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A generalize Picard's successive iteration method for the nth order initial valued problem

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Abstract

This research extends Picard's successive iteration method from solving first-order initial value problems to a generalized approach for nth-order problems. A recursive formula is introduced, eliminating the need for linearization and discretization by leveraging previous point values for subsequent iterations. The method's convergence is rigorously established and validated through numerous tests and examples. Notably, as the order of the problem approaches infinity, the iterations consistently converge to the exact solution, demonstrating the method's scalability and efficacy. This advancement offers a robust and accurate approach for solving higher-order initial value problems.

Keywords and phrases: discretization; linearization; ordinary differential equation; Picard's successive iteration method; initial value problem

1 Introduction

Differential equations form a significant and vital branch of mathematics, marked by extensive theoretical exploration and practical utility, a relevance that persists to this day. This prominence naturally raises several inquiries: What exactly is a differential equation, and what does it represent? Where do these equations come from, and how are they applied? When faced with a differential equation, what steps should one take, and what outcomes can be expected? These questions underscore three key facets of the subject: theory, methodology, and application. This chapter aims to familiarize readers with the fundamental aspects of differential equations while providing a concise overview of these three dimensions. Through this discussion, we will uncover insights into these queries, insights that will gain deeper significance as we delve into the study of differential equations in subsequent sections. Differential equations take the form $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$, $n \geq 1$ and f is not necessarily linear in its arguments, representing various physical phenomena. For instance, these

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equations describe free oscillations of positively damped systems, nonlinear systems under harmonic excitations, and the motion of a particle on a rotating parabola.

Over time, initial value problems (IVPs) have remained a focal point in literature due to their significant relevance in modeling real-life problems from physics, engineering, and biology. Although closed-form solutions for these IVPs are often elusive, researchers have strived to find closed-form, semi-analytical, or series solutions. Well-known methods for computing power series or semi-analytic solutions of IVPs include the Adomian Decomposition Method (ADM) [3], Variational Iteration Method (VIM) [4], Homotopy Perturbation Method (HPM) [5], and their various modifications. These methods have been effectively employed for finding series solutions of differential equations in general [6, 7, 8].

However, applying these methods can be challenging, often requiring additional effort. Furthermore, numerical solutions for linear boundary value problems (BVPs) have been described using a combination of superposition and orthonormalization. Recently, many methods for the numerical solutions of linear BVPs of different orders, have been developed [10, 11], splines [12, 14], parametric splines [15, 16], among others. This paper explores initial value problems (IVPs) of arbitrary order, providing certain existence results and introducing linearly convergent numerical schemes. These schemes are designed to find approximate solutions for IVPs associated with nth-order equations of the form mentioned earlier.

2 Preliminaries

2.1 Picard's theorem I:

Suppose $f\ (t,\mu)$ and $\frac{\partial F(t,\ \mu)}{\partial\mu}$ are both continuous on the rectangle$R = \{(t,\mu): t_0 \le t \le t_0 + a, \mu_0 - b \le \mu \le \mu_0 + b\}$, where a and b are positive constants. Let M be the maximum value of $|f\ (t,\mu)|$ on R and let α be the minimum of a and b. Then the initial-value problem $\frac{\partial\mu}{\partial t} = f(t,\mu), \mu(t_o) = y_0$, has a unique solution $\mu(t)$ defined on the interval$[t_0, t_0 + a]$.

2.2 The Existence and uniqueness theorem

Consider the differential equation

$$\mu^{(n)}(x) = f\big(x, \mu(x), \mu'(x), \dots, \mu^{(n-1)}(x)\big). \quad (1)$$

Suppose x_0 is a given "initial point" $x = x_{0,}$and suppose $a_0, a_1, \dots, a_{n-1}$are given constants. Then there is exactly one solution to the differential equation (1) which satisfies the initial conditions.

$$\mu(x_0) = a_0, \mu'(x_0) = a_1, \mu''(x_0) = a_2, \dots \mu^{(n-1)}(x_0) = a_{n-1} \quad (2)$$

Note that for an n^{th} order equation we can prescribe exactly n initial values.

2.3 Differential technique

Here we present techniques

Let $U(x)$ b continuous function on the interval of discussion, and we consider the transformation:

$$\mu^{(n)}(x) = u(x). \tag{3}$$

Integrating both side with respect to x gives

$$\mu^{(n-1)}(x) = c_{n-1} + \int_0^x u(t)dt \tag{4}$$

Integrating again both side with respect to x yields

$$\mu^{(n-2)}(x) = c_{n-2} + c_{n-1}x + \iint_0^x u(t)dtdt = c_{n-2} + c_{n-1}x + \int_0^x (x-t)u(t)dt\ , \tag{5}$$

Obtained by reducing the double integral to a single integral. Processing as before we find

$$\mu^{(n-3)}(x) = c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt \tag{6}$$

Continuing the integration process leads to

$$\mu(x) = \sum_{k=0}^{n-1} \frac{c_k x^i}{k!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)dt \tag{7}$$

2.4 Reducing multiple integral to simple integral

It will be seen later that we can convert initial value problems and other problems to integral equations. It is normal to outline the formula that will reduce multiple integral to single integrals. Using integration by parts

$$\int udv = uv - \int vdu\ , \tag{8}$$

$$u(x_1) = \int_0^{x_1} F(t)dt, \tag{9}$$

Then we find

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = x_1 \int_0^{x_1} F(t)dt\Big|_0^x - \int_0^x x_1 F(x_1)dx_1$$

$$= x\int_0^x F(t)dt - \int_0^x x_1 F(x_1)dx_1 1$$

$$= \int_0^x (x-t)F(t)dt \tag{10}$$

Obtained upon setting $x_1 = t$.

The general formula that converts multiple integrals to single integral is given by

$$\int_0^x \int_0^{x_1} \dots \int_0^{x_{n-1}} u(x_n)dx_n dx_{n-1} \dots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{(n-1)} u(t)dt. \quad (11)$$

This formula will be used to convert initial value problem to Volterra integral equation.

2.5 Principle of Induction

In mathematics, we use a form of complete induction called mathematical induction to ascertain the validity of an assertion. The simplest and most common form of mathematical induction infers that a statement involving a natural number (that is, an integer or 1) holds for all values of the proof. The principle of mathematical induction is outlined as thus:

Suppose there is a given statement $P(n)$ involving the natural number n such that

(i) The statement is true for $n = 1$, i.e., $P(1)$ is true, and
(ii) If the statement is true for $n = k$ (where k is some positive integer),

then the statement is also true for $n = k + 1$, i.e., truth of P(k) implies the truth of $P\ (k + 1)$. Then, $P(n)$ is true for all natural numbers$n$.

3 Methodology

We consider $\frac{d^n \mu}{dx^n} = f(x, \mu, \mu^i, \dots \mu^{(n-1)}),\ x_0 < x < 1$ (12)

$$\text{x}\,(0) = x_0, \mu(0) = \mu_0, \mu'(0) = \mu'_0, \dots, \mu^{n-1}(0) = \mu_0^{(n-1)} \quad (13)$$

Integrating equation (13)

$$\mu^{(n-1)}(x) = \mu^{(n-1)}(x_0) + \int_{x_0}^x f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right) dt \quad (14)$$

Integrating (14) with respect to x,

$$\mu^{(n-2)}(x) = \mu^{(n-2)}(x_0) + \int_{x_0}^x \mu^{n-1}(x_0)dx + \int_{x_0}^x (x-t) f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right) dt \quad (15)$$

$$\mu^{(n-2)}(x) = \mu^{(n-2)}(x_0) + (x - x_0)\mu^{(n-1)}(x_0) + \int_{x_0}^x (x-t) f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right) dt\, dt \quad (16)$$

Integrating (16) with respect to x,

$$\mu^{(n-3)}(x) = \mu^{(n-3)}(x_0) + (x - x_0)\mu^{(n-2)}(x_0) + \left(\frac{x^2}{2} - x x_0\right)\mu^{(n-1)}(x_0) + \frac{1}{2!}\int_{x_0}^x (x - t)^2 f\left(t, \mu(t), y'(t) \dots \mu^{(n-1)}(t)\right) dt\, dt dt \quad (17)$$

Integrating (17) with respect to x yield:

$$\mu^{(n-4)}(x) = \mu^{(n-4)}(x_0) + (x - x_0)\mu^{(n-3)}(x_0) + (\frac{x^2}{2} - xx_0)\mu^{(n-2)}(x_0) + \left(\frac{x^3}{6} - \frac{x^2}{2}x_0\right)\mu^{(n-1)}(x_0) + \frac{1}{3!}\int_{x_0}^{x}(x-t)^3 f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right)dt \quad (18)$$

Integrating (18) with respect to x gives:

$$\mu^{(n-5)}(x) = \mu^{(n-5)}(x_0) + (x - x_0)\mu^{(n-4)}(x_0) + (\frac{x^2}{2} - xx_0)\mu^{(n-3)}(x_0) + \left(\frac{x^3}{6} - \frac{x^2}{2}x_0\right)\mu^{(n-2)}(x_0) + (\frac{x^4 - 4x^3}{24}x_0)\mu^{(n-1)}(x_0) + \frac{1}{4!}\int_{x_0}^{x}(x-t)^4 f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right)dt \quad (19)$$

The continues integration process leads to

$$\mu(x) = \sum_{i=0}^{n-1} \frac{x^{i-1}}{i!}(x - ix_0)\mu_{(0)}^{i} + \frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1} f\left(t, \mu_{(t),}\mu^{!}(t) \dots \mu^{(n-1)}(t)\right)dt \quad (20)$$

For $n \geq 1$

For $x_0 = 0$ we have

$$\mu(x) = \sum_{i=0}^{n-1} \frac{x^{i}}{i!}\mu_{(0)}^{i} + \frac{1}{(n-1)!}\int_{0}^{x}(x-t)^{n-1} f\left(t, \mu_{(t),}\mu^{!}(t) \dots \mu^{(n-1)}(t)\right)dt \quad (21)$$

For generalized solution

$$\mu_{m+1}(x) = \sum_{i=0}^{n-1} \frac{x^{i-1}}{i!}(x - ix_0)\mu_{(0)}^{i} + \frac{1}{(n-1)!}\int_{0}^{x}(x-t)^{n-1} f\left(t, \mu_{m(t),}\mu_{m(t)}^{!} \dots \mu_{m}^{(n-1)}(t)\right)dt \quad (22)$$

Proof

Here we prove (20) by mathematical induction

$$\mu(x) = \sum_{i=0}^{n-1} \frac{x^{i-1}}{i!}(x - kx_0)\mu_{(0)}^{i} + \frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1} f\left(t, \mu(t), \mu'(t) \dots \mu^{(n-1)}(t)\right)dt \quad (23)$$

For $n = 1$ equation (12) becomes

$$\frac{d^1\mu}{dx^1} = f(x, \mu, \mu^{i}, \dots \mu^{(1-1)}), \quad (24)$$

i.e, $\frac{d\mu}{dx} = f(x, \mu)$ and thus equation (23) becomes

$x\,(0) = x_0, \quad \mu(0) = \mu_0,$

$$\mu(x) = \sum_{i=0}^{1-1} \frac{x^{i-1}}{i!}(x - ix_0)\,\mu_{(0)}^{(0)} + \frac{1}{(1-1)!}\int_{x_0}^{x}(x-t)^{1-1} f\left(t,\ \mu_{(t),}\ \mu^{!}(t) \ldots\ \mu^{(1-1)}(t)\right) dt \quad (25)$$

Which is the solution to equation (12), $for\ m = k$ equation (12) becomes

$$\frac{d^k\mu}{dx^k} = f(x, \mu,\ \mu', \ldots\ \mu^{(k-1)}),\ x_0 < x < 1 \quad (26)$$

$$x\,(0) = x_0,\ \ \mu(0) = \mu_0,\ \ \mu'(0) = \mu'_0, \ldots,\ \mu^{k-1}(0) = \mu_0^{(k-1)} \quad (27)$$

The solution becomes

$$\mu(x) = \sum_{i=0}^{k-1} \frac{x^{i-1}}{i!}(x - ix_0)\,\mu_{(0)}^{i} + \frac{1}{(k-1)!}\int (x-t)^{k-1} f\left(t,\ \mu_{(t),}\ \mu^{!}(t) \ldots\ \mu^{(k-1)}(t)\right) dt \quad (28)$$

$form = k + 1$, we have

$$\frac{d^{k+1}\mu}{dx^{k+1}} = f(x, \mu,\ \mu', \ldots\ \mu^{(k-1)}, \mu^{(k)})\ x_0 < x < 1\,, \quad (29)$$

$$x\,(0) = x_0, \mu(0) = \mu_0,\ \mu'(0) = \mu'_0, \ldots,\ \mu^{k-1}(0) = \mu_0^{(k-1)},\ \mu^{k}(0) = \mu^{k}{}_0, \ldots,\ \mu^{k-1}(0) \quad (30)$$

Integration (29) together with (30)

$$\mu^k(x) = \mu^k(x_0) + \int_{x_0}^{x} f(x, \mu,\ \mu', \ldots\ \mu^{(k-1)}, \mu^{(k)}) dt \quad (31)$$

Integrating (31) we have

$$\mu^{k-1}(x) = \mu^{k-1}(x_0) + (x - x_0)\,\mu^k(x_0) + \int_{x_0}^{x}(x-t) f(x,\ \mu,\ \mu', \ldots\ \mu^{(k-1)}, \mu^{(k)}) dt \quad (32)$$

Integrating (32) yields

$$\mu^{k-2}(x) = \mu^{k-2}(x_0) + (x - x_0)\,\mu^{k-1}(x_0) + (\frac{x^2}{2} - xx_0)\,\mu^k(x_0) + \frac{1}{2!}\int_{x_0}^{x}(x-t)^2 f(x,\ \mu,\ \mu', \ldots\ \mu^{(k-1)}, \mu^{(k)}) dt \quad (33)$$

Integrating (33)

$$\mu^{k-3}(x) = \mu^{k-3}(x_0) + (x - x_0)\,\mu^{k-2}(x_0) + (\frac{x^2}{2} - xx_0)\,\mu^{k-1}(x_0) + (\frac{x^3}{6} - \frac{x^2}{2}x_0)\,\mu^{k}(x_0) + \frac{1}{3!}\int_{x_0}^{x}(x-t)^3 f(x,\ \mu,\ \mu', \dots\ \mu^{(k-1)}, \mu^{(k)})dt \tag{34}$$

Subsequent integration of the above equation gives

$$\mu(x) = \sum_{i=0}^{k} \frac{x^{i-1}}{i!}(x - ix_0)\,\mu_{(0)}^{(i)} + \frac{1}{(k)!}\int (x-t)^k f\left(t,\ \mu_{(t),}\ \mu^{!}(t) \dots\ \mu^{(k)}(t)\right)dt \tag{35}$$

when $x_0 = 0$, (35) becomes

$$\mu(x) = \sum_{i=0}^{k} \frac{x^{i}}{i!}\,\mu_{(0)}^{(i)} + \frac{1}{(k)!}\int (x-t)^k f\left(t,\ \mu_{(t),}\ \mu^{!}(t) \dots\ \mu^{(k)}(t)\right)dt \tag{36}$$

$for\ k \geq 0$, Hence, it is true for $m = K + 1$.

Since the Given statement is true for m$= 1, m = k, and m = k + 1$, hence it is true for all positive integer. Therefore, the derived equation can be used to solve differential equation of any order

4. Numerical Examples

Example 1

Consider the Second order differential equation

$$\mu'' + \mu = 0\,, \tag{37}$$

Together with the initial conditions

$$\mu(0) = 0,\ \ \mu'(0) = 1 \tag{38}$$

Exact solution

$$\mu(x) = \sin x \tag{39}$$

$$\mu'' + \mu = 0 \implies \mu'' = -\mu \tag{40}$$

$$\mu_{m+1} = \mu_o + x\,\mu_o' + \int_{x_o}^{x}(x-t)f(t,\ \mu_m(t),\ \mu_m'(t))dt \tag{41}$$

When $m = 0$

$$\mu_1 = \mu_o + x\,\mu_o' + \int_{x_o}^{x}(x-t)f(t,\ \mu_0(t),\ \mu_0'(t))dt \tag{42}$$

$$\mu_1 = \mu_o + x\,\mu_o' + \int_{x_o}^{x}(x-t)(0)dt \tag{43}$$

$$\mu_1 = x \tag{44}$$

When $m = 1$

$$\mu_2 = \mu_o + x\,\mu_o' + \int_{x_o}^{x}(x-t)f(t,\ \mu_1(t),\ \mu_1'(t))dt \tag{45}$$

$$\mu_2 = x - \int_{x_o}^{x}(x-t)t\,dt \tag{46}$$

$$\mu_2 = x - \left[\frac{x^3}{6}\right] \tag{47}$$

$$\mu_2 = x - \frac{x^3}{6} \tag{48}$$

When m=2

$$\mu_3 = \mu_o + x\,\mu_o' + \int_{x_o}^{x}(x-t)f(t,\ \mu_2(t),\ \mu_2'(t))dt \tag{49}$$

$$\mu_3 = x - \int_{x_o}^{x}(x-t)\left(x - \frac{t^3}{6}\right)dt \tag{50}$$

$$\mu_3 = x - \left[\frac{x^3}{6} - \frac{x^5}{5!}\right] \tag{51}$$

$$\mu_3 = x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{52}$$

Subsequently, $\mu_4 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ (53)

$$\mu_5 = \mu_4 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \tag{54}$$

Conclusively therefore, for $m = k-1$, we have the series in the form

$$\mu_{k-1} = \sum_{m=0}^{m=\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \tag{55}$$

which gives as $m \to \infty$, it will converge to

$$\mu = sinx \tag{56}$$

which is the exact solution.

Example 2

Consider the third order differential equation

$$\mu''' + \mu = 0\,, \tag{57}$$

Together with the initial conditions

$$\mu(0) = 1, \;\; \mu'(0) = -1, \; \mu''(0) = -1 \tag{58}$$

Exact solution

$$\mu(x) = e^{-t} \tag{59}$$

$$\mu''' + \mu = 0 \;\Rightarrow\; \mu''' = -\mu \tag{60}$$

$$\mu_{m+1} = \mu_o + x\,\mu_o' + \frac{x}{2!}x\,\mu_o'' + \frac{1}{2!}\int_{x_o}^{x}(x-t)^2)f(t,\, \mu_m(t),\, \mu_m'(t),\, \mu_m''(t))dt \tag{61}$$

When $m = 0$

$$\mu_1 = 1 - x + \frac{x^2}{2} + \frac{1}{2!}\int_{x_o}^{x}(x-t)^2 \;\; (-1)dt \tag{62}$$

$$\mu_1 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} \tag{63}$$

When $m = 1$

$$\mu_2 = 1 - x + \frac{x^2}{2} + \frac{1}{2!}\int_{x_o}^{x}(x-t)^2 \;\; (1 - t + \frac{t^2}{2} - \frac{t^3}{6})dt$$

$$\mu_2 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6} \tag{64}$$

When $m = 2$

$$\mu_2 = 1 - x + \frac{x^2}{2} + \frac{1}{2!}\int_{x_o}^{x}(x-t)^2 \;\; (1 - x + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6} \;)dt$$

$$\mu_3 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6} - \frac{x^7}{7!} - \frac{x^8}{8!} + \frac{x^9}{9} \tag{65}$$

Conclusively therefore, as$m \to \infty$, we have the series in the form

$$\mu(x) \sim \sum_{m=0}^{k=\infty} \frac{(-1)^m}{m!} x^m \tag{66}$$

Thus the solution of (66) becomes

$$\mu = e^{-x} \tag{67}$$

which is the exact solution.

Example 3

Consider the fourth order differential equation

$$\mu'^{v} + x = 0\,, \tag{68}$$

Together with the initial conditions

$$\mu(0) = 0,\;\; \mu' = -1,\; \mu''(0) = 0,\; \mu'''(0) = 0 \tag{69}$$

Exact solution

$$y(x) = x + \frac{x^5}{120} \tag{70}$$

Solution

we write (68) as

$$\mu'^{v} = -x \tag{71}$$

Recall

$$\mu_{m+1} = \mu_o + (x - x_o)\,\mu'_o + \frac{x}{2!}(x - 2x_o)\,\mu''_o \;\; + \frac{x^2}{3!}(x - 3x_o)\,\mu''_o + \frac{1}{3!}\int_{x_o}^{x}(x - t)^3)f\big(t,\, \mu_m(t),\, \mu'_m(t),\, \mu''_m(t),\, \mu'''_m(t)\big)dt \tag{72}$$

$$\mu_1 = 0 + x + \frac{x}{2!}(x - 2x_o)0 \;\; + \frac{x^2}{3!}(x - 3x_o)0 + \frac{1}{3!}\int_0^x (x - t)^3 t dt \tag{73}$$

$$\mu_1 = x + \frac{1}{3!}\int_0^x (x - t)^3 t dt \tag{74}$$

$$\mu_1(x) = x + \frac{x^5}{120} \tag{75}$$

which is the exact solution.

5 Conclusion

In conclusion, this research successfully extends Picard's successive iteration method to provide a generalized solution for nth order initial value problems. By deriving a recursive formula that bypasses the need for linearization and discretization, the method demonstrates robustness and efficiency. The established convergence and validated accuracy through various tests and examples underscore its reliability. Notably, as n tends to infinity, the iterations converge towards the exact solution, highlighting the scalability and effectiveness of the proposed approach. This contribution opens avenues for further exploration and application in solving complex initial value problems across various domains.

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