International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 7 Issue 2, 2024

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

Picard-Mann hybrid iteration process for approximating fixed points of contraction mappings

E. Ekuma-Okereke*†, O.D. Koko†‡, C.V. Okpako§, and J.O.Ogbodu¶

Abstract

Fixed point theory has been very remarkable since its introduction and it has evolved greatly by hosting a great number of researches of many sort. Several mathematical problems have been solved by the fixed point method by transforming any of such problem into an operation of the fixed point equation. Fixed point theory is primarily concerned with finding conditions on the ambient space and also on the properties of the mapping so as to find results on the existence and uniqueness of fixed points. In this paper, we construct and introduce a new four-step iteration process called the Picard-Mann hybrid iteration process. We prove that this new iteration process converges to the unique fixed point of contraction mappings. We then show that it converges faster than several other iteration processes mentioned in the body of literature with numerical examples visualized in graphs and tables to substantiate our theoretical assertions. Data dependence and numerical stability is further established. Our results obtained in uniformly convex Banach space generalizes and extend other results cited in the literature.

Keywords and phrases: Fixed points theory; contraction mappings; Picard-Mann hybrid process; uniformly convex Banach space; convergence result.

1 Introduction

Fixed point theory has been very remarkable since its introduction and it has evolved greatly by hosting a great number of researches of many sort. Several mathematical problems have been solved by the fixed point method by transforming any of such problem into an operation of the fixed point equation. Fixed point theory is primarily concerned with finding conditions on the ambient space and also on the properties of the mapping so as to find results on the existence and uniqueness of fixed points.

Historically, Poincare (see [1]) was the first to research on fixed point theory, and was followed by Brouwer. In 1922, Banach established a principle called Banach contraction principle (BCP) in metric spaces (see, [2]). So many authors after Banach have studied fixed point theory away from metric space to Hilbert and general Banach spaces (see [3]-[5]). Notably it has become a flourishing and interesting research area for many researchers seeking approximation of fixed points of different mappings. The theory of fixed point have been applied in many fields like numerical mathematics, differential equations (see [6]).

A plethora of metrical fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and uniqueness of fixed points for a certain contractive mapping. Among these fixed point theorems, only a small number are important from a practical point of view, that is, they offer a constructive method for finding the fixed points. Among the last ones only a few give information on the rate of convergence of the method [7].

*Corresponding Author; E-mail: ekuma.okereke@fupre.edu.ng, enyiekumaokereke@gmail.com

†Department of Mathematics, Federal University of Petroleum Resources, Effurun, Nigeria

‡E-mail: kokodogheneovo@gmail.com

§Department of Mathematics, Federal University of Petroleum Resources, Effurun, Nigeria; E-mail: mcvotu@gmail.com

¶Department of Mathematics, Federal University of Petroleum Resources, Effurun, Nigeria; E-mail:ogbodujoseph1987@gmail.com

However, from a practical point of view it is important not only to know the fixed point exists (and, possibly, is unique), but also to be able to construct that fixed point(s). As the constructive methods used in metrical fixed point theory are prevailingly iterative procedures, that is, approximate methods, it is also of crucial importance to have a priori or / and a posteriori error estimates (or, alternatively, rate of convergence) for such a method, [8]. A vast literature is available on iterative methods for studying fixed point problems (FPP), variational inequality problems (VIP), monotone variational inclusion problems (MVIP), equilibrium problems (EP), etc for nonlinear mappings, separately, and on iterative methods to approximate common solutions of these problems. But to construct and study the more general iterative methods for approximating solutions (common solutions) of new generalizations of these problems in the settings of Hilbert spaces and Banach spaces, is still an unexplored field. There are substantial number of iterative methods for studying separately, the fixed point problems for nonlinear mappings, variational inequalities, and equilibrium problems; and for approximating the common solutions of these problems.

Picard, Krasnoselkii and Mann iteration processes (see [9]-[11]) were among the first processes to be introduced in this direction for the approximation of fixed points of contraction mappings in metric spaces and closed convex subset of Banach spaces respectively. Others after followed from these processes.

Ishikawa [12] introduced the Ishikawa iterative technique which is recursively defined as follows:

$$\begin{cases} \omega_1 = \omega \in K, \\ \omega_{n+1} = (1-\alpha_n)\omega_n + \alpha_n \Omega, \\ \theta_n = (1-\beta_n)\omega_n + \beta_n \Omega \omega_n, n \geq 1, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ and $\{\alpha_n\} \in (0,1)$.

Noor [13] introduced a three-step approximation processes for general variational inequalities. The iteration process is defined by the sequence $\{t_n\}$:

$$\begin{cases} t_1 = t \in C \\ t_{n+1} = (1-\alpha_n)t_n + \alpha_n \Omega y_n, \\ y_n = (1-\beta_n t_n + \beta_n \Omega z_n, \\ Z_n = (1-\gamma_n)t_n + \gamma_n \Omega t_n, n \in N, \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \in (0,1)$. This process is also called a three-step approximation process for solving variational inequality problems.

Khan [14] introduced an iterative process called Picard-Mann hybrid iterative process, which is a hybrid of Picard and Mann iterative processes and defined thus:

$$\begin{cases} m_1 = m \in K \\ m_{n+1} = \Omega z_n, \\ z_n = (1-\alpha_n)m_n + \alpha_n \Omega m_n, n \in N, \end{cases} \tag{3}$$

where $\{\alpha_n\} \in (0,1)$. They showed that the process converges faster than all of [9], [10], [11] and [12] iterative processes in the sense of [15].

Thakur [16] introduced a new iterative process defined thus:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\Omega x_n + \alpha_n \Omega y_n, \\ y_n = (1-\beta_n)z_n + \beta_n \Omega z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \Omega x_n, \end{cases} \tag{4}$$

where $x_1 \in K$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$. They showed that the process converges faster than all of iteration processes for contractions in the sense of [15]

Abbas [17], introduced a new iterative process defined thus:

$$\begin{cases} y_1 = y \in K, \\ y_{n+1} = (1-\alpha_n)\Omega w_n + \gamma_n \Omega z_n, \\ w_n = (1-\beta_n)\Omega y_n + \alpha_n \Omega z_n, \\ z_n = (1-\gamma_n) y_n + \beta_n \Omega y_n, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$. They showed that the process developed has rate of convergence that is even faster than that of [18].

[19] introduced the Picard-Krasnoselskii hybrid iterative process which is a hybrid of Picard and Krasnoselskii iteration processes and showed that for nonlinear contractive operators, the iteration process converges faster than all of Picard, Mann, Ishikawa and Krasnoselskii iteration processes for contractions in the sense of [15]. The scheme is defined thus:

$$\begin{cases} y_1 = y \in K, \\ y_{n+1} = \Omega w_n, \\ w_n = (1-\lambda) y_n + \lambda \Omega y_n, \end{cases} \tag{6}$$

where $\lambda \in (0,1)$.

[20] intrdoced a hybrid of Picard and Ishikawa iteration processes called the Picard-Ishikawa hybrid iterative process and showed that it converges faster than all of Picard, Krsanoselskii, Mann, Ishikawa, Noor, Picard-Mann and Picard-Krasnoselskii iteration processes in the sense of [15]. The iterative process is defined thus:

$$\begin{cases} g_1 = g \in K, \\ g_{n+1} = \Omega w_n, \\ w_n = (1-\alpha_n) g_n + \alpha_n \Omega z_n, \\ z_n = (1-\beta_n) g_n + \beta_n \Omega g_n, \end{cases} \tag{7}$$

where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$.

[21], intrdoced a hybrid of Picard (see [9]) and Thakur (see [16]) iteration processes called the Picard-Thakur hybrid iterative process. The iterative process is defined thus:

$$\begin{cases} j_1 = j \in K, \\ j_{n+1} = \Omega k_n, \\ k_n = (1-\alpha_n)\Omega m_n + \alpha_n \Omega l_n, \\ l_n = (1-\beta_n) m_n + \beta_n \Omega m_n, m_n = (1-\gamma_n) j_n + \gamma_n \Omega j_n, n \in N \end{cases} \tag{8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$.

[22] (cf [23]) introduced a new four-step iteration process, called the Picard-Noor hybrid iterative processis defined thus:

$$\begin{cases} j_1 = j \in K, \\ j_{n+1} = \Omega k_n, \\ k_n = (1-\alpha_n) j_n + \alpha_n \Omega l_n, \\ l_n = (1-\beta_n) j_n + \beta_n \Omega m_n, m_n = (1-\gamma_n) j_n + \gamma_n \Omega j_n, n \in N. \end{cases} \tag{9}$$

They applied their results in solving delay differential equations.

[24] introduced three step iteration process, the SP iteration process, which generates the sequence ω_n defined by:

$$\begin{cases} \omega_1 = \omega, \\ \omega_{n+1} = (1-\alpha_n) w_n + \alpha_n \Omega w_n, \\ w_n = (1-\beta_n) u_n + \beta_n \Omega u_n, \\ u_n = (1-\gamma_n)\omega_n + \gamma_n \Omega \omega_n, n \in \mathbf{N}, \end{cases} \tag{10}$$

There is no iteration process that generalizes and extends other existing processes for a contraction mapping in a uniformly convex Banach space, hence this paper constructs a new hybrid fixed point iteration of Picard and Mann methods to approximate the fixed point of any contraction mapping.

2 Methodology/Model formulation

Let Ω represent an operator (mapping) defined on ambient space. Let ω_1=ω represent an arbitrary initial value from the ambient space. Picard iteration process is defined by the sequence $\{\omega_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} \omega_1 = \omega \in K, \\ \omega_{n+1} = \Omega\omega_n, n \geq 1, \end{cases} \tag{11}$$

Independently, Mann iterative process is defined as:

$$\begin{cases} \omega_1 = \omega \in K, \\ \omega_{n+1} = (1-\alpha_n)\omega_n + \alpha_n\Omega\omega_n, n \geq 1, \end{cases} \tag{12}$$

where $\{\alpha_n\} \in [0, 1]$.

Since we are interested in having only one recursive process written with respect to ω_{n+1} and generated by an initial value $\omega_1 = \omega$, we observe that both Picard and Mann iterative processes have the recursive sequence ω_{n+1}, we therefore proceed by setting the recursive process in Mann as follows:

Step1:

$$u_n = (1-\alpha_n)\omega_n + \alpha_n\Omega\omega_n, \tag{13}$$

Step 2:

$$w_n = (1-\beta_n)u_n + \beta_n\Omega u_n \tag{14}$$

Step 3:

$$x_n = (1-\gamma_n)w_n + \gamma_n\Omega w_n \tag{15}$$

Finally, we construct a recursive process following Picard iteration process with respect to the three steps of Mann iteration process as follows:

Step 4:

$$\omega_{n+1} = \Omega x_n. \tag{16}$$

Observations

i. The three steps of Mann iteration process has three different control sequences α_n, β_n and γ_n which are elements in the given interval $[0, 1]$,

ii. If $\alpha_n = \beta_n = \gamma_n = 0$, then the three steps reduces to one step Picard iteration process,

iii. If If $\alpha_n = \beta_n = \gamma_n = 1$, then the three steps reduces to sequence of composed operators of Picard iteration process form.

Now, the coupled and compact form of these processes is defined below:

$$\begin{cases} \omega_1 = \omega, \\ \omega_{n+1} = \Omega x_n, \\ x_n = (1-\alpha_n)w_n + \alpha_n\Omega w_n, \\ w_n = (1-\beta_n)u_n + \beta_n\Omega u_n, \\ u_n = (1-\gamma_n)\omega_n + \gamma_n\Omega\omega_n, n \in \mathbf{N}, \end{cases} \tag{17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, are real sequences in $[0, 1]$.

The following definitions will be used in the sequel.

Definition 2.1. *(Uniformly convex Banach spaces) [22]*
A normed space X is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that for any $x, y \in X$ with $\|x\| = 1 = \|y\|$ and $\|x - y\| \geq \epsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.

Definition 2.2. *(Mappings in Normed Vector Space)[1]*
Let $(X, \|.\|)$ be a normed vector space. A mapping $\Omega : X \to X$ is called

(i) Lipschitzian, if $\exists\ L > 0$ satisfying $\|\Omega x - \Omega y\| \leq L.\|x - y\|, \forall x, y \in X$,

(ii) Contraction, when $0 \leq L < 1$ (see [7]),

(iii) Nonexpansive, when $L = 1$, (see [25]).

Definition 2.3. *(Fixed Point Set)[1]*
Let Ω be any mapping defined on $X \neq \emptyset$. Let the fixed point of Ω be denoted by $Fix(\Omega)$. The set of fixed point of Ω is denoted by $Fix(\Omega) = \{x \in X : \Omega(\omega) = \omega\}$. So ω is a point invariant under the transformation Ω.

Definition 2.4. *Approximate operator[1]*
Let $\Omega, \hat{\Omega} : D \to D$ be two operators. We say that $\hat{\Omega}$ is an approximate operator of Ω if for all $\omega \in D$ and for a fixed $\epsilon > 0$ we have

$$\|\Omega - \hat{\Omega}\| \leq \epsilon \tag{18}$$

Definition 2.5. *Stability[26]*
Intuitively, a fixed point iteration procedure is numerically stable if, "small" modifications in the initial data or in the data that are involved in the computation process, will produce a "small" influence on the computed value of the fixed point.

Definition 2.6. *Faster Convergence[15] Let ${a_n}_{n=0}^{\infty}$ and ${b_n}_{n=0}^{\infty}$ be two sequences of real numbers converging to a and b respectively. If*

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$$

then ${a_n}_{n=0}^{\infty}$ is said to converge to a faster than ${b_n}_{n=0}^{\infty}$ to b.

3 Analysis of the model

Overtime, we observe that improvements of some iterations can be based on better and convergence results. We have also observed that hybrid models of any iteration process significantly generate improved convergent result as compared to the normal or classical iteration process. The flexibility of fixed point theory and its application has influenced new hybrid iteration process. We have proposed a new iteration process called Picard–Mann Hybrid (HybridPicMan) iteration process in the following theorem.

Theorem 3.1. *Let K be a nonempty closed convex subset of a Banach space $\mathbb{B}$ and let $\Omega : K \to K$ be a contraction map satisfying the contractive condition and such that $Fix(\Omega) \neq \emptyset$. Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence generated by the hybrid iterative process defined as follows:*

$$\begin{cases} \omega_1 = \omega, \\ \omega_{n+1} = \Omega x_n, \\ x_n = (1 - \alpha_n)w_n + \alpha_n \Omega w_n, \\ w_n = (1 - \beta_n)u_n + \beta_n \Omega u_n, \\ u_n = (1 - \gamma_n)\omega_n + \gamma_n \Omega \omega_n, n \in \mathbf{N}, \end{cases} \tag{19}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, are real sequences in (0,1) satisfying $\sum_{n=0}^{\infty} \alpha_n = 0$. Then $\{\omega_n\}$ converges to a fixed point of Ω.

Proof. Banach's fixed point theorem guarantees the existence and the uniqueness of fixed point say ω^*. All we need do here is to show that the sequence $\{\omega_n\}_{n=1}^{\infty}$ generated by the hybrid iterative process converges to this point ω^* as $n \to \infty$. So let $\omega^* \in Fix(\Omega)$, then using equations (triangle inequality).....we obtain the following conditions (estimates):

$$\begin{aligned} \|u_n - \omega^*\| &= \|(1-\gamma_n)\omega_n + \gamma_n \Omega \omega_n - \omega^*\| \\ &= \|(1-\gamma_n)(\omega_n - \omega^*) + \gamma_n(\Omega\omega_n - \omega^*)\| \\ &\leq (1-\gamma_n)\|\omega_n - \omega^*\| + \gamma_n\|\Omega\omega_n - \omega^*\| \\ &\leq (1-\gamma_n)\|\omega_n - \omega^*\| + L.\gamma_n\|\omega_n - \omega^*\| \\ &= [1-\gamma_n(1-L)]\,\|\omega_n - \omega^*\|. \end{aligned} \tag{20}$$

In addition,

$$\begin{aligned} \|w_n - \omega^*\| &= \|(1-\beta_n)u_n + \beta_n \Omega u_n - \omega^*\| \\ &= \|(1-\beta_n)(u_n - \omega^*) + \beta_n(\Omega u_n - \omega^*)\| \\ &\leq (1-\beta_n)\|u_n - \omega^*\| + \beta_n\|\Omega u_n - \omega^*\| \\ &\leq (1-\beta_n)\|u_n - \omega^*\| + L.\beta_n\|u_n - \omega^*\| \\ &= [1-\beta_n(1-L\,[1-\gamma_n(1-L)])]\,\|\omega_n - \omega^*\| \\ &\leq [1-\beta_n(1-L)]\,\|\omega_n - \omega^*\|. \end{aligned} \tag{21}$$

Also,

$$\begin{aligned} \|x_n - \omega^*\| &= \|(1-\alpha_n)w_n + \alpha_n \Omega w_n - \omega^*\| \\ &= \|(1-\alpha_n)(w_n - \omega^*) + \alpha_n(\Omega w_n - \omega^*)\| \\ &\leq (1-\alpha_n)\|w_n - \omega^*\| + \alpha_n\|\Omega w_n - \omega^*\| \\ &\leq (1-\alpha_n)\|w_n - \omega^*\| + L.\alpha_n\|w_n - \omega^*\| \\ &= [1-\alpha_n(1-L\,[1-\beta_n(1-L)])]\,\|\omega_n - \omega^*\|. \end{aligned} \tag{22}$$

Furthermore,

$$\begin{aligned} \|\omega_{n+1} - \omega^*\| &= \|\Omega x_n - \omega^*\| \\ &= \|\Omega x_n - \Omega\omega^*\| \\ &\leq L\|x_n - \omega^*\| \\ &\leq L \times [1-\alpha_n(1-L\,[1-\beta_n(1-L)])]\,\|\omega_n - \omega^*\|. \end{aligned} \tag{23}$$

Since from our hypothesis, $\beta_n \in (0,1)$ and $0 \leq L < 1$, we then obtain from condition (23) that

$$\begin{aligned} \|\omega_{n+1} - \omega^*\| &= \|\Omega x_n - \omega^*\| \\ &= \|\Omega x_n - \Omega\omega^*\| \\ &\leq L\|x_n - \omega^*\| \\ &\leq L\,[1-\alpha_n(1-L)]\,\|\omega_n - \omega^*\|. \end{aligned} \tag{24}$$

If $n = 0$, we obtain by induction that

$$\|\omega_1 - \omega^*\| \leq L\,[1-\alpha_0(1-L)]\,\|\omega_0 - \omega^*\|. \tag{25}$$

If $n = 1$, we obtain by induction that

$$\begin{aligned} \|\omega_2 - \omega^*\| &\leq L\,[1-\alpha_1(1-L)]\,\|\omega_1 - \omega^*\| \\ &\leq L\,[1-\alpha_1(1-L)]\,.L\,[1-\alpha_0(1-L)]\,\|\omega_0 - \omega^*\| \\ &= L^2 \prod_{k=0}^{1} [1-\alpha_k(1-L)]\,\|\omega_0 - \omega^*\|. \end{aligned} \tag{26}$$

If $n = k + 1$, we obtain by induction that

$$\begin{aligned}
\|\omega_{k+1} - \omega^*\| &\leq L\left[1 - \alpha_k(1 - L)\right]\|\omega_k - \omega^*\| \\
&\leq L^k\left[1 - \alpha_1(1 - L)\right].L\left[1 - \alpha_0(1 - L)\right]\|\omega_0 - \omega^*\| \\
&= L^{k+1}\prod_{n=0}^{k}\left[1 - \alpha_n(1 - L)\right]\|\omega_0 - \omega^*\|.
\end{aligned} \tag{27}$$

So that

$$\|\omega_{n+1} - \omega^*\| \leq L^{n+1}\prod_{k=0}^{n}\left[1 - \alpha_k(1 - L)\right]\|\omega_0 - \omega^*\|. \tag{28}$$

Therefore,

$$\|\omega_{n+1} - \omega^*\| \leq \mathbb{L}\|\omega_0 - \omega^*\|. \tag{29}$$

where $0 \leq \mathbb{L} = L^{k+1}\prod_{n=0}^{k}\left[1 - \alpha_n(1 - L)\right]\| < 1$. Thus $\{\omega_n\}_{n=0}^{\infty}$ is bounded and consequently $\lim_{n\to\infty}\|\omega_n - \omega^*\| = 0$. Hence $\omega_n \to \omega^*$ as $n \to \infty$. This completes the proof of the Theorem.

In this subsection, data dependence and stability results of the Picard-Noor iterative process will be established.

Theorem 3.2. *Let Θ be an approximate mapping of Ω which satisfies the contraction mapping condition given by inequality (18). Given the iterative process of our iterative process of equation (19) whose sequences converges to some fixed point $\omega^* \in Fix(\Omega))$ and we define an iterative sequence θ_n for the mapping Θ with some fixed point $\theta^* \in Fix(\Theta)$ as follows:*

$$\begin{cases}
\theta_1 = \theta, \\
\theta_{n+1} = \Theta\hat{x_n}, \\
\hat{x_n} = (1 - \alpha_n)\hat{w_n} + \alpha_n\Theta\hat{w_n}, \\
\hat{w_n} = (1 - \beta_n)\hat{u_n} + \beta_n\Theta\hat{u_n}, \\
\hat{u_n} = (1 - \gamma_n)\theta_n + \gamma_n\Theta\theta_n, n \in \mathbf{N},
\end{cases} \tag{30}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, are real sequences in (0,1) satisfying $\sum_{n=0}^{\infty}\alpha_n = 0$ and there exists $\mu > 0$ such that $\|\Omega\omega^* - \Theta\theta^*\| \leq \mu$. Then $\|\omega^* - \theta^*\| \leq \dfrac{5\mu}{1 - L}$, where $0 \leq L < 1$.

Proof.

$$\begin{aligned}
\|u_n - \hat{u_n}\| &= \|(1 - \gamma_n)\omega_n + \gamma_n\Omega\omega_n - (1 - \gamma_n)\theta_n - \gamma_n\Theta\theta_n\| \\
&= \|(1 - \gamma_n)(\omega_n - \theta_n) + \gamma_n(\Omega\omega_n - \Theta\theta_n\| \\
&\leq (1 - \gamma_n)\|\omega_n - \theta_n\| + \gamma_n\|\Omega\omega_n - \Theta\theta_n\| \\
&= (1 - \gamma_n)\|\omega_n - \theta_n\| + \gamma_n\|\Omega\omega_n - \Theta\theta_n + \Theta\theta_n - \Theta\theta_n\| \\
&\leq (1 - \gamma_n)\|\omega_n - \theta_n\| + \gamma_n\left[\|\Omega\omega_n - \Theta\theta_n\| + \|\Theta\theta_n - \Theta\theta_n\|\right] \\
&\leq (1 - \gamma_n)\|\omega_n - \theta_n\| + \gamma_n L\|\omega_n - \theta_n\| + \gamma_n\mu \\
&= (1 - \gamma_n(1 - L))\|\omega_n - \theta_n\| + \gamma_n\mu.
\end{aligned} \tag{31}$$

In addition,

$$\begin{aligned}
\|w_n - \hat{w_n}\| &= \|(1 - \beta_n)u_n + \beta_n\Omega u_n - (1 - \beta_n)u_n - \beta_n\Theta\hat{u_n}\| \\
&= \|(1 - \beta_n)(u_n - \hat{u_n}) + \beta_n(\Omega u_n - \Theta\hat{u_n})\| \\
&\leq (1 - \beta_n)\|u_n - \hat{u_n}\| + \beta_n\|\Omega u_n - \Theta\hat{u_n}\| \\
&= (1 - \beta_n)\|u_n - \hat{u_n}\| + \beta_n\|\Omega u_n - \Theta\hat{u_n} + \Theta\hat{u_n} - \Theta\hat{u_n}\| \\
&\leq (1 - \beta_n)\|u_n - \hat{u_n}\| + \beta_n\left[\|\Omega u_n - \Theta\hat{u_n}\| + \|\Theta\hat{u_n} - \Theta\hat{u_n}\|\right] \\
&\leq \left[(1 - \beta_n) + \beta_n L\right]\|u_n - \hat{u_n}\| + \beta_n\mu.
\end{aligned} \tag{32}$$

Also,

$$\begin{aligned}
\|x_n - \hat{x_n}\| &= \|(1-\alpha_n)w_n + \alpha_n\Omega w_n - (1-\alpha_n)\hat{w_n} - \alpha_n\Theta\hat{w_n}\| \\
&= \|(1-\alpha_n)(w_n - \hat{w_n} + \alpha_n(\Omega w_n - \Theta\hat{w_n})\| \\
&\leq (1-\alpha_n)\|w_n - \hat{w_n}\| + \alpha_n\|\Omega w_n - \Theta\hat{w_n}\| \\
&= (1-\alpha_n)\|w_n - \hat{w_n}\| + \alpha_n\|\Omega w_n - \Omega\hat{w_n} + \Omega\hat{w_n} - \Theta\hat{w_n}\| \\
&\leq (1-\alpha_n)\|w_n - \hat{w_n}\| + \alpha_n\left[\|\Omega w_n - \Omega\hat{w_n}\| + \|\Omega\hat{w_n} - \Theta\hat{w_n}\|\right] \\
&\leq (1-\alpha_n)\|w_n - \hat{w_n}\| + \alpha_n L\|w_n - \hat{w_n}\| + \alpha_n\mu. \qquad (33)
\end{aligned}$$

Continuing we have

$$\|x_n - \hat{x_n}\| \leq (1-\alpha_n(1-L[1-\beta_n[1-L[1-\gamma_n(1-L)]]]))\|\omega_n - \theta_n\| + \alpha_n L\beta_n L\gamma_n\mu + \alpha_n L\beta_n\mu + \alpha_n\mu. \qquad (34)$$

Furthermore,

$$\begin{aligned}
\|\omega_{n+1} - \theta_{n+1}\| &= \|\Omega x_n - \Theta\hat{x_n}\| \\
&= \|\Omega x_n - \Omega\hat{x_n} + \Omega\hat{x_n} - \Theta\hat{x_n}\| \\
&\leq \|\Omega x_n - \Omega\hat{x_n}\| + \|\Omega\hat{x_n} - \Theta\hat{x_n}\| \\
&\leq L\|x_n - \hat{x_n}\| + \mu. \qquad (35)
\end{aligned}$$

Continuing we have

$$\begin{aligned}
\|\omega_{n+1} - \theta_{n+1}\| \leq{}& L(1-\alpha_n(1-L[1-\beta_n[1-L[1-\gamma_n(1-L)]]]))\|\omega_n - \theta_n\| + \alpha_n L\beta_n L\gamma_n\mu + \alpha_n L\beta_n\mu \\
&+ \alpha_n\mu + \mu. \qquad (36)
\end{aligned}$$

From our hypothesis, observe that

$$L < 1 \qquad (37)$$

$$\beta_n\gamma_n L^2 < 1 \qquad (38)$$

$$\beta_n L^2 < 1 \qquad (39)$$

$$(1-\beta_n[1-L[1-\gamma_n(1-L)]]) < 1 \qquad (40)$$

$$1-\alpha_n \leq \alpha_n \implies 1 = 1-\alpha_n+\alpha_n \leq \alpha_n+\alpha_n = 2\alpha_n. \qquad (41)$$

So from (36) we have that

$$\begin{aligned}
\|\omega_{n+1} - \theta_{n+1}\| &\leq (1-\alpha_n(1-L))\|\omega_n - \theta_n\| + \alpha_n 3\mu + \mu \\
&\leq (1-\alpha_n(1-L))\|\omega_n - \theta_n\| + \alpha_n 3\mu + (1-\alpha_n+\alpha_n)\mu \\
&\leq (1-\alpha_n(1-L))\|\omega_n - \theta_n\| + \alpha_n 3\mu + 2\alpha_n\mu \\
&\leq (1-\alpha_n(1-L))\|\omega_n - \theta_n\| + \alpha_n(1-L)\frac{5\mu}{(1-L)}. \qquad (42)
\end{aligned}$$

From (42), let $a_n := \|\omega_n - \theta_n\|$ so that $a_{n+1} := \|\omega_{n+1} - \theta_{n+1}\|$ be a sequence of non-negative real numbers. Let $\hat{\alpha_n} := \alpha_n(1-L) \in (0,1)$ such that $\lim_{n\to\infty}\hat{\alpha_n} = 0, \sum_{n=1}^{\infty}\hat{\alpha_n} = \infty$. Let $\mu_n = \dfrac{5\mu}{(1-L)}$ such that it is non-negative real constants. Now since limit is unique, we have that $\lim_{n\to\infty}\omega_n = \omega^* = \lim_{n\to\infty}\omega_{n+1}$. Since $\lim_{n\to\infty}\theta_n = \theta^* = \lim_{n\to\infty}\theta_{n+1}$, we then have

$$\begin{aligned}
\|\omega^* - \theta^*\| &\leq (1-\hat{\alpha_n})\|\omega^* - \theta^*\| + \hat{\alpha_n}\frac{5\mu}{1-L} \\
\hat{\alpha_n}\|\omega^* - \theta^*\| &\leq \hat{\alpha_n}\frac{5\mu}{1-L} \\
\|\omega^* - \theta^*\| &\leq \frac{5\mu}{1-L}. \qquad (43)
\end{aligned}$$

This completes the proof of data dependence of the iterative process.

4 Numerical simulations/sensitivity analysis

4.1 Stability and Convergence of our Scheme

Firstly, in this section we would numerically show the stability and convergence of our iterative process (19) of Theorem 3.1 using the following examples. Intuitively, a fixed point iteration procedure is numerically stable if, "small" modifications in the initial data or in the data that are involved in the computation process (see [26]), will produce a "small" influence on the computed value of the fixed point.

Example 4.1. Let $K = [1,6] \subset \mathbb{B} = R$ denote the reals with the usual norm $|.|$ and define the map $\Omega : K \to K$ by

$$\Omega(\omega) := \frac{1}{2}\omega + 1, \forall \omega \in [1,6]. \tag{44}$$

Then Ω is a contraction map with a constant $L = \frac{1}{2}$ and $Fix(\Omega) = \{2\}$.

Using this example, we perform the following iterations such that the tables and graphs below are generated with Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz.

Table 4.1: Numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = \frac{1}{n+3}, \beta_n = \frac{1}{n+1}, \gamma_n = \frac{1}{n+2} \in (0,1)$, initial value $x = 6 \in [1,6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	5.3333333	3.6666667	3.2500000	6.0000000	3.3750000
1	2.5468750	2.4101562	2.3417969	2.6250000	0.4541016
2	2.1538086	2.1281738	2.1121521	2.1708984	0.1148224
3	2.0514030	2.0449777	2.0404799	2.0560760	0.0358361
4	2.0187942	2.0169148	2.0155052	2.0202399	0.0124873
5	2.0072681	2.0066624	2.0061865	2.0077526	0.0046594
6	2.0029214	2.0027127	2.0025432	2.0030933	0.0018217
7	2.0012080	2.0011325	2.0010696	2.0012716	0.0007368
8	2.0005105	2.0004821	2.0004580	2.0005348	0.0003058
9	2.0002195	2.0002085	2.0001990	2.0002290	0.0001295
10	2.0000957	2.0000913	2.0000875	2.0000995	0.0000557
11	2.0000422	2.0000404	2.0000389	2.0000438	0.0000243
12	2.0000188	2.0000181	2.0000174	2.0000194	0.0000107
13	2.0000084	2.0000081	2.0000079	2.0000087	0.0000048
14	2.0000038	2.0000037	2.0000036	2.0000039	0.0000021
15	2.0000017	2.0000017	2.0000016	2.0000018	0.0000010
16	2.0000008	2.0000008	2.0000008	2.0000008	0.0000004
17	2.0000004	2.0000004	2.0000003	2.0000004	0.0000002
18	2.0000002	2.0000002	2.0000002	2.0000002	0.0000001
19	2.0000001	2.0000001	2.0000001	2.0000001	0.0000000
20	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
21	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
22	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
23	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
24	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

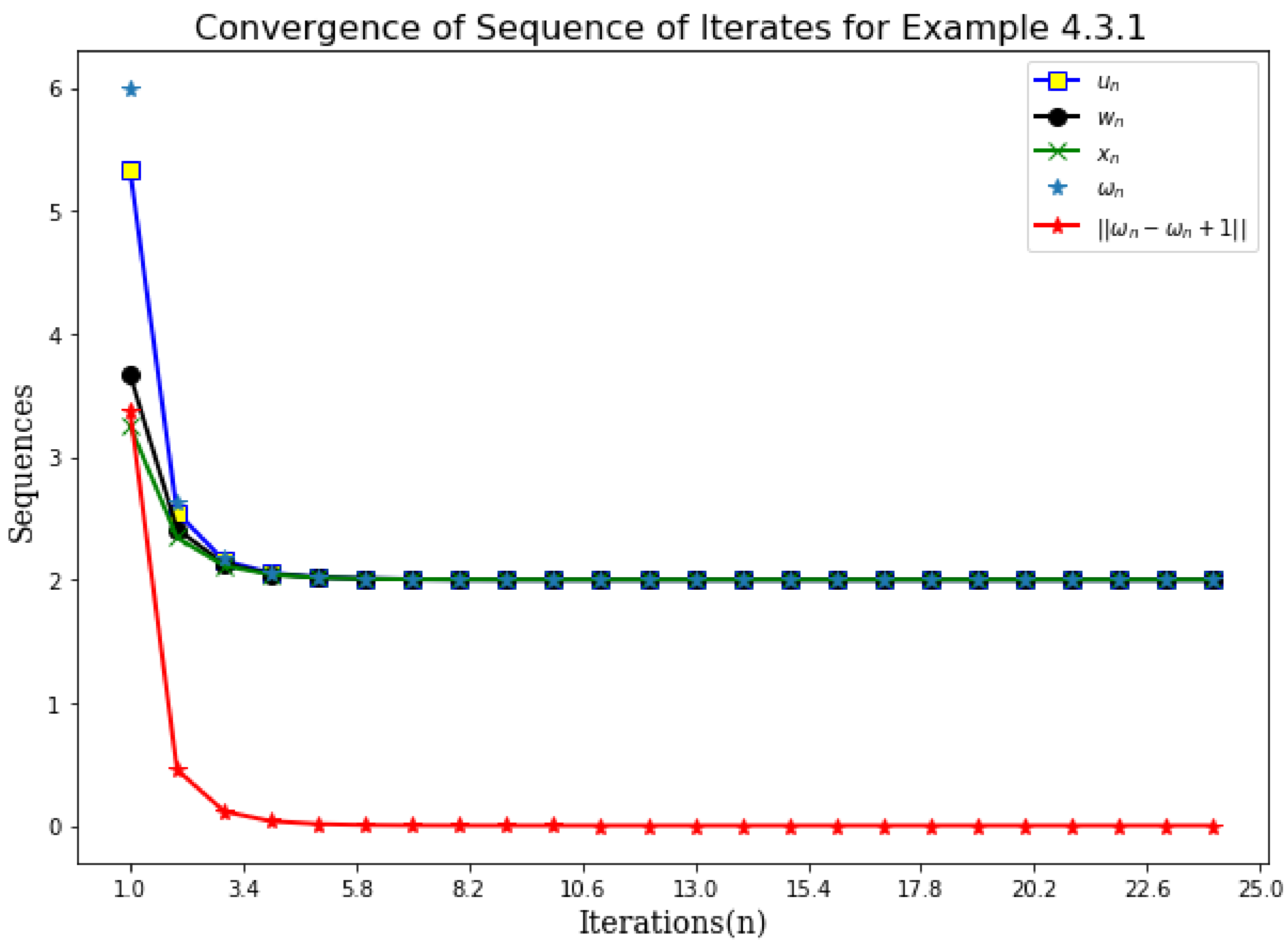

FIGURE 4.1. Graph corresponding to Table 4.1

Table 4.2: Numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25 \in (0,1)$, initial value $x = 6 \in [1,6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	5.0000000	4.5050000	4.1918750	6.0000000	2.9040625
1	2.8219531	2.6863309	2.6005395	3.0959375	0.7956677
2	2.2252023	2.1880439	2.1645384	2.3002698	0.2180005
3	2.0617019	2.0515211	2.0450810	2.0822692	0.0597287
4	2.0169054	2.0141160	2.0123515	2.0225405	0.0163647
5	2.0046318	2.0038676	2.0033841	2.0061757	0.0044837
6	2.0012690	2.0010597	2.0009272	2.0016921	0.0012285
7	2.0003477	2.0002903	2.0002540	2.0004636	0.0003366
8	2.0000953	2.0000795	2.0000696	2.0001270	0.0000922
9	2.0000261	2.0000218	2.0000191	2.0000348	0.0000253
10	2.0000072	2.0000060	2.0000052	2.0000095	0.0000069
11	2.0000020	2.0000016	2.0000014	2.0000026	0.0000019
12	2.0000005	2.0000004	2.0000004	2.0000007	0.0000005
13	2.0000001	2.0000001	2.0000001	2.0000002	0.0000001
14	2.0000000	2.0000000	2.0000000	2.0000001	0.0000000
15	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
16	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
17	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
18	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
19	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
20	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
21	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

22	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
23	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
24	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

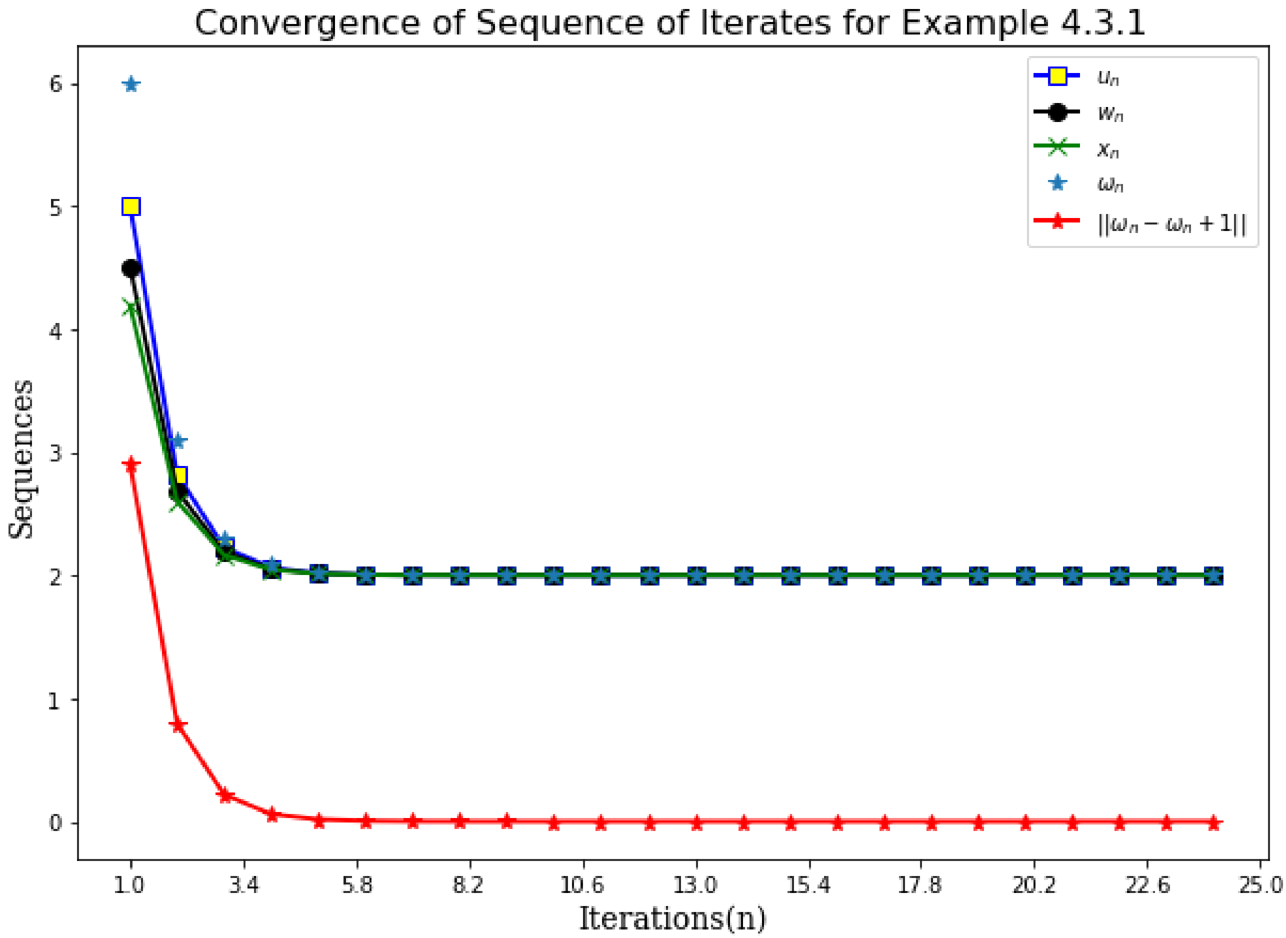

FIGURE 4.2. Graph corresponding to Table 4.2

Table 4.3: Numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25 \in (0, 1)$, initial value $x = 2.5 \in [1, 6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	2.3750000	2.3131250	2.2739844	2.5000000	0.3630078
1	2.1027441	2.0857914	2.0750674	2.1369922	0.0994585
2	2.0281503	2.0235055	2.0205673	2.0375337	0.0272501
3	2.0077127	2.0064401	2.0056351	2.0102837	0.0074661
4	2.0021132	2.0017645	2.0015439	2.0028176	0.0020456
5	2.0005790	2.0004834	2.0004230	2.0007720	0.0005605
6	2.0001586	2.0001325	2.0001159	2.0002115	0.0001536
7	2.0000435	2.0000363	2.0000318	2.0000579	0.0000421
8	2.0000119	2.0000099	2.0000087	2.0000159	0.0000115
9	2.0000033	2.0000027	2.0000024	2.0000044	0.0000032
10	2.0000009	2.0000007	2.0000007	2.0000012	0.0000009
11	2.0000002	2.0000002	2.0000002	2.0000003	0.0000002
12	2.0000001	2.0000001	2.0000000	2.0000001	0.0000001
13	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
14	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
15	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

16	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
17	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
18	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
19	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
20	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
21	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
22	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
23	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
24	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

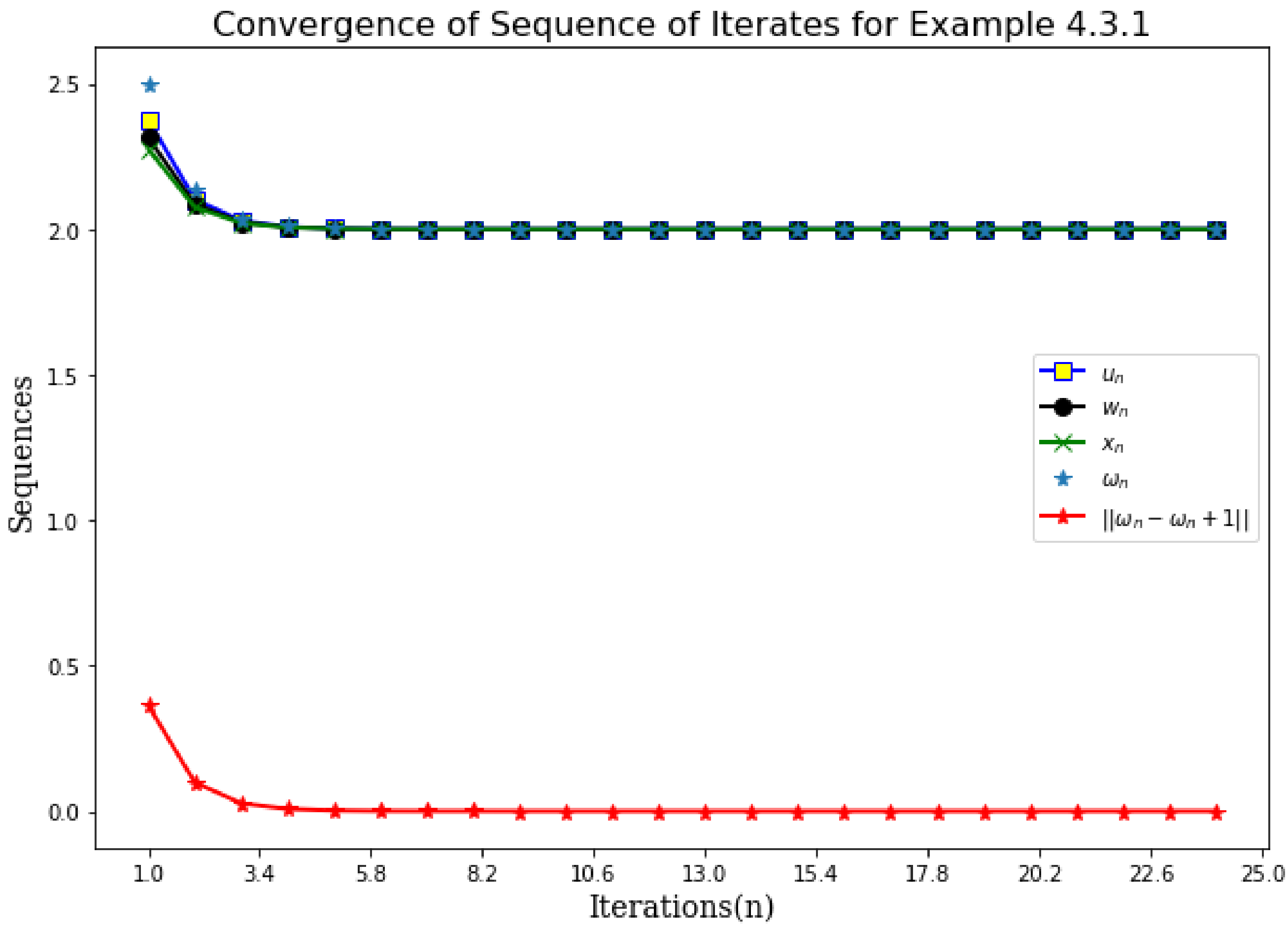

FIGURE 4.3. Graph corresponding to Table 4.3

Table 4.4: Numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = \frac{1}{n+3}, \beta_n = \frac{1}{n+1}, \gamma_n = \frac{1}{n+2} \in (0,1)$, initial value $x = 2.5 \in [1,6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	2.4166667	2.2083333	2.1562500	2.5000000	0.4218750
1	2.0683594	2.0512695	2.0427246	2.0781250	0.0567627
2	2.0192261	2.0160217	2.0140190	2.0213623	0.0143528
3	2.0064254	2.0056222	2.0050600	2.0070095	0.0044795
4	2.0023493	2.0021144	2.0019382	2.0025300	0.0015609
5	2.0009085	2.0008328	2.0007733	2.0009691	0.0005824
6	2.0003652	2.0003391	2.0003179	2.0003867	0.0002277
7	2.0001510	2.0001416	2.0001337	2.0001589	0.0000921
8	2.0000638	2.0000603	2.0000573	2.0000668	0.0000382
9	2.0000274	2.0000261	2.0000249	2.0000286	0.0000162

10	2.0000120	2.0000114	2.0000109	2.0000124	0.0000070
11	2.0000053	2.0000051	2.0000049	2.0000055	0.0000030
12	2.0000023	2.0000023	2.0000022	2.0000024	0.0000013
13	2.0000011	2.0000010	2.0000010	2.0000011	0.0000006
14	2.0000005	2.0000005	2.0000004	2.0000005	0.0000003
15	2.0000002	2.0000002	2.0000002	2.0000002	0.0000001
16	2.0000001	2.0000001	2.0000001	2.0000001	0.0000001
17	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
18	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
19	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
20	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
21	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
22	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
23	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000
24	2.0000000	2.0000000	2.0000000	2.0000000	0.0000000

==

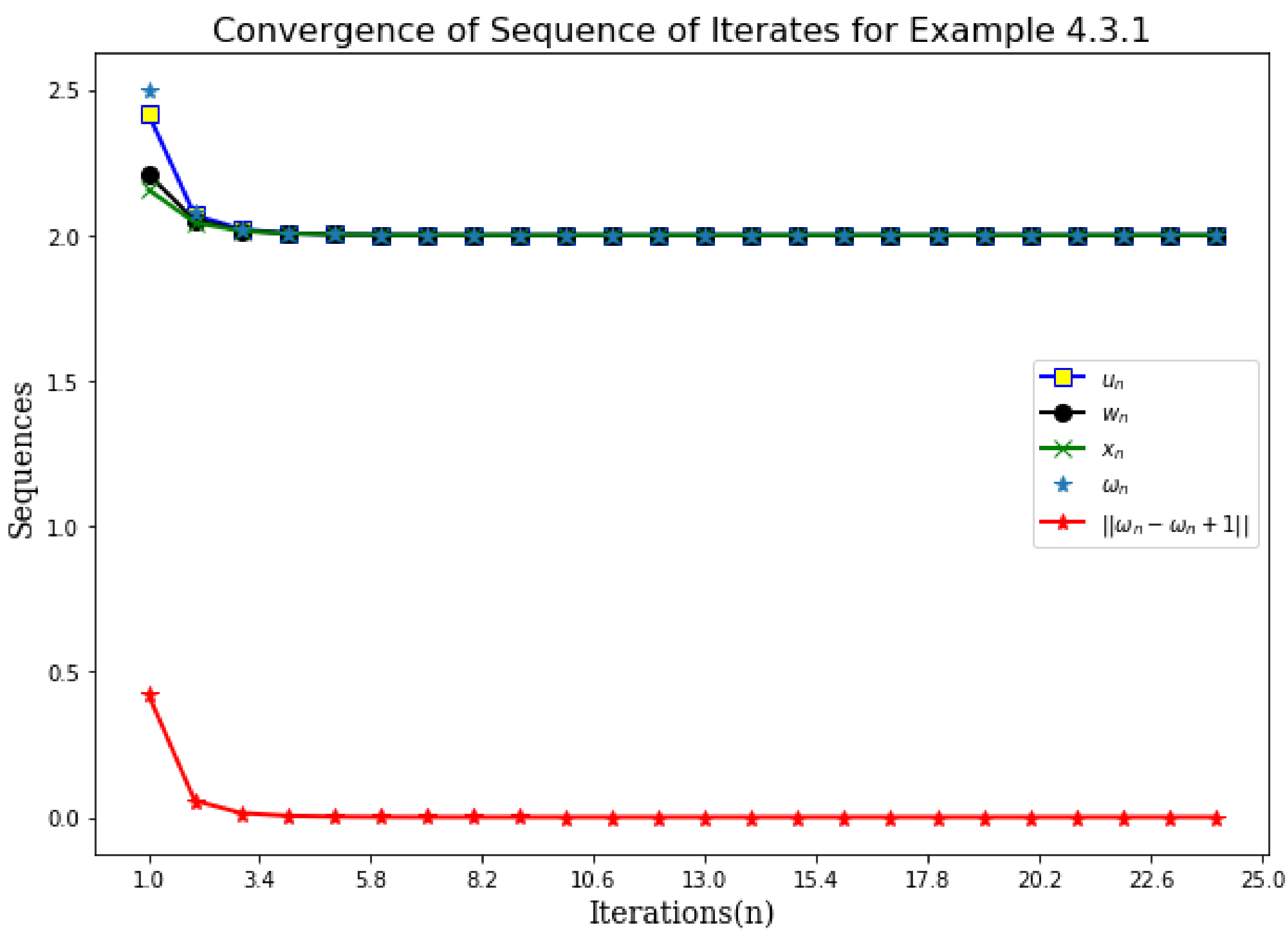

FIGURE 4.4. Graph corresponding to Table 4.4

Example 4.2. Let $K = [1, 10] \subset \mathbb{B} = R$ denote the reals with the usual norm $|.|$ and define the map $\Omega : K \to K$ by

$$\Omega(\omega) := (4\omega + 15)^{\frac{1}{3}}, \forall \omega \in [1, 10]. \tag{45}$$

Then Ω is a contraction map with a constant $L = \frac{1}{(15)^{\frac{1}{3}}}$ and $Fix(\Omega) = \{3\}$.

Proof. We first prove that the fixed point is actually the 3. To see this, we make use of the definition of fixed point as follows:

$$\begin{aligned} Fix(\Omega) &= \{\omega \in R : \Omega\omega = \omega\} \\ &= \{\omega \in R : (4\omega + 15)^{\frac{1}{3}} = \omega\} \\ &= \{\omega \in R : 4\omega + 15 = \omega^3\} \\ &= \{\omega \in R : \omega^3 - 4\omega - 15 = 0\} \\ &= \{\omega \in R : \omega = 3 \in [1, 10]\}. \end{aligned}$$

Thus $Fix(\Omega) = \{3\}$.

Next, we prove that Ω is $L = \frac{1}{(15)^{\frac{1}{3}}}$ contractive map. Now since $\mathbb{B} = R$, is the reals with the usual norm, we have that the norm becomes absolute norm. That is

$$\|\Omega\omega - \Omega\hat{\omega}\| \leq L\|\omega - \hat{\omega}\|, \forall\omega, \hat{\omega} \in R \tag{46}$$

becomes

$$|\Omega\omega - \Omega\hat{\omega}| \leq L|\omega - \hat{\omega}|, \forall\omega, \hat{\omega} \in R \tag{47}$$

where $\Omega\omega = (4\omega + 15)^{\frac{1}{3}}$ and $\Omega\hat{\omega} = (4\hat{\omega} + 15)^{\frac{1}{3}}$,$\forall\forall\omega, \hat{\omega} \in [1, 10] \subset R$. So using (47), we have

$$|\,(4\omega + 15)^{\frac{1}{3}} - (4\hat{\omega} + 15)^{\frac{1}{3}}\,| \leq \frac{1}{(15)^{\frac{1}{3}}}|\omega - \hat{\omega}|.$$

Thus $L = \frac{1}{(15)^{\frac{1}{3}}} \in [1, 10]$ as expected and that completes the proof of Example 4.2.

Using this example, we perform the following iterations such that the tables and graphs below are generated with Python software, run on HP PC with Intel(R)Core(TM)i7-3520M CPU @ 2.90GHz.

Table 4.5: Numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25 \in (0, 1)$, initial value $x = 3.5 \in [1, 10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	3.2861584	3.3487968	3.3877021	3.5000000	0.4436285
1	3.0323498	3.0393479	3.0437331	3.0563715	0.0499064
2	3.0037113	3.0045130	3.0050159	3.0064650	0.0057221
3	3.0004265	3.0005186	3.0005764	3.0007429	0.0006575
4	3.0000490	3.0000596	3.0000662	3.0000854	0.0000756
5	3.0000056	3.0000069	3.0000076	3.0000098	0.0000087
6	3.0000006	3.0000008	3.0000009	3.0000011	0.0000010
7	3.0000001	3.0000001	3.0000001	3.0000001	0.0000001
8	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
9	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
10	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
11	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
12	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
13	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
14	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000

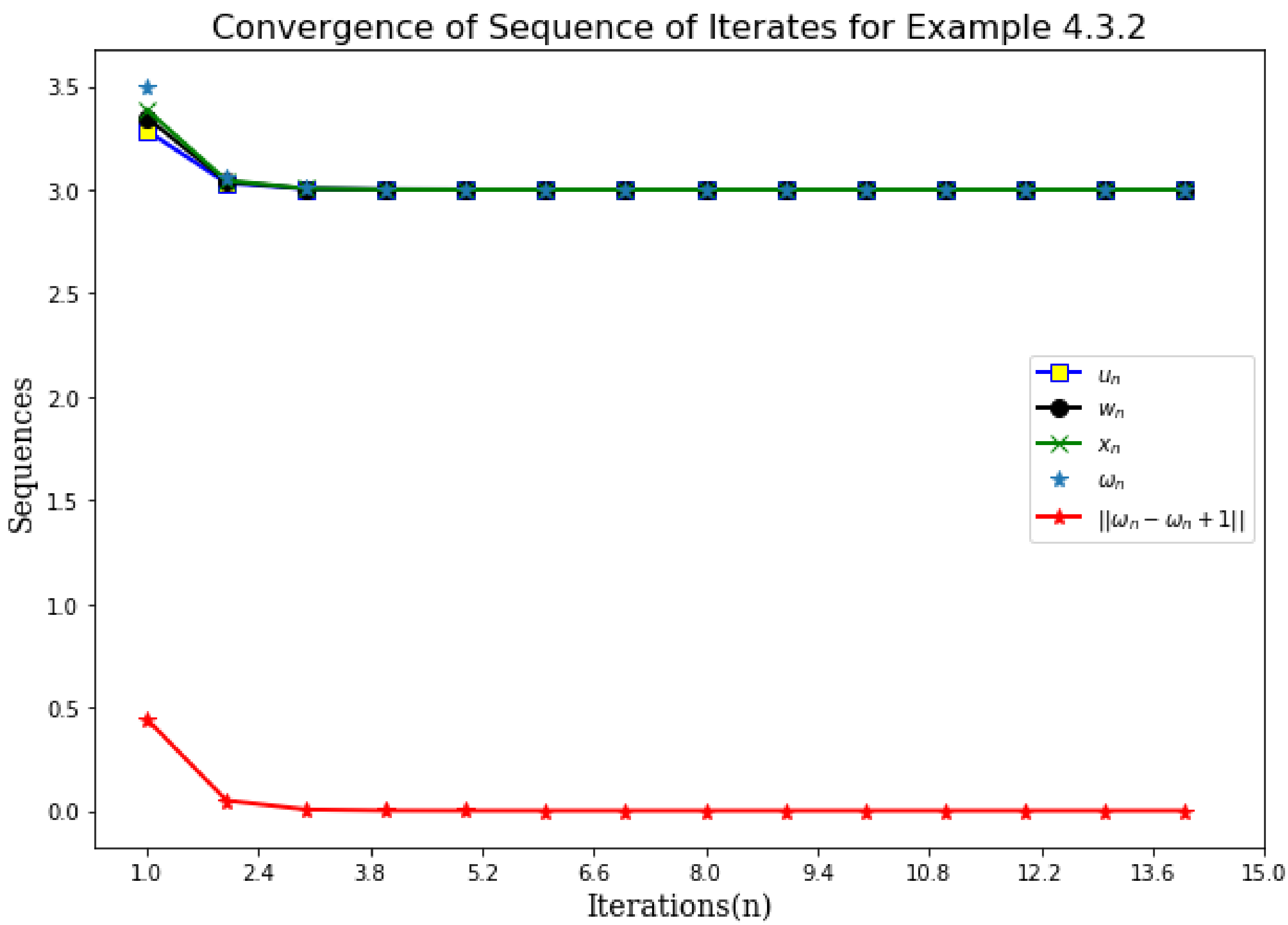

FIGURE 4.5. Graph corresponding to Table 4.5

Table 4.6: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = \frac{1}{n+3}, \beta_n = \frac{1}{n+1}, \gamma_n = \frac{1}{n+2} \in (0,1)$, initial value $\omega_1 = 3.5 \in [1, 10]$ "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	3.3574389	3.0520458	3.2538454	3.5000000	0.4628551
1	3.0292319	3.0207347	3.0257862	3.0371449	0.0333296
2	3.0031653	3.0026998	3.0029615	3.0038153	0.0033766
3	3.0003764	3.0003429	3.0003611	3.0004387	0.0003852
4	3.0000470	3.0000442	3.0000457	3.0000535	0.0000467
5	3.0000060	3.0000058	3.0000059	3.0000068	0.0000059
6	3.0000008	3.0000008	3.0000008	3.0000009	0.0000008
7	3.0000001	3.0000001	3.0000001	3.0000001	0.0000001
8	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
9	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
10	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
11	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
12	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
13	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
14	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000

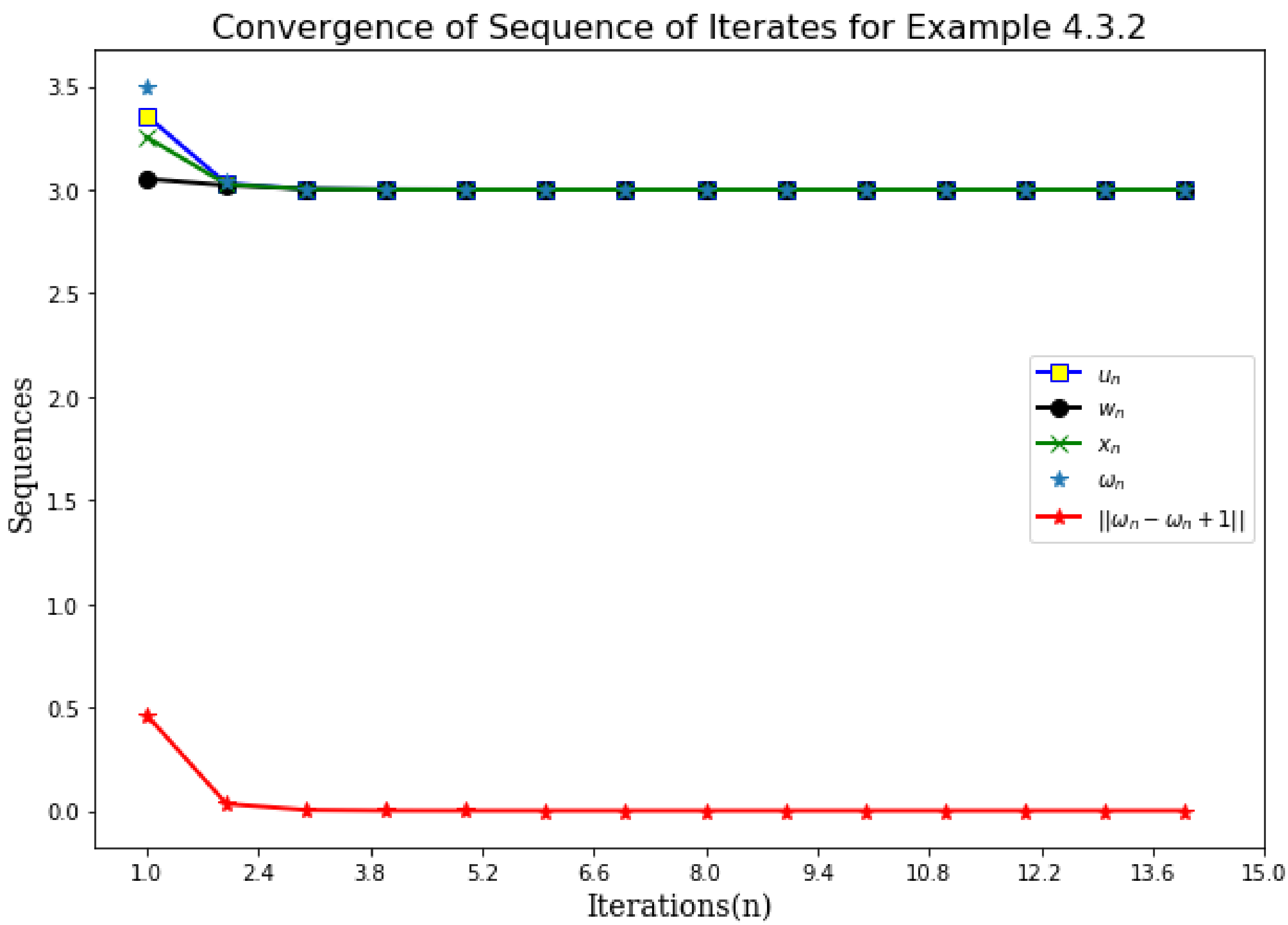

FIGURE 4.6. Graph corresponding to Table 4.6

Table 4.7: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25 \in (0,1)$, initial value $\omega = 10 \in [1, 10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	6.9014762	7.8525786	8.3984174	10.0000000	6.3508373
1	3.3712034	3.4527639	3.5032795	3.6491627	0.5763827
2	3.0417619	3.0508001	3.0564618	3.0727800	0.0644385
3	3.0047884	3.0058228	3.0064717	3.0083415	0.0073830
4	3.0005502	3.0006691	3.0007436	3.0009585	0.0008483
5	3.0000632	3.0000769	3.0000855	3.0001102	0.0000975
6	3.0000073	3.0000088	3.0000098	3.0000127	0.0000112
7	3.0000008	3.0000010	3.0000011	3.0000015	0.0000013
8	3.0000001	3.0000001	3.0000001	3.0000002	0.0000001
9	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
10	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
11	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
12	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
13	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
14	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000

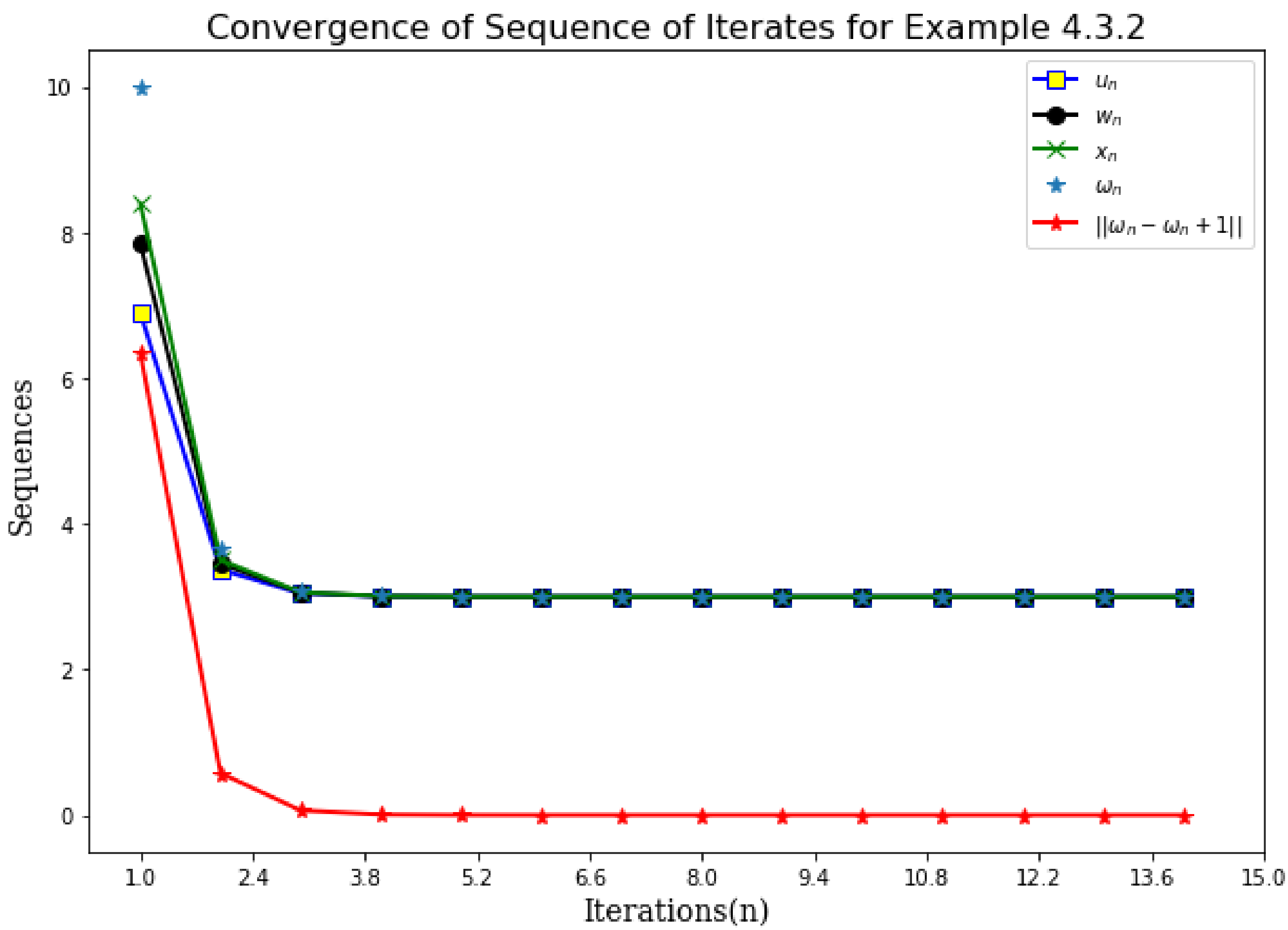

FIGURE 4.7. Graph corresponding to Table 4.7

Table 4.8: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = \frac{1}{n+3}, \beta_n = \frac{1}{n+1}, \gamma_n = \frac{1}{n+2} \in (0,1)$, initial value $\omega = 10 \in [1,10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]	err[n]
0	7.9343175	3.6020891	6.5433352	10.0000000	6.5469301
1	3.3562207	3.2524706	3.3143620	3.4530699	0.4072026
2	3.0380498	3.0324537	3.0356005	3.0458673	0.0406024
3	3.0045174	3.0041160	3.0043339	3.0052649	0.0046230
4	3.0005638	3.0005302	3.0005480	3.0006419	0.0005607
5	3.0000725	3.0000694	3.0000711	3.0000812	0.0000707
6	3.0000095	3.0000092	3.0000094	3.0000105	0.0000091
7	3.0000013	3.0000012	3.0000013	3.0000014	0.0000012
8	3.0000002	3.0000002	3.0000002	3.0000002	0.0000002
9	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
10	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
11	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
12	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
13	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000
14	3.0000000	3.0000000	3.0000000	3.0000000	0.0000000

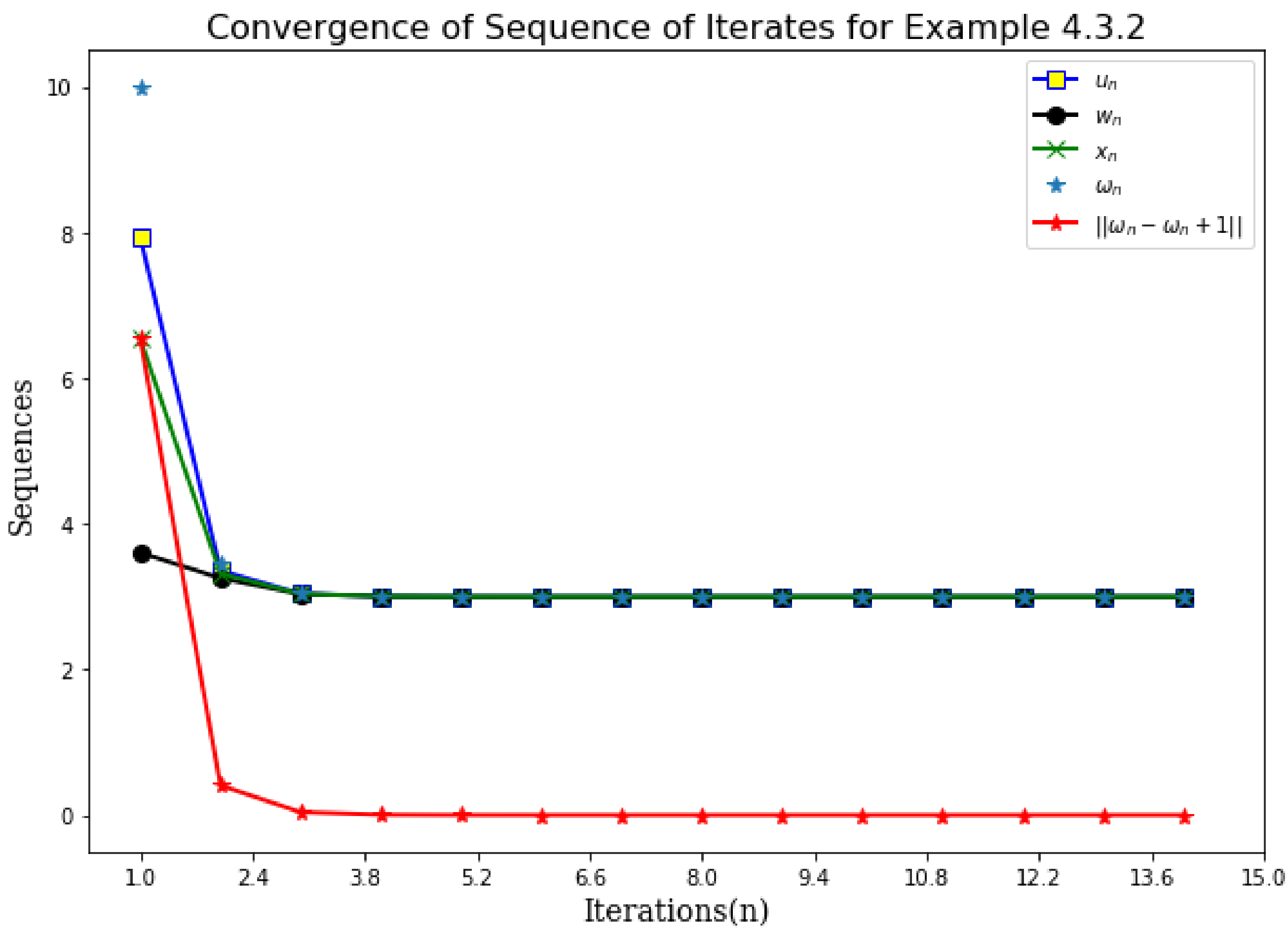

FIGURE 4.8. Graph corresponding to Table 4.8

Table 4.9: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25 \in (0,1)$, initial value $\omega = 3 \in [1,10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	u[n]	w[n]	x[n]	omega[n]
0	3.0000000	3.0000000	3.0000000	3.0000000
1	3.0000000	3.0000000	3.0000000	3.0000000
2	3.0000000	3.0000000	3.0000000	3.0000000
3	3.0000000	3.0000000	3.0000000	3.0000000
4	3.0000000	3.0000000	3.0000000	3.0000000
5	3.0000000	3.0000000	3.0000000	3.0000000
6	3.0000000	3.0000000	3.0000000	3.0000000
7	3.0000000	3.0000000	3.0000000	3.0000000
8	3.0000000	3.0000000	3.0000000	3.0000000
9	3.0000000	3.0000000	3.0000000	3.0000000
10	3.0000000	3.0000000	3.0000000	3.0000000
11	3.0000000	3.0000000	3.0000000	3.0000000
12	3.0000000	3.0000000	3.0000000	3.0000000
13	3.0000000	3.0000000	3.0000000	3.0000000
14	3.0000000	3.0000000	3.0000000	3.0000000

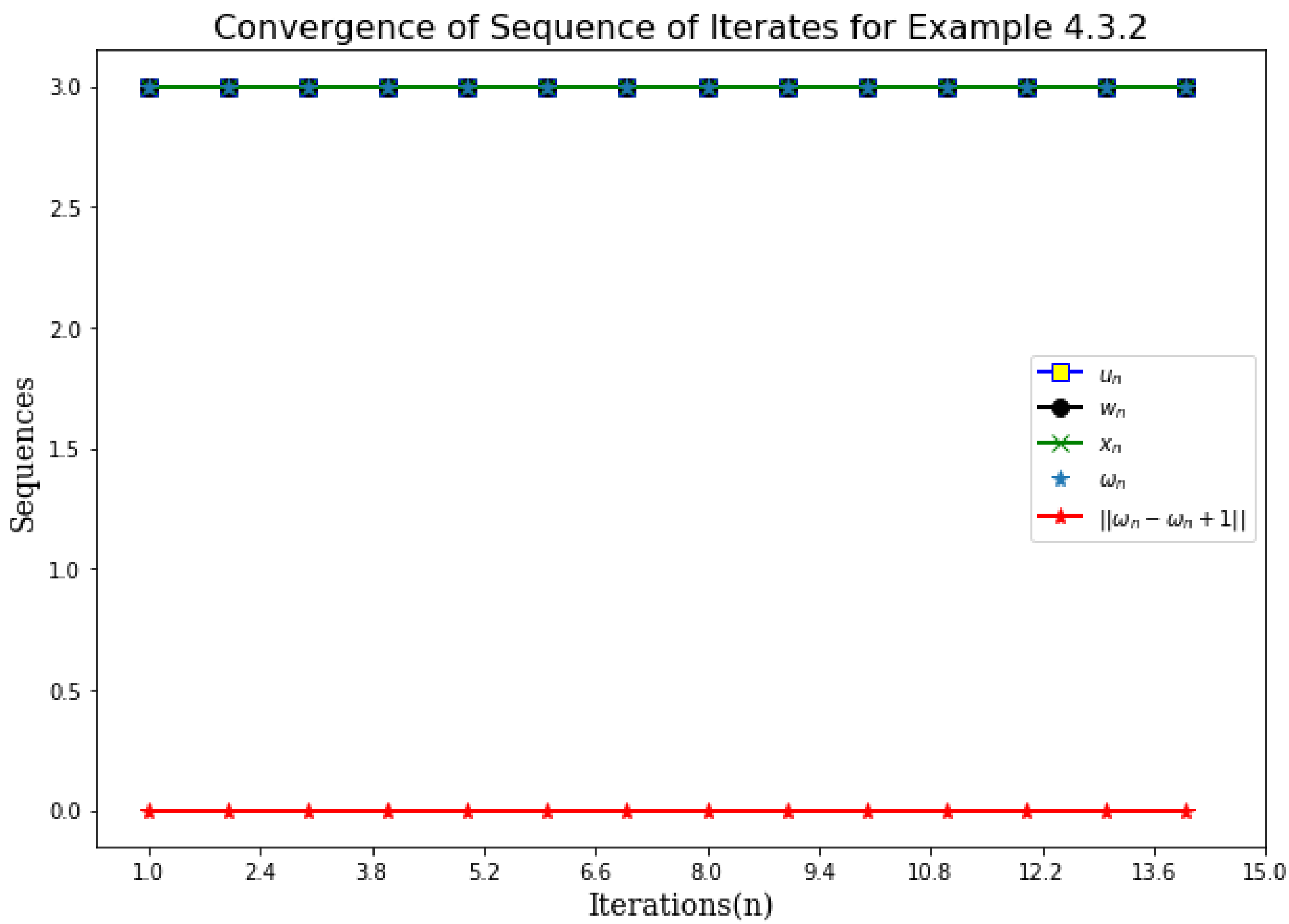

FIGURE 4.9. Graph corresponding to Table 4.9

4.2 Comparison of the rate of convergence of several iteration processes

Table 4.10: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25, \lambda = 0.5 \in (0,1)$, initial value $\omega = 3.5 \in [1,6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	HybridPicMann[n]	PicNoor[n]	Picard[n]	Mann[n]	Krasnoselskii[n]
0	3.5000000	3.5000000	3.5000000	3.5000000	3.5000000
1	2.4101562	2.6367188	2.7500000	3.1250000	3.1250000
2	2.1121521	2.2702738	2.3750000	2.8437500	2.8437500
3	2.0306666	2.1147256	2.1875000	2.6328125	2.6328125
4	2.0083854	2.0486986	2.0937500	2.4746094	2.4746094
5	2.0022929	2.0206716	2.0468750	2.3559570	2.3559570
6	2.0006270	2.0087746	2.0234375	2.2669678	2.2669678
7	2.0001714	2.0037247	2.0117188	2.2002258	2.2002258
8	2.0000469	2.0015810	2.0058594	2.1501694	2.1501694
9	2.0000128	2.0006711	2.0029297	2.1126270	2.1126270
10	2.0000035	2.0002849	2.0014648	2.0844703	2.0844703
11	2.0000010	2.0001209	2.0007324	2.0633527	2.0633527
12	2.0000003	2.0000513	2.0003662	2.0475145	2.0475145
13	2.0000001	2.0000218	2.0001831	2.0356359	2.0356359
14	2.0000000	2.0000092	2.0000916	2.0267269	2.0267269
15	2.0000000	2.0000039	2.0000458	2.0200452	2.0200452
16	2.0000000	2.0000017	2.0000229	2.0150339	2.0150339
17	2.0000000	2.0000007	2.0000114	2.0112754	2.0112754
18	2.0000000	2.0000003	2.0000057	2.0084566	2.0084566
19	2.0000000	2.0000001	2.0000029	2.0063424	2.0063424

20	2.0000000	2.0000001	2.0000014	2.0047568	2.0047568
21	2.0000000	2.0000000	2.0000007	2.0035676	2.0035676
22	2.0000000	2.0000000	2.0000004	2.0026757	2.0026757
23	2.0000000	2.0000000	2.0000002	2.0020068	2.0020068
24	2.0000000	2.0000000	2.0000001	2.0015051	2.0015051

Table 4.11: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25, \lambda = 0.5 \in (0,1)$, initial value $\omega = 3.5 \in [1,6]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	Ishikawa[n]	Noor[n]	PicMann[n]
0	3.5000000	3.5000000	3.5000000
1	3.0631250	3.0553906	2.5625000
2	2.7534898	2.7425662	2.2109375
3	2.5340359	2.5224650	2.0791016
4	2.3784980	2.3676031	2.0296631
5	2.2682604	2.2586432	2.0111237
6	2.1901296	2.1819798	2.0041714
7	2.1347543	2.1280398	2.0015643
8	2.0955071	2.0900880	2.0005866
9	2.0676907	2.0633854	2.0002200
10	2.0479758	2.0445975	2.0000825
11	2.0340028	2.0313786	2.0000309
12	2.0240995	2.0220778	2.0000116
13	2.0170805	2.0155338	2.0000044
14	2.0121058	2.0109295	2.0000016
15	2.0085800	2.0076899	2.0000006
16	2.0060811	2.0054106	2.0000002
17	2.0043100	2.0038068	2.0000001
18	2.0030547	2.0026785	2.0000000
19	2.0021650	2.0018846	2.0000000
20	2.0015344	2.0013260	2.0000000
21	2.0010875	2.0009329	2.0000000
22	2.0007708	2.0006564	2.0000000
23	2.0005463	2.0004618	2.0000000
24	2.0003872	2.0003250	2.0000000

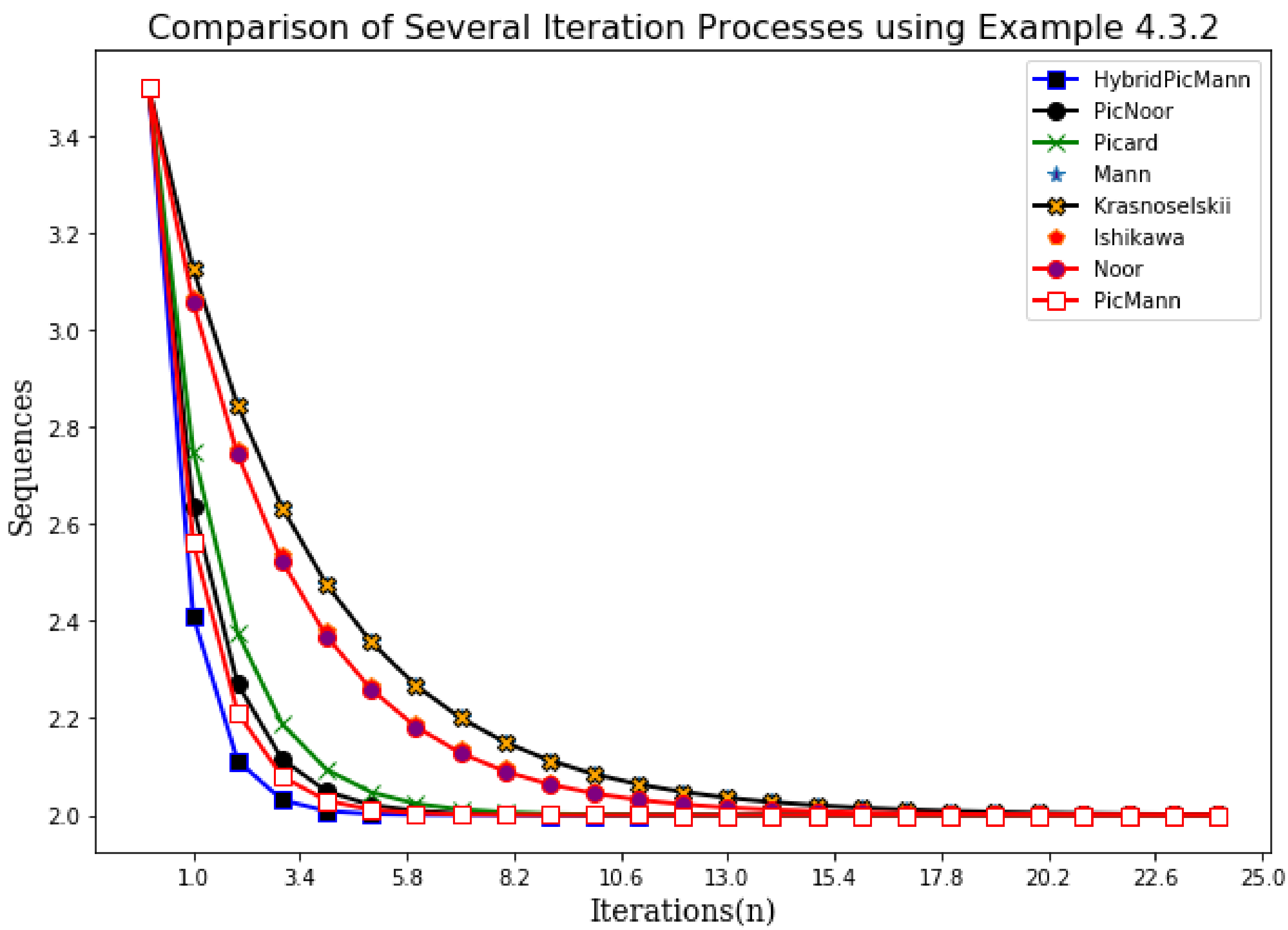

FIGURE 4.10. Graph corresponding to Table 4.10 and Table 4.11

Table 4.12: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25, \lambda = 0.5 \in (0,1)$, initial value $\omega = 3.5 \in [1,10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	HybridPicMann[n]	PicNoor[n]	Picard[n]	Mann[n]	Krasnoselskii[n]
0	3.5000000	3.5000000	3.5000000	3.5000000	3.5000000
1	3.0236702	3.0563637	3.0723168	3.2861584	3.2861584
2	3.0011338	3.0064632	3.0106756	3.1639834	3.1639834
3	3.0000543	3.0007426	3.0015807	3.0940416	3.0940416
4	3.0000026	3.0000853	3.0002342	3.0539547	3.0539547
5	3.0000001	3.0000098	3.0000347	3.0309634	3.0309634
6	3.0000000	3.0000011	3.0000051	3.0177718	3.0177718
7	3.0000000	3.0000001	3.0000008	3.0102012	3.0102012
8	3.0000000	3.0000000	3.0000001	3.0058558	3.0058558
9	3.0000000	3.0000000	3.0000000	3.0033616	3.0033616
10	3.0000000	3.0000000	3.0000000	3.0019297	3.0019297
11	3.0000000	3.0000000	3.0000000	3.0011078	3.0011078
12	3.0000000	3.0000000	3.0000000	3.0006360	3.0006360
13	3.0000000	3.0000000	3.0000000	3.0003651	3.0003651
14	3.0000000	3.0000000	3.0000000	3.0002096	3.0002096
15	3.0000000	3.0000000	3.0000000	3.0001203	3.0001203
16	3.0000000	3.0000000	3.0000000	3.0000691	3.0000691
17	3.0000000	3.0000000	3.0000000	3.0000397	3.0000397
18	3.0000000	3.0000000	3.0000000	3.0000228	3.0000228
19	3.0000000	3.0000000	3.0000000	3.0000131	3.0000131
20	3.0000000	3.0000000	3.0000000	3.0000075	3.0000075

21	3.0000000	3.0000000	3.0000000	3.0000043	3.0000043
22	3.0000000	3.0000000	3.0000000	3.0000025	3.0000025
23	3.0000000	3.0000000	3.0000000	3.0000014	3.0000014
24	3.0000000	3.0000000	3.0000000	3.0000008	3.0000008

Table 4.13: The numerical values of the sequences $\{u_n\}$, $\{w_n\}$, $\{x_n\}$ and $\{\omega_n\}$ with $\alpha_n = 0.5, \beta_n = 0.33, \gamma_n = 0.25, \lambda = 0.5 \in (0,1)$, initial value $\omega = 3.5 \in [1,10]$ and a "good" stopping criteria $\|\omega_n - \omega_{n+1}\| \leq 10^{-7}$.

n	Ishikawa[n]	Noor[n]	PicMann[n]
0	3.5000000	3.5000000	3.5000000
1	3.2761249	3.2757664	3.0418085
2	3.1526115	3.1522093	3.0035506
3	3.0843848	3.0840478	3.0003019
4	3.0466713	3.0464210	3.0000257
5	3.0258165	3.0256424	3.0000022
6	3.0142816	3.0141657	3.0000002
7	3.0079009	3.0078258	3.0000000
8	3.0043710	3.0043235	3.0000000
9	3.0024182	3.0023886	3.0000000
10	3.0013379	3.0013196	3.0000000
11	3.0007402	3.0007291	3.0000000
12	3.0004095	3.0004028	3.0000000
13	3.0002266	3.0002225	3.0000000
14	3.0001253	3.0001229	3.0000000
15	3.0000693	3.0000679	3.0000000
16	3.0000384	3.0000375	3.0000000
17	3.0000212	3.0000207	3.0000000
18	3.0000117	3.0000115	3.0000000
19	3.0000065	3.0000063	3.0000000
20	3.0000036	3.0000035	3.0000000
21	3.0000020	3.0000019	3.0000000
22	3.0000011	3.0000011	3.0000000
23	3.0000006	3.0000006	3.0000000
24	3.0000003	3.0000003	3.0000000

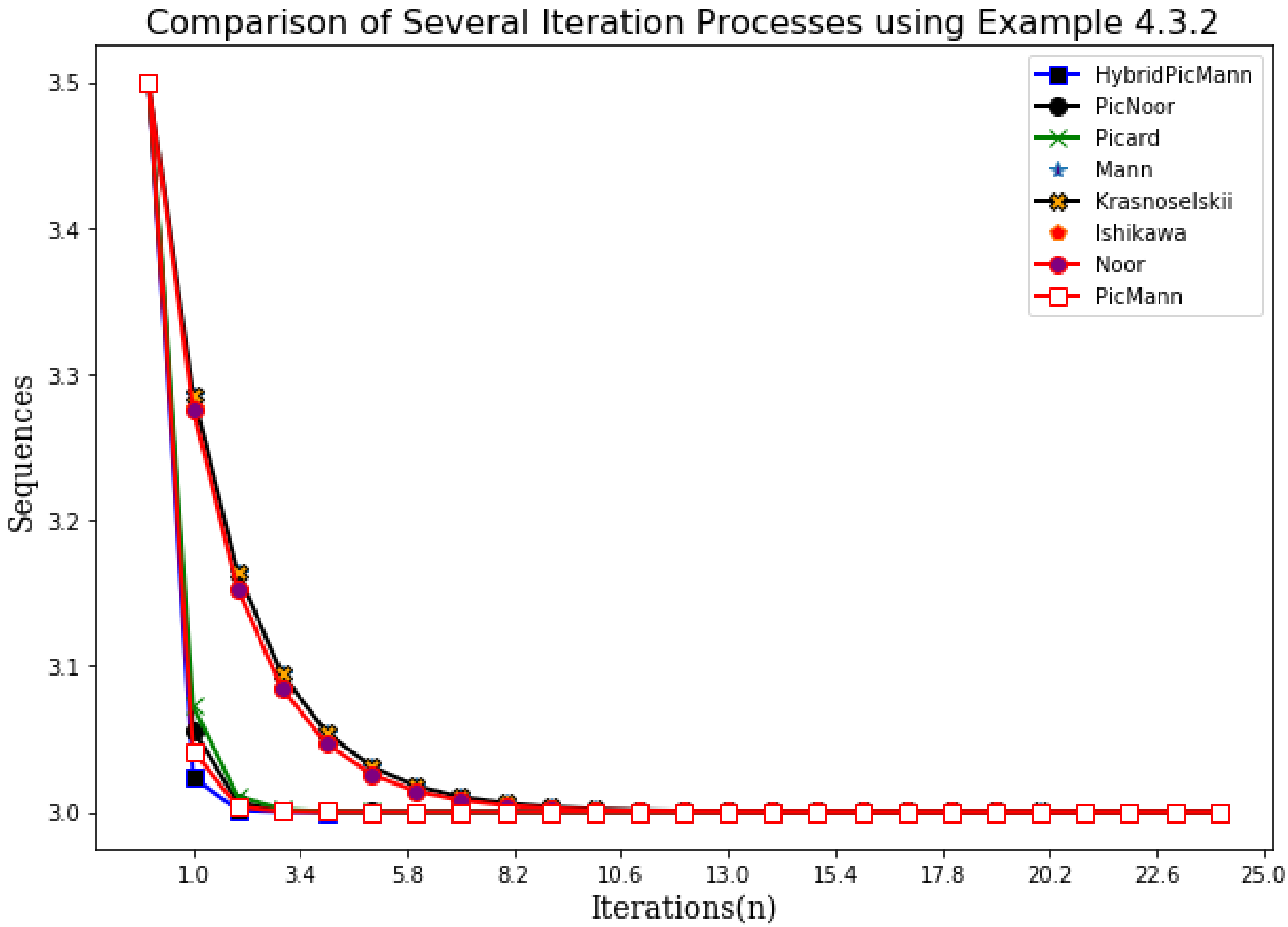

FIGURE 4.5. Graph corresponding to Table 4.12 and Table 4.13

5 Discussions and Conclusion

We introduced a new four-step iteration process called Picard-Mann hybrid iterative process. We proved that it converges to the unique fixed points of contractive mappings. We further showed with a good stopping criteria that the new scheme converges faster than several other listed and cited iteration processes with numerically performed examples using Python program. Our results showed numerical stability and data dependence. In view of the results proved in this work, it is evident that the Picard-Manns' hybrid iteration process is a generalization of the results of [9], [10], [11]-[22].

References

[1] Berinde, V. (2007). Iterative approximation of fixed points. Springer 1912.

[2] Zamfirescu, T.(1972). Fixed point theorems in metric spaces. Arch. Math. 23(1), 292–298.

[3] Browder, F.(1965). Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. U.S.A. 54(4), 1041–1044.

[4] Göhde, D.(1965). Zum Prinzip der kontraktiven Abbildung. Math. Nachr. 30(3–4), 251–258.

[5] Kirk, W. (1965). A fixed point theorem for mappings which do not increase distances. Am. Math. Mon. 72(9), 1004–1006.

[6] Okeke, G. A., Ofem, A. E. (2022). A novel iterative scheme for solving delay differential equations and nonlinear integral equations in Banach spaces. Math. Meth. Appl. Sci., 1–24.

[7] Berinde, V. (2004). On the convergence of the Ishikawa iteration in the class of quasi contractive operators. Acta Mathematica Universitatis Comenianae, 73(1), 119–126.

[8] Rhoades, B.,and Xue, Z. (2010). Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasicontractive maps. Fixed Point Theory and Applications, 2010, 1–12.

[9] Picard, È. (1890). Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. Journal de Mathématiques Pures et Appliquées, 6, 145–210.

[10] Krasnosel'skii, M. A. (1955). Two remarks on the method of successive approximations. Uspekhi matematicheskikh nauk, 10:1, 123–127.

[11] Mann, W. R. (1953). Mean value method in iteration. Proc. Amer. Math.Soc., 4, 506-510.

[12] Ishikawa, S. (1974). Fixed points by a new iteration method. Proceeding of the American Mathematical Society, 44:1, 147–150.

[13] Noor, M. A. (2000). New approximation scheme for general variational inequalities. Journal of Mathematical Analysis and Application, 251, 217–229.

[14] Khan, S. H. (2013). A Picard-Mann hybrid iterative process. Fixed Point Theory and Application, 2013, 69, doi:10.1186/1687-1812-2013-69.

[15] Berinde, V. (2002). Iterative Approximation of Fixed Points. Efemeride, Baia Mare.

[16] Thakur, B. S., Thakur, D., and Postolache, M. (2016). A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. Applied Mathematics and Computation, 275, 147–155.

[17] Abbas, M., and Nazir, T. (2014). Some new faster iteration process applied to constrained minimization and feasibility problems. Matematicki Vesnik, 66:2, 223–234.

[18] Agarwal, R. P., O'Regan, D., and Sahu, D. R. (2007). Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal., 8:1, 61–79.

[19] Okeke, G. A., & Abbas, M. (2017). A solution of delay differential equations via Picard–Krasnoselskii hybrid iterative process. Arabian Journal of Mathematics,6, 21–29.

[20] Okeke, G. A. (2019). Convergence analysis of the Picard–Ishikawa hybrid iterative process with applications. Afrika Matematika, 30, 817–835.

[21] Jia, J., Shabbir, K., Ahmad, K., Shah, N.A., and Botmart, T. (2022). Strong convergence of a new hybrid iterative scheme for nonexpansive mappings and application. Journal of Function Spaces, https://doi.org/10.1155/2022/4855173.

[22] Okeke, G. A., Anozie, E. H., Udo, A. V., and Olaoluwa, H. (2024). A novel fixed point iteration process applied in solving delay differential equations. Journal of the Nigerian Mathematical Society. 43(2), 115-143.

[23] Eke, K. and Akewe, H.(2019). Equivalence of Picard-type hybrid iterative algorithms for contractive mappings. Asian J. Sci. Res. 12(3), https://doi.org/10.3923/ajsr.2019.298.307.

[24] Phuengrattana, W., and Suantai, S.(2010). On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for countinuous functions on an arbitrary interval. J. Comput. Appl. Math. 235(9), 3006–3014.

[25] Takahashi, W.(2000). Nonlinear Functional Analysis. Yokohama Publishers, Yokohama.

[26] Yu, T.M., Shahid, A., Shabbir, K., Shah, N., and Li, Y.M.(2021). An iteration process for a general class of contractive-like operators: convergence, stability and polynomiography. AIMS Mathematics 6(7), 6699–6714.