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On a class of piecewise continuous Lyapunov functions and eventual stability for nonlinear impulsive Caputo fractional differential equations via new generalized Dini derivative

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Abstract

In this paper, the eventual stability for nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse is examined using the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions. Together with the comparison results, sufficient conditions for the eventual stability of impulsive Caputo fractional differential equations are presented. Results obtained are extension and improvements on the existing results.

Keywords and phrases: eventual stability; Caputo derivative; impulsive fractional differential equations; vector Lyapunov functions.

1 Introduction

In the qualitative theory of differential equations, one of the main properties of interest is the stability of solutions since it enables us to compare the behaviour of solutions starting at different points [20]. The theory of impulsive differential equation is richer than the corresponding theory of differential equations [26, 19], as they constitutes very important models in the description of the true state of several real life processes and phenomena.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations acts instantaneously, that is, in the form of impulses. It is known for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects [26, 18].

Now, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [31], and one of the most versatile methods in the study of the stability properties of impulsive systems is the method of Lyapunov function (Lyapunov's second method). The method was originally developed for studying the stability of a fixed point of an autonomous or nonautonomous differential equations. However, as was argued in [5], the method was then extended from fixed points to sets, from differential equation to dynamical systems and to stochastic equations.

There are several approaches in the literature in the study of the stability of solutions, one of which is the Lyapunov's second method. However, the novelty of the Lyapunov's second method over other methods of examining stability properties like the Razumikhin technique, the use of matrix inequality,

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etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential equation. Again, the method involves seeking an appropriate continuously differentiable function that is positive definite and whose time derivative along the trajectory curve or solution path is negative semidefinite. This function is called the Lyapunov function. The stability of the zero solution of impulsive differential equations have been extensively studied in [2, 3], [14], [29].

Again, the stability of solutions of differential equations via Lyapunov method has been intensively investigated in the past, and in many real cases, it is obligatory to study the stability of such sets, which are invariant with respect to a given system of differential equation which immediately excludes the stability in the sense of Lyapunov [35]. To allay the problem that will arise subsequently, [26] introduced a new concept called eventual stability, maintaining that, the set under consideration, despite not being invariant in the usual sense, is invariant in the asymptotic sense (see also [39]). Accordingly, the eventual stability of solutions of impulsive differential systems have been extensively studied (see [26], [32], [35] and the references therein).

Furthermore, the study of stability for fractional order systems is quite recent and one of the main difficulties in the application of a Lyapunov function to fractional order differential equations is the appropriate definition of its derivative among the fractional differential equations (see [20]). The stability of fractional order systems is examined in [4, 5, 6, 7, 8, 9, 11, 21, 22, 38]. Using the generalized Caputo fractional Dini derivative and scalar impulsive fractional differential equations, [1] established the comparison results together with sufficient conditions for the stability properties of impulsive fractional differential equations using the scalar Lyapunov function.

In this paper, we extend the study on the stability properties of impulsive Caputo fractional differential equations using scalar Lyapunov function in [1], and the corresponding study on the eventual stability for impulsive differential equations using the scalar Lyapunov function in [35] to the eventual stability of impulsive Caputo fractional differential equations using the vector Lyapunov functions. By adopting the new definition of the Caputo fractional Dini derivative as proposed in [20], and employing the vector Lyapunov functions which is generalized by a class of piecewise continuous functions, together with the comparison results, sufficient conditions for the eventual stability of the set $x(t) = 0$ is established with illustrative example.

2 Preliminaries, notations and definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|.\|$, let Ω be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $t_0 \in \mathbb{R}_+, t > 0$.
Let $J \subset \mathbb{R}_+$ Define the following class of functions $PC^q[J, \Omega] = \alpha : J \to \Omega, \alpha(t)$ is a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t)$ exists.
Fractional calculus being the generalization of the classical calculus to non integer order allows for the extension of the classical concepts of derivative and integral to functions with fractional orders. It allows for functions with non integer orders which makes it much more flexible in describing real world systems (see [23], [27] and [30]). There are several definitions of fractional derivatives and fractional integrals

General case

Let the number $n - 1 < q < n, q > 0$ be given, where n is a natural number and $\Gamma(.)$ denotes the gamma function.

Definition 2.1. *The Riemann-Liouville fractional derivative of order q of $x(t)$ is given by (see [30])*

$$ {}_{t_0}^{RL}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} x(s) ds, t \geq t_0 $$

Definition 2.2. *The Caputo fractional derivative of order q of $x(t)$ is defined by (see [30])*

$$ {}_{t_0}^{C}D^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, t \geq t_0 $$

The Caputo derivatives has many properties that are similar to those of the standard derivatives, which makes them easier to understand and apply. The initial conditions of fractional differential equations using the Caputo derivative are also easier to interpret in physical context, which is another reason why it is often used in applications of fractional calculus.

Definition 2.3. *The Grunwald-Letnikov fractional derivative of order q of $x(t)$ is given by (see [20])*

$$ D_0^q x(t) = \lim_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t-rh), t \geq t_0 $$

and the Grunwald-Letnikov fractional Dini derivative of order q of $x(t)$ is given by (see [20])

$$ D_0^q x(t) = \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t-rh), t \geq t_0 $$

where ${}^qC_r$ are the binomial coefficients and $[\frac{(t-t_0)}{h}]$ denotes the integer part of $\frac{(t-t_0)}{h}$.

Particular case

(when n=1). In most applications, the order of q is often less than 1, so that $q \in (0,1)$. For simplicity of notation, we will use ${}^CD^q$ instead of ${}_{t_0}^{C}D^q$ and the Caputo fractional derivative of order q of the function $x(t)$ is

$$ {}^CD^q x = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x' ds, t \geq t_0 \tag{1} $$

3 Impulses in fractional differential equations

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}^CD^q x = f(t,x), t \geq t_0, x(t_0) = x_0, \tag{2} $$

where $x \in \mathbb{R}^N$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$.
Some sufficient conditions for the existence of the global solutions to (2) are considered in [12], [15], [28], [30], [38]. The IVP for FrDE (2) is equivalent to the following Volterra integral equation (See [1, 16, 37]),

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, t \geq t_0 \tag{3} $$

Consider the initial value problem for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < q < 1$,

$$ \begin{aligned} {}^CD^q x &= f(t,x), t \geq t_0, t \neq t_k, k = 1, 2, ... \\ \Delta x &= I_k(x(t_k)), k \in N, t = t_k \\ x(t_0) &= x_0, \end{aligned} \tag{4} $$

where $x, x_0 \in \mathbb{R}^N$, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, and $t_0 \in \mathbb{R}_+$, $I_k : \mathbb{R}^N \to \mathbb{R}^N$, $k = 1, 2, ...$
under the following assumptions:

(i) $0 < t_1 < t_2 < ... < t_k < ...$, and $t_k \to \infty$ as $k \to \infty$;

(ii) $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, is continuous in $(t_{k-1}, t_k]$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...$, $\lim_{(t,y)\to(t_k^+,x)} f(t, y) = f(t_k^+, x)$ exists;

(iii) $I_k : \mathbb{R}^N \to \mathbb{R}^N$

In this paper, we assume that $f(t, 0) \equiv 0$, $I_k(0) = 0$ for all k, so that we have the trivial solution for (4), and the points $t_k, k = 1, 2, ...$ are fixed such that $t_1 < t_2 < ...$ and $\lim_{k\to\infty} t_k = \infty$. The system (4) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^N)$.

Remark 3.1. The second equation in (4) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 3.2. *Let $V : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N_+$. Then V is said to belong to class $\mathcal{L}$ if,*

1. *(i) V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^{\mathbb{N}}$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...$ $\lim_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$ exists;*

2. *(ii) V is locally Lipschitz with respect to its second argument x and $V(t, 0) \equiv 0$.*

Now, for any function $V(t, x) \in C([t_0, \infty) \times \xi, \mathbb{R}^{\mathbb{N}}_+)$ we define the Caputo fractional Dini derivative as:

$$ {}^cD^q_+V(t, x) = \limsup_{h\to0^+} \frac{1}{h^q}\{V(t, x)-V(t_0, x_0)- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t-rh, x-h^qf(t, x))-V(t_0, x_0)]\} \tag{5}$$

$t \geq t_0$ where $t \in [t_0, \infty), x, x_0 \in \xi$, and there exists $h > 0$ such that $t - rh \in [t_0, T]$

Definition 3.3. *A function $g \in C[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasimonotone nondecreasing in x, if $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$ implies $g_i(x) \leq g_i(y), \forall i$.*

Definition 3.4. *The set $x(t) \equiv 0$ of (4) is said to be:*
$(S1)$ eventually stable if for every $\epsilon > 0$ there exists a number $T = T(\epsilon) > 0$ for all $t_0 \in \mathbb{R}_+$ and $\delta = \delta(t_0, \epsilon)$ for all $x_0 \in \mathbb{R}^N$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \epsilon$ for $t \geq t_0$.

$(S2)$ eventually uniformly stable if the δ in $S1$ is independent of t.

Definition 3.5. *A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in C([0, \psi), \mathbb{R}_+], a(0) = 0$, and $a(r)$ is strictly monotone increasing in r.*

In this paper, we define the following sets:

$$\bar{S}_\psi = \{x \in \mathbb{R}^N : \|x\| \leq \psi\}$$
$$S_\psi = \{x \in \mathbb{R}^N : \|x\| < \psi\}$$

It suffices to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned} {}^c_{t_0}D^q u = g(t, u), t \geq t_0, t \neq t_k, k = 1, 2, ... \\ \Delta u = \psi_k(u(t_k)), k \in N, t = t_k \\ u(t_0^+) = u_0, \end{aligned} \tag{6}$$

existing for $t \geq t_0$, where $u \in \mathbb{R}^n, \mathbb{R}_+ = [t_0, \infty), g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, g(t, 0) \equiv 0$,
where g is the continuous mapping of $\mathbb{R}_+ \times \mathbb{R}^n$ into $\mathbb{R}^n$. The function $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ is such that for any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the system (6) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$.

Lemma 3.6. *Assume $m \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^C D^q_+ m(t^*, \alpha_1) -^C D^q_+ m(t^*, \alpha_2) > 0$ holds.*

Proof. Let $V(t, x) = m(t, \alpha_1) - m(t, \alpha_2)$.
Applying (5), we have

$$\begin{aligned}
{}^C D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&\quad - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\}
\end{aligned}$$

When $m(t^*, \alpha_1) = m(t^*, \alpha_2)$, we have

$$\begin{aligned}
{}^C D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h\to 0^+} \frac{1}{h^q}\{-[m(t_0, \alpha_1) - m(t_0, \alpha_2)] - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r \\
&\quad [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\} \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad + \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&\quad - \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q}\{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&= -\limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)]
\end{aligned}$$

Applying equation (3.8) in [1], we have

$${}^c D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = -\frac{(t - t_0)^{-q}}{\Gamma(1 - q)}[m(t_0, \alpha_1) - m(t_0, \alpha_2)]$$

By the lemma, we have that

$$m(t, \alpha_1) - m(t, \alpha_2) < 0, \text{for}\, t_0 \leq t < t^*$$

And so it follows that

$${}^c D^q_+(m(t^*, \alpha_1) - m(t^*, \alpha_2)) > 0$$

4 Fractional differential inequalities and comparison results for impulsive vector fractional differential equations

In this section, we assume that $0 < q < 1$.

Theorem 4.1. *Assume that*

(i) $g \in PC[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ *and is continuous in* $(t_{k-1}, t_k]$, $k = 1, 2, ...$ *and* $g(t, u)$ *is quasimonotone nondecreasing in* u *for each* $u \in \mathbb{R}^n$ *and* $\lim_{(t,y)\to(t_k^+,u)} g(t, u) = g(t_k^+, u)$ *exists;*

(ii) $V \in PC[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N_+]$ *and* $V \in \mathcal{L}$ *such that*
${}^C D^q_+ V(t, x) \leq g(t, V(t, x))$, $t \neq t_k$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$ *and*
$V(t_k, x + I_k(x(t_k))) \leq \rho_k(V(t, x))$, $t = t_k$, $x \in S_\psi$ *and the function* $\rho_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ *is nondecreasing for* $k = 1, 2, ...$

(iii) $r(t) = r(t; t_0, u_0) \in PC^q([t_0, T], \mathbb{R}^n)$ *is the maximal solution of the the IVP for the IFrDE (6).*

Then,

$$V(t, x(t)) \leq r(t), t \geq t_0 \tag{7}$$

where $x(t) = x(t; t_0, x_0) \in PC^q([t_0, T], \mathbb{R}^N)$ *is any solution of (4) existing on* $[t_0, \infty)$*, provided that*

$$V(t_0^+, x_0) \leq u_0. \tag{8}$$

Proof. Let $\eta \in \bar{S}_\psi =: \{\eta \in \mathbb{R}^n : \|\eta\| \leq \psi\}$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations.

$$\begin{aligned} {}^c D^q u = g(t, u) + \eta, \Delta u = \psi_k(u(t_k)), t = t_k, k = 1, 2, ... \\ u(t_0^+) = u_0 + \eta \end{aligned} \tag{9}$$

for $t \in [t_0, \infty)$.
The function $u_\eta(t, \alpha)$ is a solution of (9), where $\alpha > 0$, if and only if it satisfies the Volterra Integral equation

$$u_\eta(t, \alpha) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1}(g(s, u_\eta(s, \alpha)) + \eta) ds,\ t \in [t_0, \infty) \tag{10}$$

Let the function $m(t, \alpha) \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ be defined as $m(t, \alpha) = V(t, x^*(t))$.
We now prove that

$$m(t, \alpha) < u_\eta(t, \alpha), \quad \text{for} \quad t \in [t_0, \infty) \tag{11}$$

Observe that the inequality (11) holds for $t = t_0$ i.e

$$m(t_0, \alpha) = V(t_0, x_0) \leq u_0 < u_\eta(t_0, \alpha)$$

Assume that the inequality (11) is not true, then there exist a point $t_1 > t_0$ such that

$$m(t_1, \alpha) = u_\eta(t_1, \alpha) \quad \text{and} \quad m(t, \alpha) < u_\eta(t, \alpha) \quad \text{for} \quad t \in [t_0, t_1)$$

It follows from lemma (3.6) that

$${}^C D^q_+ m(t_1, \alpha) - {}^C D^q_+ u_\eta(t_1, \alpha) > 0$$

So that

$${}^C D^q_+ (V(t_1, x(t_1))) > {}^C D^q_+ (u_\eta(t_1, \alpha))$$

and using (9) we arrive at

$$^{C}D^{q}_{+}(V(t_1, x(t_1))) > g(t_1, u_\eta(t_1, \alpha) + \eta) > g(t_1, u(t_1, \alpha))$$

Therefore,

$$^{C}D^{q}_{+}(m(t_1, \alpha)) > g(t_1, u(t_1, \alpha)) \tag{12}$$

For $t \in [t_0, T]$, we maintain that $x^*(t)$ satisfies (4) and the equality,

$$\limsup_{h\to 0^+} \frac{1}{h^q}[x^*(t) - x_0 - S(x^*(t), h)] = f(t, x^*(t)) \tag{13}$$

holds, where $x^*(t)$ is any other solution of (4).

$$S(x^*(t), h) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[x^*(t - rh) - x_0)] \tag{14}$$

is the Grunwald Letnikov fractional derivative and $[\frac{t-t_0}{h}]$ is the integer part of $\frac{t-t_0}{h}$.

Multiply equation (13) althrough by h^q we have,

$$\limsup_{h\to 0^+}[x^*(t) - x_0 - S(x^*(t), h)] = h^q f(t, x^*(t))$$

$$x^*(t) - x_0 - \limsup_{h\to 0^+}[S(x^*(t), h)] = h^q f(t, x^*(t))$$

$$x^*(t) - x_0 - [S(x^*(t), h) + \rho(h^q)] = h^q f(t, x^*(t))$$

$$x^*(t) - h^q f(t, x^*(t)) = [S(x^*(t), h) + x_0 + \rho(h^q)] \tag{15}$$

For $t \in [t_0, T]$, we have

$$\begin{aligned} & m(t, \alpha) - m(t_0, \alpha) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[m(t - rh, \alpha) - m(t_0, \alpha)] \\ = \; & V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t - rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \\ = \; & V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t - rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \\ & + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t - rh, S(x^*(t), h) + x_0 + \rho(h^q) - V(t_0, x_0)] \\ & -[V(t - rh, x^*(t - rh) - V(t_0), x_0)] \end{aligned}$$

Since $V(t, x)$ is locally Lipschtzian in the second variable, we have

$$\leq L|(-1)^{r+1}||| \sum_{r=1}^{[\frac{t-t_0}{h}]} ({}^qC_r)[S(x^*(t), h) + x_0 + \rho(h^q) - x^*(t - rh)]||$$

where $L > 0$ is the Lipschitz constant.

$$\leq L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r[S(x^*(t),h) + \rho(h^q) - (x^*(t-rh) - x_0)]\| \tag{16}$$

Using equations (14), equation (16) becomes,

$$\leq L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[x^*(t-rh) - x_0)] + \rho(h^q) - (x^*(t-rh) - x_0)]\|$$

$$\begin{aligned} \leq \quad & L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(-1)^{r+1}(\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r[x^*(t-rh) - x_0)] + \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r\rho(h^q) \\ & - \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh) - x_0)]\| \end{aligned}$$

$$\leq L(-1)^{r+1}\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh) - x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r\rho(h^q)\| \tag{17}$$

Substituting equation (17) we obtain

$$\begin{aligned} &= V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh,x^*(t) - h^qf(t,x^*(t)) - V(t_0,x_0)] \\ &+L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh) - x_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\| \\ &= V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh,x^*(t) - h^qf(t,x^*(t)) - V(t_0,x_0)] \\ &+L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh) - x_0)[- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{rq}C_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\| \end{aligned}$$

Dividing althrough by $h^q > 0$ and taking the $limsup$ as $h \to 0^+$ we have,

$$\begin{aligned} &{}^CD_+^q m(t,\alpha) = \limsup_{h\to 0^+} \frac{1}{h^q}[V(t,x^*(t)) - V(t_0,x_0) \\ &- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r[V(t-rh,x^*(t) - h^qf(t,x^*(t)) - V(t_0,x_0)]] \\ &+ \limsup_{h\to 0^+} \frac{1}{h^q}L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r(x^*(t-rh) - x_0)[- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{rq}C_r - 1] \\ &+ \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1q}C_r\rho(h^q)\| \end{aligned}$$

Recall $\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{rq}C_r = -1$ *and* $\limsup_{h\to 0^+}\frac{1}{h^q}\rho(h^q) = 0$ so that we obtain,

$$^{C}D_{+}^{q}m(t,\alpha) = ^{C}D_{+}^{q}V(t,x^*(t)) + L\|\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^{q}C_r(x^*(t-rh)-x_0)[-(-1)-1]+0\|$$

$$^{C}D_{+}^{q}m(t,\alpha) = ^{C}D_{+}^{q}V(t,x^*(t)) + 0$$

Using condition (ii) of Theorem 4.1 we have

$$^{C}D_{+}^{q}m(t,\alpha) \leq g(t,V(t,x^*(t))) = g(t,m(t,\alpha)) \tag{18}$$

Also,

$$m(t_0^+,\alpha) \leq u_0 \; and \; m(t_k^+,\alpha) = V(t_k^+, x(t_k) + I_k(x(t_k)) \leq \rho_k(m(t_k)) \tag{19}$$

Now, equation (18) with $t = t_1$ contradicts (12), hence (11) holds.
For $t \in [t_0, T]$, we now show that whenever $\eta_1 < \eta_2$, then

$$u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \tag{20}$$

It is obvious that (20) holds for $t = t_0$. Assume the inequality (20) is not true. Then there exist a point $t_1 > t_0$ such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_1,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha)$ for $t \in [t_0, t_1)$.
By lemma (3.6), we have that

$$^{C}D_{+}^{q}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha)) > 0$$

However,

$$\begin{aligned}
^{C}D_{+}^{q}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha_)) &= {}^{C}D_{+}^{q}u_{\eta_1}(t_1,\alpha) - ^{C}D_{+}^{q}u_{\eta_2}(t_1,\alpha) \\
&= g(t_1, u(t_1,\alpha)+\eta_1)) - [g(t_1, u(t_1,\alpha)+\eta_2))] \\
&= \eta_1 - \eta_2 < 0
\end{aligned}$$

which is a contradiction and so (20) is true. Thus, equations (11) and (20) guarantee that the family of solutions $\{u_\eta(t,\alpha)\}$, $t \in [t_0, T]$ of (9) is uniformly bounded, i.e. there exists $P > 0$ with $|u_\eta(t,\alpha)| \leq P$, with bound P on $[t_0, T]$.
We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0, T]$. Assume $K = sup\{g(t,x) : (t,x) \in [t_0,T] \times [-P,P]\}$. Also, fix a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}(t)$, such that $\lim_{i\to\infty}\eta_i = 0$ and consider a sequence of functions $u_{\eta_i}(t,\alpha)$. Again let $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$, then we have the following estimate

$$\begin{aligned}
\|u_{\eta_i}(t_2,\alpha) - u_{\eta_i}(t_1,\alpha)\| &= \|u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i) \\
&- (u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i))\| \\
&= \frac{1}{\Gamma(q)}\|\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds \\
&- \int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds\| \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_2}(t_2-s)^{q-1} - \int_{t_0}^{t_1}(t_1-s)^{q-1}\right| ds \\
&= \frac{k}{\Gamma(q)}\left|-(\int_{t_0}^{t_1}(t_1-s)^{q-1} - \int_{t_0}^{t_2}(t_2-s)^{q-1})\right| ds \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1} - (\int_{t_0}^{t_1}(t_2-s)^{q-1} + \int_{t_1}^{t_2}(t_2-s)^{q-1})\right| ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}-\int_{t_0}^{t_1}(t_2-s)^{q-1}-\int_{t_1}^{t_2}(t_2-s)^{q-1}\right|ds\\
&\le \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}-\int_{t_0}^{t_1}(t_2-s)^{q-1}\right|ds+\left|\int_{t_1}^{t_2}(t_2-s)^{q-1}\right|ds\\
&= \frac{k}{\Gamma(q)}\left|\frac{(t_1-t_0)^q}{q}+\frac{(t_2-t_1)^q}{q}-\frac{(t_2-t_0)^q}{q}\right|+\frac{(t_2-t_1)^q}{q}\\
&\le \frac{k}{\Gamma(q+1)}(t_1-t_0)^q+(t_2-t_1)^q-(t_2-t_0)^q+(t_2-t_1)^q\\
&= \frac{k}{\Gamma(q+1)}(t_1-t_0)^q-(t_2-t_0)^q+2(t_2-t_1)^q\\
&\le \frac{2k}{\Gamma(q+1)}(t_2-t_1)^q<\epsilon
\end{aligned}$$

provided $\|t_2-t_1\|<\delta=(\frac{\epsilon\Gamma(q+1)}{2k})^{\frac{1}{q}}$, proving that the family of solutions $\{u_{\eta_i}(t;\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t;\alpha)\}$ has a subsequence $\{u_{\eta_{i_j}}(t;\alpha)\}$ which converges uniformly to a function $r(t)$ on $[t_0,T]$. We then show that $r(t)$ is a solution of (10). Equation (10) becomes

$$u_{\eta_{i_j}}(t,\alpha) = u_{0_{i_j}}+\eta_{i_j}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g_{i_j}(s,u_{i_j}(s,\eta_{i_j}))+\eta_{i_j})ds \tag{21}$$

Taking the limit as $i_j\to\infty$ in (21), yields

$$r(t)=u_0+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g(s,r(t)))ds \tag{22}$$

Thus, $r(t)$ is a solution of (6) on $[t_0,T]$. We claim that $r(t)$ is the maximal solution of (6). Then from (11), we have that $m(t,\alpha)<u_\eta(t,\alpha)\le r(t)$ on $[t_0,T]$.

5 Main results

In this section, we will obtain sufficient conditions for the eventual stability of the system (3.3).

Theorem 5.1. *Assume the following*

(i) $g\in PC[\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n]$ *satisfies* (A_0) *(ii) and* $g(t,u)$ *is quasi-monotone nondecreasing in* u *with* $g(t,0)\equiv 0$.

(ii) $V\in PC[\mathbb{R}_+\times\mathbb{R}^N,\mathbb{R}^N_+]$ *and* $V\in\mathcal{L}$ *such that*

$$^{C}D^q_+V(t,x)\le g(t,V(t,x)),t\ne t_k,(t,x)\in\mathbb{R}_+\times\mathbb{R}^N \tag{23}$$

holds for all $(t,x)\in\mathbb{R}_+\times S_\psi$.

(iii) there exists a $\psi_0>0$ *such that* $x_0\in S_\psi$ *implies that* $x+I_k(x)\in S_\psi$ *and* $V(t_k,x+I_k(x))\le\psi_k(V(t,x))$, $t=t_k,x\in S_\psi$ *and the function* $\psi_k:\mathbb{R}^N_+\to\mathbb{R}^N_+$ *is nondecreasing for* $k=1,2,...$

(iv) $a(\|x\|)\le V_0(t,x)$, *where* $a\in\mathcal{K}$ *and* $V_0(t,x)=\sum_{i=1}^N V_i(t,x)$
Then the eventual stability of the set of trivial solution $u=0$ *of the IFrDE (6) implies the eventual stability of the set of trivial solution* $x=0$ *of (4).*

Proof. Since $V(t,x)$ is positive definite, then

$$a(\|x\|)\le V_0(t,x),(t,x)\in J\times S_\psi,\ where\ J\subset\mathbb{R}_+ \tag{24}$$

and $a(r)$ is in the k-class. Let $0 < \epsilon < \psi$ and $t_0 \in \mathbb{R}_+$ be given.
Assume that the solution $u = 0$ of (6) is stable. Then given each $a(\epsilon) > 0$, and $t_0 \in \mathbb{R}_+$, there exist a positive function $\delta = \delta(t_0, \epsilon) > 0$ continuous in t_0 such that whenever

$$u_0 = \sum_{i=1}^{n} u_{i0} < \delta, \ we \ \ have \sum_{i=1}^{n} u_i(t; t_0, u_0) < a(\epsilon), t \geq t_0 \tag{25}$$

where $u(t; t_0, u_0)$ is any solution of (6).
Choose $u_0 = V(t_0^+, x_0)$
Since $V(t, x)$ is continuous, then by the property of continuity, given $\delta > 0$ there exists a positive function $\delta_1 = \delta_1(t_0, \delta) > 0$ that is continuous in t_0 for each δ, such that

$$\|V(t, x) - V(t_0, x_0)\| < \delta \ implies \ \|x - x_0\| < \delta_1$$

and as $\|V(t, x)\| \to 0$ as $\|x\| \to 0$, then the inequalities,

$$||x_0|| < \delta_1 \ and \ V(t_0, x_0) < \delta \tag{26}$$

hold simultaneously.
We claim that if

$$\|x_0\| < \delta_0, \ then \ \|x(t, t_0, x_0)\| < \delta.$$

Suppose that this claim is false, then there would exists a point $t_1 > t_0$ and a solution $x(t)$ with $\|x_0\| < \delta$ such that

$$\|x(t_1)\| = \epsilon \ and \ \|x(t)\| \leq \epsilon, t \in [t_0, t_1) \tag{27}$$

so that (24) becomes

$$a(\|x(t_1)\|) \leq V_0(t_1, x(t_1)). \tag{28}$$

This means that $\|x(t)\| < \psi$ for $t \in [t_0, t_1)$
Combining equations (7), (25) and (28) we obtain the estimate

$$a(\|x(t_1)\|) \leq V_0(t_1, x(t_1)) \leq \sum_{i=0}^{n} r_i(t; t_0, u_0)$$

where $r(t) = \sum_{i=0}^{n} r_i(t; t_0, u_0)$ is the maximal solution of (6).

$$a(\epsilon) \leq V_0(t_1, x(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < a(\epsilon)$$

which leads to a contradiction. Hence, the eventual stability of the trivial solution $u = 0$ of (6) implies the eventual stability of the set of trivial solution $x = 0$ of (4).

6 Application

Let the points $t_k, t_k < t_{k+1}$, $lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive Caputo fractional differential equations

$$\begin{aligned} {}^C D^q x_1(t) &= -10x_1 - \frac{x_2^2 cos x_1}{2x_1} + x_1 sin x_2 + \frac{x_2^2 tan x_1}{x_1} \\ {}^C D^q x_2(t) &= \frac{3x_1^2}{x_2} - x_2 sin x_1 - 5x_2 cosec x_1 - x_1^2 cos x_2 \\ \Delta x_1 &= s_k(x(t_k)), \Delta x_2 = n_k(x(t_k)), k = 1, 2, .. \end{aligned} \tag{29}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_{10} \quad \text{and} \quad x_2(t_0^+) = x_{20}$$

Consider a vector $V = (V_1, V_2)^T$, where
$V_1(t, x_1, x_2) = x_1^2$ and $V_2(t, x_1, x_2) = x_2^2$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$.
Now

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = x_1^2 + x_2^2$$

and so $b(\|x\|) \leq V_0(t, x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, implying that $a, b \in \mathcal{K}$. From (5), we compute the Caputo fractional Dini derivative for $V_1(t, x_1, x_2) = x_1^2$ for $t > 0, t \neq t_k$ as follows:

$$\begin{aligned}
{}^{C}D_+^q V_1(t; x_1, x_2) &= \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ V(t, x) - V(t_0, x_0) \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[V(t - rh, x - h^q f(t, x)) - V(t_0, x_0)] \big\} \\
&= \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[(x_1 - h^q f_1(t; x_1, x_2)^2 - x_{10}^2] \big\} \\
&\leq \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ x_1^2 - x_{10}^2| + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[x_1^2 - 2x_1 h^q f_1(t; x_1, x_2) \\
&\quad + h^{2q} f_1(t, x_1) - x_{10}^2] \big\} \\
&= \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2 \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) 2x_1 h^q f_1(t; x_1, x_2) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^{2q} f_1(t; x_1, x_2) \big\} \\
&= \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ x_1^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2 - x_{10}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{10}^2 \\
&\quad - 2x_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_1(t; x_1, x_2)| \big\} \\
&= \limsup_{h \to 0^+} \frac{1}{h^q} \big\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_1^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{10}^2 \\
&\quad - 2x_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_1(t; x_1, x_2) \big\}
\end{aligned}$$

Recall that,

$$\limsup_{h \to 0^+} \frac{1}{h^q} \big\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) = \frac{x_1^2}{t^q \Gamma(1-q)}, \tag{30}$$

and

$$\lim_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^{r\,q}C_r = -1 \tag{31}$$

Substituting (30) and (31), we obtain

$$\begin{aligned}
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)} - \frac{x_{10}^2}{t^q\Gamma(1-q)} + 2x_1f_1(t;x_1,x_2)\\
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1f_1(t;x_1,x_2)
\end{aligned}$$

$$\begin{aligned}
{}^{C}D_{+}^{q}V_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1(-10x_1 - \frac{x_2^2cosx_1}{2x_1} + x_1sinx_2 + \frac{x_2^2tanx_1}{x_1})\\
&= \frac{x_1^2}{t^q\Gamma(1-q)} - 20x_1^2 - x_2^2cosx_1 + 2x_1^2sinx_2 + x_2^2tanx_1
\end{aligned}$$

As $t\to\infty$, $\frac{x_1^2}{t^q\Gamma(1-q)}\to 0$, so that we have

$$\begin{aligned}
&\leq -20x_1^2 - x_2^2cosx_1 + 2x_1^2sinx_2 + x_2^2tanx_1\\
&= 2x_1^2(-10+sinx_2) + x_2^2(2tanx_1 - cosx_1)\\
&\leq 2x_1^2(-10+|sinx_2|) + x_2^2(2\frac{|sinx_1|}{|cosx_1|} - |cosx_1|)\\
&\leq 2x_1^2(-10+1) + x_2^2(2-1)\\
&= 2x_1^2(-9) + x_2^2(1)\\
&= x_1^2(-18) + x_2^2(1)
\end{aligned}$$

Therefore,

$${}^{C}D_{+}^{q}V_1(t;x_1,x_2) \leq -18V_1 + V_2 \tag{32}$$

Also, for $x_0\in S_\psi$, for $t=t_k, k=1,2,..$, we have

$$V(t,x(t)+c_k) = |c_k + x(t)| \leq V(t,x(t))$$

Similarly, using (5), we compute the Caputo fractional Dini derivative for $V_2(t,x_1,x_2)=x_2^2$ as follows:

$$\begin{aligned}
{}^{C}D_{+}^{q}V_2(t;x_1,x_2) &= \limsup_{h\to 0^+}\frac{1}{h^q}\{V(t,x)-V(t_0,x_0)\\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)[V(t-rh,x-h^qf(t,x))-V(t_0,x_0)]\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)[(x_2-h^qf_2(t;x_1,x_2))^2 - x_{20}^2]\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2| + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r)[x_2^2 - 2x_2 h^q f_2(t;x_1,x_2) \\
&\quad + h^{2q} f_2(t,x_2) - x_{20}^2]\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 - x_{20}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_2^2 \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) 2x_2 h^q f_2(t;x_1,x_2) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^{2q} f_2(t;x_1,x_2)\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_2^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_2^2 - x_{20}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{20}^2 \\
&\quad -2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_2(t;x_1,x_2)|\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\{\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_2^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) x_{20}^2 \\
&\quad -2x_2 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^qC_r) h^q f_2(t;x_1,x_2)\}
\end{aligned}
$$

$$
\begin{aligned}
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} - \frac{x_{20}^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t;x_1,x_2) \\
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2 f_2(t;x_1,x_2) \\
{}^C D_+^q V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2(\frac{3x_1^2}{x_2} - x_2 sinx_1 - 5x_2 cosecx_1 - \frac{x_1^2 cosx_2}{x_2}) \\
&= \frac{x_1^2}{t^q\Gamma(1-q)} - 2x_2^2 sinx_1 + 6x_1^2 - 10x_2^2 cosecx_1 - 2x_1^2 cosx_2 \\
&\leq -2x_2^2 sinx_1 - 10x_2^2 cosecx_1 + 6x_1^2 - 2x_1^2 cosx_2 \\
&= -2x_2^2(sinx_1 + 5cosecx_1) + 2x_1^2(3 - cosx_2) \\
&\leq -2x_2^2(|sinx_1| + \frac{10}{|sinx_1|}) + 2x_1^2(3 - |cosx_2|) \\
&\leq -2x_2^2(1+5) + 2x_1^2(3-1) \\
&= -2x_2^2(6) + 2x_1^2(2) \\
&= -12V_2 + 4V_1
\end{aligned}
$$

Therefore,

$$ {}^C D_+^q V_1(t;x_1,x_2) \leq 4V_1 - 12V_2 \tag{33} $$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$ V(t, x(t) + d_k) = |d_k + x(t)| \leq V(t, x(t)) $$

Combining (33) and (32), we have

$$ {}^C D_+ V \leq \begin{pmatrix} -18 & 1 \\ 4 & -12 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V) \tag{34} $$

Now consider the comparison system

$$^{C}D^{q}u = g(t,u) = Au \tag{35}$$

where $A = \begin{pmatrix} -18 & 1 \\ 4 & -12 \end{pmatrix}$.

The vectorial inequality (34) and all other conditions of Theorem (5.1) are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (35) is eventually stable. Therefore, the set $x(t) = 0$ for the system of IFrDE (29) is eventually stable.

7 Discussions and conclusion

This study provides a significant extension to the theory of nonlinear impulsive Caputo fractional differential equations, particularly in the context of eventual stability with fixed moments of impulse. By employing vector Lyapunov functions generalized through piecewise continuous functions, a broader and more flexible framework for analyzing the stability of such systems was established. The incorporation of comparison results further enriched the analytical approach, facilitating the derivation of sufficient conditions for eventual stability. Compared to existing methods, our approach offers notable improvements by generalizing stability conditions and extending their applicability to a wider class of impulsive fractional systems. The results bridge the gap between classical stability analysis and the unique challenges posed by fractional impulsive systems. This is particularly important for real-world systems where impulsive effects and fractional-order dynamics coexist, such as in biological systems, control engineering, and economic models.

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