_n) + \ldots$$

with $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_{p+1} = 0$, and $\bar{C}_{p+2} \neq 0$. The $\bar{C}_i$ are vectors, and the $\bar{C}_{p+2}$ is called the error constant. For the above method we have that $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_4 = 0$, and

$$\bar{C}_5 = \left(\frac{-89}{720}, \frac{-1}{6}, \frac{-33}{160}, \frac{-21}{128}, \frac{-13}{45}, \frac{-1}{6}\right)^T$$

revealing that the proposed method has at least order $p = 3$, and this is a sufficient condition for the associated block formula to be consistent i.e., $p \geq 1$, see [35]. Note that zero-stability focuses on the stability of the difference system (11) as the step size h approaches zero. Therefore, when $h \to 0$, the method in (11) can be reformulated in a more suitable vector form for analyzing zero-stability as $A^{(0)}\bar{Y}_n - A^{(1)}\bar{Y}_{n-2} = 0$, where $\bar{Y}_n = (y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2})^T$ and $\bar{Y}_{n-2} = (y_{n-1}, y_{n-\frac{1}{2}}, y_n)^T$. Here, $A^{(0)}$ and $A^{(1)}$ are constant matrices,

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The block method is considered zero-stable if the roots w_j of the first characteristic polynomial $\rho(w) = \det[A^{(0)}w - A^{(1)}]$ satisfy $|w_j| \leq 1$. Furthermore, for any roots where $|w_j| = 1$, their multiplicity must not exceed 2 (see [27], and [35]). Given that $\rho(w) = (w-1)w^3 = 0$, we can conclude that the block method in (11) is zero-stable. Consequently, this implies that the block method (11) is convergent.

3 Analysis

As previously noted, zero-stability pertains to the behavior of a numerical method as h approaches zero. To assess whether a numerical method yields satisfactory results for a given positive value of h, we need a different concept of stability that goes beyond zero stability. In many numerical approaches designed for solving second-order problems, stability is typically examined using the linear test equation introduced by [36]:

$$y''(x) = -\mu^2 y(x), \quad \mu > 0. \tag{13}$$

While zero stability is solely dependent on the method itself, linear stability, in general for finite h, is influenced by the problem being addressed as well. Given that our method applies to general second-order equations and the Lambert-Watson test does not involve the first derivative, we will instead consider the following linear problem to evaluate linear stability:

$$y''(x) = -2\mu^2 y'(x) - \mu^2 y(x). \tag{14}$$

We choose this particular problem because it yields bounded solutions for $\mu \geq 0$ that approach zero as xxx tends to infinity. Our goal is to identify the region in which the numerical method replicates the behavior of the actual solutions.

Now, let us detail the procedure to determine this region. Our method involves six equations, incorporating five distinct derivative terms: y_n', y_{n+1}', $y_{n+\frac{3}{2}}'$, y_{n+2}', and an intermediate value $y_{n+\frac{3}{2}}$. By employing the Mathematica program, we have successfully eliminated these terms from the system of equations, resulting in a polynomial of the form:

$$\pi(w,z) = 6 + w(180 - 108z + z^3) + 4w^c(-96 + 9z^2 + z^3) + w^2(198 + 108z + 18z^2 + z^3),$$

where $c = v = \frac{3}{2}$, $z = \mu h$. We analyze the boundedness of their solutions to the characteristic equation to identify the stability region. For the method to maintain stability, it is essential that the roots of the characteristic equation lie within the unit circle, meaning they must be less than 1 in magnitude. The root(s) of the characteristic equation are thus critical to this assessment. Using the boundary locus technique, we obtain the stability region of the formula in (11) for $k = 2$, see Figure 1: The stability region, however, depends on the nature of μ. If μ belongs to the complex domain ($\mu \in C$),

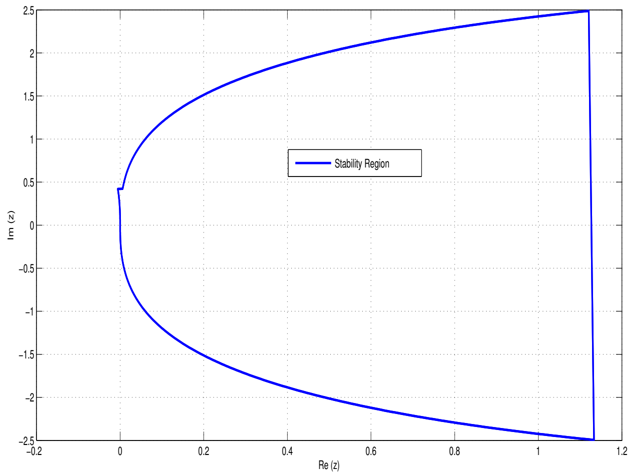

Figure 1: The stability region (is the interior part of the curve) of the method in (11).

the stability region forms within the complex μh-plane. On the other hand, if μ is a real value ($\mu \in R$), the corresponding stability region becomes a real interval known as the interval of stability. The interval of stability for the method under consideration is shown in Fig. 1, and it is found to be $(0, 1.121)$. This region represents the set of values in the complex z-plane where the roots of $\pi(w, z)$ are bounded by unity in modulus. In other words, it is the collection of z values in the complex plane for which the solution of the stability polynomial decays in response to the test equation (14).

4 Numerical simulations

Here, we report the obtained numerical solutions for the problem (1)–(3) utilizing the proposed implicit method in (11), and to implement this method, we resolve the arising implicitness in the algorithm using the Newton-Raphson iterative scheme. We evaluate the accuracy of the method mentioned above using the maximum absolute errors of the approximate solution at the grid points within the integration interval.

Example 1

Consider the well-established problem, which has been discussed frequently in the literature, including references, [27] and [34].

$$y'' = x(y')^2, \quad y(0) = 1, \quad y'(0) = \frac{1}{2},$$

whose exact solution is $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$. The results, presented in Table 1, demonstrate that our method yields impressive results despite having a lower order. The accuracy of the methods was measured using maximum absolute error, $|y(x_n) - y_n|$, see Table 1.

Table 1: The errors for Example 1 with step-size h = 0.003125

x	Exact Solution $y(x_n)$	Numerical Solution y_n	Error in the new method $\lvert y(x_n) - y_n \rvert$
0.1	1.053175062421496	1.053175062422460	9.6322E-13
0.2	1.103492920586838	1.103492920588777	1.9386E-12
0.3	1.154338911580413	1.154338911583481	3.0675E-12
0.4	1.205989894564679	1.205989894569136	4.4571E-12
0.5	1.258748937687529	1.258748937693783	6.2536E-12
0.6	1.312957225169341	1.312957225178018	8.6763E-12
0.7	1.369009467368560	1.369009467380633	1.2072E-12
0.8	1.427374729847812	1.427374729864824	1.7012E-11
0.9	1.488625715831921	1.488625715856404	2.4482E-11
1.0	1.553481533942765	1.553481533979034	3.6268E-11

Example 2

We also consider the following IVP,

$$y'' = -\frac{6}{x}y' - \frac{4}{x^2}y, \quad y(0) = 1, \quad y'(0) = 1,$$

whose exact solution is $y(x) = \frac{5x^3-2}{3x^4}$. The results, presented in Table 2, demonstrate the accuracy of our method. The accuracy of the methods is measured using maximum absolute error, $|y(x_n) - y_n|$, see Table 2.

Table 2: The errors for Example 2 with step-size h = 0.003125

x	Exact Solution $y(x_n)$	Numerical Solution y_n	Error in the new method $\lvert y(x_n) - y_n \rvert$
1.003125	1.008944995088838	1.008951677188663	6.6820E-06
1.006250	1.011741018167988	1.011749605027926	8.5868E-06
1.009375	1.014447542686414	1.014457987637015	1.0444E-05
1.012500	1.017066494235672	1.017078751814261	1.2257E-05
1.015625	1.019599754756288	1.019613780672991	1.4025E-05
1.018750	1.022049163629432	1.022064914734252	1.5751E-05
1.021875	1.024416518738403	1.024433952989302	1.7434E-05
1.025000	1.026703577500806	1.026722653932765	1.9076E-05
1.028125	1.028912057872337	1.028932736567357	2.0678E-05
1.031250	1.031043639323018	1.031065881381039	2.2242E-05

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