International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 7 Issue 2, 2024

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

A fractional-order transmission dynamics for Diphtheria and Pertussis co-infection

J.O. Omolehin[*†], **O.S. Johnson**[†‡], and **I.O. Babarinsa**[†§]

Abstract

Resurgence of vaccine preventable infectious illnesses like diphtheria and pertussis place a heavy burden on global health especially in developing countries. In this article, a modified SIR-type fractional-order compartmental model was used to study the transmission dynamics of pertussis and diphtheria co-infection. Fractional-order models through fractional calculus provides heuristic approach to improving flexibility, convergence, and accuracy in long-term disease predictions. In Particular, the Laplace-Adomian decomposition method (LADM) is applied to solve the complexity of diphtheria-pertussis co-dynamics model by adding fractional derivatives to account for non-integer order dynamics. The approximate solutions of the model is obtained in the form of an infinite series which shows convergence to equilibrium point. The efficiency and reliability of (LADM) is demonstrated in different time intervals by the numerical simulations.

Keywords and phrases: Co-infection; fractional-order; Laplace-Adomian decomposition method; Caputo fractional derivative.

1 Introduction

The rise in vaccine-preventable respiratory infections is a global health concern, worsened by disruptions to routine immunization during the COVID-19 pandemic [1–4]. In 2021, the WHO and UNICEF report [5] found that about 25 million children missed their DTP vaccines. Respiratory diseases remain a significant threat to children's health. It is crucial to understand how different pathogens interact within a host, as co-infections are common and can influence disease severity and outcomes. [7–10].

Diphtheria and pertussis are bacterial respiratory diseases that can be prevented with the DTaP vaccine [11–14]. Diphtheria, caused by Corynebacterium species, can cause serious complications such as paralysis or death and spreads through droplets or close contact with an infected person [15]. In addition, Whooping cough (pertussis) is extremely contagious and spreads easily through the air. It poses a serious health risk, particularly in infants, and can be difficult to diagnose [17–19]. Studies show that asymptomatic adolescents and adults are the main carriers, spreading the disease to infants.

Recently, mathematicians have focused on studying the interaction between these pathogens using a fractional-order compartmental approach to account for non-integer dynamics, convergence, and long-term prediction accuracy [20, 21]. For example, [20] applied Laplace Adomian Decomposition Method (LADM) to study COVID-19 and Influenza dynamics, while [21] used it for COVID-19 and Monkeypox. [22] used fractional calculus in an SEIR model without vaccination, showing the importance of timely vaccination. [23] developed a fractional order model for malaria and COVID-19 co-infection, suggesting that preventive measures could reduce the risk of co-infection and potentially eliminate both diseases.

[*]Corresponding Author; E-mail: oladipuposamueljohnson@gmail.com
[†]Department of Mathematics, Federal University Lokoja, Kogi, Nigeria
[‡]E-mail: oladipuposamueljohnson@gmail.com
[§]E-mail: olayiwola.babarinsa@fulokoja.edu.ng

2 Model formulation

In this article, the Caputo derivative is used as a differential operator. This section outlines the key terms and concepts used in this study.

2.1 Preliminary Definitions

Definition [24] Let $f(t)$ be a function defined on the interval $0 < t < \infty$, the following integral converges in a certain region of s, where $F(s)$ is the Laplace transform of $f(t)$, $\mathscr{L}$ is Laplace operator and s is a complex parameter.

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

Proposition [25,26]. Let $0 < q < 1$, then the Laplace transformation of the Caputo derivative is given by

$$\mathscr{L}\{{}^C\mathcal{D}^q f(t)\} = \mathscr{L}[f(t)] - s^{q-1}f(0), \tag{2}$$

Definition [27] Let $q \in [0, 1]$ The Caputo fractional time derivative of a function f of order q is defined by:

$$\mathscr{L}\{{}^C\mathcal{D}_0^q f(t)\} = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}f^{(n)}(s)ds \tag{3}$$

Where q represents the integer part of q. Therefore, when $n = 1$ the Caputo derivative becomes:

$$\mathscr{L}\{{}^C\mathcal{D}_0^q f(t)\} = \frac{1}{\Gamma(1-q)}\int_0^t \frac{f(s)}{(t-s)^q}ds \tag{4}$$

The diphtheria-pertussis dual infection model with total population $N(t)$ consist of eight sub-population given by

$$N(t) = S(t) + I_{AD}(t) + I_{SD}(t) + I_{AP}(t) + I_{SP}(t) + I_{SDP}(t) + Q_{SD}(t) + R(t) \tag{5}$$

The following Table 1 gives a summary Definition of model parameters and variables:

Table 1: Values of parameters used in simulation.

Variables	Description	Unit
$S(t)$	susceptible	People
$I_{AD}(t)$	asymptomatic diphtheria infected	People
$I_{SD}(t)$	symptomatic diphtheria infected	People
$I_{AP}(t)$	asymptomatic pertussis infected	People
$I_{SP}(t)$	symptomatic pertussis infected	People
$I_{SDP}(t)$	diphtheria-pertussis co-infected	People
$Q_{SD}(t)$	quarantined diphtheria infected	People
$R(t)$	Recovered	People

This article models a population where the susceptible group (S) grows due to two factors: the addition of unvaccinated individuals $bN(1-V)$ and the movement of vaccinated newborns directly into the recovered group VbN.

Furthermore, S get infected with diphtheria only with infection force $\Phi_D = \frac{\beta_D(\alpha_D I_{AD}+I_{SD})}{N}$, Pertussis only with infection force $\Phi_P = \frac{\beta_P(\alpha_P I_{AP}+I_{SP})}{N}$, and diphtheria- Pertussis dual infection force $\Phi_{DP} = \frac{\beta_{DP}I_{SDP}}{N}$ respectively. Individuals exhibiting symptoms of diphtheria or pertussis are

Table 2: Values of parameters used in simulation.

Parameters	Description	Values	Cite
β_D	Contact rate for Diphtheria	0.57/day	[30]
β_P	Contact rate for Pertussis	0.5-1/day	[28]
β_{DP}	Diphtheria and Pertussis dual transmission	0.2/day	fitted
Ω	Recovery rate of Quarantine	0.1-0.9	Assumed
η_D	Diphtheria Recovery rate.	0.5	[28]
η_P	Pertussis Recovery rate	$[\frac{1}{14}, 0.0206$	[28]
η_{DP}	Dual infection Recovery rate	0.15/day	[29]
ϖ_D	Disease induced death rate of Diphtheria	0.05	[30]
ϖ_P	Disease induced death rate of Pertussis	0.0309	[28]
ϖ_{DP}	Disease induced death rate of Dual infection	0.005/day	[30]
μ	Natural death rate	0.006	[30]
θ	Diphtheria Quarantined rate	0.1/day	Assumed
V	Vaccination	0.1-1.2	Assumed
α_D, α_P	Modification parameter	0.5	[28]
γ_1, γ_2	Modification parameter	1	[29]
b	Birth rate	0.019	[30]
ϕ_D, ϕ_P	Progression to symptomatic infected stage	$[\frac{3}{7}]$/day , $[\frac{7}{7}]$/day	Fitted

susceptible to secondary infections at rates $\gamma_1 \frac{\beta_P(\alpha_P I_{AP}+I_{SP})}{N} I_{SD}$ and $\gamma_2 \frac{\beta_D(\alpha_D I_{AD}+I_{SD})}{N} I_{SP}$.The model parameters are listed in Table 2, with the governing equations in equation (6).

$$
\begin{aligned}
\frac{dS}{dt}(t) &= bN(1-V) - (\Phi_D + \Phi_P + \Phi_{DP})\, S - \mu S,\\
\frac{dI_{AD}}{dt}(t) &= \Phi_D S - (\phi_D + \mu) I_{AD},\\
\frac{dI_{SD}}{dt}(t) &= \phi_D I_{AD} - (\theta + \eta_D + \varpi_D + \mu) I_{SD} - \gamma_1 \Phi_P I_{SD},\\
\frac{dI_{AP}}{dt}(t) &= \Phi_P S - (\phi_P + \mu) I_{AP},\\
\frac{dI_{SP}}{dt}(t) &= \phi_P I_{AP} - (\eta_P + \varpi_P + \mu) I_{SP} - \gamma_2 \Phi_D I_{SP},\\
\frac{dI_{SDP}}{dt}(t) &= \phi_{DP} S + \gamma_1 \Phi_P I_{SD} + \gamma_2 \Phi_D I_{SP} - (\eta_{DP} + \varpi_{DP} + \mu) I_{SDP},\\
\frac{dQ_{SD}}{dt}(t) &= \theta I_{SD} - (\Omega + \mu) Q_{SD},\\
\frac{dR}{dt}(t) &= VbN + \eta_P I_{AP} + \eta_D I_{AD} + \eta_{SDP} I_{SDP} - \mu R.
\end{aligned}
\tag{6}
$$

$$S(0) > 0,\ I_{AD}(0) > 0,\ I_{SD}(0) > 0,\ I_{AP}(0) > 0,\ I_{SP}(0) > 0,\ I_{SDP}(0) > 0,\ Q_{SD}(0) > 0$$

3 Fractional-order of Diphtheria-Pertussis co-infection model

This section incorporates the Caputo fractional derivative into our diphtheria-pertussis co-infection model. Fractional calculus is applied because it effectively models systems and processes exhibiting memory and hereditary characteristics [22]. Therefore, the fractional-order model is constructed as follows:

$$\begin{aligned}
{}^{C}\mathcal{D}^{q}S(t) &= bN(1-V) - (\Phi_D + \Phi_P + \Phi_{DP})\,S - \mu S,\\
{}^{C}\mathcal{D}^{q}I_{AD}(t) &= \Phi_D S - (\phi_D + \mu)I_{AD},\\
{}^{C}\mathcal{D}^{q}I_{SD}(t) &= \phi_D I_{AD} - (\theta + \eta_D + \varpi_D + \mu)I_{SD} - \gamma_1\Phi_P I_{SD},\\
{}^{C}\mathcal{D}^{q}I_{AP}(t) &= \Phi_P S - (\phi_P + \mu)I_{AP},\\
{}^{C}\mathcal{D}^{q}I_{SP}(t) &= \phi_P I_{AP} - (\eta_P + \varpi_P + \mu)I_{SP} - \gamma_2\Phi_D I_{SP},\\
{}^{C}\mathcal{D}^{q}I_{SDP}(t) &= \Phi_{DP}S + \gamma_1\Phi_P I_{SD} + \gamma_2\Phi_D I_{SP} - (\eta_{DP} + \varpi_{DP} + \mu)I_{SDP},\\
{}^{C}\mathcal{D}^{q}Q_{SD}(t) &= \theta I_{SD} - (\Omega + \mu)Q_{SD},\\
{}^{C}\mathcal{D}^{q}R(t) &= VbN + \eta_P I_{SP} + \eta_D I_{SD} + \eta_{SDP}I_{SDP} + \Omega Q_{SD} - \mu R.
\end{aligned} \tag{7}$$

With the following initial conditions;

$$S(0) > 0,\ I_{AD}(0) > 0,\ I_{SD}(0) > 0,\ I_{AP}(0) > 0,\ I_{SP}(0) > 0,\ I_{SDP}(0) > 0,\ Q_{SD}(0) > 0,\ R_(0) = 0.$$

3.1 Applying Laplace-Adomian Decomposition method solution

Here, we will apply Laplace transformation to convert the governing system of differential equations (6) into a system of algebraic equations. Furthermore, apply the Laplace Adomian Decomposition Method (LADM) to solve equation (7), and obtain the solution in terms of series;

$$\begin{aligned}
\mathscr{L}\{{}^{C}\mathcal{D}^{q}S(t)\} &= \mathscr{L}\{bN(1-V) - (\Phi_D + \Phi_P + \Phi_{DP})\,S - \mu S\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}I_{AD}(t)\} &= \mathscr{L}\{\Phi_D S - (\phi_D + \mu)I_{AD}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}I_{SD}(t)\} &= \mathscr{L}\{\phi_D I_{AD} - (\theta + \eta_D + \varpi_D + \mu)I_{SD} - \gamma_1\Phi_P I_{SD}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}I_{AP}(t)\} &= \mathscr{L}\{\Phi_P S - (\phi_P + \mu)I_{AP}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}I_{SP}(t)\} &= \mathscr{L}\{\phi_P I_{AP} - (\eta_P + \varpi_P + \mu)I_{SP} - \gamma_2\Phi_D I_{SP}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}I_{SDP}(t)\} &= \mathscr{L}\{\Phi_{DP}S + \gamma_1\Phi_P I_{SD} + \gamma_2\Phi_D I_{SP} - (\eta_{DP} + \varpi_{DP} + \mu)I_{SDP}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}Q_{SD}(t)\} &= \mathscr{L}\{\theta I_{SD} - (\Omega + \mu)Q_{SD}\},\\
\mathscr{L}\{{}^{C}\mathcal{D}^{q}R(t)\} &= \mathscr{L}\{VbN + \eta_P I_{SP} + \eta_D I_{SD} + \eta_{SDP}I_{SDP} + \Omega Q_{SD} - \mu R\}.
\end{aligned} \tag{8}$$

which is equivalent to;

$$\begin{aligned}
s^q\mathscr{L}\{S(t)\} - s^{q-1}S(0) &= \mathscr{L}\{bN(1-V) - (\Phi_D + \Phi_P + \Phi_{DP})\,S - \mu S\},\\
s^q\mathscr{L}\{I_{AD}(t)\} - s^{q-1}I_{AD}(0) &= \mathscr{L}\{\Phi_D S - (\phi_D + \mu)I_{AD}\},\\
s^q\mathscr{L}\{I_{SD}(t)\} - s^{q-1}I_{SD}(0) &= \mathscr{L}\{\phi_D I_{AD} - (\theta + \eta_D + \varpi_D + \mu)I_{SD} - \gamma_1\Phi_P I_{SD}\},\\
\mathscr{L}\{I_{AP}(t)\} - s^{q-1}I_{AP}(0) &= \mathscr{L}\{\Phi_P S - (\phi_P + \mu)I_{AP}\},\\
s^q\mathscr{L}\{I_{SP}(t)\} - s^{q-1}I_{SP}(0) &= \mathscr{L}\{\phi_P I_{AP} - (\eta_P + \varpi_P + \mu)I_{SP} - \gamma_2\Phi_D I_{SP}\},\\
s^q\mathscr{L}\{I_{SDP}(t)\} - s^{q-1}I_{SDP}(0) &= \mathscr{L}\{\Phi_{DP}S + \gamma_1\Phi_P I_{SD} + \gamma_2\Phi_D I_{SP} - (\eta_{DP} + \varpi_{DP} + \mu)I_{SDP}\},\\
s^q\mathscr{L}\{Q_{SD}(t)\} - s^{q-1}Q_{SD}(0) &= \mathscr{L}\{\theta I_{SD} - (\Omega + \mu)Q_{SD}\},\\
s^q\mathscr{L}\{R(t)\} - s^{q-1}R(0) &= \mathscr{L}\{VbN + \eta_P I_{SP} + \eta_D I_{SD} + \eta_{SDP}I_{SDP} + \Omega Q_{SD} - \mu R\}.
\end{aligned} \tag{9}$$

Substituting the initial conditions and Dividing equation (9) by (s^q) we have

$$
\begin{aligned}
\mathscr{L}\{S(t)\} &= \frac{S_0}{s} + \frac{\kappa}{s^{q+1}} - [\frac{1}{s^q}\mathscr{L}\{(\Phi_D + \Phi_P + \Phi_{DP})\, S - \mu S\}],\\
\mathscr{L}\{I_{AD}(t)\} &= \frac{I_{AD_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\Phi_D S - (\phi_D + \mu) I_{AD}\}],\\
\mathscr{L}\{I_{SD}(t)\} &= \frac{I_{SD_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\phi_D I_{AD} - (\theta + \eta_D + \varpi_D + \mu) I_{SD} - \gamma_1 \Phi_P I_{SD}\}],\\
\mathscr{L}\{I_{AP}(t)\} &= \frac{I_{AP_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\Phi_P S - (\phi_P + \mu) I_{AP}\}],\\
\mathscr{L}\{I_{SP}(t)\} &= \frac{I_{SP_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\phi_P I_{AP} - (\eta_P + \varpi_P + \mu) I_{SP} - \gamma_2 \Phi_D I_{SP}\}],\\
\mathscr{L}\{I_{SDP}(t)\} &= \frac{I_{SDP_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\Phi_{DP} S + \gamma_1 \Phi_P I_{SD} + \gamma_2 \Phi_D I_{SP} - (\eta_{DP} + \varpi_{DP} + \mu) I_{SDP}\}],\\
\mathscr{L}\{Q_{SD}(t)\} &= \frac{Q_{SD_0}}{s} + [\frac{1}{s^q}\mathscr{L}\{\theta I_{SD} - (\Omega + \mu) Q_{SD}\}],\\
\mathscr{L}\{R(t)\} &= \frac{R_0}{s} + [\frac{1}{s^q}\mathscr{L}\{VbN + \eta_P I_{SP} + \eta_D I_{SD} + \eta_{SDP} I_{SDP} + \Omega Q_{SD} - \mu R\}].
\end{aligned} \tag{10}
$$

Expressing the obtained solutions as an infinite series,

$S(t) = \sum_{n=0}^{\infty} S_n(t)$, $I_{AD}(t) = \sum_{n=0}^{\infty} I_{AD_n}(t)$, $I_{SD}(t) = \sum_{n=0}^{\infty} I_{SD_n}(t)$ $I_{AP}(t) = \sum_{n=0}^{\infty} I_{AP_n}(t)$, $I_{SP}(t) = \sum_{n=0}^{\infty} I_{SP_n}(t)$, $I_{SDP}(t) = \sum_{n=0}^{\infty} I_{SDP_n}(t)$, $Q_{SD}(t) = \sum_{n=0}^{\infty} Q_{SD_n}(t))$, $R(t) = \sum_{n=0}^{\infty} I_{R_n}(t)$.

$S(t)I_{AD}(t)$, $S(t)I_{DP}(t)$, $S(t)I_{AP}(t)$, $S(t)I_{SD}(t)$, and $S(t)I_{SDP}(t)$, are the non-linear term in the model and it can be decomposed by Adomian polynomial.

$I_{AD}(t)S(t) = \sum_{n=0}^{\infty} A_n$, $I_{SD}(t)S(t) = \sum_{n=0}^{\infty} B_n$, $I_{AP}(t)S(t) = \sum_{n=0}^{\infty} C_n$, $I_{SP}(t)S(t) = \sum_{n=0}^{\infty} D_n$, $I_{SDP}(t)S(t) = \sum_{n=0}^{\infty} E_n$.

where A_n, B_n, C_n, D_n and E_n are Adomian polynomials [21].

$$
\begin{aligned}
A_n &= \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{\infty} \lambda^k I_{AD}(k) \sum_{k=0}^{\infty} \lambda^k S(k)\Big]_{\lambda=0}\\
B_n &= \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{\infty} \lambda^k I_{SD}(k) \sum_{k=0}^{\infty} \lambda^k S(k)\Big]_{\lambda=0}\\
C_n &= \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{\infty} \lambda^k I_{AP}(k) \sum_{k=0}^{\infty} \lambda^k S(k)\Big]_{\lambda=0}\\
D_n &= \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{\infty} \lambda^k I_{SP}(k) \sum_{k=0}^{\infty} \lambda^k S(k)\Big]_{\lambda=0}\\
E_n &= \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{\infty} \lambda^k I_{SDP}(k) \sum_{k=0}^{\infty} \lambda^k S(k)\Big]_{\lambda=0}.
\end{aligned} \tag{11}
$$

The first three terms of polynomials $A(n), B(n), C(n), D(n)$ and $E(n)$ calculated as

$$
A(n) = \left\{
\begin{array}{l}
A_0 = S_0 I_{AD_0},\\
A_1 = S_1 I_{AD_0} + S_0 I_{AD_1},\\
A_2 = S_2 I_{AD_0} + S_1 I_{AD_1} + S_0 I_{AD_2}\\
\vdots
\end{array}\right.
\qquad
B(n) = \left\{
\begin{array}{l}
B_0 = S_0 I_{SD_0},\\
B_1 = S_1 I_{SD_0} + S_0 I_{SD_1},\\
B_2 = S_2 I_{SD_0} + S_1 I_{SD_1} + S_0 I_{SD_2}\\
\vdots
\end{array}\right.
$$

$$C(n) = \begin{cases} C_0 = S_0 I_{AP_0}, \\ C_1 = S_1 I_{AP_0} + S_0 I_{AP_1}, \\ C_2 = S_2 I_{AP_0} + S_1 I_{AP_1} + S_0 I_{AP_2} \\ \vdots \end{cases} \quad D(n) = \begin{cases} D_0 = S_0 I_{SP_0}, \\ D_1 = S_1 I_{SP_0} + S_0 I_{SP_1}, \\ D_2 = S_2 I_{SP_0} + S_1 I_{SP_1} + S_0 I_{SP_2} \\ \vdots \end{cases}$$

$$E(n) = \begin{cases} E_0 = S_0 I_{SDP_0}, \\ E_1 = S_1 I_{SDP_0} + S_0 I_{SDP_1}, \\ E_2 = S_2 I_{SDP_0} + S_1 I_{SDP_1} + S_0 I_{SDP_2} \\ \vdots \end{cases}$$

Substituting the infinite series into equation (9), by matching the terms on both sides we obtain the following iterative algorithm

$$
\begin{aligned}
\mathscr{L}\{\sum_{n=0}^{\infty} S_{n+1}\} &= \frac{S_n}{s} + \frac{1}{s^q}\mathscr{L}\Big[bN(1-V) - \Big(\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \\
&\quad + \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} + \frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N}\Big) - \mu \sum_{n=0}^{\infty} S_n\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{AD_{n+1}}\} &= \frac{I_{AD_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} - (\phi_D + \mu)\sum_{n=0}^{\infty} I_{AD_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SD_{n+1}}\} &= \frac{I_{SD_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\phi_D \sum_{n=0}^{\infty} I_{AD_n} - (\theta + \eta_D + \varpi_D + \mu)\sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{AP_{n+1}}\} &= \frac{I_{AP_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} - (\phi_P + \mu)\sum_{n=0}^{\infty} I_{AP_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SP_{n+1}}\} &= \frac{I_{SP_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\phi_P \sum_{n=0}^{\infty} I_{AP_n} - (\eta_P + \varpi_P + \mu)\sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SDP_{n+1}}\} &= \frac{I_{SDP_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N} + \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad + \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_n} - (\eta_{DP} + \varpi_{DP} + \mu)\sum_{n=0}^{\infty} I_{SDP_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} Q_{SD_{n+1}}(t)\} &= \frac{Q_{SD_n}}{s} + \frac{1}{s^q}\mathscr{L}\Big[\theta \sum_{n=0}^{\infty} I_{SD_n} - (\Omega + \mu)\sum_{n=0}^{\infty} Q_{SD_n}\Big], \\
\mathscr{L}\{\sum_{n=0}^{\infty} R_{n+1}\} &= \frac{R_0}{s} + \frac{1}{s^q}\mathscr{L}\Big[VbN + \eta_P \sum_{n=0}^{\infty} I_{AP_n} + \eta_D \sum_{n=0}^{\infty} I_{AD_n} + \eta_{SDP} \sum_{n=0}^{\infty} I_{SP_n} - \mu \sum_{n=0}^{\infty} R_n\Big].
\end{aligned} \tag{12}
$$

Evaluating the Laplace transform of the second equation at the RHS of equation (12), [21] we get

$$\begin{aligned}
\mathscr{L}\{\sum_{n=0}^{\infty} S_{n+1}\} &= \frac{S_n}{s} + \Big[bN(1-V) - \Big(\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \\
&\quad + \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} + \frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N}\Big) - \mu \sum_{n=0}^{\infty} S_n\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{AD_{n+1}}\} &= \frac{I_{AD_n}}{s} + \Big[\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} - (\phi_D + \mu) \sum_{n=0}^{\infty} I_{AD_n}\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SD_{n+1}}\} &= \frac{I_{SD_n}}{s} + \Big[\phi_D \sum_{n=0}^{\infty} I_{AD_n} - (\theta + \eta_D + \varpi_D + \mu) \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_{n+1}}\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{AP_{n+1}}\} &= \frac{I_{AP_n}}{s} + \Big[\frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} - (\phi_P + \mu) \sum_{n=0}^{\infty} I_{AP_n}(t)\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SP_{n+1}}\} &= \frac{I_{SP_n}}{s} + \Big[\phi_P \sum_{n=0}^{\infty} I_{AP_n} - (\eta_P + \varpi_P + \mu) \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_{n+1}}\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} I_{SDP_{n+1}}\} &= \frac{I_{SDP_n}}{s} + \Big[\frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N} + \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad + \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_n} - (\eta_{DP} + \varpi_{DP} + \mu) \sum_{n=0}^{\infty} I_{SDP_n}\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} Q_{SD_{n+1}}\} &= \frac{Q_{SD_n}}{s} + \Big[\theta \sum_{n=0}^{\infty} I_{SD_n} - (\Omega + \mu) \sum_{n=0}^{\infty} Q_{SD_n}\Big] \frac{1}{s^{q+1}}, \\
\mathscr{L}\{\sum_{n=0}^{\infty} R_{n+1}\} &= \frac{R_n}{s} + \Big[VbN + \eta_P \sum_{n=0}^{\infty} I_{SP_n} + \eta_D \sum_{n=0}^{\infty} I_{SD_n} + \eta_{SDP} \sum_{n=0}^{\infty} I_{SP_n} - \mu \sum_{n=0}^{\infty} R_n\Big] \frac{1}{s^{q+1}}.
\end{aligned} \tag{13}$$

Taking the inverse Laplace transform of equation (13)

$$
\begin{aligned}
\sum_{n=0}^{\infty} S_{n+1} &= S_n + \Big[bN(1-V) - \Big(\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \\
&\quad + \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} + \frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N}\Big) - \mu \sum_{n=0}^{\infty} S_n\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} I_{AD_{n+1}} &= I_{AD_n} + \Big[\frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} - (\phi_D + \mu) \sum_{n=0}^{\infty} I_{AD_n}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} I_{SD_{n+1}} &= I_{SD_n} + \Big[\phi_D \sum_{n=0}^{\infty} I_{AD_n} - (\theta + \eta_D + \varpi_D + \mu) \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_1}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} I_{AP_{n+1}} &= I_{AP_n} + \Big[\frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} - (\phi_P + \mu) \sum_{n=0}^{\infty} I_{AP_n}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} I_{SP_{n+1}} &= I_{SP_n} + \Big[\phi_P \sum_{n=0}^{\infty} I_{AP_n} - (\eta_P + \varpi_P + \mu) \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad - \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_1}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} I_{SDP_{n+1}} &= I_{SDP_n} + \Big[\frac{\beta_{DP} \sum\limits_{n=0}^{\infty} E(n)}{N} + \gamma_1 \frac{\beta_P(\alpha_P \sum\limits_{n=0}^{\infty} C(n) + \sum\limits_{n=0}^{\infty} D(n))}{N} \sum_{n=0}^{\infty} I_{SD_n} \\
&\quad + \gamma_2 \frac{\beta_D(\alpha_D \sum\limits_{n=0}^{\infty} A(n) + \sum\limits_{n=0}^{\infty} B(n))}{N} \sum_{n=0}^{\infty} I_{SP_n} - (\eta_{DP} + \varpi_{DP} + \mu) \sum_{n=0}^{\infty} I_{SDP_n}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} Q_{SD_{n+1}} &= Q_{SD_n} + \Big[\theta \sum_{n=0}^{\infty} I_{SD_n} - (\Omega + \mu) \sum_{n=0}^{\infty} Q_{SD_n}\Big] \frac{t^q}{\Gamma(q+1)}, \\
\sum_{n=0}^{\infty} R_{n+1} &= R_n + \Big[VbN + \eta_P \sum_{n=0}^{\infty} I_{AP_n} + \eta_D \sum_{n=0}^{\infty} I_{AD_n} + \eta_{SDP} \sum_{n=0}^{\infty} I_{SP_n} - \mu \sum_{n=0}^{\infty} R_n\Big] \frac{t^q}{\Gamma(q+1)}.
\end{aligned} \tag{14}
$$

for $n = 0$

$$
\begin{aligned}
S_1 &= S_0 + \Big[bN(1-V) - \Big(\frac{\beta_D(\alpha_D A_0 + B_0)}{N} + \frac{\beta_P(\alpha_P C_0 + D_0)}{N} + \frac{\beta_{DP}E_0}{N}\Big) \\
&\quad - \mu S_0\Big]\frac{t^q}{\Gamma(q+1)}, \\
I_{AD_1} &= I_{AD_0} + \Big[\frac{\beta_D(\alpha_D A_0 + B_0)}{N} - (\phi_D + \mu)I_{AD_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
I_{SD_1} &= I_{SD_0} + \Big[\phi_D I_{AD_0} - (\theta + \eta_D + \varpi_D + \mu)I_{SD_0} - \gamma_1\frac{\beta_P(\alpha_P C_0 + D_0)}{N}I_{SD_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
I_{AP_1} &= I_{AP_0} + \Big[\frac{\beta_P(\alpha_P C_0 + D_0)}{N} - (\phi_P + \mu)I_{AP_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
I_{SP_1} &= I_{SP_0} + \Big[\phi_P I_{AP_0} - (\eta_P + \varpi_P + \mu)I_{SD_0} - \gamma_2\frac{\beta_D(\alpha_D A_0 + B_0)}{N}I_{SP_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
I_{SDP_1} &= I_{SDP_0} + \Big[\frac{\beta_{DP}E_0}{N} + \gamma_1\frac{\beta_P(\alpha_P C_0 + D_0)}{N}I_{SD_0} \\
&\quad + \gamma_2\frac{\beta_D(\alpha_D A_0 + B_0)}{N}I_{SP_0} - (\eta_{DP} + \varpi_{DP} + \mu)I_{SDP_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
Q_{SD_1} &= Q_{SD_0} + \Big[\theta I_{SD_0} - (\Omega + \mu)Q_{SD_0}\Big]\frac{t^q}{\Gamma(q+1)}, \\
R_1 &= R_0 + \Big[VbN + \eta_P I_{AP_0} + \eta_D I_{AD_0} + \eta_{SDP}I_{SP_0} - \mu R_0\Big]\frac{t^q}{\Gamma(q+1)}.
\end{aligned} \tag{15}
$$

4 Numerical Simulations

In this section, we will use the parameters in Table 2 and assign initial values to the compartments. The Figure 1 illustrates Effect of Varying q on the System Variables at $n = 0$ using equation (15) with $S(0) = 990,\ I_{AD}(0) = 10,\ I_{SD}(0) = 5,\ I_{AP}(0) = 7,\ I_{SP}(0) = 4,\ I_{SDP}(0) > 3,\ Q_{SD}(0) = 2$ respectively.

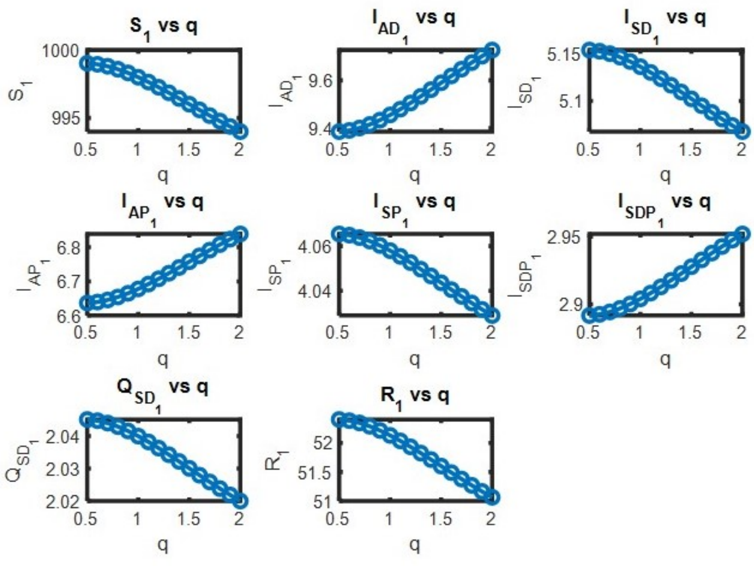

Figure 1: Effect of Varying q on the System Variables at $n = 0$.

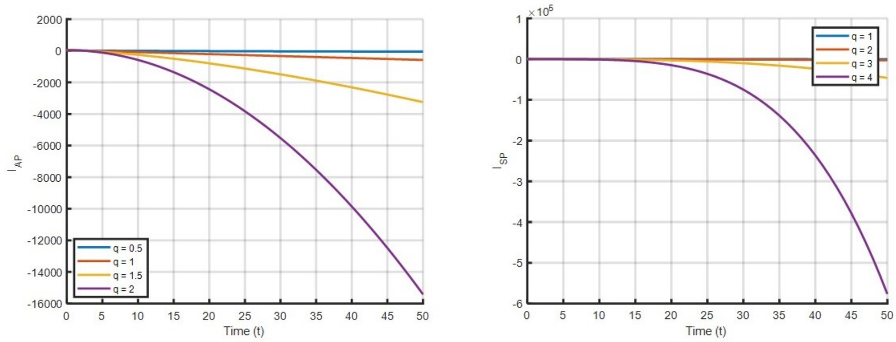

Figure 2: Plots on the effect of varying q on I_{AP} and I_{SP} compartments over time.

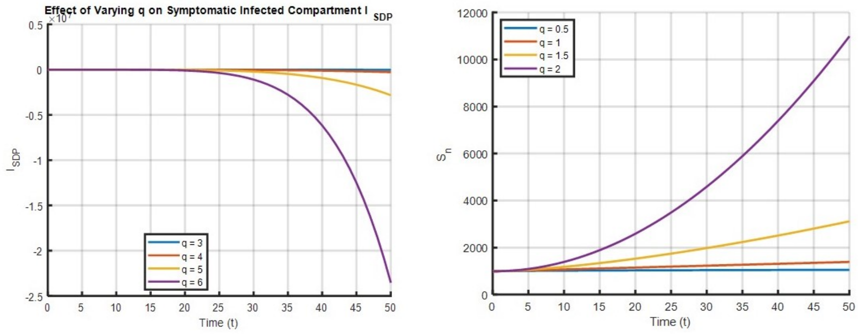

Figure 3: Plots on the effect of varying q on I_{SDP} and S compartments over time.

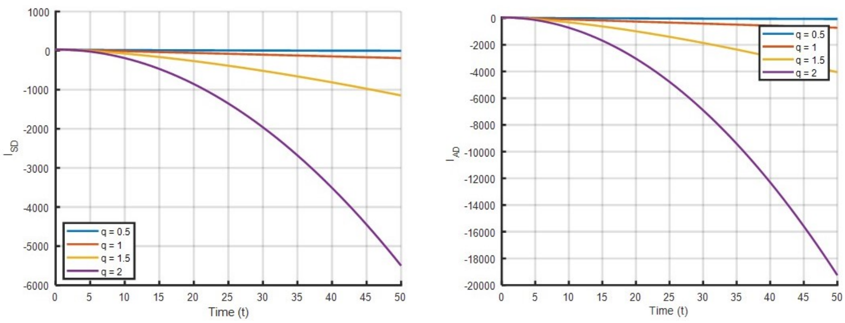

Figure 4: Plots on the effect of varying q on I_{SD} and I_{AD} compartments over time.

5 Conclusion

In this article, we developed a fractional-order compartmental model to investigate the transmission dynamics of Diphtheria and Pertussis co-infections within the human population. To analyze and solve the model, we applied the Laplace-Adomian Decomposition Method, yielding a series of solutions that demonstrably converge to exact values. Our findings highlight the critical role of fractional-order models in understanding and controlling disease outbreaks.

References

[1] Khatiwada, A. P., Shrestha, N., and Shrestha, S. (2021). Will COVID-19 lead to a resurgence of vaccine-preventable diseases?. Infection and Drug Resistance, 119-124.

[2] Agüero, M. L., Castillo, J. B. D., Falleiros-Arlant, L. H., Berezín, E., de Moraes, J. C., Torres-Martínez, C., ... and López-Medina, E. (2023). Risks of low vaccination coverage and strategies

to prevent the resurgence of vaccine-preventable diseases in infants in the COVID-19 pandemic scenario: recommendations for Latin America and the Caribbean by the group of experts on infant immunization for Latin America. Expert Review of Vaccines, 22(1), 1091-1101.

[3] Johnson, O. S., Edogbanya, H. O., Wakili, A., and John, A. T. (2024). Diphtheria Disease Transmission Dynamics In Low Vaccine Coverage Setting. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(2), 79-106.

[4] Johnson, O. S., Edogbanya, H. O., Emmanuel, J., and Olukanni, S. E. (2023). Stability Analysis of COVID-19 Model with Quarantine. International Journal of Mathematical Sciences and Computing, 9(3), 26-45.

[5] Organization, W.H.: COVID-19 Pandemic Fuels Largest Continued Backslide in Vaccinations in Three Decades, https://www.who.int/news/item/15-07-2022- covid-19-pandemic-fuels-largest-continued-backslide-in-vaccinations-in-three decades. Accessed: 2024-05-26 (2022).

[6] Gouveia, N., and Fletcher, T. (2000). Respiratory diseases in children and outdoor air pollution in Sao Paulo, Brazil: a time series analysis. Occupational and environmental medicine, 57(7), 477-483.

[7] Clay, P. A., Duffy, M. A., and Rudolf, V. H. (2020). Within-host priority effects and epidemic timing determine outbreak severity in co-infected populations. Proceedings of the Royal Society B, 287 (1922), 20200046.

[8] Cox, F. E. G. (2001). Concomitant infections, parasites and immune responses. Parasitology, 122 (S1), S23-S38.

[9] Glidden, Caroline K., Courtney AC Coon, Brianna R. Beechler, Chase McNulty, Vanessa O. Ezenwa, and Anna E. Jolles. “Co-infection best predicts respiratory viral infection in a wild host.” Journal of Animal Ecology 90, no. 3 (2021): 602-614.

[10] Griffiths, E. C., Pedersen, A. B., Fenton, A., and Petchey, O. L. (2011). The nature and consequences of coinfection in humans. Journal of Infection, 63(3), 200-206.

[11] Liang, J. L. (2018). Prevention of pertussis, tetanus, and diphtheria with vaccines in the United States: recommendations of the Advisory Committee on Immunization Practices (ACIP). Recommendations and reports, 67.

[12] Johnson, S. O., Edogbanya, H. O., Omolehin, J. O., and Johnson, O. (2024). Characteristics and control strategies for Diphtheria-Pertussis co-infection risk in the post COVID-19 era. International Journal of Mathematical Analysis and Modelling, 7(2).

[13] Murphy, T. V., Slade, B. A., Broder, K. R., Kretsinger, K., Tiwari, T., Joyce, M. P., ... and Moran, J. S. (2008). Prevention of pertussis, tetanus, and diphtheria among pregnant and postpartum women and their infants. Recomm Rep, 57, 1-51.

[14] Kuchar, E., Karlikowska-Skwarnik, M., Han, S., and Nitsch-Osuch, A. (2016). Pertussis: history of the disease and current prevention failure. Pulmonary dysfunction and disease, 77-82.

[15] Sadoh, A. E., and Oladokun, R. E. (2012). Re-emergence of diphtheria and pertussis: implications for Nigeria. Vaccine, 30 (50), 7221-7228.

[16] Calderón, T. A., Coffin, S. E., and Sammons, J. S. (2015). Preventing the spread of pertussis in pediatric healthcare settings. Journal of the Pediatric Infectious Diseases Society, 4(3), 252-259.

[17] Brooks, D. A., and Clover, R. (2006). Pertussis infection in the United States: role for vaccination of adolescents and adults. The Journal of the American Board of Family Medicine, 19 (6), 603-611.

[18] Edogbanya, H. O., and Johnson, O. S. 1.20 Sensitivity Analysis of Pertussis Transmission Dynamics. DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCE, 27.

[19] Helen O. E., and O. S. Johnson. (2024). Mathematical Modeling for Mitigating Pertussis Resurgence in the Post-COVID-19 Era: A Sensitivity Analysis and Intervention Strategies. International Journal of Novel Research in Physics Chemistry and Mathematics, 11(1), 12–28. https://doi.org/10.5281/zenodo.10721146

[20] Meenakshi, A., Renuga, E., Čep, R., and Karthik, K. (2024). Analysis of Caputo Fractional-Order Co-Infection COVID-19 and Influenza SEIR Epidemiology by Laplace Adomian Decomposition Method. Mathematics, 12(12), 1876.

[21] Acheneje, G. O., Omale, D., Agbata, B. C., Atokolo, W., Shior, M. M., and Bolarinwa, B. (2024). Approximate Solution of the Fractional Order Mathematical Model on the Transmission Dynamics on The Co-Infection of COVID-19 and Monkeypox Using the Laplace-Adomian Decomposition Method. International Journal of Mathematics and Statistics Studies, 12(3), 17-51.

[22] Zeng, Y. (2022). A Laplace Adomian Decomposition Method for Fractional Order Infection Model. J. Appl. Math. Comput, 6, 529-534.

[23] Abioye, A. I., Peter, O. J., Ogunseye, H. A., Oguntolu, F. A., Ayoola, T. A., and Oladapo, A. O. (2023). A fractional-order mathematical model for malaria and COVID-19 co-infection dynamics. Healthcare Analytics, 4, 100210.

[24] Schiff, J. L. (2013). The Laplace transform: theory and applications. Springer Science and Business Media.

[25] Caputo, M. (1969). Elasticita e dissipazione. Zanichelli.

[26] Caputo, M., and Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications, 1(2), 73-85

[27] Meenakshi, A., Renuga, E., Čep, R., and Karthik, K. (2024). Analysis of Caputo Fractional-Order Co-Infection COVID-19 and Influenza SEIR Epidemiology by Laplace Adomian Decomposition Method. Mathematics, 12(12), 1876.

[28] Yakubu, A. A., Abdullah, F. A., and Yatim, Y. M. (2022). Mathematical Model of Pertussis and Pneumonia Co-Infection in Infants with Maternally Derived Immunity. Sains Malaysiana, 51(7), 2197-2209.

[29] Omame, A., and Abbas, M. (2023). Modeling SARS-CoV-2 and HBV co-dynamics with optimal control. Physica A: Statistical Mechanics and its Applications, 615, 128607.

[30] Izzati, N., and Andriani, A. (2021, April). Dynamical analysis of diphtheria epidemic model with natural immunity rate on exposed individuals. In Journal of Physics: Conference Series (Vol. 1869, No. 1, p. 012117). IOP Publishing.