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Algorithm for approximating common fixed points of family of strictly pseudo-contractive mappings in Hilbert spaces

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Abstract

It is our purpose in this paper to construct and study a Halpern-type algorithm with inertial and error terms for approximating common fixed points of strictly pseudo-contractive mappings in the framework of real Hilbert spaces. We prove strong convergence theorem for the sequence generated by the algorithm under certain mild conditions on the control sequences. Numerical examples are given to illustrate the effectiveness of this algorithm. Our result is intended to complement and extend some well known results in the literature.

Keywords and phrases: Common fixed points; strictly pseudo-contractive mappings; Halpern-type algorithm; inertial and error terms; strong convergence theorem

1 Introduction

Let H be a real Hilbert space. Let $T_i : H \to H, i = 1, 2, ..., \mathbb{N}$ be any mapping for each i defined on H. Let $\cap_{i=1}^{N} Fix(T_i)$ represent finite intersection of a non-empty, closed and convex family of mappings which is a subset of H, where

$$Fix(T_i) := \{x \in H : x = T_i(x), \text{ for each } i \geq 1\}, \tag{1}$$

is the fixed point set of a family of mappings and $\mathbb{N}$ is the set of all natural numbers. Let $P_{\cap_{i=1}^{N} Fix(T_i)}(x)$ denote a metric projection of H onto $\cap_{i=1}^{N} Fix(T_i)$ defined by

$$\|x - P_{\cap_{i=1}^{N} Fix(T_i)}(x)\| = \inf\{\|x - z\| : z \in \cap_{i=1}^{N} Fix(T_i)\}, \forall \tag{2}$$

and the metric projection holds true if and only if it satisfies the variational inequality

$$\langle x - P_{\cap_{i=1}^{N} Fix(T_i)}(x), z - P_{\cap_{i=1}^{N} Fix(T_i)}(x)\rangle \leq 0. \tag{3}$$

Using this characterization, we get that a metric projection $P_{\cap_{i=1}^{N} Fix(T_i)}(x)$ is nonexpansive, that is $\|P_{\cap_{i=1}^{N} Fix(T_i)}(x) - P_{\cap_{i=1}^{N} Fix(T_i)}(y)\|^2 \leq \|x - y\|^2, \forall x, y \in H.$

A mapping $T : H \to H$ is called

(a) Lipschitzian if there exists $L \geq 0$ such that

$$\|Tx - Ty\|^2 \leq L\|x - y\|^2, \forall x, y \in H, \tag{4}$$

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(b) quasi nonexpansive if

$$\|Tx - p\|^2 \leq \|x - p\|^2, \forall x \in H, p \in Fix(T). \tag{5}$$

(c) strictly pseudocontractive [1] if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in H. \tag{6}$$

or equivalently, if there exists a constant $\lambda > 0$ such that

$$\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2 - \lambda\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in H. \tag{7}$$

(d) demicontraction (or k-demicontractive) (see [2]) if there exists a constant $k \in [0, 1)$ and $Fix(T) \neq \emptyset$ such that

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in H, p \in Fix(T). \tag{8}$$

The mapping T is called a contraction if $0 \leq L < 1$. It is called nonexpansive if $L = 1$. It is demiclosed at a point $p \in Fix(T)$ if $Fix(T)$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to $\bar{x} \in Fix(T)(x_n \rightharpoonup \bar{x})$ and Tx_n converges strongly to p $(Tx_n \to p)$ and $T\bar{x} = p$.

Many authors have proposed and studied different iterative schemes for the approximation of fixed points of contractive, nonexpansive and pseudo-contractive mappings in Hilbert space, (see for example, [2], [3], [4]).

If given contraction mapping (i.e) $T : H \to H$, and if there exists a constant $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq L.d(x, y), \text{ for all } x, y \in H, \tag{9}$$

then T has a unique fixed point $p \in H$ and the n^{th} iterate of x under T, (i.e) the Picard iteration (see [5])for some $x_0 \in H$, the sequence $\{x_n\}$ given by

$$x_n = T^n(x_0), n \geq 1, \tag{10}$$

converges to the fixed point p as $n \to \infty$ and for each $x \in H$. However, we observed that if the mapping T is not a (strict) contraction, that is, if there is no $L \in [0, 1)$, then the Picard's iterative technique fails to converge to its fixed point. Hence the need for more consideration of some other iterative techniques. In the literature, some authors have considered mappings with compactness conditions to guarantee strong convergence since strong convergence has advantage over weak convergence.

In the iterative approximation of fixed points of nonexpansive maps and strictly pseudocontractive mappings, the iterative schemes of Mann [6]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \geq 0, \tag{11}$$

where $\{\alpha_n\} \subset [0, 1]$. and Ishikawa [7]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n\left[(1 - \beta_n)x_n + \beta_n Tx_n\right], n \geq 0, \tag{12}$$

where $\{\alpha_n\}, \{\beta_n\} \subset$ are suitable control sequences in $[0, 1]$, play an important role. However, the two iteration schemes yield weak convergence only, and require the "compactness" assumption either on the mappings or the domain of the mappings or even both to yield the desired strong convergence. Often strong conditions are needed to impose on the fixed point point set, $Fix(T)$, to obtain the strong convergence by using Mann or the Ishikawa iteration process. For example, Qilhou in [8] required that $Fix(T)$ is finite where T is a continuous pseudocontractive-type self-mapping of a nonempty convex compact of a Hilbert space and in [9] required that the interior of $Fix(T)$ is nonempty. Recently many new algorithms have been constructed and studied to achieve strong convergence with mild assumptions on the mappings, their ambient domain, their fixed point sets and other necessary components, see ([2],

[10]). Meanwhile, the Halpern ([11]), iterative method generated by $\{x_n\}$ and for $x_0 \in H$ is defined recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, n \geq 0. \tag{13}$$

where $\alpha_n = 0$ satisfies certain necessary conditions to guarantee strong convergence of the sequence generated by (13).

Several authors have studied nonexpansive mapping and pseudocontractive mappings and proposed different iterative methods to approximate the fixed points of their mappings in Hilbert spaces but without the inertial and error terms (see for example [12], [13], [21], [15], [16], [17]). To be sure, the study of fixed point of nonexpansive mappings and its generalization is important in many areas of Mathematical applications (see for example [4]). Besides, there have been a growing interest among authors to speed up the rate of convergence of algorithms. The inertial method introduced by Polyak (see [10], [18], [19]) is one of the methods for speeding up the convergence rates. This method was firstly introduced by Polyak in 1964 as acceleration process which solves a smooth convex minimization problem (see [?]). It is a two-step iterative technique in which the next iterate is defined recursively with the use of the previous two iterates as follows: initialize $x_{-1}, x_0 \in X$ and let $\{x_n\}$ be generated by

$$x_{n+1} = x_n - \alpha P(x_n) + \beta(x_n - x_{n-1}), n \geq 0, \tag{14}$$

where P is an operator usually linear defined on a Banach space X, and $\alpha, \beta \in (0,1)$. It is easy to observe that when $P \equiv 0$, then (14) reduces to

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}), n \geq 0, \tag{15}$$

Recently, [4], studied fixed point of nonexpansive mappings using the following algorithm:

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \beta_n y_n + \eta_n T y_n + e_n, n \geq 1, \end{cases} \tag{16}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\eta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$, $\{\theta_n\}_{n=1}^{\infty}$ is a positive sequences and e_n is an error sequence satisfying some convergence conditions. The sequence $\{x_n\}$ generated by the algorithm (16) converges strongly to metric projection of H onto $Fix(T)$, which is the fixed point of nonexpansive mapping. They applied it to solve compressed sensing problem.

Similarly, the authors in citeagb, studied fixed point of strictly pseudo-contractive mapping and zeros of inverse strongly monotone operators using the same algorithm as (16) and the sequences satisfying suitable convergence conditions. The sequence $\{x_n\}$ converges strongly to metric projection of H onto $Fix(T)$, which is the fixed point of strictly pseudo-contractive mapping.

Consequently, Ekuma-Okereke, et.al [20] proposed and studied a modified Mann-type iterative procedure with inertial term for the approximation of fixed point of strictly pseudo-contractive mappings defined on a closed and convex subset of real Hilbert spaces. below is their algorithm: Let the sequence $\{x_n\} \in C$ be generated by choosing $x_1 < x_0 \in C$ and using the procedure

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ x_{n+1} = P_C(\beta_n y_n + (1-\beta_n)T y_n), n \geq 1, \end{cases} \tag{17}$$

where $\{\beta_n\}_{n=1}^{\infty} \in (k,1)$ for any $k \in [0,1)$, $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\theta_n\}_{n=1}^{\infty}$ is a positive sequence satisfying required convergence conditions and P_C is the metric projection onto a nonempty, closed and convex subset subset C of H. Then $\{x_n\}$ generated by (17)converges strongly to metric projection of C onto $Fix(T)$.

Question 1. Is it possible to construct an algorithm with inertial and error terms that converges strongly to a common fixed point of a family of strictly pseudocontractive mappings more general than nonexpansive mappings?

Thus, we are inspired by the above results to construct a Halpern-type iterative with inertial and error terms algorithm for approximating a common fixed point of a family of strictly pseudocontractive mappings in a real Hilbert space. It is intended that our method complement and extend some results cited here.

2 Methodology/Model formulation

This section formulate a Halpern algorithm with inertial-like and error terms for approximating a common fixed point of a finite family of strictly pseudo-contractive mappings and then discuss its convergence analysis in the next section.

Model/Algorithm Hypotheses

(C_1) Let H be a real Hilbert Space.

(C_2) Let $T_i : H \to H, i = 1, 2, ..., N$, represent a strictly pseudo-contractive mappings.

(C_3) Let $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$.

(C_4) Let $\{\beta_n^0\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \in (0, 1)$ for each i, $\{\alpha_n\}_{n=1}^{\infty} \in (0, 1)$ be positive sequences and $\{\zeta_n\}_{n=1}^{\infty}$ be an error sequence which positive.

Let the sequences satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \zeta_n < \infty, \lim_{n\to\infty}, \dfrac{\zeta_n}{\alpha_n} = 0,$

(ii) $\liminf_{n\to\infty} \gamma_n^i[\beta_n - (1 - \alpha_n)k] > 0$, where $\beta_n = (1 - \alpha_n)\beta_n^0, \gamma_n^i = (1 - \alpha_n) \sum_{i=1}^{N} \beta_n^i$

(iii) $\alpha_n + \beta_n + \gamma_n^i = 1, \sum_{i=0}^{N} \beta_n^i = 1$

(iv) $\tau \in [0, 1)$ such that $0 \leq \tau_n \leq 1$ with

$$\tau_n := \begin{cases} \min\{\frac{\zeta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \text{whenever } x_n \neq x_{n-1} \\ \tau, & \text{otherwise .} \end{cases}$$

Algorithm 2.1

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Initialization: Let $x_1, x_0 \in H$. Given x_{n-1}, x_n, then calculate x_{n+1} as follows:

Step 1: Compute

$$z_n = x_n + \zeta_n(x_n - x_{n-1}) \tag{18}$$

where

$$\tau_n := \begin{cases} \min\{\frac{\theta_n}{\|x_n - x_{n-1}\|}, \tau\}, & \text{whenever } x_n \neq x_{n-1} \\ \tau, & \text{otherwise .} \end{cases}$$

Step 2: Compute

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \left(\beta_n^0 z_n + \sum_{i=1}^{N} \beta_n^i T_i z_n + \frac{\zeta_n}{1 - \alpha_n} \right), n \geq 1. \tag{19}$$

Set $n := n + 1$ and go to **Step 1.** Stop if $x_{n+1} = x_n = x_{n-1} = z_n$.

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The following Lemmas are needed in the sequel.

Lemma 2.1. *cf.([21]) Let H be a real Hilbert space. Given $x, y \in H$, then the inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{20}$$

Lemma 2.2. *cf.([21]) Let H be a real Hilbert space. Given $x, y \in H$, then the equality hold:*

$$\|\alpha x+\beta y\|^2=\alpha(\alpha+\beta)\|x\|^2+\beta(\alpha+\beta)\|y\|^2-\alpha\beta\|x-y\|^2, \forall x, y \in H, \alpha, \beta \in \mathbb{R} \tag{21}$$

Lemma 2.3. *cf.([22]) Let H be a real Hilbert space. For all $x_i \in H, \alpha_i \in [0,1], i=0,1,2,...,n$ satisfying $\alpha_0+\alpha_1+\alpha_2+...+\alpha_n=1$, then we have the equality condition as follows:*

$$\|\alpha_0 x_0+\alpha_1 x_1+\alpha_2 x_2+...+\alpha_n x_n\|^2=\sum_{i=0}^{n}\alpha_i\|x_i\|^2-\sum_{0\leq i,j\leq n}\alpha_{i,j}\|x_i-x_j\|^2. \tag{22}$$

Lemma 2.4. *([2]) Let C represent a non-empty, closed and convex subset of a real Hilbert space H. Let $T: C \to C$, represent a strictly pseudo-contractive mapping. Then*

(i) $(1-T)$ is demiclosed at 0 and $x_n - Tx_n \to 0$, then $x = Tx$,

(ii) $Fix(T)$ is closed and convex subset of C.

Lemma 2.5. *([23]) Let $\{a_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{e_n\}_{n=1}^{\infty}$ be sequences contain in $[0,\infty)$, $\{b_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{d_n\}_{n=1}^{\infty}$ be a sequence contain in $(-\infty,+\infty)$ such that*

$$a_{n+1} \leq [1-b_n+c_n]a_n+d_n+e_n, n \geq 1. \tag{23}$$

If $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} e_n\infty$, then

(i) if $d_n \leq Mb_n$ for some $M \geq 0$, then $\{a_n\}_{n=1}^{\infty}$ is bounded.

(ii) if $\lim_{n\to\infty} b_n = 0, \sum_{n=1}^{\infty} c_n = \infty$ and $\limsup_{n\to\infty} \dfrac{d_n}{b_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. *([24]) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n+\alpha_n\delta_n, n \geq n_0 \tag{24}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$, $\{\delta_n\}$ is a sequence in $(-\infty,+\infty)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $\lim_{n\to\infty} sup\delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. *([25]) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$ that is $a_{n_i} < a_{n_{i+1}}$ $\forall i \in N$. Then there exists a nondecreasing subsequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large number) $k \in N$: $a_{m_k} \leq a_{m_{k+1}}$ and $a_k \leq a_{m_{k+1}}$. In fact, $m_k = max\{j \leq k : a_j \leq a_{j+1}\}$.*

3 Analysis of the model

Theorem 3.1. *Suppose that conditions $C_1 - C_4$ are satisfied. Then the sequence $\{x_n\}$ generated by **Algorithm 2.1** converges strongly to a point $x^* \in F$.*

Proof. We first show that the sequence $\{x_n\}$ generated by **Algorithm 2.1** is bounded. To do this, we set $\beta_n = (1-\alpha_n)\beta_n^0, \gamma_n^i = (1-\alpha_n)\sum_{i=1}^{N}\beta_n^i$ with $\alpha_n+\beta_n+\gamma_n^i = 1$. Also, we let $p \in F$. In view of Lemma 2.3 and the property of T_i for each i, we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n x_0 + (1-\alpha_n)\left(\beta_n^0 z_n + \sum_{i=1}^{N}\beta_n^i T_i z_n + \frac{\zeta_n}{1-\alpha_n}\right) - p\|^2 \\
&= \|\alpha_n x_0 + (1-\alpha_n)\beta_n^0 z_n + (1-\alpha_n)\sum_{i=1}^{N}\beta_n^i T_i z_n + \zeta_n - p\|^2 \\
&= \|\alpha_n(x_0-p) + (1-\alpha_n)\beta_n^0(z_n-p) + (1-\alpha_n)\sum_{i=1}^{N}\beta_n^i(T_i z_n - p)\|^2 + \|\zeta_n\|^2 \\
&+ 2\langle \zeta_n, \alpha_n(x_0-p) + (1-\alpha_n)\beta_n^0(z_n-p) + (1-\alpha_n)\sum_{i=1}^{N}\beta_n^i(T_i z_n - p)\rangle \\
&\leq \alpha_n\|x_0-p\|^2 + \beta_n\|z_n-p\|^2 + \gamma_n^i\|T_i z_n - p\|^2 + \|\zeta_n\|^2 \\
&- \alpha_n\beta_n\|x_0-z_n\|^2 - \alpha_n\gamma_n^i\|x_0 - T_i z_n\|^2 - \beta_n\gamma_n^i\|z_n - T_i z_n\|^2 \\
&+ 2\|\zeta_n\|\left[\alpha_n\|x_0-p\| + \beta_n\|z_n-p\| + \gamma_n^i\|T_i z_n - p\|\right] \\
&\leq \alpha_n\|x_0-p\|^2 + \beta_n\|z_n-p\|^2 + \gamma_n^i\left(\|z_n-p\|^2 + k\|z_n - T_i z_n\|^2\right) + \|\zeta_n\|^2 \\
&- \alpha_n\beta_n\|x_0-z_n\|^2 - \alpha_n\gamma_n^i\|x_0 - T_i z_n\|^2 - \beta_n\gamma_n^i\|z_n - T_i z_n\|^2 \\
&+ 2\|\zeta_n\|\left[\alpha_n\|x_0-p\| + \beta_n\|z_n-p\| + \gamma_n^i L\|z_n - p\|\right]. \\
&\leq \alpha_n\|x_0-p\|^2 + \left[\beta_n + \gamma_n^i\right]\|z_n-p\|^2 - \gamma_n^i(\beta_n - k)\|z_n - T_i z_n\|^2 + \|\zeta_n\|^2 \\
&- \alpha_n\beta_n\|x_0-z_n\|^2 - \alpha_n\gamma_n^i\|x_0 - T_i z_n\|^2 - \beta_n\gamma_n^i\|z_n - T_i z_n\|^2 \\
&+ 2\|\zeta_n\|\left[\alpha_n\|x_0-p\| + \beta_n\|z_n-p\| + \gamma_n^i L\|z_n - p\|\right].
\end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned}
\|z_n-p\|^2 &= \|x_n + \tau_n(x_n - x_{n-1}) - p\|^2 \\
&= \|(x_n-p) + \tau_n(x_n - x_{n-1})\|^2 \\
&= \|x_n-p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\langle\tau_n(x_n - x_{n-1}, x_n - p)\rangle \\
&\leq \|x_n-p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|.\|x_n-p\|^2 + 2\tau_n\|x_n - x_{n-1}\| \\
&= [1 + 2\tau_n\|x_n - x_{n-1}\|].\|x_n-p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|.
\end{aligned} \tag{26}$$

Substituting (26) into (25), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|x_0-p\|^2 + \left[\beta_n + \gamma_n^i\right] \\
&\times [1 + 2\tau_n\|x_n - x_{n-1}\|].\|x_n-p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\| \\
&- \gamma_n^i(\beta_n - k)\|z_n - T_i z_n\|^2 + \|\zeta_n\|^2 \\
&- \alpha_n\beta_n\|x_0-z_n\|^2 - \alpha_n\gamma_n^i\|x_0 - T_i z_n\|^2 - \beta_n\gamma_n^i\|z_n - T_i z_n\|^2 \\
&+ 2\|\zeta_n\|\left[\alpha_n\|x_0-p\| + [\beta_n + \gamma_n^i L] \times [1 + 2\tau_n\|x_n - x_{n-1}\|].\|x_n-p\|^2 + \tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - \right. \\
&\leq [1 - \alpha_n + 2(\beta_n + \gamma_n^i\tau_n)\|x_n - x_{n-1}\| \\
&+ 2(\beta_n + \gamma_n^i L)\|\zeta_n\|(1 + 2\tau_n\|x_n - x_{n-1})\|x_n - p\|^2 \\
&+ [(\beta_n + \gamma_n^i) + 2(\beta_n + \gamma_n^i L)\|\zeta_n\|] \times [\tau_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\|x_n - x_{n-1}\|] \\
&+ \alpha_n\|x_n - p\|^2 + 2\alpha_n\|\zeta_n\|\|x_0 - p\| + \|\zeta_n\|^2 + 2(\beta_n + \gamma_n^i L)\|\zeta_n\|.
\end{aligned} \tag{27}$$

If $\lim_{n\to\infty}\dfrac{\|\zeta_n\|}{\alpha_n}=0$, the above inequality (27) can be expressed thus

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\alpha_n+2\alpha_n(\beta_n+\gamma_n^i)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &+2(\beta_n+\gamma_n^iL)\alpha_n\times\frac{\|\zeta_n\|}{\alpha_n}(1+2\tau_n\|x_n-x_{n-1}\|)]\|x_n-p\|^2\\ &+[\alpha_n(\beta_n+\gamma_n^i)+2(\beta_n+\gamma_n^iL)\alpha_n\times\frac{\|\zeta_n\|}{\alpha_n}]\times[\alpha_n^2\times\frac{\tau_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2+2\alpha_n\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|]\\ &+\alpha_n\|x_n-p\|^2+2\alpha_n^2\frac{\|\zeta_n\|}{\alpha_n}\|x_0-p\|+\alpha_n^2\frac{\|\zeta_n\|^2}{\alpha_n^2}+2(\beta_n+\gamma_n^iL)\alpha_n\frac{\|\zeta_n\|}{\alpha_n}. \end{aligned}\tag{28}$$

By our hypothesis,there exists a positive integer N such that

$$\{2(\beta_n+\gamma_n^i)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|+2(\beta_n+\gamma_n^iL)\times\frac{\|\zeta_n\|}{\alpha_n}(1+2\tau_n\|x_n-x_{n-1}\|)\}\leq\frac{1}{2},\forall n\geq N.\tag{29}$$

In addition, we can find a positive constant M such that

$$\begin{aligned}&2\{(\beta_n+\gamma_n^i)+2(\beta_n+\gamma_n^iL)\frac{\|\zeta_n\|}{\alpha_n}[\frac{\tau_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2+2\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|]\\ &+\|x_n-p\|^2+2\alpha_n\frac{\|\zeta_n\|}{\alpha_n}\|x_0-p\|+\alpha_n\frac{\|\zeta_n\|^2}{\alpha_n^2}+2(\beta_n+\gamma_n^iL)\frac{\|\zeta_n\|}{\alpha_n}\}\leq M\end{aligned}\tag{30}$$

Substituting (29) and (30) into (28), we get

$$\|x_{n+1}-p\|^2\leq[1-\frac{\alpha_n}{2}]\|x_n-p\|^2+\frac{\alpha_n}{2}M,\forall n\geq N.\tag{31}$$

Following from Lemma (2.5), the sequence $\{x_n\}$ is bounded. On the other hand, if $\sum_{n=1}^{\infty}\|\zeta_n\|<\infty$, let N be a positive integer such that

$$2(\beta_n+\gamma_n^i)\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|\leq\frac{1}{2},\forall n\geq N.$$

In addition, let M_1, M_2 and M_3 be positive integers such that

$$+2(\beta_n+\gamma_n^i)(1+2\tau_n\|x_n-x_{n-1}\|)\}\leq M_1,$$

$$2(\beta_n+\gamma_n^i)[\frac{\tau_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2+2\frac{\tau_n}{\alpha_n}\|x_n-x_{n-1}\|]+\|x_n-p\|^2+2\|\zeta_n\|\|x_0-p\|\leq M_2,$$

and

$$[2(\beta_n+\gamma_n^iL)\left(\tau_n^2\|x_n-x_{n-1}\|^2+2\tau_n\|x_n-x_{n-1}+1\right)+\|\zeta_n\|]\|\zeta_n\|\leq M_3.$$

From (27), we get

$$\|x_{n+1}-p\|^2\leq[1-\frac{\alpha_n}{2}+M_1\|\zeta_n\|]\|x_n-p\|^2+\frac{\alpha_n}{2}M_2+\|\zeta_n\|M_3,\forall n\geq N.\tag{32}$$

Following from Lemma 2.5 the sequence $\{x_n\}$ is bounded. Now, from Lemma 2.1

$$\begin{aligned}\|z_n-p\|^2&=\|x_n+\tau_n(x_n-x_{n-1})-p\|^2\\ &=\|x_n-p\|^2+\tau_n^2\|x_n-x_{n-1}\|^2+2\langle\tau_n(x_n-x_{n-1},x_n-p)\rangle\\ &\leq\|x_n-p\|^2+\tau_n\|x_n-x_{n-1}\|^2+2\tau_n\langle x_n-x_{n-1},x_n-p\rangle.\end{aligned}\tag{33}$$

Note that,

$$\begin{aligned}\|x_{n-1}-p\|^2 &= \|x_{n-1}-x_n+x_n-p\|^2\\ &= \|x_n-p\|^2 - 2\langle x_n-x_{n-1}, x_n-p\rangle + \|x_n-x_{n-1}\|,\end{aligned} \tag{34}$$

(34) implies that

$$2\langle x_n-x_{n-1}, x_n-p\rangle = -\|x_{n-1}-p\|^2 + \|x_n-p\|^2 + \|x_n-x_{n-1}\|. \tag{35}$$

Plugging (35) into (33) gives

$$\begin{aligned}\|z_n-p\|^2 &\le \|x_n-p\|^2 + \tau_n\|x_n-x_{n-1}\|^2 + \tau_n[-\|x_{n-1}-p\|^2 + \|x_n-p\|^2 + \|x_n-x_{n-1}\|]\\ &= \|x_n-p\|^2 + 2\tau_n\|x_n-x_{n-1}\|^2 + \tau_n[\|x_n-p\|^2 - \tau_n\|x_{n-1}-p\|^2].\end{aligned} \tag{36}$$

On the other hand, we show that the sequence $\{x_n\}$ generated by **Algorithm 2.1** converges to the metric projection mapping of H onto $\cap_{i=1}^N Fix(T_i)$. To do this, we let $p = P_{\cap_{i=1}^N Fix(T_i)} x_0$. In view of the construction of algorithm (18)-(19), Lemmas 2.1, 2.2 and the approach used to get inequality (27), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n x_0 + \beta_n z_n + \gamma_n^i T_i z_n + \zeta_n - p\|^2\\ &= \|\alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}) + \beta_n(z_n-p) + \gamma_n^i(T_i z_n - p)\|^2\\ &\le \|\beta_n(z_n-p) + \gamma_n^i(T_i z_n - p)\|^2 + 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\\ &= \beta_n(\beta_n+\gamma_n^i)\|(z_n-p)\|^2 + \gamma_n^i(\beta_n+\gamma_n^i)\|T_i z_n - p\|^2 - \beta_n\gamma_n^i\|T_i z_n - z_n\|^2\\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\\ &\le \beta_n(\beta_n+\gamma_n^i)\|(z_n-p)\|^2 + \gamma_n^i(\beta_n+\gamma_n^i)[\|z_n-p\|^2 + k\|z_n - T_i z_n\|^2] - \beta_n\gamma_n^i\|T_i z_n - z_n\|^2\\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\\ &= (\beta_n+\gamma_n^i)^2\|(z_n-p)\|^2 - \gamma_n^i[\beta_n - (\beta_n+\gamma_n^i)k]\|z_n - T_i z_n\|^2 + 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\\ &\le (1-\alpha_n)\|z_n-p\|^2 - \gamma_n^i[\beta_n - (\beta_n+\gamma_n^i)k]\|z_n - T_i z_n\|^2\\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle.\end{aligned} \tag{37}$$

Plugging (36) into (37), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)[\|x_n-p\|^2 + 2\tau_n\|x_n-x_{n-1}\|^2 + \tau_n[\|x_n-p\|^2 - \tau_n\|x_{n-1}-p\|^2]\\ &- \gamma_n^i[\beta_n - (\beta_n+\gamma_n^i)k]\|z_n - T_i z_n\|^2\\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\\ &= (1-\alpha_n)\|x_n-p\|^2 - \gamma_n^i[\beta_n - (\beta_n+\gamma_n^i)k]\|z_n - T_i z_n\|^2\\ &+ 2\tau_n(1-\alpha_n)\|x_n-x_{n-1}\|^2 + \tau_n[\|x_n-p\|^2 - \|x_{n-1}-p\|^2]\\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1}-p)\rangle\end{aligned} \tag{38}$$

Next, we consider two cases to complete the proof of our Theorem.

Case 1. If we suppose to find $n_0 \in N$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is decreasing, then we can get $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is convergent and hence

$$\|x_n-p\| - \|x_{n+1}-p\| \to 0, n \to \infty. \tag{39}$$

Consequently,

$$\|x_{n-1} - p\| - \|x_n - p\| \to 0, n \to \infty \tag{40}$$

and

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|x_n - p + p - x_{n-1}\| \\ &\leq \|x_n - p\| + \|x_{n-1} - p\| \to 0, n \to \infty\end{aligned} \tag{41}$$

In view of (38) we get

$$\begin{aligned}\gamma_n^i[\beta_n - (\beta_n + \gamma_n^i)k]\|z_n - T_i z_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \alpha_n\|x_n - p\|^2 \\ &+ 2\tau_n(1 - \alpha_n)\|x_n - x_{n-1}\|^2 + \tau_n[\|x_n - p\|^2 - \|x_{n-1} - p\|^2] \\ &+ 2\langle \alpha_n(x_0 - p + \frac{\zeta_n}{\alpha_n}, x_{n+1} - p)\rangle.\end{aligned} \tag{42}$$

Invoking (39), (40), (41) and conditions (i) and (ii) of our hypothesis, we get

$$\|z_n - T_i z_n\| \to 0, \text{ for each i=1,2,3,...,N}, n \to \infty. \tag{43}$$

Now from the definition of z_n and conditions (i) and (v), we get

$$\begin{aligned}\|z_n - x_n\|^2 &= \|x_n + \tau_n(x_n - x_{n-1}) - x_n\|^2 \\ &= \|\tau_n(x_n - x_{n-1})\|^2 \\ &\leq \tau_n\|x_n - x_{n-1}\|^2 \\ &\leq \alpha_n \frac{\theta_n}{\alpha_n} \to 0, n \to \infty.\end{aligned}$$

Thus,

$$\|z_n - x_n\| \to 0, n \to \infty. \tag{44}$$

Since $\{z_n\}$ is bounded, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in H$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle x_0 - p, z_n - p\rangle &= \limsup_{i\to\infty}\langle x_0 - p, z_{n_i} - p\rangle \\ &= \langle x_0 - p, z - p\rangle.\end{aligned} \tag{45}$$

Now, in view of (44), it follows that $x_{n_i} \rightharpoonup z \in H$. From (43) and Lemma 2.4, we get $z \in \cap_{i=1}^N F(T_i)$. By invoking (38) and Lemma 2.6 we find that $\lim_{n\to\infty}\|x_n - p\| = 0$. Consequently, $x_n \to p = P_{\cap_{i=1}^N F(T_i)} x_0$.

Case 2. Suppose we find a subsequence n_i of $n \neq n_0$ such that $\{\|x_n - p\|\}_{n=n_i}^{\infty}$ is decreasing, that is

$$\|x_{n_i} - p\| \leq \|x_{n_i+1} - p\|, \forall i \in N.$$

Now invoking Lemma 2.7, and following the same line of argument as in Case 1, we find that $x_n \to p = p = P_{\cap_{i=1}^N F(T_i)} x_0$.

Corollary 3.2. *Suppose that $T_i : H \to H$, $i = 1$ represent strictly pseudo-contractive self mapping and if conditions C_1, C_3, C_4 are satisfied. Then the sequence $\{x_n\}$ generated by iterative algorithm 2.1 converges strongly to an element of the set of fixed point of a single strictly pseudo-contractive mapping.*

Proof. Set $i = 1$, then the result holds for a single pseudo-contractive mapping and this reduces to the results of [2] and [19] respectively.

Corollary 3.3. *Suppose that $T_i : H \to H$, $i = 1, 2, ..., N$ represent nonexpansive self mappings and if conditions C_1, C_3, C_4 are satisfied with $k = 0$. Then the sequence $\{x_n\}$ generated by iterative algorithm 2.1 converges strongly to a common element of the set of fixed points of a family of nonexpansive mappings.*

Proof. If we set $k = 0$ and $T_i : H \to H, i = 1, 2, ..., N$ such that T_i represent nonexpansive self mappings in the proof of Theorem 3.1, then we get the conclusion of Corollary 3.3.

Corollary 3.4. *Suppose that $T_i : H \to H$, $i = 1$ represent a single nonexpansive self mapping and conditions C_1, C_3, C_4 are satisfied. Then the sequence $\{x_n\}$ generated by iterative algorithm 2.1 converges strongly to an element of the set of fixed point of a single nonexpansive mapping.*

Proof. If we set $k = 0$ and $T : H \to H$ such that T represent a single nonexpansive self mapping in the proof of Theorem 3.1, then we get the conclusion of Corollary 3.4 and our result reduces to that of [4].

4 Numerical simulations/sensitivity analysis

In this section we would demonstrate the effectiveness, implementation and strong convergence results of our algorithm (18)-(19) of Theorem 3.1 as discussed. All codes are written in Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz.

Example 4.1. (See [2])
Let $H = R$ denote the reals with the usual norm $|.|$ and the inner product $\langle x, y\rangle = xy$, let R denote the reals and define $T : R \to R$ by

$$T(x) := \begin{cases} -3x + 1 & \text{if } x \in (-\infty, 0] \\ \frac{1}{2}(x + 2) & \text{if } x \in (0, \infty). \end{cases} \tag{46}$$

Then T is $\frac{1}{2}$ strictly pseudo-contractive mapping with $Fix(T) = \{2\}$.

Proof. We first prove that the fixed point is actually the 2. To see this, we make use of the definition of fixed point as follows:

$$\begin{aligned} Fix(T) &= \{x \in R : Tx = x\} \\ &= \{x \in R : \frac{1}{2}(x + 2) = x\} \\ &= \{x \in R : \frac{1}{2}x + 1 = x\} \\ &= \{x \in R : x + 2 = 2x\} \\ &= \{x \in R : x = 2 \in (0, \infty)\}. \end{aligned}$$

Thus $Fix(T) = \{2\}$.

Next, we prove that T is $\frac{1}{2}$-strictly pseudo-contractive mapping. Now since $H = R$, is the reals with the usual norm, we have that the norm becomes absolute norm. That is

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \forall x, y \in R \tag{47}$$

becomes

$$|Tx - Ty|^2 \leq |x - y|^2 + k|(x - y) - (Tx - Ty)|^2, \forall x, y \in R \tag{48}$$

where $Tx = -3x+1$ and $Ty = -3y+1, \forall x, y \in (-\infty, 0]$. So using (48), we have

$$\begin{aligned}
|-3x+1-(-3y+1)|^2 &\leq |x-y|^2 + k|(x-y)-(-3x+1-(-3y+1))|^2, \forall x, y \in R\\
|-3x+3y|^2 &\leq |x-y|^2 + k|(x-y)+3x-3y|^2, \forall x, y \in R\\
|-3(x-y)|^2 &\leq |x-y|^2 + k|(x-y)+3(x-y)|^2, \forall x, y \in R\\
9|x-y|^2 &\leq |x-y|^2 + k|(x-y)(1+3)|^2, \forall x, y \in R\\
9|x-y|^2 &\leq |x-y|^2 + 16k|x-y|^2, \forall x, y \in R\\
9|x-y|^2 &\leq (1+16k)|x-y|^2, \forall x, y \in R\\
9 &\leq (1+16k)\\
8 &\leq 16k\\
1 &\leq 2k\\
k &= \frac{1}{2}.
\end{aligned}$$

Thus $k = \frac{1}{2} \in [0, 1)$ as expected and that completes the proof of the example.

Example 4.2. Two Strictly Pseudo-contrative mappings with a common fixed points
Let $H = R$ denote the reals with the usual norm $|.|$ and the inner product $\langle x, y\rangle = xy$, let R denote the reals and define $T_i : R \to R$ for each $i = 1, 2$ by

$$T_1(x) := \begin{cases} -3x+1 & \text{if } x \in (-\infty, 0] \\ \frac{1}{2}(x+2) & \text{if } x \in (0, \infty). \end{cases} \tag{49}$$

and

$$T_2(x) := \begin{cases} 2 & \text{if } x \in [0, \infty) \\ -3x & \text{if } x \in (-\infty, 0). \end{cases} \tag{50}$$

Then T_1 and T_2 are strictly pseudo-contractive mappings with constant $k = \dfrac{1}{2}$ and $\cap_{i=1}^{2} Fix(T_i) = \{2\}$, for each $i = 1, 2$.

Next, we implement our method in Python Program using this example to show the convergence of the method to the common fixed point of these mappings. Now we construct the following initialization. Let $\alpha_n = \dfrac{1}{n+1}$ so that $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\zeta_n = \dfrac{1}{1+n^2}$ such that $\lim_{n\to\infty} \zeta_n = 0, \lim_{n\to\infty}, \dfrac{\zeta_n}{\alpha_n} = 0$.Let $\tau_n = \min\{\dfrac{1}{3}, 1\} = \dfrac{1}{3}$. Let $\beta_n^i = \dfrac{2^{i-1}}{3^i}, \forall i = 1, 2, ..., N$ which implies that $\sum_{i=1}^{N} \beta_n^i = \sum_{i=1}^{N} \dfrac{2^{i-1}}{3^i} < 1$. This implies that $\beta_n^0 = 1 - \sum_{i=1}^{N} \dfrac{2^{i-1}}{3^i}$, such that our hypothesis are satisfied. With a stopping criterion of $\|x_n - x_{n+1}\| \leq 10^{-7}$, the results of Algorithm 2.1 are displayed:

n	z[n]	x[n]	err[n]
1	1.3300000	1.0000000	0.2208333
2	1.2937083	1.2208333	0.0798843
3	1.1145873	1.1409491	0.0943320
4	1.0154875	1.0466170	0.0441352
5	0.9879172	1.0024818	0.0001889
6	1.0022306	1.0022929	0.0263274
7	1.0373084	1.0286203	0.0394170
8	1.0810450	1.0680373	0.0444176
9	1.1271128	1.1124550	0.0450747

10	1.1724044	1.1575297	0.0435867
100	1.8830614	1.8826797	0.0011343
500	1.9761197	1.9761040	0.0000475
1000	1.9880298	1.9880259	0.0000119
10000	1.9988005	1.9988005	0.0000001
10003	1.9988007	1.9988006	0.0000001
10004	1.9988008	1.9988007	0.0000001
10005	1.9988009	1.9988009	0.0000001
10006	1.9988010	1.9988010	0.0000001
10007	1.9988011	1.9988011	0.0000001
10008	1.9988013	1.9988012	0.0000001
10009	1.9988014	1.9988013	0.0000001
10010	1.9988015	1.9988015	0.0000001
10011	1.9988016	1.9988016	0.0000001
13392	1.9991041	1.9991041	0.0000001
15487	1.9992253	1.9992253	0.0000001
15488	1.9992253	1.9992253	0.0000001
15489	1.9992254	1.9992254	0.0000000

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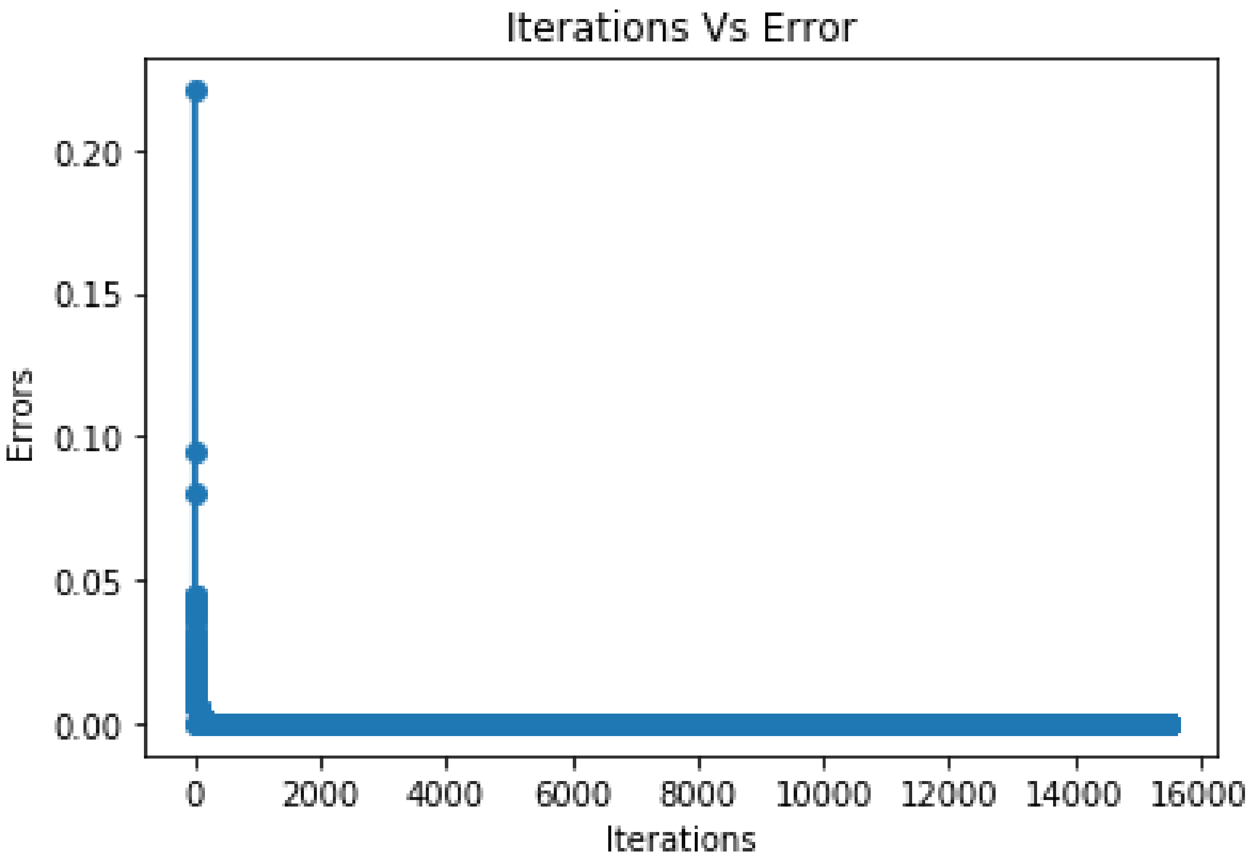

Fig. 1

n	z[n]	x[n]	err[n]
1	1.3300000	1.0000000	0.2766667
2	1.3679667	1.2766667	0.0242370

3	1.2444314	1.2524296	0.0302139
4	1.2122451	1.2222157	0.0164719
5	1.2441233	1.2386876	0.0465091
6	1.3005447	1.2851967	0.0564273
7	1.3602450	1.3416240	0.0551856
8	1.4150208	1.3968096	0.0496985
9	1.4629086	1.4465081	0.0434322
10	1.5042729	1.4899403	0.0377019
50	1.8837069	1.8829398	0.0022357
51	1.8859134	1.8851756	0.0021521
52	1.8880379	1.8873277	0.0020731
53	1.8900849	1.8894007	0.0019983
54	1.8920585	1.8913990	0.0019275
55	1.8939626	1.8933266	0.0018604
56	1.8958009	1.8951870	0.0017967
57	1.8975766	1.8969837	0.0017363
58	1.8992929	1.8987200	0.0016788
59	1.9009528	1.9003988	0.0016241
60	1.9025589	1.9020229	0.0015721
61	1.9041138	1.9035950	0.0015225
62	1.9056199	1.9051175	0.0014752
63	1.9070795	1.9065927	0.0014301
64	1.9084948	1.9080228	0.0013870
65	1.9098676	1.9094099	0.0013459
66	1.9111999	1.9107558	0.0013065
67	1.9124934	1.9120623	0.0012689
68	1.9137499	1.9133312	0.0012328
69	1.9149708	1.9145640	0.0011983
70	1.9161577	1.9157622	0.0011652
71	1.9173119	1.9169274	0.0011334
72	1.9184348	1.9180608	0.0011029
73	1.9195276	1.9191637	0.0010736
74	1.9205916	1.9202373	0.0010455
75	1.9216279	1.9212829	0.0010185
76	1.9226375	1.9223014	0.0009925
77	1.9236214	1.9232938	0.0009675
78	1.9245806	1.9242613	0.0009434
79	1.9255161	1.9252047	0.0009202
80	1.9264286	1.9261249	0.0008979
81	1.9273191	1.9270228	0.0008763
82	1.9281883	1.9278991	0.0008555
83	1.9290370	1.9287546	0.0008355
84	1.9298658	1.9295901	0.0008161
85	1.9306756	1.9304063	0.0007974
86	1.9314668	1.9312037	0.0007794
87	1.9322403	1.9319831	0.0007619
88	1.9329964	1.9327450	0.0007451
89	1.9337359	1.9334901	0.0007287
90	1.9344593	1.9342188	0.0007130
91	1.9351670	1.9349318	0.0006977
92	1.9358597	1.9356294	0.0006829

93	1.9365377	1.9363123	0.0006686
94	1.9372015	1.9369809	0.0006547
95	1.9378516	1.9376356	0.0006412
96	1.9384884	1.9382768	0.0006282
97	1.9391123	1.9389050	0.0006155
98	1.9397236	1.9395205	0.0006033
99	1.9403228	1.9401237	0.0005913
100	1.9409102	1.9407151	0.0005798
10950	1.9994521	1.9994521	0.0000001
10951	1.9994522	1.9994522	0.0000001
10952	1.9994522	1.9994522	0.0000001

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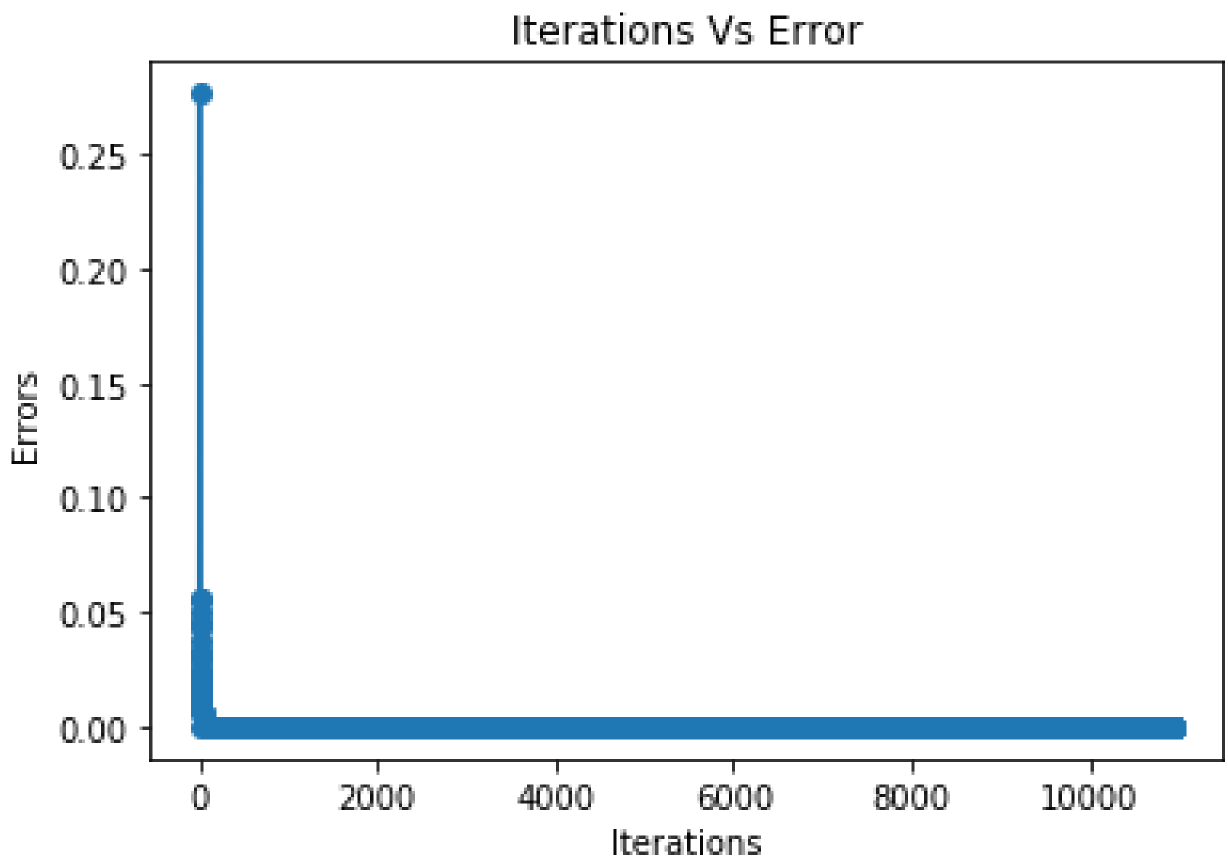

Fig. 2

5 Discussions and Conclusion

We observed that the choice of the control sequences of $\{\alpha_n\}, \{\beta_n^i\}$, $\{\zeta_n\}$ and $\{\tau_n\}$ satisfying the convergence hypothesis, influenced to a great deal, the strong convergence of the Algorithm 2.1 to the common fixed point of the family of mappings. With a stopping criteria of $\|x_{n+1} - x_n\| \leq 10^{-7}$, the computation can be truncated after many iterations. If the stooping criteria is higher at $\|x_{n+1} - x_n\| \leq 10^{-5}$, then truncating the computation early may have a negligible computational error. Consequently, $\|x_n - p\| = \|x_n - 1\| \to F = \cap_{i=1}^{N} F(T_i)$ as $n \to \infty$. Thus, we introduced the Halpern type algorithm with inertial and error terms for approximating the common fixed point of a family of strictly pseudo-contractive mappings defined on a real Hilbert spaces. We proved strong convergence theorem for it. We supported our paper with a some examples to illustrate the effectiveness of our algorithm. We observed that our algorithm performed favourably with a "good" stopping criteria $\|x_n - x_{n+1}\| \leq 10^{-7}$. Therefore, our algorithm is a good fit for approximating fixed points of strictly pseudo-contractive mappings in Hilbert spaces.

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