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Application of Jensen's integral formulae on polynomials with entire functions of positive real parts and Gauss mean value theorem

G.N. Emenogu *†, I.C. Yemisi ‡, E.I. Edugbe †§, C. Chigozie †**, and B.O. Osu †††

Abstract

In this study, new theorems with proofs for new inequalities on polynomials which are of the importance in the theory of transcendental numbers, the product of polynomials and their associated roots were developed. Jensen's integral formula was applied to obtain new results about the bounds on the poles and roots of complex functions with positive real parts and Gauss Mean Value Theorem through the relationship between the coefficients and zeros of polynomials. It was shown that if the k-polynomials have $n_i\ roots, i = 1,2,3, \ldots, k,$ then their products f_1(z)f_2(z)… f_{k-1}(z)f_k(z) have distinct roots of $n_1 + n_2 + n_3 \ldots + n_{k-1} + n_k$.

Keywords and phrases: Jensen's formulae; polynomials; complex variable; meromorphic functions; zeros and roots of multiplicity

1 Introduction

Polynomials are a natural type of function to consider: a generalization of linear, quadratic and cubic functions. They can sometimes be solved exactly. They graphs can be sketched. Many quantities in the real world are related by polynomial functions. Importantly, any smooth function can be approximated by polynomials. So proficiency in dealing with polynomial is also important. In [1, 2, 3], a polynomial of degree 'n' is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} +$

†††Corresponding Author; E-mail: Osu.bright@abiastateuniversity.edu.ng (https://orcid.org/0000-0003-2463-430X)
†††Department of Mathematics, Abia State University, Uturu, Nigeria.
*†Department of of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria.
*†E-mail: ng.emenogi@mouau.edu.ng
‡Department of Mathematics, National Open University, Nigeria
§E-mail: cishola@noun.edu.ng
†§Department of Mathematics, Abia State University, Uturu, Nigeria
E-mail: iedugbe01@gmail.com
†** Department of Insurance, University of Jos, Jos, Nigeria
†E-mail: chigoziec@unijos.edu.ng (https://orcid.org/0000- 0002-3174-7751)

$\cdots + a_2x^2 + a_1x + a_0$ where the *a*'s are real numbers (sometimes called the coefficient of the polynomial and n is an integer). The degree of a polynomial is the highest of x in its expression. Polynomial of degree 1, 2, 3, 4, 5 are respectively are called linear, quadratic, cubic, quartic and quintic. A degree zero polynomial is just a constant function. A root or a zero polynomial *f(x)* is a number r such that *f(r) =0*. According to [4, 5], a polynomial function is a function such as a quadratic, a cubic etc involving only non-negative-integer power of x. The highest degree term a_nx^n is called the leading term and its coefficient a_n is called the leading coefficient. If the leading coefficient is 1, the polynomial is called Monic. The term a_0 is called the constant term. However, the leading coefficient is always non-zero as it is the coefficient of the highest power of *x* which actually appears. Jensen's formula has been applied by many scholars to study roots and poles of meromorphic function which is only analytic everywhere in the finite plane. [6,7,8,9,10] applied Jensen's formula in geometry of the zeros of the derivative of a polynomial a central role is taken by luca's theorem in which the derivative lie in the smallest convex polygon containing the zeros of the original polynomial under certain conditions. [11,12,13] studied multivariate polynomial with complex coefficients with concerns in plurisubharmonic functions with very small order to determine area of the intersection of the analytic set and inequality similarly to the case of p complex variable and where the unit disk in C is replaced by the unit ball of C^P. [14, 15, 16,] used theory of transcendental number to obtain an inequality for a pair of relatively prime polynomials.

These researchers encountered a setback in the generalization of its inequality which involves several numbers and weights of measure theory in probability form. In order to overcome this setback, this research aims at applying Jensen's Integral formula on meromorphic function which is not only analytic everywhere in the finite plane but also at a finite number of poles and applying it to polynomials of entire function with positive real parts and Gauss Mean Value Theorem to determine the relationship between the coefficients and zeros of polynomials and give theorems about the product polynomials and their associated roots. Entire function is a function that is analytic everywhere in the finite plane except at ∞. To achieve the aim of this research, this paper focuses on functions of one complex variable z, $Q(z)$ which are regular inside and on the circle $|z| = r$; when $Q(0) \neq 0$ and all the zeros of $Q(z)$ satisfying $|a_k| \leq r$ where multiple zeros are counted as many times as their multiplicity with $Q(z)$ having the exact degree of $d(Q) = n$ to obtain new proofs, new results about the number of roots of the product of polynomial and new results on functions with positive real parts for new inequalities on polynomials through the application of Jensen's integral formula developed by [17]

2 Methodology/Model formulation

In [18], Jensen's integral formula was adapted and presented below

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(pe^{i\theta})\right| d\theta = ln|Q(0)| + \sum_{k=1}^{N} ln\frac{p}{|a_k|} \qquad (1)$$

Let $\Omega \subseteq \mathbb{C}$ be a region (connected open set). A a function U: $\Omega \rightarrow R$ is said to be harmonic if u∈$C^2(\Omega)$ and $\Delta u(z) = 0$ for all z ∈ Ω. Here,

$$\Delta u(z) = u_{x,x}(z) + v_{y,y}(z); z = x + iy \qquad (2)$$

2.1 Basic Properties of Meromorphic or harmonic Function [19, 20, 21]

i). Mean value property: let u∈$C^2(\Omega)$. Then u is harmonic if and only if it has the mean value property (MVP). Recall that u is said to have MVP if

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta \qquad (3)$$

for all z ∈ Ω *and* $r > 0$ such that

$\overline{D}$(z, r) ⊆ Ω. Hence we have averaged u over a circle centered at z. one can also average over disk. In the other words, MVP is equivalent to saying

$$u(z) = \frac{1}{\pi r^2}\int_{D(z,r)} u(w)dm(w) \qquad (4)$$

for all z ∈ Ω *and* $r > 0$ such that $\overline{D}$(z, r) ⊆ Ω.

ii). Maximum principle: let Ω be a bounded region, $u \in C(\bar{\Omega})$ and $\Delta u = 0$ in Ω. Then $\max_{\Omega} u = \max_{\partial\Omega} u$. Further, if $u(z) = \max_{\bar{\Omega}} u$ for some z ∈ Ω, then the u must be constant. Since $-u$ is harmonic whenever u is the same holds for the minimum too.

iii). (relationship analytic function): If $f: \Omega \rightarrow c$ is holomorphic, then $u = Re\, f$ is harmonic in Ω. The converse is true if Ω is a disk.

i.e. if Ω is a disk and u is a harmonic on Ω, then there is a holomorphic function on Ω whose real part is equal to u. An algebra is of the form;

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_1 z + a_0 = 0 \qquad (5)$$

where a_i is the complex number and the degree of the equation is $n(if\ a_n \neq 0)$.

2.2.1 Zeroes and Roots

A number r is a zero of multiplicity m of a polynomial $P(z)$ if there is a polynomial $Q(z)$ with $Q(r) \neq 0$ such that $P(z) = (z - r)^m Q(z)$. Also, r is a root of multiplicity m of the equation $P(z)$ =0. If r is a root of multiplicity $m(m \geq 1)$ of the equation $P(z)$ =0, then r is a root of multiplicity $m - 1$of the equation $P'(z) = 0$.

2.2.2 The Factor Theorem

$P(z)$ contains the factor $z - r \leftrightarrow P(r) = 0$

$P(z)$ contains the factor $(z - r)^m \leftrightarrow P(r) = P'(r) = P^{(m-1)}(r) = 0$

2.2.3 The Fundamental Theorem of Algebra

An algebra equation $P(z) = 0$ of degree n has exactly n roots (including multiplicity) in $C.$ if the roots are $r_1, \dots, r_n$, then

$$P(z) = a_n(z - r_1), \dots, (z - r_n) \tag{6}$$

2.2.4 Relationship between Roots and Coefficients

If $r_1, \dots, r_n$, are the roots (4), then

$$\begin{cases} r_1 + r_2, \dots, + r_n = -\frac{a_{n-1}}{a_n} \\ r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = \sum_{i<j} r_i r_j = \frac{a_{n-2}}{a_n} \\ \cdots \\ r_1 r_2 \dots r_n = (-1)^n \frac{a_0}{a_n} \end{cases} \tag{7}$$

3 Discussions and conclusion

In this work new proofs are obtained for new inequalities on polynomials following Jensen's Integral formula which are of importance in the theory of transcendental numbers different from the initial results carried-out by other researchers which are not fully connected to the basic properties of meromorphic or harmonic function, we shall deduce the new proofs of inequalities on polynomials from the same source, viz from Jensens integral formula in the theory of analytic functions.

Link between Coefficient and Roots of Polynomials

Theorem 1

Let $a_1, a_2, \dots, a_n$ be all the zeroes of the n^{th} degree polynomial

$$Q_n(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-2} z^2 + \alpha_{n-1} z + \alpha_n \tag{8}$$

Where $\alpha_k, k = 0,1,2, \dots, n$ are arbitrary real or complex coefficients. With $\alpha_n \neq 0$ and $\alpha_0 \neq 0$ then

(i) $Q_n(o) = (-1)^n, a_1.a_2.a_3 \dots a_{n-3}.a_{n-2}.a_{n-1}.a_n$ (8a)

$(ii) Q_n(z)$ has exact degree $d(Q_n(z) = n$ and does not vanish for $z = 0$. (8b)

Proof

Let $a_1, a_2, \dots, a_n$ be the only zeroes of $Q_n(z)$, these are all different from 0.

Since $Q_n(z)$ is of exact degree n, the fundamental theorem of algebra affirms that $Q_n(z)$ has exactly n zeroes $a_1, a_2, a_3 \dots a_{n-2} a_{n-1}, a_n$ and can be written as

$$Q_n(z) = (z - a_1)(z - a_2)(z - a_3) \dots (z - a_{n-2})(z - a_{n-1})(z - a_n)$$

$$= a_1 \left(\frac{z}{a_1} - 1\right) a_2 \left(\frac{z}{a_2} - 1\right) a_3 \dots a_{n-2} \left(\frac{z}{a_2} - 1\right) a_{n-1} \left(\frac{z}{a_1} - 1\right) a_n \left(\frac{z}{a_n} - 1\right)$$

$$= (-1)^n a_1 a_2 a_3 \dots a_{n-2} a_{n-1} a_n \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) \tag{9}$$

In the form (9) $Q_n(z)$ is expressed in an finite product and we need not bother about its convergence from (8) it follows that

$$|Q_n(o)| = |a_1||a_2||a_3| \dots |a_{n-2}||a_{n-1}||a_n| \tag{10}$$

From (10)

$$|Q_n(z)| = |a_1||a_2||a_3| \dots |a_{n-2}||a_{n-1}||a_n| \prod_{k=1}^{n} \left|\left(1 - \frac{z}{a_k}\right)\right|$$

$$|Q_n(o)| \prod_{k=1}^{n} \left|\left(1 - \frac{z}{a_k}\right)\right|$$

(11)

It will be convenient to order the zeroes so that

$$|a_1| \leq |a_2| \leq |a_3| \leq \dots \leq |a_N| \leq 1 \leq |a_{N+1}| \leq |a_{N+2}| \leq \dots \leq |a_n|.$$

Let $Q(z)$ be a function of a complex variable z, which is regular inside and on the circle $|z| = r$, let $Q(0) \neq 0$ and let $a_1, a_2, \dots, a_N$ be all the zeroes of Q(z) satisfying $|a_k| \leq r$ where the multiple zeroes are counted as many times as their multiplicity.

A polynomial of degree n is written as

$$Q_n(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \dots + \alpha_{n-2} z^2 + \alpha_{n-1} z + \alpha_n \tag{12}$$

where

$\alpha_k, k = 0,1,2, \dots, n$ are arbitrary real or complex coefficients. We shall assume that

$\alpha_0 \neq 0$, $\alpha_n \neq 0$. so that $Q(z)$is of exact degree n there is no restriction in assuming that the zeroes have been numbered so that some are less than 1 and others greater than 1.That is:

$$|a_1| \leq |a_2| \leq \cdots \leq |a_N| \leq 1 < |a_{N+1}| \leq |a_{N+2}| \leq \cdots \leq |a_n| \quad (13)$$

Suppose $r = 1$ in Jensen's integral formula developed by [18], then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right| d\theta = ln|Q(0)| + \sum_{k=1}^{N} ln\frac{1}{|a_k|} \quad (14)$$

It follows then that

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right| d\theta = ln|a_n| + \sum_{v=1}^{N} ln\frac{1}{|a_k|} = ln\left|\frac{\alpha_0 a_1 a_2 a_3 \dots a_N a_{N+1} a_{N+2} \dots a_n}{a_1 a_2 a_3 \dots a_N}\right| \quad (15)$$

Thus (15) may also be written as

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right| d\theta = ln|\alpha_0 a_{N+1} a_{N+2} \dots a_n| \quad (16)$$

Let

$i_1, i_2 \dots i_m$ be an arbitrary set of not more than n distinct suffixes 1, 2, … n; neither of the cases when $m = 0,$ and $m = n$ is excluded. From the numbering of the zeros, it is obvious that

$$\left|\alpha_0 a_{i_1} a_{i_2} \dots a_{i_m}\right| \leq |\alpha_0 a_{N+1} a_{N+2} \dots a_n|$$

Hence, the identity (16) implies the inequality

$$ln\left|\alpha_0 a_{i_1} a_{i_2} \dots a_{i_m}\right| \leq \frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right| d\theta \quad (17)$$

Apart from a factor ± 1, each coefficient α_m of $Q(z)$ is a sum of $\binom{n}{m}$ terms of same form

$$\alpha_0 a_{i_1} a_{i_2} \dots a_{i_m}$$

From (17)

$$|\alpha_0| + |\alpha_1| + \cdots + |\alpha_n| \leq \sum\left|\alpha_0 a_{i_1} a_{i_2} \dots a_{i_m}\right|$$

where the summation extends over all possible sets of suffixes

$i_1, i_2 \dots i_m$ since m can have the value 0,1,…,n, the sum has $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ terms

More ever, each term satisfies the inequality (16), we have

$$\ln(|\alpha_0| + |\alpha_1| + \cdots + |\alpha_n|) \leq d(Q)ln2 + \frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right|d\theta \qquad (18)$$

The upper estimate for the integral

$$\left|Q(e^{i\theta})\right| \leq |\alpha_0| + |\alpha_1| + \cdots + |\alpha_n| \; for \; 0 \leq \theta \leq 2\pi$$

So also

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right|d\theta \leq ln|\alpha_0| + |\alpha_1| + \cdots + |\alpha_n| \qquad (19)$$

From Feldman and Gelfond inequalities, we have by combining (12) and (14), we find that

$$\left|\alpha_0 a_{i_1} a_{i_2} \dots a_{i_m}\right| \leq |\alpha_0| + |\alpha_1| + \cdots + |\alpha_n| \qquad (20)$$

Then the strengthened form of Feldman's inequality, let $Q(z)$ be written as a product of polynomials

$$Q(z) = \prod_{\sigma=1}^{s} Q_\sigma(z)$$

We use the abbreviation

$$S(Q) = |\alpha_0| + |\alpha_1| + \cdots + |\alpha_n| \qquad (21)$$

(21) implies therefore the s relation

$$S(Q_\sigma) = d(Q_\sigma)ln2 + \frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right|d\theta \quad (\sigma = 1,2,\dots,5) \qquad (22)$$

where

$$\sum_\sigma^s d(Q_\sigma) = d(Q)$$

Hence, adding these inequalities

$$ln\prod_{\sigma=1}^{s} S(Q_\sigma) \leq d(Q)ln\,2 + \frac{1}{2\pi}\int_0^{2\pi} ln\left|Q(e^{i\theta})\right|d\theta$$

Hence, it follows from (17) that Gelfond's inequality is;

$$\prod_{\sigma=1}^{s} S(Q_\sigma) \leq 2^{d(Q)}S(Q). \qquad (23)$$

Application of Jensens's integral formula to entire functions with positive real part

order axioms of set positive numbers

Let R be the set of real numbers, then R is a ring and three exits a subset IR^+ of IR whose elements are called positive real numbers, such that the following statements are true.

Axioms (a) (close of IR^+): If $a\ and\ b$are positive real numbers, then so are $a+b$ and ab (juxtapositions means ordinary multiplication).

Axioms (b) (Trichotomy axiom of IR^+): if a is a real number then one and only one of the following three statement is true: $a\epsilon IR^+, -a\epsilon IR^+, or\ a=0$

Axioms (c) (cancellation law of IR^+): if $a>b$ then $a+c>b+c$ for any a,b in $IR^+$and $c\ in\ IR$.

Lemma 1

If a, b, c, and d are in IR^+ and $a<b$ and $c<d$, then

$(i)\ \ a+c<b+d$

$(ii)\ \ ac<bd$

Proof

If $a<b$ then $b-a>0$ i.e $b-a\epsilon IR^+$.

If $c<d$ then $d-c>0$ i.e $d-c\epsilon IR^+$

By the closure property of IR^+

$(b-a)+(d-c)>0$ hence,

$$(b+d)-(a+c)>0$$

Or

$$a+c<b+d$$

(ii). $(b-a)c>0$ and $b(d-c)>0$

Or

$$(bc-ac)>0 \text{ and } (bd-bc)>0$$

or

$$ac < bc \text{ and } bc < bd$$

Hence,

$$ac < bd$$

Lemma 2

If $0 < A < B < C$ then

(i).$e^A < e^B < e^C$

(ii) $InA < InB < InC$

Proof

(i). the positive number axiom for this expression is: if $\varphi > 0$ then $e^\varphi > 1$

Therefore,

$$A < B \text{ implies } B - A > 0$$

So,

$$e^{B-A} > 1$$

Or

$$e^B e^{-A} > 1$$

That is,

$$e^B > e^A \tag{24}$$

Similarly,

$$B < C \text{ implies } C - B > 0$$

Then,

$$e^{C-B} > 1$$

Or

$$e^C e^{-B} > 1$$

That is

$$e^C > e^B \tag{25}$$

Then equation (24) and (25) lead to

$$e^A < e^B \text{ and } e^B < e^C$$

Or,

$$e^A < e^B < e^C$$

(ii). The IR^+ axiom for this case is if $\emptyset > 1$ then, $In\emptyset > 0$,

Now, if $A < B\ then\ \frac{B}{A} > 1$

and by the axiom

$$In\frac{B}{A} > 0 \text{ or } InB - InA > 0$$

That is

$$InA < InB \tag{26}$$

On the other hand, if $B < C$ then, $\frac{C}{B} > 1$ and by the axiom

$$In\frac{C}{B} > 0 \text{ or } InC - InB > 0$$

that is

$$InB < InC \tag{27}$$

By equation (26) and (27)

$$InA < InB < InC \tag{28}$$

Definition 1

The set $Q = \{f(z): z \in E\}$ is called a set of functions with positive real parts if its members satisfying the following conditions,

(i). $f(z) = 1 + q_1 z + q_2 z^1 + \cdots + q_n z^n + \cdots = +\sum_{n=1}^{\infty} q_n z^n$ (29a)

that are regular in E= $\{z: |z| < 1\}$ and such that for $z\ in\ E$

(ii). $Re(f(z)) > 0$ (29b)

Examples following Definition 1

i).$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots |z| < \infty$ (30)

(ii). $\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$ (31)

(iii). $\frac{\sin^{-1} z}{z} = 1 + \frac{1}{z}\frac{z^2}{3} + \frac{1.3}{2.4}\frac{z^4}{5} + \frac{1.3.5}{2.4.6}\frac{z^6}{7} + \cdots + F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2};\ z^2\right)$ (32)

(iv). $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}z^n + \cdots |z| < 1$ (33)

(v). $\frac{\tan^{-1} z}{z} = 1 - \frac{z^2}{2} + \frac{z^4}{4} - \cdots (-1)^{n-1}\frac{z^{2n-1}}{2n-1} + \cdots |z| < 1$ (34)

Any function in Q is called a function with positive real part in E.

Thus$f(z) = 1 + z^n$ is in Q for any integer $n \geq 0$. it is clear that with sufficient labour, any theorem about the class Q can be transformed into a suitable theorem about a fraction $g(z)$ that carries E into some prescribed half plane.

The mobius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n \quad (35)$$

Plays a central role in the class Q, it is regular in E, and maps E onto the half-plane H^+, and maximizes $|q_n|$ in the Q.

Some operations that carry functions in Q into function in Q are given (33).

Definition 2

Let $f(z)$ and $F(z)$ be given by the power series

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ (36)

and $F(z) = \sum_{n=0}^{\infty} A_n Z^n$ (37)

Convergent in some disk $E_R: |z| < R, R > 0$. e say that $f(z)$ is dominated by $F(z)$ or $F(z)$ dominates $f(z)$, and we write $f(z) \ll F(z)$ if for each integer $n \geq 0$.

$|a_n| \leq A_n$ (38)

Theorem 2

Suppose that $f(z) \ll F(z)$. then $A_n \geq 0, n = 0,1,2, \ldots$

$$|f(z)| \leq F(r), 0 \leq |z| \equiv r < R \tag{39}$$

$$f'(z) \ll F'(z)$$

$$\int_0^z f(\zeta)d\zeta \ll \int_0^z F(\zeta)d\zeta$$

$$e^{f(z)} \ll e^{F(z)}$$

And if $|F(z)| < 1$ in some disk Ethen in that disk $-In(-f(z)) \ll -In(1 - F(z))$

Further, if $|b| \leq B$ and $k \geq 0$ is any integer, then $bz^k f(z) \ll BZ^k F(z)$

and $[f(z)]^k \ll [F(z)]^k$

Finally, if $f(z) \ll F(z)$ and $g(z) \ll G(z)$, then $f(z) + g(z) \ll F(z) + G(z)$

And $f(z)g(z) \ll F(z)G(z)$

Theorem 3

Suppose that $f(z)$ is in Q, then the function $g(z)$ is in Q, where,

$$g(z) = f(e^{i\alpha}z), \alpha\ real. \tag{40}$$

$$g(z) = [f(z)]^t, or g(z) = f(tz) \qquad -1 < t < 1 \tag{41}$$

$$g(z) = \frac{1}{f(z)} \tag{42}$$

Theorem 4

Let $N \geq 1$ be a fixed integer. If $f(z)$, given by (4.4.4(a)) is in Q, then

$$|q_n| \leq 2 \tag{43}$$

This inequality is sharp, if $\eta = e^{\frac{2\pi i}{N}}$ and

$$f(z) = \sum_{k=1}^{N} \mu_k \frac{1+\eta^k z}{1-\eta^k z} = 1 + \sum_{n=1}^{\infty} q_n z^n \tag{44}$$

where $\mu_k \geq 0\ for\ k = 1, 2, \dots, N$ and

$$\sum_{k=1}^{N} \mu_k = 1 \tag{45}$$

Then, $f(z)$ is in Q and qN=2

Theorem 5

If $f(z)$ is in Q and $z = re^{i\theta}$, then

$$\frac{1-r}{1+r} \leq |f(z)| \leq \frac{1+r}{1-r} \tag{46}$$

$$|f'(z)| \leq \frac{2}{(1-r)^2}$$

(47)

Theorem 6

If $f(z)$ is in Q and $z = re^{i\theta}, r < 1$ then

$$\frac{1-r}{1+r} \leq \prod_{i=1}^{m} \frac{r^m}{|a_i|} - \prod_{j=1}^{n} \frac{r^n}{|b_j|} \leq \frac{1+r}{1-r}$$

where a_i, b_j, $i = 1,2,3 \dots, m$ and $j = 1,2,3, \dots, n$ are the roots and poles respectively of $f(z)$ and $f(z)$ has neither zeros nor poles on $|z| = 1$

Proof

(i) From lemma 1

$$ln\frac{1-r}{1+r} \leq ln|f(z)| \leq ln\frac{1+r}{1-r}$$

Taking the integral

$$\frac{1}{2\pi}\int_0^{2\pi} ln\frac{1-r}{1+r}d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} ln\,|f(z)|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} ln\frac{1+r}{1-r}d\theta$$

$$ln\frac{1-r}{1+r} \leq \frac{1}{2\pi}\int_0^{2\pi} ln\,|f(z)|d\theta \leq ln\frac{1+r}{1-r}$$

By lemma 2

$$ln\frac{1-r}{1+r} \leq ln|f(0)| + \sum_{i=1}^{m} ln\frac{r}{|a_i|} - \sum_{j=1}^{n} ln\frac{r}{|b_j|} \leq ln\frac{1+r}{1-r}$$

$$\exp\left(ln\frac{1-r}{1+r}\right) \leq \exp\left(ln|f(0)|\right) + \exp\left(\sum_{i=1}^{m} ln\frac{r}{|a_i|}\right) - \exp\left(\sum_{j=1}^{n} ln\frac{r}{|b_j|}\right) \leq \exp\left(ln\frac{1+r}{1-r}\right)$$

$$\frac{1-r}{1+r} \leq e^{\left(ln\frac{r}{|a_1|}+\frac{r}{|a_2|}\dots+\frac{r}{a_m}\right)} - e^{\left(ln\frac{r}{|b_1|}+\frac{r}{|b_2|}\dots+\frac{r}{|b_n|}\right)} \leq \frac{1+r}{1-r}$$

$$\frac{1-r}{1+r} \leq \prod_{i=1}^{m} \frac{r^m}{|a_i|} - \prod_{j=1}^{n} \frac{r^n}{|b_j|} \leq \frac{1+r}{1-r}$$

Theorem 7

If $f(z)$ is in Q and $z = re^{i\theta}, r < 1$ then

$$N - P \leq \frac{2r(1+r)}{(1-r)^3}$$

Proof

If $|f'(z)| \leq \frac{2}{(1-r)^2}$ then

$$\frac{1}{|f(z)|} \leq \frac{1+r}{1-r}$$

By lemma 2

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2(1+r)}{(1-r)^3}$$

$$\int_0^{2\pi} \left|\frac{f'(z)}{f(z)}\right| dz \leq \int_0^{2\pi} \frac{2(1+r)}{(1-r)^3} dz$$

But

$$\int_0^{2\pi} \left|\frac{f'(z)}{f(z)}\right| dz = \int_0^{2\pi} \left|\frac{f'(z)}{f(z)}\right| r d\theta$$

Since

$$|dz| = r d\theta$$

$$\int_c \left|\frac{f'(z)}{f(z)}\right| r d\theta \leq \int_0^{2\pi} \frac{2(1+r)}{(1-r)^3} r d\theta$$

But from 18

$$\frac{1}{2\pi i} \int_c \left|\frac{f'(z)}{f(z)}\right| dz = N - P$$

Then

$$N - P \leq \frac{2r(1+r)}{(1-r)^3}$$

Theorem 8

If $f(z) and\ g(z)$ are polynomials of degree n and m respectively then the number of roots of fg is the number of roots of f plus the number of roots of g.

Proof

$$(fg)' = f'g + fg'$$

Hence

$$\frac{(fg)'}{fg} = \frac{f'g}{fg} + \frac{fg'}{fg}$$

$$= \frac{f'}{f} + \frac{g'}{g}$$

So

$$\frac{1}{2\pi i}\oint \frac{(fg)'}{fg}dz = \frac{1}{2\pi i}\oint \frac{f'}{f}dz + \frac{1}{2\pi i}\oint \frac{g'}{g}dz$$

$$N_{fg} = N_f + N_g$$

Theorem 9

The number of the roots of the product of k- polynomials is the sum of the number of the roots of each of the polynomials in the product

Proof

$$(f_1(z)\, f_2 \dots f_{k-1}(z)f_k(z))' = f'_1(z)f_2(z) \dots f_{k-1}(z)f_k(z) + f_1(z)\, f'_2(z) \dots f_{k-1}(z)f_k(z) + \cdots . + f_1(z)\, f_2 \dots f_{k-1}(z)f'_k(z)$$

$$\frac{(f_1(z)\, f_2 \dots f_{k-1}(z)f_k(z))'}{(f_1(z)\, f_2 \dots f_{k-1}(z)f_k(z))} = \frac{f'_1(z)f_2(z) \dots f_{k-1}(z)f_k(z)}{f_1(z)\, f_2 \dots f_{k-1}(z)f_k(z)} + \frac{f_1(z)\, f'_2(z) \dots f_{k-1}(z)f_k(z)}{f_1(z)f_2(z) \dots f_{k-1}(z)f_k(z)} + \cdots + \frac{f_1(z)\, f_2 \dots f_{k-1}(z)f'_k(z)}{f_1(z)\, f_2 \dots f_{k-1}(z)f_k(z)}$$

$$\frac{1}{2\pi i}\oint\frac{(f_1(\mathrm{z})\, f_2 \dots f_{k-1}(z)f_k(z))'}{(f_1(\mathrm{z})\, f_2 \dots f_{k-1}(z)f_k(z))}dz$$
$$=\frac{1}{2\pi i}\oint\frac{f'_1(z)}{f_1(\mathrm{z})}dz+\frac{1}{2\pi i}\oint\frac{f'_2(z)}{f_2(z)}dz+\cdots+\frac{1}{2\pi i}\oint\frac{f'_k(z)}{f_k(z)}dz$$

$$N_{f_1(\mathrm{z})\, f_2 \dots f_k(z)} = N_{f_1} + N_{f_2} + \dots + N_{f_K}$$

Application of Jensens's Integral Formula to Gauss Mean Value Theorem

Theorem 10

Suppose $f(z)$ is analytic inside and on circle C with center at a and radius r. Then $f(a)$is the mean of the value of $f(z)$ on C for

$$|z-a|<r,\ z=a+re^{i\theta}|a|<r\ and$$
$$f(a)=\frac{1}{2\pi}\int_0^{2\pi}f(a+re^{i\theta})d\theta. \quad (48)$$

Lemma 3

Harmonicity of $ln|f(z)|$

If $f(z)$ is analytic and non-zero for $|z|\le R$. Then $\ln|f(z)|$ is harmonic for

$$|z|\le R. \quad (49$$

Proof

If $f(z)\neq 0, |z|\le R$.

There exists an analytic function $g(z)$ for which

$$f(z)=e^{g(z)}$$

Let

$$g(z)=u(z)+iv(z)$$

Then

$$f(z)=e^{u(z)+iv(z)}$$
$$=e^{u(z)}e^{iv(z)}$$

So

$$|f(z)| = e^{u(z)}$$

Therefore

$$ln|f(z)| = lne^{u(z)} = u(z)$$

Hence

$ln|f(z)|$ is a harmonic function for $|z| \leq R$

Lemma 4

MVP for $ln|h(z)|$

If $h(z)$ is analytic and non-zero for all z such that $|z| \leq R$, then.

$$ln|h(z)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|h\left(z + Re^{i\theta}\right)\right| d\theta \quad (50)$$

Proof

$$ln|h(a)| = \frac{1}{2\pi}\int_c ln\left|\frac{h(z)}{z-a}\right| dz$$

Let $z - a = Re^{i\theta}$ then $dz = iRe^{i\theta}, 0 \leq \theta \leq 2\pi$

So

$$ln|h(a)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{ln\left|h(a+Re^{i\theta})\right|}{Re^{i\theta}} iRe^{i\theta} d\theta$$

$$ln|h(a)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|h\left(a + Re^{i\theta}\right)\right| d\theta$$

Hence

$$ln|h(z)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|h\left(z + Re^{i\theta}\right)\right| d\theta$$

Corollary 1

If $h(z)$ is analytic and non-zero for all z such that $|z| \leq R$

Then

$$ln|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|h\left(Re^{i\theta}\right)\right| d\theta \quad (51)$$

Proof

Let $z = 0$ in equation (50) from lemma 4

Then

$$ln|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|h\left(Re^{i\theta}\right)\right| d\theta$$

Theorem 11

Let f(z) be analytic inside and on a simple closed curve C except for a pole $z = \alpha$

of order (multiplicity) p inside C. suppose also that inside C, f(z) has only one zero $z = \beta$

of order (multiplicity) n and no zero on C then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\, dz = n - p. \tag{52}$$

Proof

If f(z) has a pole at $z = \alpha$ of order p and a zero at $z = \beta$ of order n

Let c_1 *and* Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ *and*

$z = \beta$ respectively as shown below

Then

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{c_1} \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i}\int_{c_2} \frac{f'(z)}{f(z)} dz \tag{53}$$

Now

Since f(z) has a pole of order P at $z = \alpha$ we have

$$f(z) = \frac{f(z)}{(z-\alpha)^p} \tag{54}$$

Upon differentiation

$$f'(z) = \frac{f'(z)}{(z-\alpha)^p} - \frac{pf(z)}{(z-\alpha)}$$

Dividing through by equation (54)

$$\frac{f'(z)}{f(z)} = \frac{f'(z)}{(z-\alpha)^p} \cdot \frac{(z-\alpha)^p}{f(z)} - \frac{pf(z)}{(z-\alpha)^{p+1}} \cdot \frac{(z-\alpha)^p}{f(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{f'(z)}{f(z)} - \frac{p}{(z-\alpha)}$$

Upon integration we have

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int\frac{f'(z)}{f(z)}dz - \frac{p}{2\pi i}\int\frac{dz}{(z-\alpha)}$$

By CIF

$$\frac{1}{2\pi i}\int\frac{f'(z)}{f(z)}dz = -p$$

If $f(z)$ has a zero of multiple n at $z = \beta$ we have

$$f(z) = (z-\beta)^n g(z), g(\beta) \neq 0 \tag{55}$$

Differentiating equation (55)

$$f'(z) = g'(z)(z-\beta)^n + ng(z)(z-\beta)^{n-1} \tag{56}$$

Dividing through by equation (54) we have

$$\frac{f'(z)}{f(z)} = \frac{g'(z)(z-\beta)^n}{g(z)(z-\beta)^n} + \frac{ng(z)(z-\beta)^{n-1}}{g(z)(z-\beta)^n}$$

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{n}{(z-\beta)}$$

Upon integration

$$\int_{C_2}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int\frac{g'(z)}{(z)}dz + \frac{n}{2\pi i}\int\frac{dz}{z-\beta}dz$$

By CIF

$$\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz = n \tag{57}$$

Adding equation (53) and (54) and from (55) give

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)} = n - p$$

Lemma 5

For all $z \in C$ and any $r > 0$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = \begin{cases} ln|z| \; if \; |z| < r \\ lnr \; if \; |z| = r \\ -ln|z| \; if |z| > r \end{cases}$$

Proof

If $|z| < r$

$$z - re^{i\theta} = -(re^{i\theta} - z)$$

$$= -e^{i\theta}\left(r - \frac{z}{e^{i\theta}}\right)$$

$$= -e^{i\theta}\left(\frac{rz}{z} - \frac{rz}{re^{i\theta}}\right)$$

$$= -e^{i\theta} z\left(\frac{1}{z} - \frac{1}{re^{i\theta}}\right)$$

Hence

$$\left|z - re^{i\theta}\right| = |r||z|\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right|$$

and

$$ln\left|z - re^{i\theta}\right| = ln|z| + lnr + ln\left|\frac{1}{z} - \frac{1}{r}e^{-i\theta}\right|$$

So integrating over θ from $\theta = 0 \; to \; \theta = 2\pi$

$$\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = \int_0^{2\pi} ln|z| d\theta + \int_0^{2\pi} lnr d\theta + \int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right| d\theta$$

So that

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = ln|z| + lnr + \int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right| d\theta$$

Set

$$ln\left|\frac{1}{z} - \frac{1}{r}e^{-i\theta}\right| = ln\left|Z - Re^{i\emptyset}\right|, \mathrm{Z} = \frac{1}{z}, \mathrm{R} = \frac{1}{r}, \emptyset = -\theta, \; d\theta = -d\emptyset$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{r}e^{-i\theta}\right| d\theta = \frac{1}{2\pi}\int_{-2\pi}^{0} ln\left|Z - Re^{i\emptyset}\right| d\emptyset$$

$$= \frac{1}{2\pi}\int_0^{2\pi} ln\left|Z - Re^{i\emptyset}\right| d\emptyset = ln|Z|$$

Since

$$Z = \frac{1}{z}, \; R = \frac{1}{r}, \text{ and}$$

$$|Z| < R. then \; \left|\frac{1}{z}\right| < \frac{1}{r}$$

Hence if $|z| > r$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = ln\frac{1}{|z|} = -ln|z|$$

Now, suppose

$$|z| < r$$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = lnr$$

Proof

Set

$$|z| < r$$

We have that

$$z - re^{i\theta} \quad = -(re^{i\theta} - z)$$

$$= -e^{i\theta}(r - \frac{z}{e^{i\theta}})$$

$$= -e^{i\theta}\left(\frac{rz}{z} - \frac{rz}{re^{i\theta}}\right)$$

$$= -e^{i\theta}z\left(\frac{1}{z} - \frac{1}{re^{i\theta}}\right)$$

Hence

$$\left|z - re^{i\theta}\right| = |r||z|\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right|$$

and

$$ln\left|z - re^{i\theta}\right| = ln|z| + lnr + ln\left|\frac{1}{z} - \frac{1}{r}e^{-i\theta}\right|$$

Integrating over θ

$$\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = \int_0^{2\pi} ln|z| d\theta + \int_0^{2\pi} lnr d\theta + \int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right| d\theta$$

$$2\pi\, ln|z| + 2\pi lnr + \int_0^{2\pi} lnrd\theta + \int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right| d\theta$$

So

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = ln|z| + lnr + \int_0^{2\pi} ln\left|\frac{1}{z} - \frac{1}{re^{i\theta}}\right| d\theta$$

The last integral, $ln\left|\frac{1}{z}\right|$. Adding the first five terms we have

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = \frac{1}{|z|} < \frac{1}{r}$$

If $|z| < r$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|z - re^{i\theta}\right| d\theta = ln\mathrm{r}$$

Lemma 6 (Jensen's Lemma)

Let *f* be a meromorphic function on the whole plane. If $f(0) \neq 0, \infty$ then , for any $r > 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta})\right| d\theta = ln|f(0)| + \sum_{i=1}^{m} ln\frac{r}{|a_i|} - \sum_{j=1}^{n} ln\frac{r}{|b_j|} \qquad (58)$$

where $a_i \ldots a_m$ are the zeros of *f* in $|z| \leq r$ and $b_1 \ldots b_n$ are the poles of *f* in $|z| \leq r$.

Proof

Let

$$f(z) = g(z)\prod_{i=1}^{m}(z - a_i)\left[\prod_{j=1}^{n}(z - b_j)\right]^{-1}$$

where $g(z)$ is an analytic function on the whole plane that has no zeros or poles in $|z| \leq r$. Then

$$|f(z)| = \left|g(z)\frac{(z-a_1)(z-a_2)\ldots(z-a_m)}{(z-b_1)(z-b_2)\ldots(z-b_n)}\right|$$

$$|f(0)| = |g(0)|\left|\frac{(-a_1)(-a_2)\ldots(-a_m)}{(-b_1)(-b_2)\ldots(-b_n)}\right|$$

and

$$ln|f(z)| = ln\left|g(z)\frac{(z-a_1)(z-a_2)\ldots(z-a_m)}{(z-b_1)(z-b_2)\ldots(z-b_n)}\right|$$

Thus

$$ln|f(z)| = ln|g(z)| + ln\{|z - a_1||z - a_2| \dots |z - a_m|\}$$
$$-ln\{|z - b_1||z - b_2| \dots |z - b_n|\}$$

Or

$$ln|f(z)| = ln|g(z)| + \sum_{i=1}^{m} ln\,|z - a_i| - \sum_{j=1}^{n} ln\left|z - b_j\right|$$

and

$$ln|f(0)| = ln|g(0)| + \sum_{i=1}^{m} ln\,|a_i| - \sum_{j=1}^{n} ln\left|b_j\right| \tag{59}$$

Upon integration for $|z| = r,$ we get

$$\frac{1}{2\pi}\int_0^{2\pi} ln|f(z)|d\theta = \frac{1}{2\pi}\int_0^{2\pi} ln|g(z)|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^{m} ln|z - a_i|d\theta$$
$$-\frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{n} ln\left|z - b_j\right|d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta = \frac{1}{2\pi}\int_0^{2\pi} ln\left|g(re^{i\theta}\right|d\theta + \frac{1}{2\pi}\int_0^{2\pi} ln\left|re^{i\theta} - a_i\right|d\theta$$
$$-\frac{1}{2\pi}\int_0^{2\pi} ln\left|re^{i\theta} - b_j\right|d\theta$$

But by lemma 4 and equation (50)

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta = ln|g(0)| + mlnr - nlnr$$

By application of equation (57), this become

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta = ln|f(0)| - \sum_{i=1}^{m} ln\,|a_i| + \sum_{j=1}^{n} ln\left|b_j\right| + mlnr - nlnr$$
$$ln|f(0)| + lnr + lnr + \cdots + lnr - (ln|a_1| + ln|a_2| + \cdots + ln|a_m|)$$
$$-lnr - lnr - \cdots - lnr + (ln|b_1| + ln|b_2| + \cdots ln|b_n|)$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta = ln|f(0)| + ln\frac{r}{|a_1|} + ln\frac{r}{|a_2|} + \cdots + ln\frac{r}{|a_m|}$$
$$-(ln\frac{r}{|b_1|} + ln\frac{r}{|b_2|} + \cdots + ln\frac{r}{|b_n|})$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta = ln|f(0)| + \sum_{i=1}^{m} ln\frac{r}{|a_i|} - \sum_{j=1}^{n} ln\frac{r}{|b_j|}$$

Hence,

$$ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} ln\left|f(re^{i\theta}\right|d\theta - \sum_{i=1}^{m} ln\frac{r}{|a_i|} + \sum_{j=1}^{n} ln\frac{r}{|b_j|}$$

Theorem 12

Let $f(z)$and $g(z)$ be analytic inside and on simple closed curve C except that $f(z)$ has zero at $a_1, a_2, \dots a_m$ and poles at $b_1, b_2, \dots . b_n$ of orders (multiplicity) $p_1, p_2, \dots p_m$ and $q_1, q_2, \dots q_n$ respectively. Then

$$\frac{1}{2\pi i}\oint_c g(z)\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{m} p_k g(a_k) - \sum_{k=1}^{n} q_k g(b_k)$$

Theorem 13

Suppose $f(z)$ is analytic inside and on the $|z| = R$ except for zeros at $a_1, a_2, \dots, a_m$ of multiplicities $\alpha_1, \alpha_2, \dots, \alpha_m$ and poles at $b_1, b_2, \dots, b_n$ of multiplicities $\beta_1, \beta_2, \dots, \beta_n$ respectively and suppose $f(0)$ is finite and different from zero. Then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|d\theta = ln|f(0)| + \sum_{k=1}^{m} \alpha_k ln\left(\frac{R}{|a_k|}\right) - \sum_{k=1}^{n} \beta_k ln\left(\frac{R}{|b_k|}\right)$$

Proof

Let

$$f(z) = g(z)\prod_{i=1}^{m}(z-a_m)^{\alpha_m}\left[\prod_{j=1}^{n}(z-b_n)^{\beta_n}\right]^{-1}$$

$$f(z) = g(z)\frac{(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\dots(z-a_m)^{\alpha_m}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\dots(z-b_n)^{\beta_n}}$$

Where $g(z)$ is analytic and has no zero on $|z| \leq R$

and

$$ln|f(0)| = |g(0)|\left|\frac{(-a_1)^{\alpha_1}(-a_2)^{\alpha_2}\dots(-a_m)^{\alpha_m}}{(-b_1)^{\beta_1}(-b_2)^{\beta_2}\dots(-b_n)^{\beta_n}}\right|$$

$$ln|f(z)| = ln\left|g(z)\frac{(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\dots(z-a_m)^{\alpha_m}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\dots(z-b_n)^{\beta_n}}\right|$$

$$ln|f(z)| = ln|g(z)| + ln|(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\dots(z-a_m)^{\alpha_m}|$$

$$-ln\left|(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\dots(z-b_n)^{\beta_n}\right|$$

$$ln|f(z)| = ln|g(z)| + \alpha_1 ln|z-a_1| + \alpha_2 ln|z-a_2| + \cdots + \alpha_m ln|z-a_m|$$

$$-(\beta_1 ln|z-b_1| + \beta_2 ln|z-b_2| + \cdots + \beta_n ln|z-b_n|)$$

$$ln|f(z)| = ln|g(z)| + \sum_{k=1}^{m}\alpha_k ln\,|z-a_k| - \sum_{k=1}^{n}\beta_k ln\,|z-b_k|$$

and

$$ln|f(0)| = |g(0)| + \sum_{k=1}^{m}\alpha_k ln\,|a_k| - \sum_{k=1}^{n}\beta_k ln\,|b_k| \tag{60}$$

Upon integration for $z = R$ we get

$$\frac{1}{2\pi}\int_0^{2\pi} ln|f(z)|d\theta = \frac{1}{2\pi}\int_0^{2\pi} ln\,|g(z)|d\theta + \sum_{k=1}^{m}\frac{1}{2\pi}\int_0^{2\pi}\alpha_k ln|z-a_k|d\theta$$

$$-\sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\beta_k ln|z-b_k|d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} ln|f(z)|d\theta = \frac{1}{2\pi}\int_0^{2\pi} ln\,\left|g(Re^{i\theta})\right|d\theta + \sum_{k=1}^{m}\frac{1}{2\pi}\int_0^{2\pi}\alpha_k ln\left|Re^{i\theta}-a_k\right|d\theta$$

$$-\sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\beta_k ln\left|Re^{i\theta}-b_k\right|d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi}\alpha_1\ln|z-a_m|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\alpha_2 ln|z-a_2|d\theta + \cdots + \frac{1}{2\pi}\int_0^{2\pi}\alpha_m ln|z-a_m|d\theta$$

$$-(\frac{1}{2\pi}\int_0^{2\pi}\beta_1\ln|z-b_1|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\beta_2 ln|z-b_2|d\theta + \cdots + \frac{1}{2\pi}\int_0^{2\pi}\beta_n ln|z-b_n|d\theta$$

$$= \alpha_1 lnR + \alpha_2 lnR + \cdots + \alpha_m lnR - (\beta_1 lnR + \beta_2 lnR + \cdots + \beta_n lnR)$$

But by lemma 6 and equation (57)

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|d\theta = ln|g(0)| + \sum_{k=1}^{m}\alpha_k lnR - \sum_{k=1}^{n}\beta_k lnR$$

Thus, using equation (58) to eliminate $ln|g(0)|$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f\left(Re^{i\theta}\right)\right|d\theta = ln|f(0)| - \sum_{k=1}^{m}\alpha_k ln\,|a_k|$$

$$+\sum_{k=1}^{n}\beta_k ln\left|b_j\right| + \sum_{k=1}^{m}\alpha_k lnR - \sum_{k=1}^{n}\beta_k lnR$$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f\left(Re^{i\theta}\right)\right|d\theta = ln|f(0)| + \alpha_k[lnR + lnR + \cdots -$$

$$\begin{pmatrix} lnR + ln|a_1| + ln|a_2| + \cdots + ln|a_k|) \\ -\beta_k[(lnR + lnR + \cdots + lnR \end{pmatrix} + (ln|b_1| + ln|b_2| + \cdots + ln|b_k|)]$$

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|d\theta = ln|f(0)| + \alpha_k(ln\frac{R}{|a_1|} + ln\frac{R}{|a_2|} + \cdots + ln\frac{R}{|a_k|})$$

$$-\beta_k(ln\frac{R}{|b_1|} + ln\frac{R}{|b_2|} + \cdots + ln\frac{R}{|b_k|})$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|d\theta = ln|f(0)| + \sum_{k=1}^{m}\alpha_k \ln\left(\frac{R}{|a_i|}\right) - \sum_{k=1}^{n}\beta_k \ln\left(\frac{R}{|b_k|}\right)$$

Theorem 14

Let a complex variable be given by $z = re^{i\theta}$, then

(i) The function $f(z)$ is analytical in the region $|z| \leq R$

(ii) $f(z)$ has no zeroes on $|z| = R$, and

(iii) $f(0) = 1$ then

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|dr = \int_0^R \frac{n(r)}{r}dr \tag{61}$$

Where the function $n(r)$ is the number of zeroes of $f(z)$ inside the disc $|z| = r$.

Proof

Let

$$z = re^{i\theta}\ for\ |z| \leq R, 0 \leq r \leq R,$$

Since $f(z)$ has no zero on the circumference of the circle of radius R, we have that a function $y(z)$ exist such that

$$f(z) = e^{y(z)}$$

that is

$$y(z) = lnf(z)$$

Then

$$\frac{dy}{dz} = \frac{d}{dz}lnf(z)$$

$$= \frac{d}{df(z)} lnf(z). \frac{df(z)}{dz}$$

$$= \frac{1}{f(z)} \frac{df(z)}{d(z)} = \frac{f'(z)}{f(z)}$$

That is, on the circumference of the circle, $0 \leq r \leq R$

$$\frac{d}{dz} lnf(z) = \frac{f'(z)}{f(z)}$$

Integrating with respect to r

$$\int_0^R dlnf\left(re^{i\theta}\right) = \int_0^R \frac{f'\left(re^{i\theta}\right)}{f(z)} dz = \int_0^R \frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} e^{i\theta} dr$$

$$lnf\left(re^{i\theta}\right) /_{r=0}^{r=R} = \int_0^R \frac{f'(z)}{f(z)} dz$$

or

$$lnf\left(re^{i\theta}\right) - \ln f(0) = \int_0^R \frac{f'(z)}{f(z)} dz$$

Hence, considering the pole, θ from 0 to 2π, we integrate with respect to θ we get

$$\int_0^{2\pi} lnf\left(Re^{i\theta}\right) d\theta - \int_0^{2\pi} lnf(o) d\theta = \int_0^{2\pi} \int_0^R \frac{f'(z)}{f(z)} dz$$

$$\ln f(0) = 0.\ If\ f(0) = 1$$

So,

$$\int_0^{2\pi} lnf\left(Re^{i\theta}\right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^R \frac{f'(z)}{f(z)} dz \right) d\theta$$

Now, if $z = re^{i\theta}$, then since $r < R$, we consider points in the interior of the circle of circumference $2\pi R$. Then

$$dz = e^{i\theta} dr \text{ and } dz = rie^{i\theta} d\theta$$

So, the polar and radial variable are connected through

$$e^{i\theta} dr = rie^{i\theta} d\theta$$

$$\frac{rie^{i\theta}}{ri} dr = rie^{i\theta} d\theta$$

Implies,

$$\frac{dr}{ir} = d\theta$$

Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} lnf\left(Re^{i\theta}\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^R \frac{f'(z)}{f(z)}dz\right)\frac{dr}{ir}$$

Since, $\frac{f'(z)}{f(z)}$ is continous on (0,R) we can interchange the double integrals to get

$$\frac{1}{2\pi}\int_0^{2\pi} lnf\left(Re^{i\theta}\right)d\theta = \int_0^R\left\{\frac{1}{2\pi ir}\int_0^{2\pi}\frac{f'(z)}{f(z)}dz\right\}dr$$

$$= \int_0^R\left\{\frac{1}{2\pi ir}\int_{|z|=r}\frac{f'(z)}{f(z)}dz\right\}dr$$

But $$\frac{1}{2\pi ir}\int_{|z|=R}\frac{f'(z)}{f(z)}dz = n(r),$$

where n(r) is the number of zeroes of $f(z)$ in the circle of radius r.

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} ln\left|f(Re^{i\theta})\right|d\theta = \int_0^R \frac{n(r)}{r}dr$$

Thus,

$$\int_0^R \frac{n(r)}{r}dr = \left(\int_0^{r_1} + \int_{r_1}^{r_2} + \int_{r_2}^{r_3} + \cdots + \int_{r_{n-1}}^{r_n} + \int_{r_n}^{R}\right)\frac{n(r)}{r}dr$$

Suppose that the zeros of the function $f(z)$ are at the points $z_1, z_2, z_3, \ldots, z_n$. These zeroes are isolated .Let

$z_i \in (r_i, r_{i+1})$ $\quad i = 1,2,\ldots,n$ without any zero in $(0, r_i)$ or$z_0 \in (r_0, r_1)$ and n$(z_i) = i$

Then

$$\int_0^R \frac{n(r)}{r}dr = \left(\int_0^{r_1}\frac{n(r)}{r}dr\int_0^{r_2}\frac{n(r)}{r}dr\int_0^{r_3}\frac{n(r)}{r}dr + \cdots + \int_{r_{n-1}}^{r_n}\frac{n(r)}{r}dr + \int_{r_n}^{R}\frac{n(r)}{r}dr\right)$$

$$= \int_0^{r_1}\frac{0}{r}dr\int_{r_1}^{r_2}\frac{1}{r}dr\int_2^{r_3}\frac{2}{r}dr + \cdots + \int_{r_{n-1}}^{r_n}\frac{n-1}{r}dr + \int_{r_n}^{R}\frac{n}{r}dr$$

$$= 0 + lnr|_{r_1}^{r_2} + 2lnr|_{r_2}^{r_3} + \cdots + n - 1lnr|_{r_{n-1}}^{r_n} + nlnr|_{r_n}^{R}$$

$$= lnr_2 - lnr_1 + 2(lnr_3 - lnr_2) + 3(lnr_4 - lnr_3) + \cdots + n - 1(lnr_n - lnr_{n-1}) + n(lnR - lnr_{n-1})$$

$$= lnr_2 - lnr_1 + 2lnr_3 - 2lnr_2 + 3lnr_4 - 3lnr_3 + \cdots + (n - 1lnr_n) - (n - 1lnr_{n-1} + nlnR - lnr_n$$

$$= -lnr_1 - lnr_2 - lnr_3 - lnr_4 + \cdots nlnR - lnr_n$$

$$= nlnR - (lnr_1 + lnr_2 + lnr_3 + lnr_4 + \cdots + lnr_n$$

But

$$nlnR = \left(\frac{1+1+1+\cdots+1+1}{n-times}\right) lnR$$

Then

$$= -lnr_1 + lnR - lnr_2 + lnR - \cdots - lnr_n + lnR$$

$$= -ln\frac{r_1}{R} - ln\frac{r_2}{R} - \cdots - ln\frac{r_n}{R}$$

$$= ln\frac{R}{r_1} + ln\frac{R}{r_2} + \cdots + ln\frac{R}{r_n}$$

$$= ln\left|\frac{R}{r_1}\right| + ln\left|\frac{R}{r_2}\right| + \cdots + ln\left|\frac{R}{r_n}\right|$$

$$= ln\frac{R^n}{|r_1 r_2 r_3 \ldots r_n|}$$

From our analysis $In|Q(z)|$ is harmonic in any region in which $Q(z)$ does not vanish, in particular when $Q(z)$ is analytic and non zero for $|z| \le R$ and the Jensen's formulae tells us that the average value of $In|Q(z)|$ on the circumference of the circle $|Z| = R$ is increased by the presence of zeros $Q(z)$ within the circle, the closer these zeros are to the origin, the more this average value is increased. Also the larger the number of zeros within the circle $|Z| = R$, the more rapidly the average value of $In|Q(z)|$ on the circumference of the circle grows as R increases. Also Q(z) which is analytic inside and on the circle of center r attains its mean value at the evaluation of Q(z) at z = r which is regular inside and on the circle $|z| = r, Q(0) \neq 0$ and $a_{1,} a_2, \ldots, a_N$ are zeros of $Q(z)$ satisfying $|a_k| \le r$ with multiple zeros counted as times as their multiplicity. Also the new inequalities on polynomials are obtained using Jensen's Integral formula and are of the fundamental importance in the theory of transcendental numbers different from the initial results carried-out by other researchers which are not fully connected to the basic properties of meromorphic or harmonic function.

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