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Eight-order second derivative block linear multistep method for chaotic initial value problems

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Abstract

This paper aims to enhance the accuracy and stability of eight-order second derivative linear multistep methods (LMMs) for solving chaotic initial value problems (IVPs). The method is developed using a multistep collocation approach. The newly derived method is applied to plot solution curves, solid lines, and phase portraits, which are compared with those obtained from well-established ordinary differential equations (ODEs) solvers. The results show excellent agreement between the proposed method and the established solvers. The proposed block method demonstrates efficiency and accuracy, making it a suitable choice for solving chaotic and nonlinear ordinary differential equations.

Keywords and phrases: multistep; chaotic; numerical; derivative

1 Introduction

Chaos theory was first reported in the work of Lorenz [1], who discovered it when studying the earth's atmosphere and weather conditions. Since then, chaos theory has attracted the attention of many researchers because of its applicability in numerous everyday activities. The initial value problem is given by

$$y'(x) = f(y(x), y(x)), y(x_0) = y_0 \tag{1}$$

on the interval $I = [x_0, x_N]$ where $y : [x_0, x_N] \to \Re^m$ and $f : [x_0, x_N] \times \Re^m \to \Re^m$ is continuous and differentiable.

Solving chaotic systems (1) has long been a challenge to researchers and remains so due to their complex and dynamic nature. These systems are highly sensitive to tiny changes in the initial conditions, and their solutions can rapidly transition from being stable to instability. This sensitivity

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and unpredictability make chaotic systems difficult to model and analyze, posing a persistent challenge to researchers.

Chaotic systems exhibit distinct characteristics, including the butterfly effect, non-periodic order, strange attractors, bifurcation, and fractals.

However, chaotic ODEs often deferred analytical solutions, necessitating the use of numerical methods. While, analytical and semi- iterative methods are commonly employed to solve chaotic ODEs, offering advantages over linear multistep methods [2-3], we aim to demonstrate that linear multistep methods can also be successfully utilized to solve chaotic ODEs.

Accordingly, this study focuses on the derivation of an eight-order second derivative linear multistep method and its implementation as a block integrator for solving renowned chaotic initial value problems. By deriving and applying this method, we aim to demonstrate its effectiveness in tackling chaotic systems, which are notoriously challenging to solve numerically.

Theorem 1.1 (Existence and Uniqueness of the solution): Let f(x, y) be defined and continuous for all points (x, y) in region D defined by $a \le x \le b, -\infty < y < \infty$, a and b finite, and let there exist a constant L such that, for every x, y such that (x, y) and (x, y^*) are both in D, $|f(x,y) - f(x,y^*)| \le L|y - y^*|$. Then, if y^* is any given number, there exist a unique solution y(x) of the IVP (1), where y(x) is continuous and differentiable for all (x, y) in D [4].

This work is organized as follows: Section 2 presents the derivation of the eight-order second derivative linear multistep method. Section 3 provides the convergence analysis and plots the regions of absolute stability of the newly derived block integrators. Section 4 offers numerical simulations/sensitivity analysis in graphical form, accompanied by a discussion Finally, Section 5 concludes the work with a summary of the key findings.

2 Derivation of the method

Building on the approach and definition presented in [5-6] for the continuous formulation of the order k-step second derivative linear multistep methods, we have derived the continuous form of the eight-order second derivative linear multistep method, which is presented below and with $\eta = (x - x_n)$.

$$
\begin{aligned}
y(x) = {} & \frac{1}{34560h_n^9}[-\eta^9 + 32h_n\eta^8 - 446h_n^2\eta^7 + 3548h_n^3\eta^6 - 17717h_n^4\eta^5 + 57438h_n^5\eta^4 - 12036432h_n^6\eta^3 + \\
& 156432h_n^7\eta^2 - 113472h_n^4\eta + 34560h_n^9]y_n - \frac{1}{2160h_n^9}[\eta^9 - 31h_n\eta^8 + 415h_n^2\eta^7 - 3133h_n^3\eta^6 + 14584h_n^4\eta^5 - 42844h_n^5\eta^4 + 77520h_n^6\eta^3 \\
& -78912h_n^7\eta^2 + 34560h_n^4\eta]y_{n+1} + \frac{1}{192h_n^9}[-\eta^9 + 30h_n\eta^8 - 386h_n^2\eta^7 + 2776h_n^3\eta^6 - 12165h_n^4\eta^5 + 33098h_n^5\eta^4 - 54168h_n^6\eta^3 \\
& +48096h_n^7\eta^2 - 117280h_n^4\eta]y_{n+2} - \frac{1}{36h_n^9}[2\eta^9 - 57h_n\eta^8 + 692h_n^2\eta^7 - 4661h_n^3\eta^6 + 18980h_n^4\eta^5 - 47612h_n^5\eta^4 + 71376h_n^6\eta^3 \\
& -57920h_n^7\eta^2 + 19200h_n^4\eta]y_{n+3} - \frac{1}{69h_n^9}[-445\eta^9 + 12688h_n\eta^8 - 154246h_n^2\eta^7 + 1041484h_n^3\eta^6 - 4256737h_n^4\eta^5 + 10732084h_n^5\eta^4 - 16190076h_n^6\eta^3 \\
& +13230288h_n^7\eta^2 - 44150h_n^4\eta]y_{n+4} + \frac{1}{240h_n^9}[\eta^9 - 27h_n\eta^8 + 311h_n^2\eta^7 - 1993h_n^3\eta^6 + 7752h_n^4\eta^5 - 18668h_n^5\eta^4 + 27024h_n^6\eta^3 \\
& -21312h_n^7\eta^2 + 6912h_n^4\eta]y_{n+5} - \frac{1}{8640h_n^9}[\eta^9 - 26h_n\eta^8 + 290h_n^2\eta^7 - 1808h_n^3\eta^6 + 6869h_n^4\eta^5 - 16214h_n^5\eta^4 + 23080h_n^6\eta^3 \\
& -17952h_n^7\eta^2 + 5760h_n^4\eta]y_{n+6} - \frac{1}{576h_n^8}[-19\eta^9 + 544h_n\eta^8 - 6634h_n^2\eta^7 + 44884h_n^3\eta^6 - 183631h_n^4\eta^5 + 463036h_n^5\eta^4 - 698196h_n^6\eta^3 \\
& +570096h_n^7\eta^2 - 190080h_n^8\eta]f_{n+4} + \frac{1}{96h_n^7}[-\eta^9 + 28h_n\eta^8 - 334h_n^2\eta^7 + 2212h_n^3\eta^6 - 8869h_n^4\eta^5 + 21952h_n^5\eta^4 - 32556h_n^6\eta^3 + 26208h_n^7\eta^2 \\
& -8640h_n^4\eta]g_{n+4}
\end{aligned}
\qquad (2)
$$

Evaluating (2) at $\eta = 7h_n$ and its first derivative at $\eta = \{0, h_n, 7h_n, 4h_n, 5h_n, 6h_n\}$ yield the block integrators represented in block matrix using finite difference form as:

$$AY_m = BY_{m-1} + CF_m + DF_{m-1} + WG_m + UG_{m-1} \qquad (3)$$

where,

$$A=\begin{vmatrix} 1 & \frac{225}{49} & \frac{900}{49} & \frac{4475}{196} & \frac{45}{49} & \frac{-1}{49} & 0 \\ \frac{-16}{465} & 1 & \frac{64}{31} & \frac{595}{186} & \frac{-16}{155} & \frac{1}{465} & 0 \\ \frac{-13}{2545} & \frac{-139}{2036} & 1 & \frac{9625}{8144} & \frac{-316}{2545} & \frac{527}{91620} & 0 \\ \frac{-64}{2555} & \frac{72}{511} & \frac{1200}{1533} & 1 & \frac{576}{12775} & \frac{-8}{7665} & 0 \\ \frac{5}{681} & \frac{-25}{227} & \frac{500}{227} & \frac{9025}{2724} & 1 & \frac{-5}{681} & 0 \\ \frac{-48}{227} & \frac{675}{227} & \frac{11200}{227} & \frac{-27525}{454} & \frac{2160}{227} & 1 & 0 \\ 14 & -189 & 2835 & \frac{13475}{4} & -378 & 21 & 1 \end{vmatrix}, B=\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{-5}{196} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{930} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-101}{366480} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{197}{38325} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2724} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{454} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3}{3} \end{vmatrix},$$

$$C=\begin{vmatrix} \frac{-20}{49} & \frac{600}{49} & \frac{575}{49} & 0 & 0 & 0 & 0 \\ 0 & \frac{-12}{31} & \frac{-64}{31} & -\frac{50}{31} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1369}{3054} & \frac{-10205}{18324} & 0 & 0 & -\frac{1}{9162} \\ 0 & 0 & \frac{256}{511} & \frac{264}{511} & 0 & 0 & 0 \\ 0 & 0 & \frac{-200}{227} & \frac{-175}{227} & \frac{60}{227} & 0 & 0 \\ 0 & 0 & \frac{4800}{227} & \frac{5850}{227} & \frac{60}{227} & 0 & 0 \\ 0 & 0 & -1260 & -1565 & 0 & 0 & 0 \end{vmatrix}, D=\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{2555} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$W = \begin{vmatrix} 0 & 0 & 0 & \frac{-150}{49} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{12}{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{667}{3054} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{72}{511} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{150}{227} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2700}{227} & 0 & 0 & 0 \\ 0 & 0 & 0 & -630 & 0 & 0 & 0 \end{vmatrix}, U = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

is a 7-dimentional vector Y_m, Y_{m-1}, F_m, F_{m-1}, G_m, *and* G_{m-1}having interpolation and collocation points as:

$$\begin{aligned} Y_m &= \left[y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7} \right] \\ Y_{m-1} &= \left[y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n \right] \\ F_m &= \left[f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7} \right] \\ F_{m-1} &= \left[f_{n-6}, f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n \right] \\ G_m &= \left[g_{n+1}, g_{n+2}, g_{n+3}, g_{n+4}, g_{n+5}, g_{n+6}, g_{n+7} \right] \\ G_{m-1} &= \left[g_{n-6}, g_{n-5}, g_{n-4}, g_{n-3}, g_{n-2}, g_{n-1}, g_n \right] \end{aligned}$$

3 Stability analysis of the method

The newly derived block integrators are thoroughly examined, with a focus on their local truncations error, order, consistency and zero stability. Additionally, the regions of absolute stability for these bloc integrators will be potted, providing a comprehensive understanding of their numerical properties and performance.

3.1 Convergence analysis

The convergence of the method derived block integrators, presented in Section 2, is conducted as shown in [1, 4] the method (3) is of order greater than one. Furthermore, in [7], it is zero-stable, in [8], is consistent. Therefore, according [11], the method is convergent. A summary of the finding is provided in Table 1. The convergence analysis was performed to determine whether the newly derived block integrators are suitable for solving the chaotic ODEs.

Table 1. Convergence analysis of the new method in block form

Step No.	Evaluation points	Order	Error Constant
Case k=7	$y'(x_n), y'(x_{n+1}),$ $y'(x_{n+3}), y'(x_{n+4}),$ $y'(x_{n+5}), y'(x_{n+6}),$ x_{n+7}	8, 8, 8, 8, 8, 8, 8	$\frac{1}{4116}, -\frac{1}{48825}, \frac{737}{38480400}, \frac{4}{268275}, \frac{1}{57204}, \frac{1}{1589}, \frac{1}{20}$

3.1 Region of absolute stability of the new method

Following the approach outlined in [9], the newly derived bock integrators presented in Section 2 are transformed into general linear methods. The elements of matrices A, B, C, D, W, and U are obtained from equation (2) and substituted into the characteristic equation:

$$del(r(A - Cz - Wz^2) - B - Dz - Uz^2) = 0 \tag{11}$$

By solving the characteristic equation, we obtain the stability polynomial and its derivative. These will be substituted into MATLAB code to plot the region of absolute stability. According to [10], the method is A(α)-stable, and this is illustrated in Fig. 1, which shows the region of absolute stability.

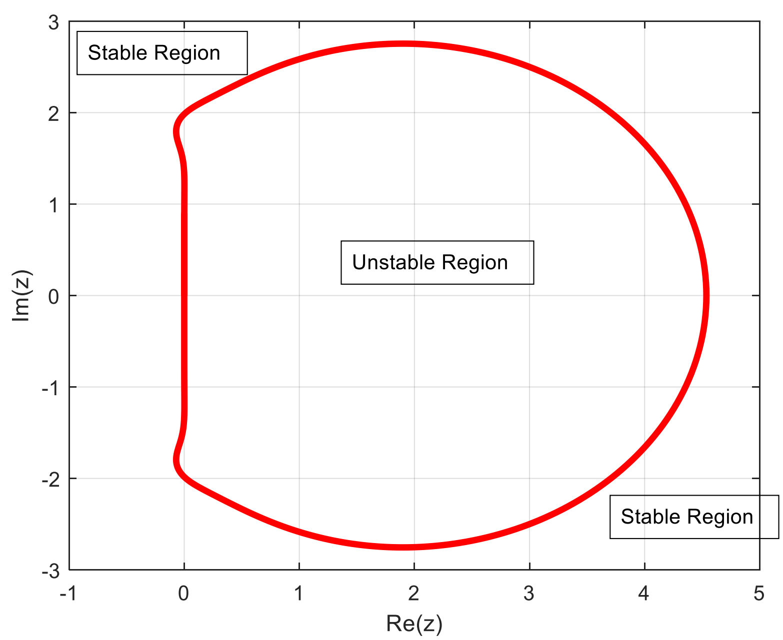

Fig. 1. Region of absolute stability for eight-order second derivative blocks linear multistep method.

4 Numerical simulations/sensitivity analysis

In this section, we apply our newly derived block integrators to solve three real-life problems exhibiting chaotic behaviour, as reported in the literature. Specially, we consider the Lorenz system, Genesio-Tesi and the Rössler system. We then compare the results obtained using our block integrators with those achieved by traditional ODE solvers, demonstrating the effectiveness of our approach in capturing the complex dynamics of these chaotic systems.

The step size, h, was varied across different systems: between 0.1 and 0.0001 for the Lorenz system, 0.1 and 0.001 for the Genesio-Tesi system, and 0.1 and 0.00001 for the Rössler system. The accuracy level set to six decimal places. The results were compared with MATLAB in-built solvers for IVPs of ODEs, specifically: ode45, ode113, ode23s and ode15s. Each of these solvers has its advantages depending on the type of problem type and required order of accuracy.

Model 1: Lorenz system

The Lorenz system [1] was derived from modeling a two-dimensional fluid cell confined between two parallel plates at different temperatures. The variables y_1, y_2, and y_3 represent the rate of convergence of overturning the horizontal temperature difference, and the departure gradient, respectively. The bounded regions of the trajectories converge to an attractor, which is a fractal with a dimension approximately 2.06.

$$
\begin{aligned}
y_1' &= a(y_2 - y_1) \\
y_2' &= -y_1 y_3 + b y_1 - y_2 \\
y_3' &= y_1 y_2 - c y_3
\end{aligned}
$$

where $a = 10, b = 28, c = \frac{8}{3}$ $y_1(0) = 1, y_2(0) = 5, y_3(0) = 10,$ for $x \in [0,10]$ and $h = 10^{-1}$ to 10^{-3}.

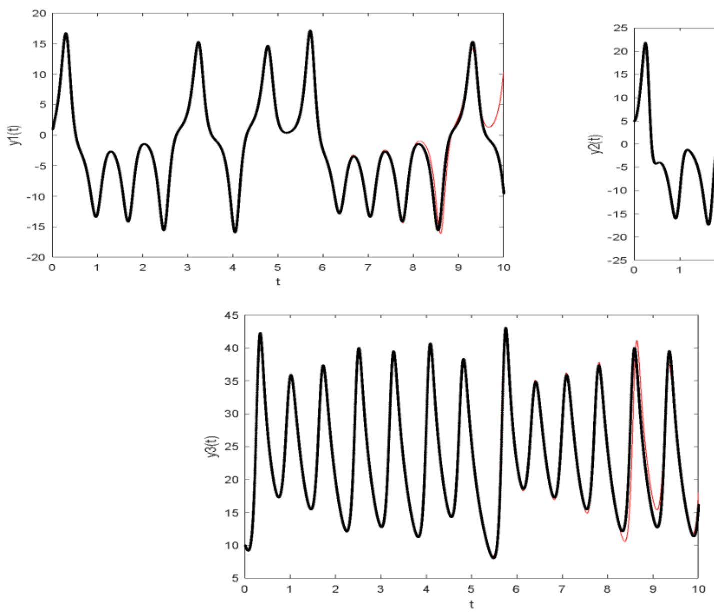

Fig. 2. Comparison of each component of the solutions as a function of time t, showing the plot of the coordinates of the Lorenz system using eight-order second derivative block linear multistep method (red solid lines) and ode45(black dots).

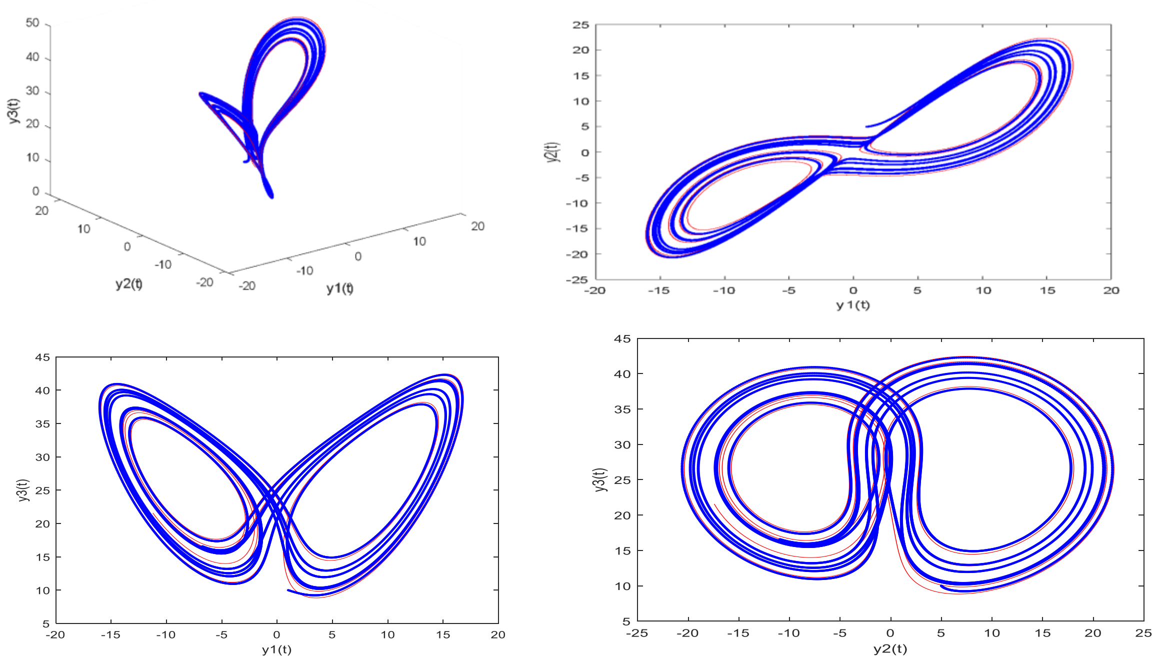

Fig. 3. Comparison of the phase portrait of Lorenz system using the eight-order second derivative block linear multistep method (red solid lines) and ode23s (blue dots).

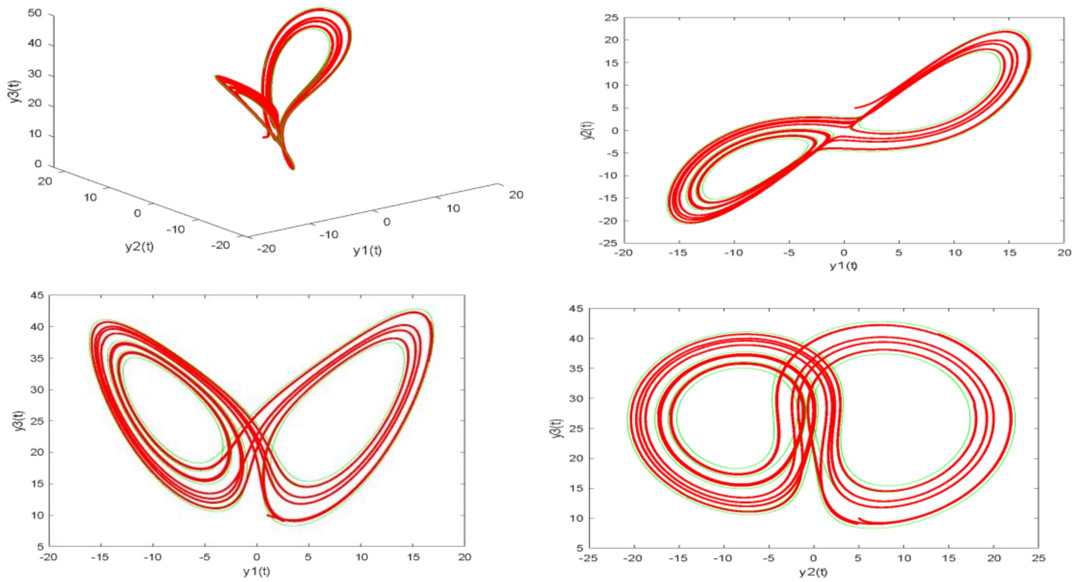

Fig. 4. Comparison of the phase portrait of the Lorenz system using the order eight-order second derivative block linear multistep method(red solid lines) and ode113(green dots).

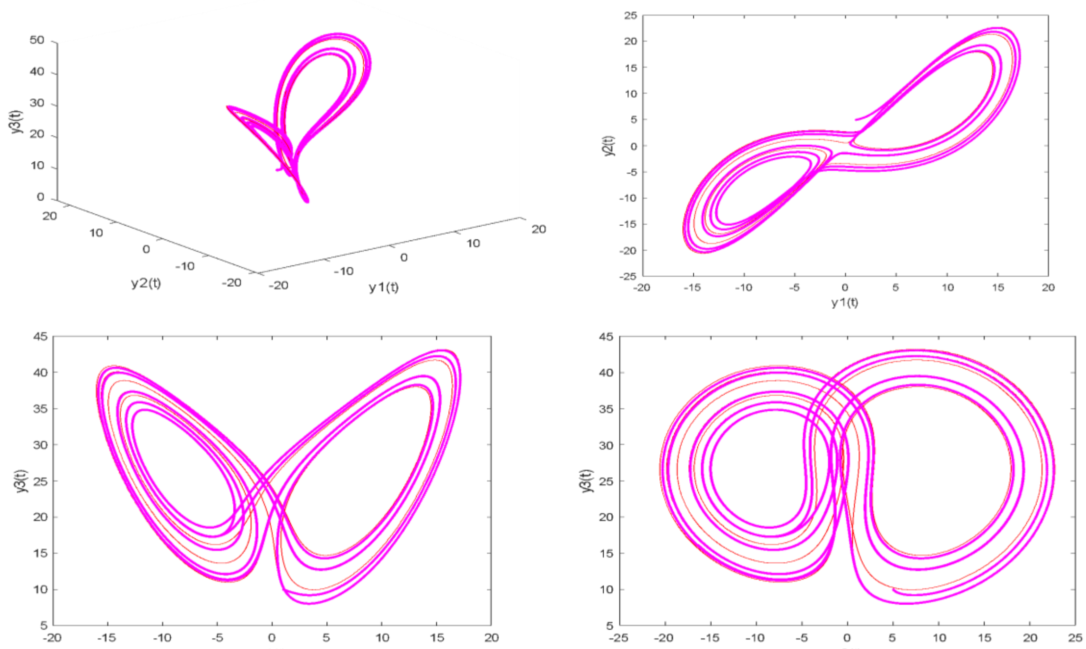

Fig. 5. Comparison of the phase portrait of the Lorenz system using the eight-order second derivative block linear multistep method (red solid lines) and ode45(marble dots).

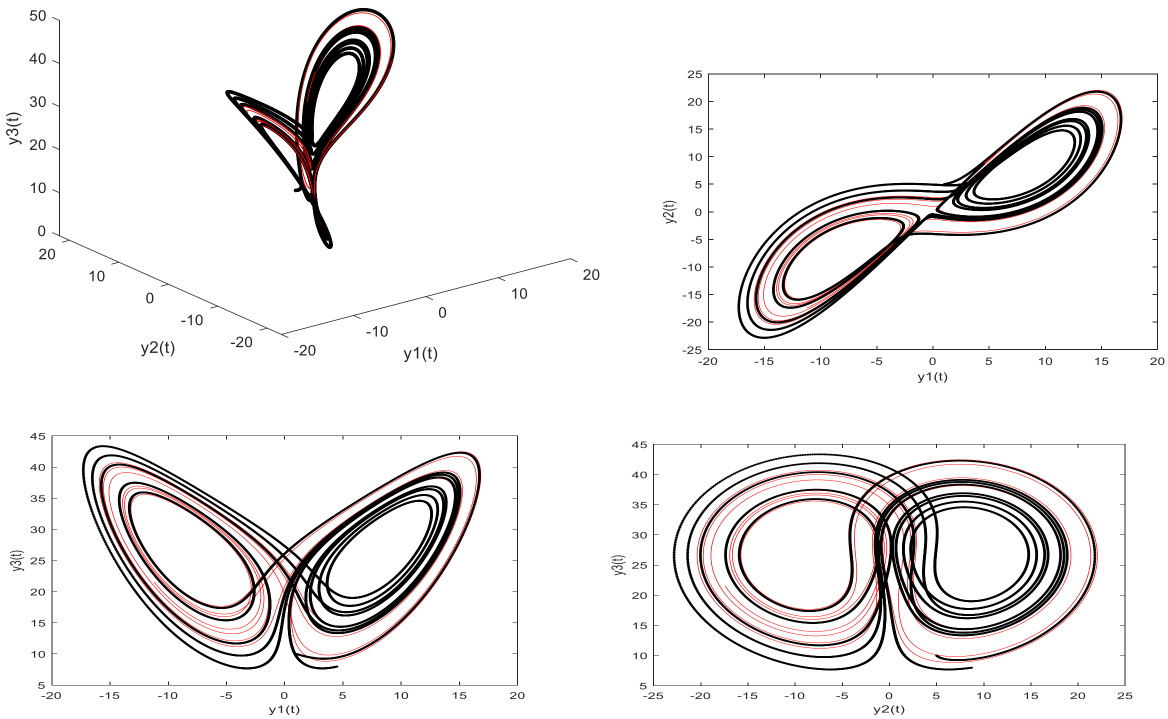

Fig. 6. Comparison of the phase portrait of the Lorenz system using eight-order second derivative block linear multistep method (red solid lines) and ode15s (black dots).

The time series and phase portraits of the approximate solutions obtained from order eight second derivative block linear multistep method for [1] are plotted in 3D and 2D in Figs. 2-6 respectively. The solid lines and the phase portraits compared with results from ODEs solvers(ode15s, ode23s, ode45 and ode113).The excellent agreement between these plots demonstrates the accuracy of our newly derived block integrator with each other. Hence, these show the accuracy of our newly derived integrators. According to the existence and uniqueness theorem (Theorem 1), trajectories cannot merge or cross, ensuring that the strange attractors are unique. In all cases, the solid lines and phase portraits show good agreement with the ODE solvers, further validating our approach.

Model 2: Genesio-Tesi system[12] is another exemplary model that embodies the paradigm chaotic systems. The system is modeled by the following set of IVPs:

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= cy_1 + by_2 + ay_3 + y_1^2
\end{aligned}
$$

This system was solved for the parameters $a = -1.2, b = -2.92, c = -6$, $y_1(0) = 0.2, y_2(0) = -0.3, y_3(0) = 0.1$ for $x \in [0, 70]$, and $h = 10^{-1}$ to 10^{-3}.

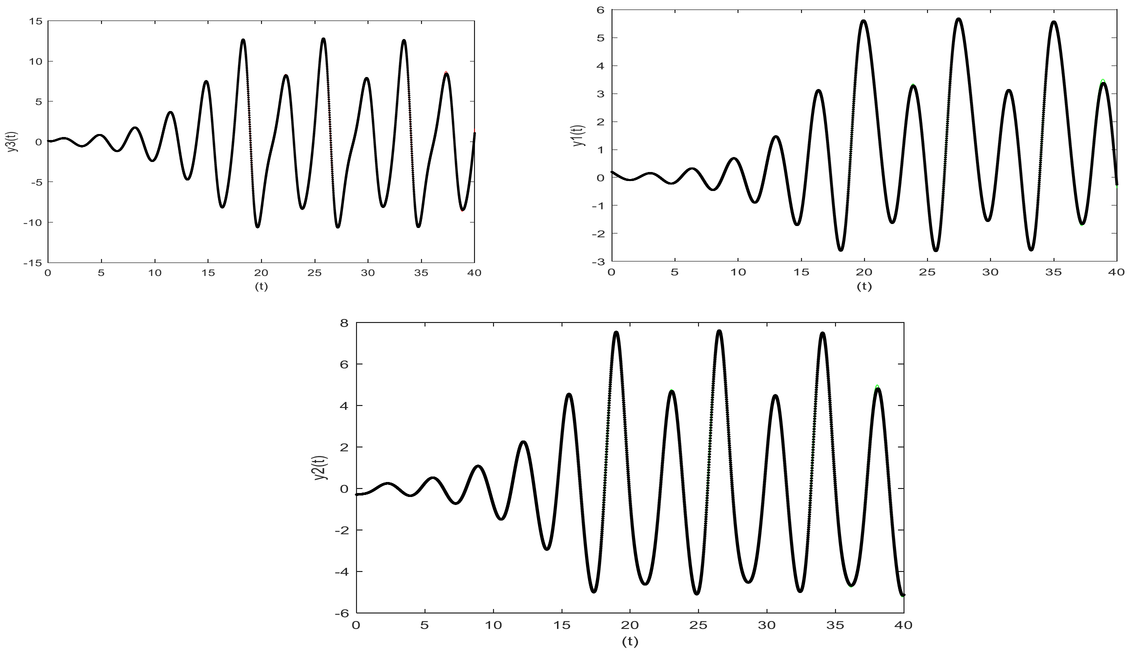

Fig. 7. Comparison of each component of the solutions as a function of time t, showing the plot of the coordinates of the Genesio-Tesi system using the eight-order second derivative block linear multistep method (green solid lines) and ode45(black dots).

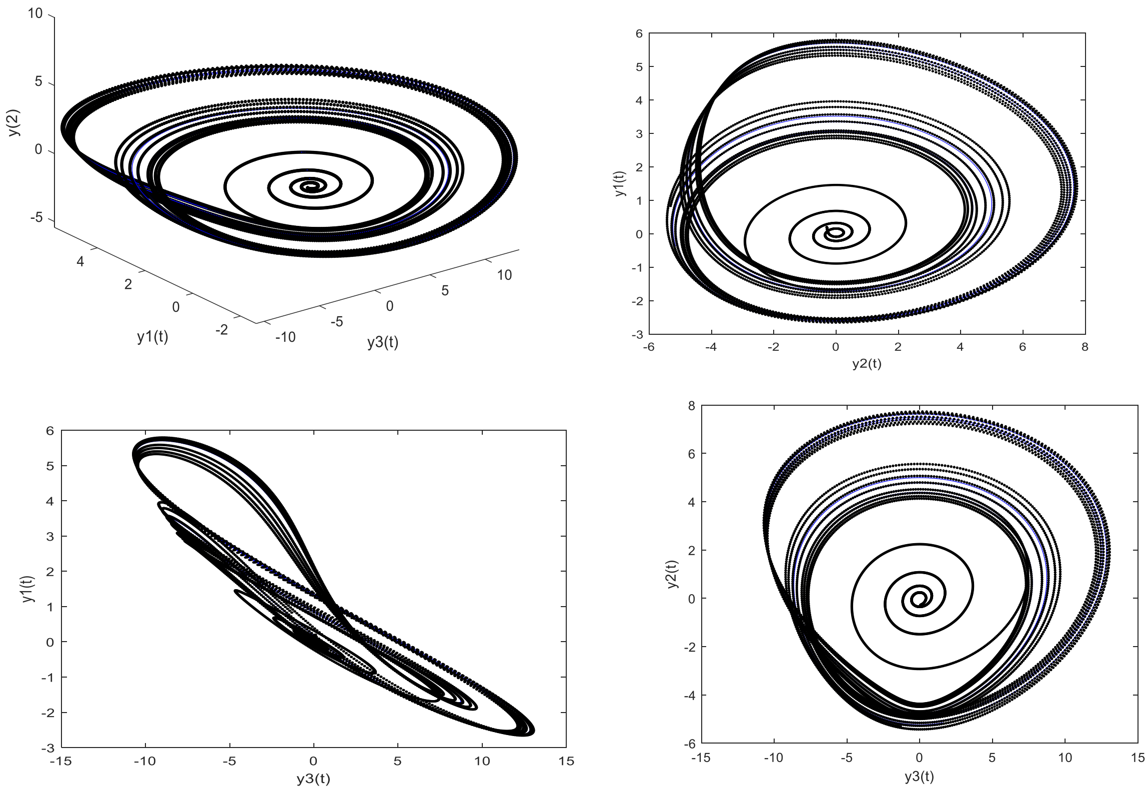

Fig. 8. Comparison of the phase portrait of the Genesio -Tesi system using the order eight second derivative block linear multistep method (blue solid lines) and ode23s (black dots).

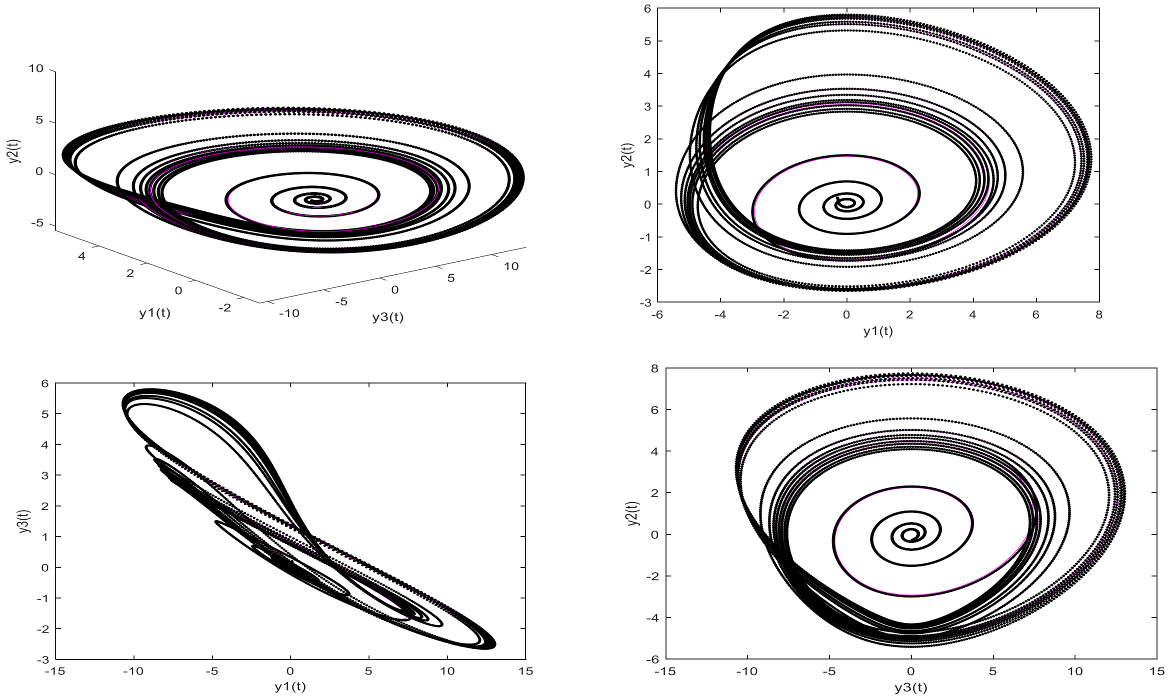

Fig. 9. Comparison of the phase portrait of the Genesio -Tesi system using the eight-order second derivative block linear multistep method (marble solid lines) and ode15s (black dots).

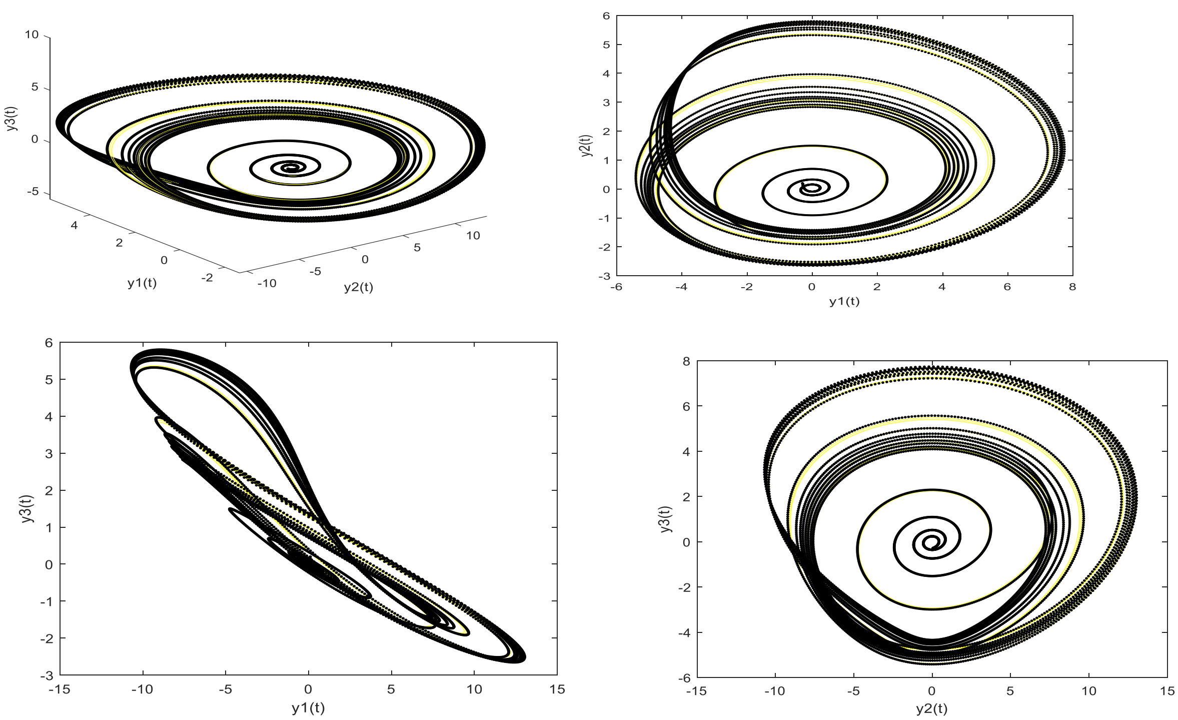

Fig. 10. of the phase portrait of the Genesio -Tesi system using the eight-order second derivative block linear multistep method (yellow solid lines) and ode113(black dots).

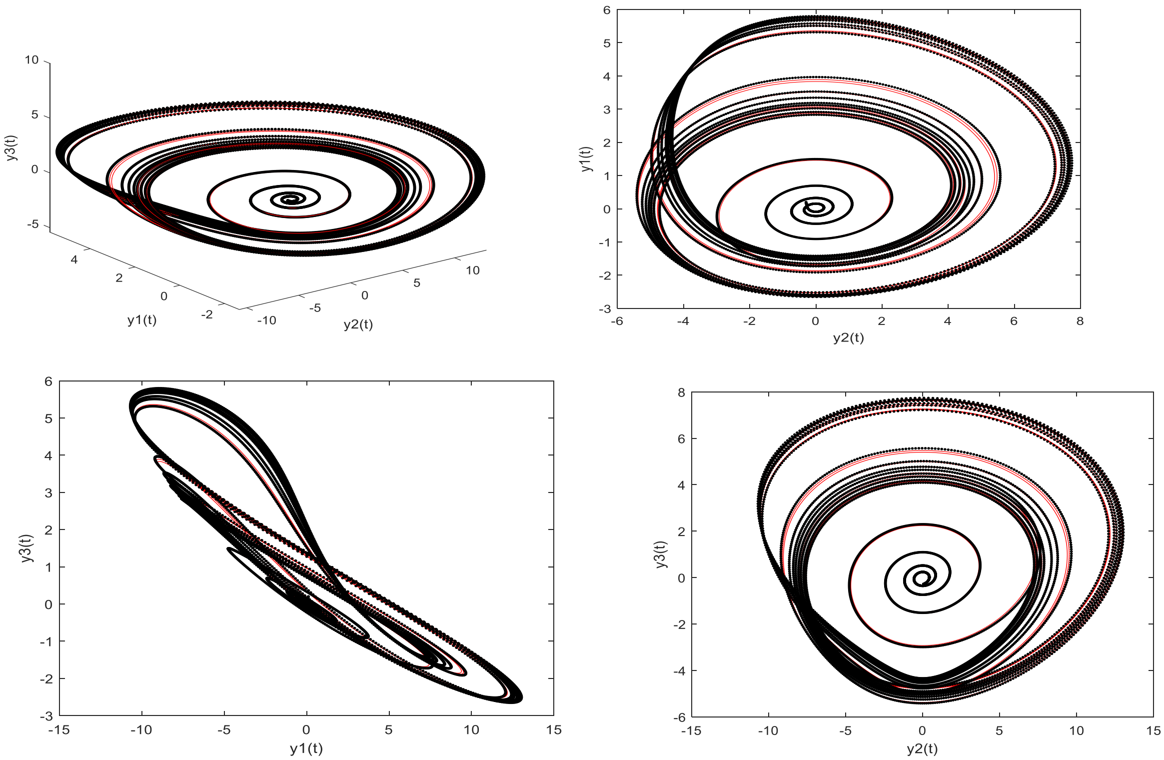

Fig. 11. Comparison of the phase portrait of the Genesio -Tesi system using the eight-order second derivative block linear multistep method(red solid lines) and ode15s(black dots).

The time series and phase portraits of the approximate solutions obtained from order eight second derivative block linear multistep method for[12] are plotted in 3D and 2D in Figs. 7-11 respectively. Similarly, the time series and the phase portraits of the approximate solutions for the [12] are compared with results from ODEs solvers(ode15s, ode23s, ode45 and ode113) in Figs 7-11. The solid lines and phase portraits show excellent agreement between our newly derived block integrator and the ODE solvers, demonstrating their accuracy. According to the existence and uniqueness theorem (Theorem 1), trajectories cannot merge or cross, ensuring that the strange attractors are unique. In all cases, the solid lines and phase portraits show good agreement with the ODE solvers, further validating our approach.

Model 3: Rössler system [13]

$$\begin{aligned} y_1' &= -(y_2 - y_1) \\ y_2' &= y_1 + a y_2 \\ y_3' &= b + y_1 y_3 - c y_3 \end{aligned}$$

where, $a = 0.2, b = 0.2, c = 5.7$, $y_1(0) = 0, y_2(0) = 0, y_3(0) = 0$ for $x \in [0,80]$, and $h = 10^{-1}$ to 10^{-3}.

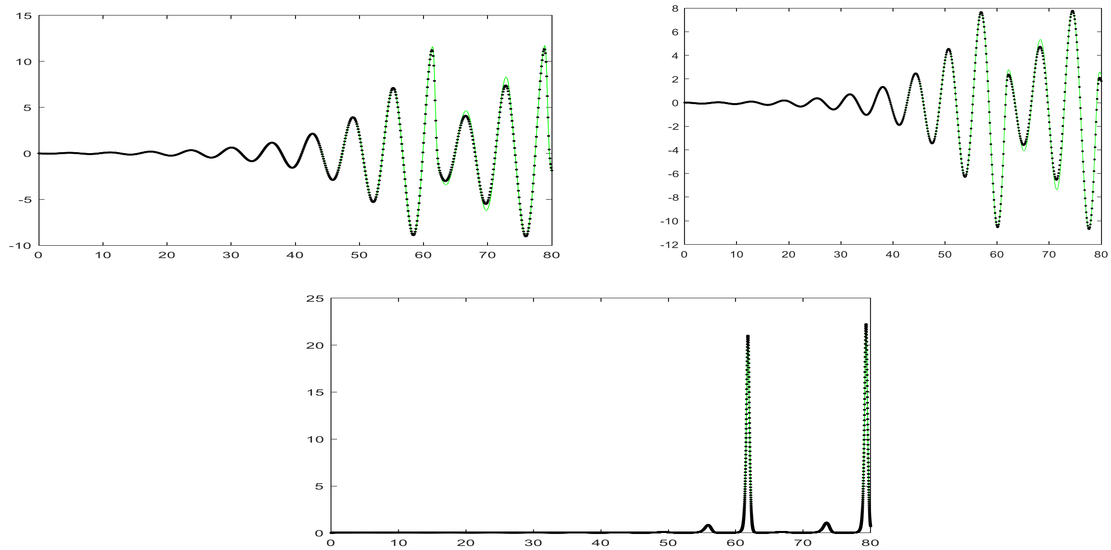

Fig. 12. Comparison of the phase portrait of Rössler system using eight-order second derivative block linear multistep method (green solid lines) and ode23s (black dots).

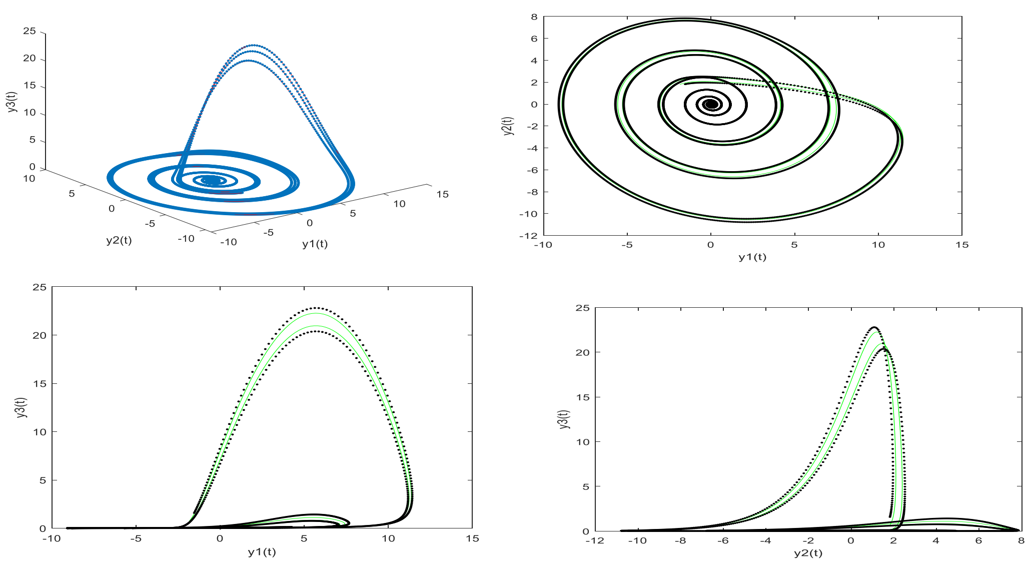

Fig. 13. Comparison of the phase portrait of Rössler system using eight-order second derivative block linear multistep method (solid lines) and ode23s (black dots).

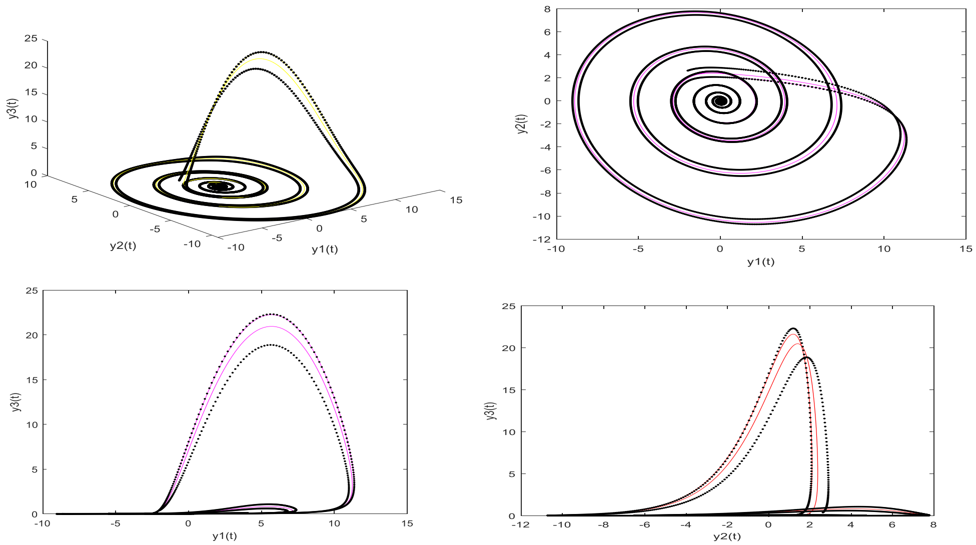

Fig. 14. Comparison of the phase portrait of Rössler system using the eight-order second derivative block linear multistep method (solid lines) and ode15s (dots).

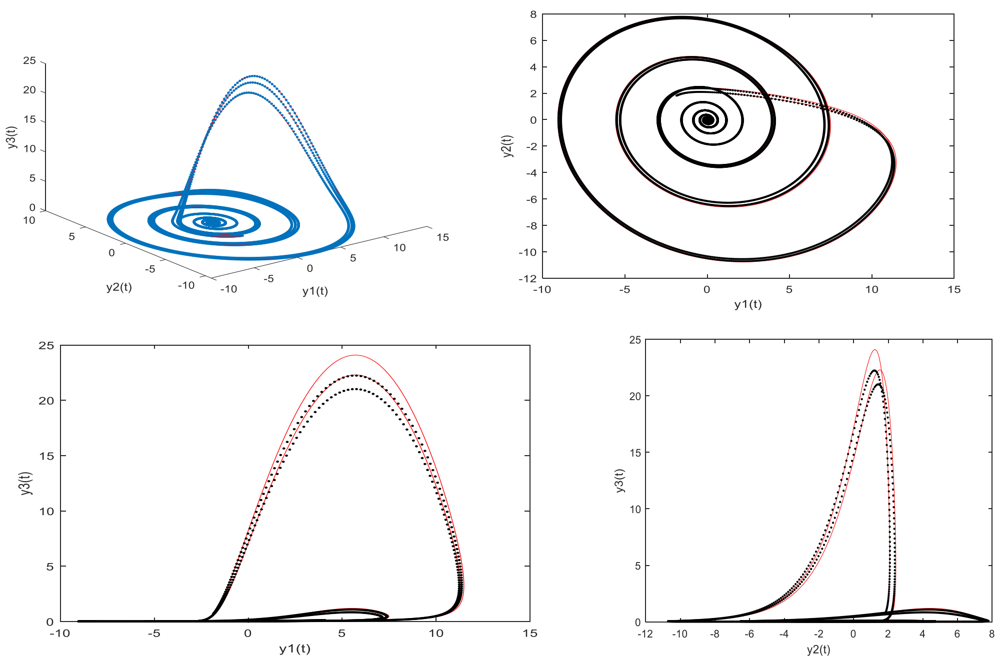

Fig. 15. Comparison of the phase portrait of Rössler system using the order eight second derivative block linear multistep method (solid lines) and ode45 (dots).

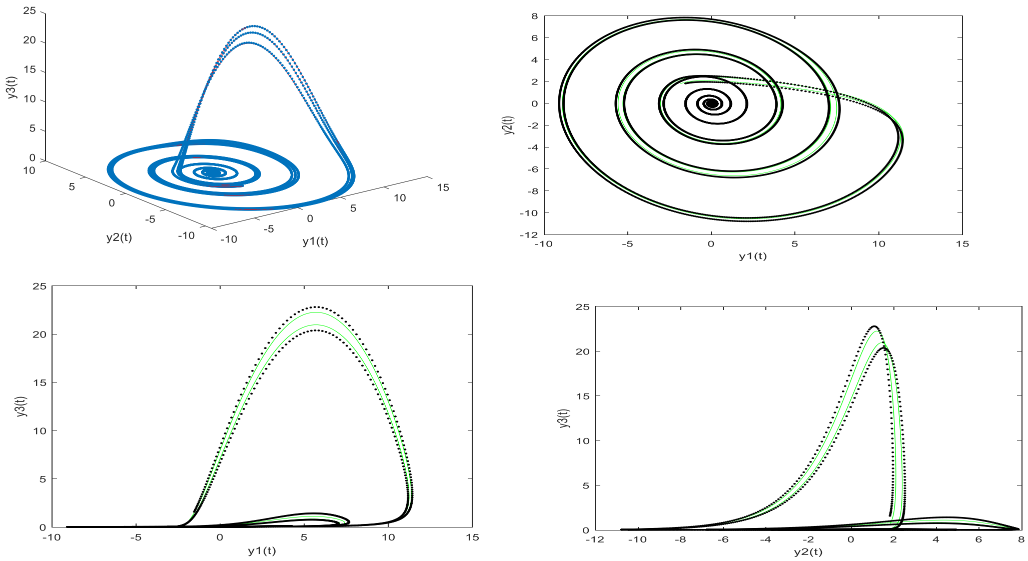

Fig. 16. Comparison of the phase portrait of Rössler system using the order eight second derivative block linear multistep method (solid lines) and ode113 (dots).

The solution curves obtained from the numerical simulations using eight-order second derivative block linear multistep method for[13] are plotted in 3D and 2D in Figs 12-16, respectively. Similarly, the time series and phase portraits of the approximate solutions for[13] are compared with the results from ODEs solvers(ode15s, ode23s, ode45 and ode113) in Figs 12-16. The solid lines and phase portraits show excellent agreement between our newly derived block integrators and the ODE solvers, demonstrating their accuracy. According to the existence and uniqueness theorem (Theorem 1), trajectories cannot merge or cross, ensuring that the strange attractors are unique. In all cases, the solid lines and phase portraits show good agreement with the ODE solvers, further validating our approach.

5 Conclusion

In this study, we have developed an eight-order second derivative block linear multistep method with variable step size for solving nonlinear IVPs of ODEs with chaotic properties. The derived method is convergent and A(α)-stable, is self-starting nature eliminates the use of starting values. Numerical simulations conducted in [12-13] demonstrate the method's efficiency and attractiveness in solving various chaotic, nonlinear, and linear ODEs, as evident from the time series and phase portraits compared with MATLAB ODEs solvers (ode45, ode113, ode23s, and ode15s) notably, our method successfully overcomes the limitations mentioned in [14].

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