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On the monoid of partial isometries with fixed set

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Abstract

Denote $[n]$ to be a finite n-chain $\{1, 2, \ldots, n\}$, and let $\mathcal{DP}_n$ denote the semigroup of all partial isometries on $[n]$. For a nonempty subset D of $[n]$, let $\text{Fix}([n], D)$ be the set of all maps in $\mathcal{DP}_n$ which fix all elements of D. Then $\text{Fix}([n], D)$ is a subsemigroup of $\mathcal{DP}_n$. In this paper, we show that the semigroup $\text{Fix}([n], D)$ is a semilattice of idempotents, and further, we obtain the cardinality and the rank of the semigroup.

Keywords and phrases: partial isometries; fixed sets; semilattice; rank.

1 Introduction and Preliminaries

In the study of transformation semigroups, many authors have investigated the algebraic, combinatorial, and rank properties of the semigroup of full transformations with fixed subsets of a chain. In many instances, the subsemigroup of a transformation semigroup, when considering fixed subsets of a chain, exhibits different algebraic, combinatorial, and rank properties. This fact has led many authors to study various subsemigroups with fixed subsets within different transformation semigroups; for examples, see [6, 13, 14, 15, 18]. It has now become customary that, upon encountering any class of transformation semigroup, one of the questions usually raised is the structural properties of its subsemigroup with fixed sets.

Denote $[n]$ to be a finite n chain $\{1, 2, \ldots, n\}$, $\mathcal{P}_n$ to be the semigroup of partial transformations and $\mathcal{I}_n$ to be an inverse subsemigroup of $\mathcal{P}_n$ (consisting of all partial injective maps in $\mathcal{P}_n$). For any transformation α, we shall denote the domain of α, image set of α and $|\text{Im } \alpha|$ by $\text{Dom } \alpha$, $\text{Im } \alpha$, $h(\alpha)$, respectively. Let

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : \text{ (for all } x, y \in \text{Dom } \alpha) \ |x\alpha - y\alpha| = |x - y|\} \tag{1}$$

be the semigroup of partial isometries on $[n]$ and

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : \text{ (for all } x, y \in \text{Dom } \alpha) \ x \leq y \Rightarrow x\alpha \leq y\alpha\} \tag{2}$$

be the semigroup of order-preserving partial isometries of $[n]$. Clearly $\mathcal{ODP}_n$ is a subsemigroup of $\mathcal{DP}_n$.

Among the algebraic and combinatorial properties of the semigroups $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ obtained include the number of their maximal subsemigroups [12], the combinatorial properties [2], the rank properties [1], the presentation [16], and the natural partial orders [11], among other results. Moreover, [5] obtained the rank of the two-sided ideals $\mathcal{DP}_{n,r} = \{\alpha \in \mathcal{DP}_n : |\text{Im } \alpha| \leq r\}$ and $\mathcal{ODP}_{n,r} = \{\alpha \in \mathcal{ODP}_n : |\text{Im } \alpha| \leq r\}$ $(2 \leq r \leq n-1)$ to be $\text{rank}(\mathcal{DP}_{n,r}) = \text{rank}(\mathcal{ODP}_{n,r}) = \binom{n}{r}$.

The authors of [2] obtained $F(n;p)$ to be $\frac{2(2n-p+1)}{p+1}\binom{n}{p}$ (where $F(n,p) = \{\alpha \in \mathcal{DP}_n : |\text{Im } \alpha| = p\}$). In particular, they obtained the order of the semigroup to be $|\mathcal{DP}_n| = 3 \cdot 2^{n+1} - (n+2)^2 - 1$.

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Now let D be a non empty subset of $[n]$ and $\mathcal{DP}_n$ be as defined in equation (1), and let

$$\text{Fix}([n], D) = \{\alpha \in \mathcal{DP}_n : d\alpha = d, \text{for all } d \in D\}. \tag{3}$$

Notice that for all $\alpha, \beta \in \text{Fix}([n], D)$ and $d \in D$, $d(\alpha\beta) = (d\alpha)\beta = d\beta = d$. Thus, $\text{Fix}([n], D)$ is a subsemigroup of $\mathcal{DP}_n$. Moreover, it is clear that $Id_{[n]} \in \text{Fix}([n], D)$ (where $Id_{[n]}$ denote the identity map on $[n]$) for every nonempty subset $D \subseteq [n]$. Thus, $\text{Fix}([n], D)$ is a monoid. Notice also that if $D = [n]$ then $\text{Fix}([n], D) = \{Id_{[n]}\} \cong \mathbb{Z}_1$. So, for the remaining content of the paper, we shall assume $\emptyset \neq D \subsetneq [n]$.

On this note, we investigate the algebraic, combinatorial, and rank properties of $\text{Fix}([n], D)$. Section 1 of this paper contains the definitions of basic terms so as to give the reader a proper understanding. In Section 2, we show that the semigroup $\text{Fix}([n], D)$ is a semilattice of idempotents, and in Section 3, we obtain the cardinality and the rank of the semigroup. For $\alpha, \beta \in \text{Fix}([n], D)$, we shall write the composition of α and β as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all $x \in \text{Dom }\alpha$.

Let S be a semigroup, for a subset A of S, A is said to be a *generating set of* S if every element of S can be written as a product of elements of A. If A generates S we simply write $\langle A \rangle = S$. The rank of a finite semigroup S is defined and denoted by

$$\text{rank}(S) = \min\{|A| : A \subset S, \langle A \rangle = S\}.$$

An element $a \in S$ is an idempotent if $a^2 = a$. The collection of all idempotent of S is usually denoted by $E(S)$. A map $\alpha \in \text{Fix}([n], D)$ is an idempotent if and only if $\text{Im }\alpha = F(\alpha) = \{x \in \text{Dom }\alpha : x\alpha = x\}$. We shall denote $f(\alpha)$ to be $|F(\alpha)|$. A semigroup S is said to be a *semilattice* if for all $a, b \in S$: $a^2 = a$ and $ab = ba$. For basic concept in semigroup theory, we refer the reader to [4, 7, 10].

To begin our investigation, we record the following results from [1].

Lemma 1 ([1], Lemma 1.2). *Let $\alpha \in \mathcal{DP}_n$. Then we have the following conditions:*

(a) The map α is either order-preserving or order-reversing. Equivalently, α is either a translation or a reflection.

(b) If $h(\alpha) = p$, then $f(\alpha) = 0$ or 1 or p.

(c) If α is order-preserving and $f(\alpha) \geq 1$, then α is a partial identity.

Throughout this paper, unless otherwise specified, we shall consider $D = \{d_1, d_2, \ldots, d_r\} \subseteq [n]$ with $0 < r < n$, and $d_i < d_j$ if and only if $i < j$. Obviously, since $D \neq \emptyset$, then by our assumption, we shall have $1 \leq r < n$. Moreover, as a consequence of Lemma 1, we have the following lemma.

Lemma 2. *Every $\alpha \in \text{Fix}([n], D)$ can be expressed as*

$$\alpha = \begin{pmatrix} d_1 & \cdots & d_r & b_1 & \cdots & b_p \\ d_1 & \cdots & d_r & b_1 & \cdots & b_p \end{pmatrix},$$

where $r + p < n$ and $\{b_j : 1 \leq j < r + p\} \subseteq [n] \setminus D$.

Notice that $\{b_j : 1 \leq j < r + p\}$ may not necessarily be part of $\text{Dom }\alpha$ and also if $r + p = n$, then $\alpha = Id_{[n]}$. Consequently by Lemma 1, we readily have the following remark.

Remark 1.1. Let $\alpha \in \text{Fix}([n], D)$. Then α is a partial identity and hence every element of $\text{Fix}([n], D)$ is an idempotent.

Now for the remaining content of the paper, unless or otherwise specified we shall refer to $\alpha, \beta \in \text{Fix}([n], D)$ to be expressed as

$$\alpha = \begin{pmatrix} d_1 & \cdots & d_r & b_1 & \cdots & b_p \\ d_1 & \cdots & d_r & b_1 & \cdots & b_p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} d_1 & \cdots & d_r & c_1 & \cdots & c_m \\ d_1 & \cdots & d_r & c_1 & \cdots & c_m \end{pmatrix}, \tag{4}$$

where $r + p < n$, $r + m < n$.

The following result shows that the monoid $\text{Fix}([n], D)$ is commutative.

Lemma 3. *The monoid* $\mathrm{Fix}([n], D)$ *is commutative.*

Proof. Let $\alpha, \beta \in \mathrm{Fix}([n], D)$ be as expressed in equation (4). Then we may assume without loss of generality that $m \leq p$. Thus, there are two cases to consider.

Case 1: If $\{b_i : 1 \leq i \leq p\} \cap \{c_i : 1 \leq i \leq m\} = \emptyset$. Then,

$$\alpha\beta = \begin{pmatrix} d_1 & \cdots & d_r \\ d_1 & \cdots & d_r \end{pmatrix} = \beta\alpha.$$

Case 2: If $\{b_i : 1 \leq i \leq p\} \cap \{c_i : 1 \leq i \leq m\} \neq \emptyset$. Then $\{b_i : 1 \leq i \leq p\} \cap \{c_i : 1 \leq i \leq m\} = \{a_i : 1 \leq i \leq t \text{ for some } t \leq m\}$. Thus,

$$\alpha\beta = \begin{pmatrix} d_1 & \cdots & d_r & a_1 & \cdots & a_t \\ d_1 & \cdots & d_r & a_1 & \cdots & a_t \end{pmatrix} = \beta\alpha.$$

The result now follows.

Hence, we have actually proved the following result.

Theorem 1. *Let* $\mathrm{Fix}([n], D)$ *be as defined in equation* (3), *then* $\mathrm{Fix}([n], D)$ *is a semilattice of idempotents.*

2 Green's relations on the monoid $\mathrm{Fix}([n], D)$

The main aim of this section is to describe Green's relations on the monoids $\mathrm{Fix}([n], D)$. For the definitions of Green's relations, we refer the reader to [7, 10].

Theorem 2. *Let* $\alpha, \beta \in \mathrm{Fix}([n], D)$ *be as expressed in equation* (4) *with* $m = p$. *Then*

(i) $(\alpha, \beta) \in \mathcal{L}$ *if and only* $\alpha = \beta$;

(ii) $(\alpha, \beta) \in \mathcal{R}$ *if and only* $\alpha = \beta$.

Proof. (i) Let $\alpha, \beta \in \mathrm{Fix}([n], D)$ be as expressed in equation (4) such that $(\alpha, \beta) \in \mathcal{L}$. This means that there exist $\gamma_1, \gamma_2 \in \mathrm{Fix}([n], D)$ such that

$$\alpha = \gamma_1\beta \text{ and } \beta = \gamma_2\alpha.$$

Notice that $\mathrm{Dom}\ \alpha = \{d_1, \ldots, d_r, b_1, \ldots, b_p\}$ and $\mathrm{Dom}\ \beta = \{d_1, \ldots, d_r, c_1, \ldots, c_p\}$. Now for $d_i \in \mathrm{Dom}\ \alpha$ $(1 \leq i \leq r)$ (which is also in $\mathrm{Dom}\ \beta$)

$$d_i\alpha = d_i\gamma_1\beta = (d_i\gamma_1)\beta = d_i\beta.$$

Also, for $b_i \in \mathrm{Dom}\ \alpha$ and $c_i \in \mathrm{Dom}\ \beta$ $(1 \leq i \leq p)$,

$$b_i\alpha = (b_i\gamma_1)\beta = b_i\beta \text{ and}$$

$$c_i\beta = (c_i\gamma_2)\alpha = c_i\alpha.$$

Thus $\alpha = \beta$. The converse follows easily.

(ii) Let $\alpha, \beta \in \mathrm{Fix}([n], D)$ be as expressed in equation (4) such that $(\alpha, \beta) \in \mathcal{R}$. This means that there exists $\gamma_1, \gamma_2 \in \mathrm{Fix}([n], D)$ such that

$$\alpha = \beta\gamma_1 \text{ and } \beta = \alpha\gamma_2.$$

Notice that $\mathrm{Dom}\ \alpha = \{d_1, \ldots, d_r, b_1, \ldots, b_p\}$ and $\mathrm{Dom}\ \beta = \{d_1, \ldots, d_r, c_1, \ldots, c_p\}$. Now for $d_i \in \mathrm{Dom}\ \alpha$ $(1 \leq i \leq r)$

$$d_i\alpha = d_i\beta\gamma_1 = (d_i\beta)\gamma_1 = d_i\gamma_1 = d_i,$$

$$d_i\beta = d_i\alpha\gamma_2 = (d_i\alpha)\gamma_2 = d_i\gamma_2 = d_i.$$

Thus, we have $d_i\alpha = d_i\beta$. Moreover, for $b_i \in \text{Dom } \alpha$ and $c_i \in \text{Dom } \beta$ $(1 \leq i \leq p)$,

$$b_i\alpha = b_i\beta\gamma_1 = (b_i\beta)\gamma_1 = b_i\gamma_1 = b_i. \tag{5}$$

Notice that β is defined on b_i, i.e., $b_i \in \text{Dom } \beta$. Hence,

$$b_i\beta = (b_i\alpha)\gamma_2 = b_i\gamma_2 = b_i. \tag{6}$$

From equation (5) and (6) we have $b_i\alpha = b_i\beta$.

In a similar way, one can show that $c_i\beta = c_i\alpha$. Thus $\alpha = \beta$.

Consequently we have the following corollary.

Corollary 1. *On the semigroup* $\text{Fix}([n], D)$, $\mathcal{L} = \mathcal{R} = \mathcal{J} = \mathcal{D} = \mathcal{H}$.

Now as a consequence of corollary 1, we readily have the following result.

Theorem 3. *Every* H_α $(\alpha \in \text{Fix}([n], D))$ *is a maximal subgroup of* $\text{Fix}([n], D)$ *isomorphic to the trivial group* $\mathbb{Z}_1$.

3 The combinatorial and rank properties of $\text{Fix}([n], D)$

In this section, we determine the cardinality and the rank of the monoid $\text{Fix}([n], D)$. These properties heavily depend on the size of the subset D, as we shall see below.

Now for $0 \leq k \leq n-r$, let $F(n;\, k) = |J_k|$ where $J_k = \{\alpha \in \text{Fix}([n], D) : |\text{Im } \alpha \setminus D| = k\}$. We now record the following results.

Proposition 1. *For* $0 \leq k \leq n-r$,

$$F(n, k) = \binom{n-r}{k}.$$

Proof. Let $D \subsetneq [n]$ such that $|D| = r$, i.e., we fixed r elements from the chain $[n]$. Out of the remaining $n-r$ elements we can select k $(0 \leq k \leq n-r)$ elements to formulate a partial identity from the $n-r$ elements. This can be done in $\binom{n-r}{k}$ ways, as required.

Theorem 4. *Let* $\text{Fix}([n], D)$ *be as defined in equation* (3). *Then* $|\text{Fix}([n], D)| = 2^{n-r}$.

Proof. Using Proposition 1, we see that

$$|\text{Fix}([n], D)| = \sum_{k=0}^{n-r} F(n,\, k) = \sum_{k=0}^{n-r} \binom{n-r}{k} = 2^{n-r},$$

as required.

Now for $0 \leq p \leq n-r-1$, let

$$L(n-r,\, p) = \{\alpha \in \text{Fix}([n], D) : |\text{Im } \alpha \setminus D| \leq p\} \tag{7}$$

and recall that

$$J_k = \{\alpha \in \text{Fix}([n], D) : |\text{Im } \alpha \setminus D| = k\}.$$

Obviously, $J_0 \cup J_1 \cup \ldots \cup J_p = L(n-r,\, p)$, which is a two sided ideal of $\text{Fix}([n], D)$ for all p.

We now state the following lemma.

Lemma 4. *Every element in* J_k $(0 \leq k \leq p \leq n-r-2)$ *can be express as a product of elements in* J_{k+1}.

Proof. Let $\alpha \in J_k$. Then α can be expressed as

$$\alpha = \begin{pmatrix} d_1 & \cdots & d_r & b_1 & \cdots & b_k \\ d_1 & \cdots & d_r & b_1 & \cdots & b_k \end{pmatrix}.$$

Now, since $p \leq n-r-2$, we can choose two elements say $x, y \in [n] \setminus \text{Dom } \alpha$ such that $x \neq y$ and let

$$\beta = \begin{pmatrix} d_1 & \cdots & d_r & b_1 & \cdots & b_k & x \\ d_1 & \cdots & d_r & b_1 & \cdots & b_k & x \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} d_1 & \cdots & d_r & b_1 & \cdots & b_k & y \\ d_1 & \cdots & d_r & b_1 & \cdots & b_k & y \end{pmatrix}.$$

Then clearly $\beta, \gamma \in J_{k+1}$ and obviously $\alpha = \beta\gamma$.
Consequently, we have the following corollary.

Corollary 2. $J_k \subseteq \langle J_{k+1} \rangle$ $(0 \leq k \leq p \leq n - r - 2)$.

We now use the following Lemmas to deduce the rank of the two sided ideal $L(n - r,\, p)$.

Lemma 5. *For* $0 \leq p \leq n - r - 1$, *the ideal* $L(n - r,\, p)$ *has* $\binom{n-r}{p}$ $\mathcal{D}-$*classes of height* p.

Proof. The result follows from Proposition 1.

Lemma 6. *Let* $0 \leq p \leq n-r-1$ *and* $G = \{\alpha \in L(n-r,\, p) : |\text{Im } \alpha \setminus D| = p\}$. *Then* G *is the minimum generating set of* $L(n - r,\, p)$.

Proof. Notice that $J_p = G$. Now by Corollary 2, we see that $J_{p-1} \subseteq \langle J_p \rangle$. Similarly, $J_{p-2} \subseteq \langle J_{p-1} \rangle, \ldots, J_0 \subseteq \langle J_1 \rangle$. Consequently, $J_0 \cup J_1 \cup J_2 \cup \ldots \cup J_p \subseteq \langle J_p \rangle$, i.e., $L(n - r,\, p) \subseteq \langle J_p \rangle$. Thus G is a generating set of $L(n - r,\, p)$. Notice that $L(n - r,\, p)$ is $\mathcal{D}$ trivial as such every $\mathcal{D}-$class in J_p contains one and only one element since $\mathcal{D} = \mathcal{J}$. This element is maximal and hence J_p is the unique minimum generating set.

Proposition 2. *For* $0 \leq p \leq n - r - 1$, *rank* $(L(n - r,\, p)) = \binom{n-r}{p}$.

Proof. The proof follows directly from Lemma 5 and 6.
Consequently, we deduce the following results.

Lemma 7. *rank* $(\text{Fix}([n], D) \setminus \{id_{\{n\}}\}) = n - r$.

Proof. Notice that if $p = n - r - 1$ then $\text{Fix}([n], D) \setminus \{id_{[n]}\} = L(n - r,\, n - r - 1)$. Thus the results now follows easily from Proposition 2.
We now have the following result.

Theorem 5. *Let* $\text{Fix}([n], D)$ *be as expressed in equation* (3). *Then rank* $(\text{Fix}([n], D)) = n - r + 1$.

Proof. Notice that $\text{Fix}([n], D) = \text{Fix}([n], D) \setminus \{id_{[n]}\} \cup \{id_{[n]}\}$. Thus rank $(\text{Fix}([n], D)) =$ rank $(\text{Fix}([n], D) \setminus \{id_{[n]}\})+$ rank $(\{id_{[n]}\})$. Now the result follows using Lemma 7.

3.1 Isomorphism property in $\text{Fix}([n], D)$

In this subsection, we give a necessary and sufficient conditions for the monoids $\text{Fix}([n], D_1)$ and $\text{Fix}([n], D_2)$ to be isomorphic, where $D_1, D_2 \subseteq [n]$.

Lemma 8. *Let* D_1 *and* D_2 *be nonempty subsets of* $[n]$. *Then* $\text{Fix}([n], D_1) \cong \text{Fix}([n], D_2)$ *if and only if* $|D_1| = |D_2|$.

Proof. Let $|D_1| = r$, $|D_2| = k$ and suppose that there exists an isomorphism $\varphi : \text{Fix}([n], D_1) \to \text{Fix}([n], D_2)$. This means that $|\text{Fix}([n], D_1)| = |\text{Fix}([n], D_2)|$. This implies

$$2^{n-r} = 2^{n-k}, \text{ and as such } r = k.$$

Thus $|D_1| = |D_2|$, as required.

Conversely, suppose $|D_1| = |D_2|$. Then $|[n] \setminus D_1| = |[n] \setminus D_2|$. Now let $D_1 = \{d_1 < \ldots < d_r\}, D_2 = \{d'_1 < \ldots < d'_r\}, [n] \setminus D_1 = \{a_1 < \ldots < a_r\}$ and $[n] \setminus D_2 = \{a'_1 < \ldots < a'_r\}$. Define θ as

$$\theta = \begin{pmatrix} d_1 & \cdots & d_r & a_1 & \cdots & a_k \\ d'_1 & \cdots & d'_r & a'_1 & \cdots & a'_k \end{pmatrix}.$$

Clearly θ is a bijection and hence

$$\theta^{-1} = \begin{pmatrix} d'_1 & \cdots & d'_r & a'_1 & \cdots & a'_k \\ d_1 & \cdots & d_r & a_1 & \cdots & a_k \end{pmatrix}.$$

Now, define $\phi : \text{Fix}([n], D_1) \to \text{Fix}([n], D_2)$ by $\alpha\phi = \theta^{-1}\alpha\theta$ for all $\alpha \in \text{Fix}([n], D_1)$. Clearly, ϕ is an isomorphism.

4 Conclusion

In this paper, we successfully studied the subsemigroup $\text{Fix}([n], D)$ of the semigroup $\mathcal{DP}_n$, consisting of all partial isometries of a finite chain $[n]$. We established that the subsemigroup is a monoid and forms a semilattice of idempotents. Furthermore, we determined its cardinality and rank. These results provide a deeper understanding of the algebraic structure of the monoid $\text{Fix}([n], D)$.

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