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A hybrid method of block pulse with Bernstein polynomial for numerical solution of linear ill-posed first kind Fredholm Integal equation

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Abstract

This study presents an effective method for approximating solutions to first-kind Fredholm integral equations using a hybrid approach combining Block-Pulse and Bernstein polynomials. The method transforms the integral equation into a system of linear algebraic equations by exploiting the properties of the hybrid functions. Additionally, we estimate the error bound under the L_2 norm to assess the accuracy of the approach. To validate the method's effectiveness, we apply it to some test problems with known exact solutions, demonstrating its reliability and efficiency.

Keywords and phrases: First kind Fredholm integral equation; Hybrid functions; Convergence analysis;

1 Introduction

Integral equations are vital mathematical tools extensively used in applied sciences, including mathematics, physics, and engineering, often serving as alternative formulations for differential equations. Among these, Fredholm integral equations are particularly significant due to their applications in various scientific fields. First-kind Fredholm integral equations arise in radiography, spectroscopy, cosmic radiation studies, image processing, and signal processing. In physics, they are instrumental in plasma diagnostics, physical electronics, nuclear physics, and optical imaging. Furthermore, several mathematical problems, such as harmonic continuation, the backward heat equation, numerical inversion of the Laplace transform, and numerical differentiation, can be reformulated as first-kind Fredholm integral equations [1, 2, 3]. These equations are characterised by the presence of the unknown function $f(t)$ solely within the integral sign. Furthermore, first-kind Fredholm integral equations typically exhibit the following form:

$$\int_a^b k(t,s)f(s)ds = g(s) \qquad s \in D, \tag{1}$$

The presence of $f(t)$ within the integral sign introduces specific challenges. First-kind Fredholm integral equations often manifest as ill-posed problems, which can either lack a solution entirely or, if a solution exists, it may not be unique and may not exhibit continuous dependence on the data $g(s)$ [4]. It is widely recognized that analytically solving first-kind Fredholm integral equations is typically difficult, and exact solutions are rare. Consequently, a computational approach to solving these equations becomes crucial in scientific investigation.

Tikhonov introduced various regularisation methods to address first-kind integral equations [5, 6] with Rajan [7] analyzing its convergence under the supremum norm. Wazwaz [4] utilised a regularisation method in conjunction with existing techniques to manage ill-posed first-kind Fredholm integral

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equations. Molabahrami [8] combined regularisation and integral mean value theorem methods to address ill-posedness in both linear and nonlinear first-kind Fredholm integral equations. Suliman *et.al* [9] introduced a constrained perturbation regularization for linear least-squares problems termed constrained perturbation regularisation. Additionally,[10] proposed the homotopy regularization method for solving both linear and nonlinear first-kind Fredholm integral equations, demonstrating its effectiveness in overcoming ill-posedness. Despite these advancements, there remains a need for methods that combine the computational efficiency of basis functions with the robustness of regularization techniques. Block-Pulse functions were originally introduced in electrical engineering by Harmuth, after which numerous researchers explored their applications [11, 12]. Bernstein polynomials have found applications across various fields of mathematics [13, 14].

Recently, hybrid functions have seen extensive use in addressing diverse types of integral equations [15, 16], integro-differential equations [17, 18], and control problems [19, 20]. One of the key attributes of these hybrid functions is their efficiency and broad applicability. Examples include hybrid Chebyshev and block-pulse functions [21, 22], hybrid Legendre and block-pulse functions [23, 24, 25], hybrid Taylor and block-pulse functions [26], hybrid Fourier and block-pulse functions [27], hybrid Bernstein polynomials and block-Pulse functions [28, 29], hybrid improved block pulse and Bernstein polynomials [30].

However,the hybridization of block-pulse and Bernstein polynomial have been successful in other contexts but remains underexplored, particularly in the context of ill-posed first-kind Fredholm integral equations. Hybrid block pulse and Bernstein polynomial have independently demonstrated strengths in numerical computations. Their combination could provide a synergistic effect, enhancing both accuracy and computational efficiency.

1.1 Preliminaries

In this section, we revisit certain definitions and characteristics of Bernstein polynomials and Block-Pulse functions.

Definition 1.1. ***Definition 1.1*** *(Bernstein polynomials) The Bernstein polynomials of m^{th} degree are defined on the interval $[a, b]$ as:*

$$B_{i,m}(t) = \binom{m}{i} \frac{(t-a)^i (b-t)^{m-i}}{(b-a)^m} \quad a \le t \le b, \;\; i = 0, 1, 2, \cdots m, \tag{2}$$

where;

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

The Bernstein polynomials constitute a basis on $L^2[a, b]$ comprisig of $m + 1$ polynomials of degree m. Consequently any polynomial of $m - th$ degree can be expressed as linear combination of $B_{i,m}(t)$ for $i = 0, 1, \cdots m$ [31].

Lemma 1

Consider the N-set of block-pulse function $\{b_n(t)n = 1, 2, \cdots N\}$ be defined over the interval $[0, 1)$ as:

$$b_n(t) = \begin{cases} 1 & \frac{n-1}{N} \le t < \frac{n}{N} \\ \\ 0 & otherwise \end{cases}, \tag{3}$$

where N is a positive integer and $b_n(t)$ is the n^{th} block-pulse function. The fundamental characteristics include disjointness, orthogonality, and completeness [32].

Proof.

The property of disjointness can be readily derived from the definition of block-pulse functions, as outlined below:

$$b_n(t)b_m(t) = \begin{cases} b_n(t) & n = m \\ \\ 0 & n \neq m \end{cases}, \tag{4}$$

where $n, m = 1, 2, \cdots, N$.

The block pulse functions also exhibit orthogonality within the interval $t \in [0, T)$:

$$\int_0^T b_n(t)b_m(t)dt = \begin{cases} h & n = m \\ \\ 0 & n \neq m \end{cases}, \tag{5}$$

where $n, m = 1, 2, \cdots, N$.
This characteristic is directly inferred from the disjointness of block pulse functions. The orthogonality property of block pulse functions serves as the foundation for expanding functions into their block pulse series.

For the completeness property, as the number of block pulse functions N approaches infinity, the set of block pulse functions becomes complete. For any function f belonging to $L^2([0, 1))$Parseval's identity holds:

$$\int_0^1 f(t)dt = \sum_{n=0}^{\infty} f_n^2 ||b_n(t)||_2,$$

where;

$$f_n = \frac{1}{h}\int_0^1 f(t)b_n(t)dt. \tag{6}$$

Definition 1.2. *Definition 1.2 (Hybrid block pulse and Bernstein polynomial HBB [33]). The combination of block-pulse and Bernstein polynomial functions are defined over the interval* $[0, 1)$ *for* $i = 0, 1, 2 \cdots m$ *and* $n = 1, 2 \cdots N$ *as:*

$$h_{n,i} = \begin{cases} B_{i,m}(Nt - n + 1) & \frac{n-1}{N} \leq t < \frac{n}{N} \\ \\ 0 \quad otherwise \end{cases}. \tag{7}$$

2 Methodology

2.1 Function approximation

A function $f(t) \in L^2[0, 1)$ can be expanded in terms of hybrid functions as follows:

$$f(t) \approx \sum_{n=1}^{N}\sum_{i=0}^{m} c_{n,i}h_{n,i} = C^T\Omega(t), \tag{8}$$

where;

$\Omega^T(t) = [h_{1,0}(t), h_{1,1}(t) \cdots h_{1,m}(t) \quad \cdots \quad h_{N,0}(t), h_{N,1}(t) \ \cdots \ h_{N,m-1}(t), h_{N,m}(t)]$ and

$$C^T = [c_{1,0}, c_{1,1} \cdots c_{1,m} \quad \cdots \quad c_{N,0}, c_{N,1} \ \cdots \ c_{N,m-1}, c_{N,m}],$$

and we have

$$C^T = \frac{\langle f, \Omega(t)\rangle}{\langle \Omega(t), \Omega(t)\rangle},$$

then

$$C = J^{-1}\left(\int_0^1 f(t)\Omega(t)^T dt\right).$$

Similarly, the function $k\,(t,s) \in L^2([0,1)\times[0,1)$ can be estimated as:

$$k\,(t,s) \approx \Omega^T(t)K\Omega(s), \tag{9}$$

where K is an $N(m+1)\times N(m+1)$ matrix called the kernel matrix and enteries of the matrix is given by:

$$k_{i,j} = \frac{\langle h_i(t), (k(t,s)h_j(s))\rangle}{\langle h_i(t), h_i(t)\rangle\,\langle h_j(s)h_j(s)\rangle} \quad i,j = 1,2\cdots N(m+1), \tag{10}$$

$$K = J^{-1}\,\langle \Omega(t), k(t,s), \Omega(s)\rangle\, J^{-1}. \tag{11}$$

2.2 Integration of cross product

The integration of the cross product of two hybrid function vectors $\Omega(t)$ is defined by:

$$J = \int_0^1 \Omega(t)\Omega(t)^T dt, \tag{12}$$

where;

$$J = \frac{1}{N}\begin{bmatrix} Q_1 & \vec{0} & \dots & \vec{0} \\ \vec{0} & Q_2 & \dots & \vec{0} \\ \vdots & & & \\ \vec{0} & \vec{0} & \dots & Q_N \end{bmatrix}. \tag{13}$$

and $\vec{0}$ is an $(m+1)\times(m+1)$ matrix, also $Q_n(n = 1,2,\cdots N)$ is an $(m+1)\times(m+1)$ defined by;

$$\begin{aligned} (Q_n) &= \begin{bmatrix} \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \cdots & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} \\ \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \cdots & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} \\ \vdots & & & \\ \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} & \cdots & \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} \end{bmatrix} n = 1,2,\cdots N, \\ i,j &= 0,1,2\cdots m. \end{aligned} \tag{14}$$

We now present a numerical direct approach for solving equation (1) using a hybrid function comprised of block pulse and Bernstein polynomial. For this purpose, we approximate the functions in equation(1) , as mentioned earlier, in the form;

$$f(t) \approx F^T\Omega(t), \tag{15}$$

$$k(t,s) \approx \Omega^T(t)K\Omega(s), \tag{16}$$

and

$$g\,(t) \approx G^T\Omega(t), \tag{17}$$

where K is an $N(m+1) \times N(m+1)$ dimensional matrix and G is a known $N(m+1) \times 1$ vector. In equation (15), F is an unknown $N(m+1) \times 1$ vector. By substituting equation(15)-(17) in to equation (1), we have;

$$\int_0^1 \Omega^T(t) K \Omega(s) \Omega^T(s) F ds = \Omega^T(t) G, \tag{18}$$

or

$$\Omega^T(t) K \int_0^1 \Omega(s) \Omega^T(s) F ds = \Omega^T(t) G. \tag{19}$$

using equation (12), equation (19) can be replaced by

$$\Omega^T(t) KJF = \Omega^T(t) G \tag{20}$$

Therefore, we have the following linear system of equations:

$$(KJ)F = G \tag{21}$$

by solving the linear system (21) we find the unknown vector F,where K is an $N(m+1) \times N(m+1)$ matrix and J is the dual operational matrix of $\Omega(t)$ which is an $N(m+1) \times N(m+1)$ matrix derived from the previous section, F is an unknown $N(m+1) \times 1$ vector and G is an known $N(m+1) \times 1$ vector.

3 Error analysis

The objective of this section is to outline the convergence characteristics of the proposed numerical method. We aim to demonstrate that, under suitable conditions, the approximate solution derived from our approach converges to the exact solution of Equation (1). To achieve this, we provide the following theorem and lemma for convergence analysis.

Lemma 2. Let $f \in C^{m+1}[0,1)$ is an $m+1$ times continuously differentiable function such that $f \approx \sum_{n=1}^{N} f_n$, and let $Y_n = span\{h_{n0}(t), h_{n1}(t) \cdots h_{nm}(t)\}$ $n = 1, 2, \cdots N$. If $C_n^T \Omega_n(t)$ is the best approximation to f_n from Y_n , where $C_n = [c_{n0}, c_{n1} ... c_{nm}]^T$
and $\Omega_n(t) = [h_{n0}(t), h_{n1}(t) \cdots h_{nm}(t)]^T$ then $p_{N(m+1)}(f)(t) = C^T \Omega(t)$ approximates $f(t)$ with the following error bound:

$$\begin{aligned} \left\| f(t) - p_{N(m+1)}(f)(t) \right\|_2 &\leq \frac{\mu}{(m+1)! N^{m+1} \sqrt{(2m+3)}}. \\ where\ \mu &= \max_{t \in [0,1)} \left| f^{m+1}(t) \right| \end{aligned} \tag{22}$$

Proof. The Taylor expansion for the function $f_n(t)$ is given as:

$$\begin{aligned} f_n(t) &= f_n\left(\frac{n-1}{N}\right) + f_n'\left(\frac{n-1}{N}\right)\left(t - \frac{(n-1)}{N}\right) + ... + \frac{f_n^m}{m!}\left(\frac{n-1}{N}\right)\left(t - \frac{(n-1)}{N}\right)^m \\ &+ \frac{f_n^{m+1}}{m+1!}\left(\frac{n-1}{N}\right)\left(t - \frac{(n-1)}{N}\right)^{m+1} + ... \\ t &\in \left[\frac{n-1}{N}, \frac{n}{N}\right), \end{aligned} \tag{23}$$

for which it is known that:

$$\left| f_n(t) - \tilde{f}_n(t) \right| \leq \left| f^{m+1}(\theta) \right| \frac{\left(t - \frac{n-1}{N}\right)^{m+1}}{(m+1)!}, \quad \theta \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \quad n = 1, 2, \cdots N.$$

Since $C_n^T\ \Omega_n(t)$ is the best approximation to $\tilde{f}_n$ from Y_n we have;

$$\left\|f_n(t) - C_n^T\Omega_n(t)\right\|_2^2 = \left\|f_n(t) - \tilde{f}_n(t)\right\|_2^2 = \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left|f_n(t) - \tilde{f}_n(t)\right|^2 dt.$$

Now;

$$\begin{aligned}\left\|f_n(t) - C_n^T\Omega_n(t)\right\|_2^2 &\le \left(\frac{\mu}{(m+1)!}\right)^2 \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left(t - \frac{n-1}{N}\right)^{2m+2} dt, \\ &= \left(\frac{\mu}{(m+1)!}\right)^2 \frac{1}{(2m+3)\,N^{2m+3}}, \end{aligned} \tag{24}$$

where $\mu = \max\left|f^{m+1}(\theta)\right| \quad \theta \in [0,1)$.
Now,

$$\begin{aligned}\left\|f - C^T\Omega(t)\right\|_2^2 &\le \sum_{n=1}^{N} \left\|\left(f_n(t) - C_n^T\Omega_n(t)\right)\right\|_2^2 \\ &\le \frac{\mu^2}{\left((m+1)!\right)^2 N^{2m+2}(2m+3)}. \end{aligned} \tag{25}$$

By taking the square root of equation (25) we have the bound;

$$\left\|f - C^T\Omega(t)\right\|_2 \le \frac{\mu}{(m+1)!N^{m+1}\sqrt{(2m+3)}}.$$

Theorem 3.1. Theorem 3.1 *In equation (1) suppose that $k(t,s)$ is continuous on the square $[0,1)^2$ and f belong to $C^{m+1}[0,1)$ with $m \ge 1$. Furthermore assume the approximate solution*

$$f(t) \simeq \sum_{n=1}^{N}\sum_{i=0}^{m} f_{ni}h_{ni}(t) = F^T\Omega(t)$$

is given by the hybrid block pulse and Bernstein polynomial method, Now if A is non-singular then

$$\left\|f(t) - \sum_{n=1}^{N}\sum_{i=0}^{m} f_{ni}h_{ni}(t)\right\|_2 \le \left\|A^{-1}\right\|_2 \frac{N(m+1)c_2}{(m+1)!N^{m+1}} \frac{\sqrt{\frac{4m^3-m}{N(m+1)}}}{m\sqrt{(2m+3)}}. \tag{26}$$

Proof. The estimation provided in (26) is derived as follows:
Let

$$f(t) \simeq f_{N(m+1)}(t) = \sum_{n=1}^{N}\sum_{i=0}^{m} f_{ni}h_{ni}(t), \tag{27}$$

where f_{ni} are unknown coefficients determined by solving the linear system in equation (21). Also suppose;

$$p_{N(m+1)}(f)(t) = \sum_{n=1}^{N}\sum_{i=0}^{m} c_{ni}h_{ni}(t), \tag{28}$$

our goal is to find a bound for $\left\|f(t) - f_{N(m+1)}(t)\right\|_2$, so

$$\left\|f(t) - f_{N(m+1)}(t)\right\|_2 \le \left\|f(t) - p_{N(m+1)}(f)(t)\right\|_2 + \left\|p_{N(m+1)}(f)(t) - f_{N(m+1)}(t)\right\|_2, \tag{29}$$

from lemma 2 we have

$$\begin{aligned} \left\| f(t) - p_{N(m+1)}(f)(t) \right\|_2 &\leq \frac{\mu}{(m+1)! N^{m+1} \sqrt{(2m+3)}} , \\ where \ \mu &= \max \left| f^{m+1}(t) \right| \quad t \in [0,1) . \end{aligned} \tag{30}$$

We now find a bound for $\left\| p_{N(m+1)}(f)(t) - f_{N(m+1)}(t) \right\|_2$,

$$\begin{aligned} \left\| p_{N(m+1)}(f)(t) - f_{N(m+1)}(t) \right\|_2^2 &= \int_0^1 \left| p_{N(m+1)}(f)(t) - f_{N(m+1)}(t) \right|^2 dt \\ &= \int_0^1 \left| \sum_{n=1}^{N} \sum_{i=0}^{m} (c_{ni} - f_{ni}) h_{ni}(t) \right|^2 dt \\ &\leq \int_0^1 \left(\sum_{n=1}^{N} \sum_{i=0}^{M} |(c_{ni} - f_{ni})| \, |h_\gamma(t)| \right)^2 dt, \end{aligned} \tag{31}$$

where $|h_\gamma(t)| = \sup |h_{ni}(t)| \quad n = 1, 2, \cdots N \quad i, j = 0, 1, \cdots m$ then;

$$\begin{aligned} \left\| p_{N(m+1)}(f)(t) - f_{N(m+1)}(t) \right\|_2^2 &\leq \int_0^1 \sum_{n=1}^{N} \sum_{i=0}^{M} |(c_{ni} - f_{ni})|^2 \, |h_\gamma(t)|^2 dt \\ &= \sum_{n=1}^{N} \sum_{i=0}^{M} |(c_{ni} - f_{ni})|^2 \alpha \\ &= \|C - F\|_2^2 \alpha, \end{aligned} \tag{32}$$

where $\alpha = \int_0^1 |h_\gamma(t)|^2 dt$.

This implies,

$$\left\| p_{N(m+1)}(f)(t) - f_{N(m+1)}(t) \right\|_2 \leq \|C - F\|_2 \sqrt{\alpha}, \tag{33}$$

if we substitute $f_{N(m+1)}$ as an approximate solution in equation (1) then;

$$\int_0^1 k(t,s) f_{N(m+1)}(s) ds = g(t),$$

but if we use $p_{N(m+1)}(f)(t)$ as an approximate solution in equation (1) we have:

$$\int_0^1 k(t,s) p_{N(m+1)}(f)(s) ds = \tilde{g}(t),$$

by using the numerical method of hybrid block and Bernstein polynomial in equation (21) we have;

$$F = (KJ)^{-1} G = A^{-1} G, \tag{34}$$

$$C = (KJ)^{-1} \tilde{G} = A^{-1} \tilde{G}, \tag{35}$$

where

$$F = [f_{10} \dots f_{1m} \quad f_{20} \dots f_{2m} \quad \dots \quad f_{N0} \dots f_{Nm}]^T ,$$

$$C = [c_{10}...c_{1m} \quad c_{20}...c_{2m} \ \dots \ c_{N0}...c_{Nm}],$$
$$G = [g_{10}...g_{1m} \quad g_{20}...g_{2m} \ \dots \ g_{N0}...g_{Nm}]^T,$$
$$\tilde{G} = [\tilde{g}_{10}...\tilde{g}_{1m} \quad \tilde{g}_{20}...\tilde{g}_{2m} \ \dots \ \tilde{g}_{N0}...\tilde{g}_{Nm}]^T,$$

Also, we observe $C - F = A^{-1}\left(\tilde{G} - G\right)$,
We have the following bound of $\|C - F\|_2$ as

$$\|C - F\|_2 \leq \left\|A^{-1}\right\|_2 \left\|\left(\tilde{G} - G\right)\right\|_2 \tag{36}$$

then it is enough to find a bound for $\left\|\left(\tilde{G} - G\right)\right\|_2$ but before then we need a bound for $\|(\tilde{g}(t) - g(t))\|_2$, Now from equation 1;

$$\begin{aligned} g(t) &= \int_0^1 k(t,s) f(s)\, ds - \int_0^1 k(t,s)\, p_{N(m+1)}(f)(s)\, ds + \int_0^1 k(t,s)\, p_{N(m+1)}(f)(s)\, ds, \\ \tilde{g}(t) &= g(t) - \int_0^1 k(t,s)\left(f(s) - p_{N(m+1)}(f)(s)\right) ds, \end{aligned}$$

therefore;

$$\begin{aligned} \|(\tilde{g}(t) - g(t))\|_2^2 &= \int_0^1 \left| \int_0^1 k(t,s)\left(f(s) - p_{N(m+1)}(f)(s)\right) ds \right|^2 dt, \\ &\leq \int_0^1 \left(\int_0^1 \left| k(t,s)\left(f(s) - p_{N(m+1)}(f)(s)\right) ds \right| \right)^2 dt \\ &= \int_0^1 \left(\int_0^1 |k(t,s)| \left|\left(f(s) - p_{N(m+1)}(f)(s)\right)\right| ds \right)^2 dt, \end{aligned}$$

If we assume $c_0 = \sup |k(t,s)| \quad t, s \in [0,1)$ we have;

$$\begin{aligned} \|(\tilde{g}(t) - g(t))\|_2^2 &\leq c_0^2 \left(\int_0^1 \left|\left(f(s) - p_{N(m+1)}(f)(s)\right)\right| ds \right)^2 \\ &\leq c_0^2 \left(\left\|\left(f(s) - p_{N(m+1)}(f)(s)\right)\right\|_2 \right)^2 \\ &\leq c_0^2 \left(\frac{\mu}{(m+1)! N^{m+1} \sqrt{(2m+3)}} \right)^2, \end{aligned} \tag{37}$$

taking square roots of both sides of equation (37) we have;

$$\|(\tilde{g}(t) - g(t))\|_2 \leq \frac{\mu c_0}{(m+1)! N^{m+1} \sqrt{(2m+3)}}. \tag{38}$$

Now we find a bound for $\left\|\tilde{G} - G\right\|_2$ since;

$$g_{ni} = \frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \int_0^1 g(t)\, h_{ni}(t)\, dt \quad n = 1, 2, \cdots N \quad i, j = 0, 1, \cdots m, \tag{39}$$

and

$$\tilde{g}_{ni} = \frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \int_0^1 \tilde{g}(t)\, h_{ni}(t)\, dt \quad n = 1, 2, \cdots N \quad i, j = 0, 1, \cdots m. \tag{40}$$

Then we have;

$$\begin{aligned}
\left\|\tilde{G} - G\right\|_2^2 &= \sum_{n=1}^{N}\sum_{i=0}^{m} \left|(\tilde{g}_{ni} - g_{ni})\right|^2 \\
&\leq N(m+1) \sup_{1\leq n\leq .N;0\leq i,j\leq m} \left(\left|\frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \int_0^1 h_{ni}(t)\left(\tilde{g}(t) - g(t)\right) dt\right|^2\right) \\
&\leq N(m+1) \sup_{1\leq n\leq .N;0\leq i,j\leq m} \left(\frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \int_0^1 \left|h_{ni}(t)\left(\tilde{g}(t) - g(t)\right)\right| dt\right)^2 \\
&= N(m+1) \sup_{1\leq n\leq .N;0\leq i,j\leq m} \left(\frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \int_0^1 \left|h_{ni}(t)\right| \left|(\tilde{g}(t) - g(t))\right| dt\right)^2 \\
&\leq N(m+1) \sup_{1\leq n\leq .N;0\leq i,j\leq m} \left(\frac{(2m+1)\binom{2m}{i+j}}{\binom{m}{i}\binom{m}{j}} \left\|h_{ni}(t)\right\|_2 \left\|(\tilde{g}(t) - g(t))\right\|_2\right)^2 \\
&= N(m+1) \sup_{1\leq n\leq .N;0\leq i,j\leq m} \left(\frac{1}{\sqrt{\left(\frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}\right)}} \left\|(g(t) - \tilde{g}(t))\right\|_2\right)^2.
\end{aligned} \tag{41}$$

Substitute equation (38) in equation (41) we have;

$$\left\|\tilde{G} - G\right\|_2^2 \leq \frac{c_1 N(m+1)}{(m+1)!N^{m+1}} \frac{\sqrt{\frac{4m^3-m}{N(m+1)}}}{m\sqrt{(2m+3)}}. \tag{42}$$

Consequently, by substituting equation (42) in the inequality (36) we can conclude:

$$\|C - F\|_2 \leq \left\|A^{-1}\right\|_2 \frac{N(m+1)c_1}{(m+1)!N^{m+1}} \frac{\sqrt{\frac{4m^3-m}{N(m+1)}}}{m\sqrt{(2m+3)}} \qquad where\ c_1 = \mu c_0, \tag{43}$$

so by substituting equation (43) in equation (33) we have;

$$\left\|P_{N(m+1)}(f)(t) - f_{N(m+1)}(t)\right\|_2 \leq \left\|A^{-1}\right\|_2 \frac{N(m+1)c_2}{(m+1)!N^{m+1}} \frac{\sqrt{\frac{4m^3-m}{N(m+1)}}}{m\sqrt{(2m+3)}}. \tag{44}$$

Finally substitute equation (30)and equation (44) in equation (29) we have;

$$\begin{aligned}
\left\|f(t) - f_{N(m+1)}(t)\right\|_2 &= \left\|f(t) - p_{N(m+1)}(f)(t)\right\|_2 + \left\|p_{N(m+1)}(f)(t) - f_{N(m+1)}(t)\right\|_2 \\
&\leq \frac{\mu}{(m+1)!N^{m+1}\sqrt{(2m+3)}} + \left\|A^{-1}\right\|_2 \frac{N(m+1)c_2}{(m+1)!N^{m+1}} \frac{\sqrt{\frac{4m^3-m}{N(m+1)}}}{m\sqrt{(2m+3)}} \\
&\leq \frac{1}{(m+1)!N^{m+1}\sqrt{(2m+3)}} \left(\mu + \left\|A^{-1}\right\|_2 N(m+1)c_2\sqrt{\frac{4m^3-m}{N(m+1)}}\right).
\end{aligned}$$

4 Numerical examples

In this section, we provide some examples demonstrating the approximation of solutions for first-kind Fredholm integral equations using the numerical method detailed in the preceding section. We conduct these numerical experiments using Maple 18 software.

Example 1. Consider the first kind Fredholm integral equation

$$\int_0^1 \sin(ts) f(s)\, ds = \frac{\sin(t) - t\cos(t)}{t^2}$$

with exact solution $f(t) = t$. The absolute errors of improved block pulse method for $t \in [0,1]$ is shown in Table (1).

Table 1. Numerical results yeilded by the proposed method for Example 1.

t_i	$exact$	$HBBM\ N=3,\ M=3$
0	0	
0.1	0.1	0.6674604728
0.2	0.2	0.5592445614
0.3	0.3	0.3329531755
0.4	0.4	0.1257379315
0.5	0.5	0.1700910529
0.6	0.6	0.2859762479
0.7	0.7	0.0824780603
0.8	0.8	0.2093223047
0.9	0.9	0.0010252419

Example 2 Consider the first kind Fredholm integral equation

$$\int_0^1 \sin(t+s) f(s)\, ds = \frac{1}{4}\left(1 - \cos(2)\cos(t) + (2 + \sin(t)\sin(t))\right)$$

with exact solution $f(t) = \cos(t)$. The absolute errors of improved block pulse method for $t \in [0,1]$ is shown in Table (2).

Table 2. Numerical results yeilded by the proposed method for Example 2.

t_i	$exact$	$HBBM\ N=2,\ M=2$
0.1	0.9950041653	0.9290675127
0.2	0.9800665778	0.3244888834
0.3	0.9553364891	1.567859402
0.4	0.9210609940	0.5496316061
0.5	0.8775825619	0.3792593930
0.6	0.8253356149	0.3395967964
0.7	0.7648421873	0.6352944497
0.8	0.6967067093	0.2229385196
0.9	0.6216099683	0.2822716825

5 Discussions and conclusion

In this research work, we introduced the hybrid block-pulse Bernstein method to solve linear ill-posed first-kind Fredholm integral equations. By leveraging the properties of hybrid functions, we transformed the problem into a solvable linear system of algebraic equations, which were effectively solved using Maple 18 software. Our error bound estimation and convergence analysis demonstrated the robustness of the method. The numerical examples further confirmed the effectiveness of this approach, and the proposed method offers a promising tool for solving complex Fredholm integral equations in applied sciences. Future work could explore incorporating regularization techniques with the hybrid functions to further enhance solution stability. Additionally, the use of preconditioning methods or multigrid techniques may improve computational efficiency. Extending the hybrid approach to nonlinear illposed first kind Fredholm integral equation could also open new avenues for solving complex real-world problems more efficiently.

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