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An equivalence in a selected family of cyclic subsemigroups and algorithm for generating systems of semigroup

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Abstract

This paper focuses on generators of cyclic subsets of semigroups which determines the whole semigroup through algebraic closure. We call a generating set minimal if it does not have a proper generating subset. A set is independent if no element of the set can be generated by the remaining members of the set. Independent subsystems of cyclic semigroups are intersected to obtain the generating set of cyclic semigroups. From intersecting subsystems of cyclic semigroups, a certain equivalence relation on the set of all cyclic Subsemigroups are obtained. The paper also shows how such equivalence relation is a tool for partitioning cyclic subsemigroups into generating subsystems. Algorithm to find the minimal generating set of a semigroups is given.

Keywords and phrases: Semigroup, Cyclic, Independent, Minimal, Generating sets

1 Introduction: somethings about cyclic semigroups

In this section, we investigate some behaviour of cyclic semigroups and discuss about the problem of intersecting semigroups/groups. We show some connections between members in such intersection and also show how some selected family of connected cyclic semigroups can be embedded into a cyclic semigroup. These provide tools for obtaining algorithms for finding generating sets. We can also determine minimal generating sets as we can intersect independent subsystems to obtain the generating set of such semigroups. We begin with definitions of Semigroups, Subsemigroups, Cosets and other relevant terminologies. Where they are not defined, the reader can read up the definitions in [1], [2], [3], [5], [6], [8], [11]. and [12].

Definition 1.1 (Semigroup). Let S be a set with a mapping $*: S \times S \Longrightarrow S$. Then S forms a semigroup $(S,*)$ if $\forall\, a, b, c \in S \Longrightarrow\ (a * b) * c = a * (b * c)$ holds.

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Definition 1.2 (Subsemigroup). Let $(S, *)$ be a semigroup. If a subset $H \subset S$ forms a semigroup for the same operation $* : H \times H \to H$ as in S, then H is a subsemigroup of S.
Definition 1.3 (Monoid). Let $(S, *)$ be as semigroup. If there is a neutral element $e \in S$, such that $e * a = a * e = a,\ \forall a \in S$, then S is a Monoid.

Lemma 1.4 [5]. Let $(S,*)$ be a semigroup. If S is not a monoid, if there exists a set $M_S := S \cup \{e\}$ and an operation $\circ$ such that $e \circ e = e$; $e \circ a = a \circ e = a$, $\forall\, a \in S$ and $a \circ b := a * b$, $\forall\, a, b \in S$. Then $\{M_S, \circ\}$ is a monoid.
Proof. If S is not a monoid, adjoining an identity e to S produces $M_S := S \cup \{e\}$. If multiplication in M_S is defined as given: $e \circ e = e$; $e \circ a = a \circ e = a, \forall\, a \in S$, then replacing $\circ$ by $*$ accords $\{M_S, \circ\}$ the properties of $(S, *)$ in definition 1.3. Hence $\{M_S, \circ\}$ is a monoid.

Remarks 1.5. There is no essential difference between a monoid and a semigroup. A monoid $\{M,*\}$ with a neutral element $e \in M$ represents a semigroup if $M \backslash \{e\} =: S$, is a subsemigroup of M. In other words, $S \subset M$ is a subsemigroup of M if S is an invariant subset of M for the semigroup operation, which means S does not have any invertible element. Any semigroup is a subsemigroup of a monoid by the lemma 1.4.

Theorem 1.6. Let $\{S,*\}$ be a semigroup. Let $H_\alpha \subset S$ be a subsemigroup $\forall \alpha \in A \neq \emptyset$, then

$$\emptyset \neq H = \bigcap_{\alpha \in A} H_\alpha$$

is a subsemigroup of S.
Proof. Let $u, v \in H$. Then $u, v \in H_\alpha, \forall \alpha \in A \implies$

$$u * v \in H_\alpha, \forall \alpha \in A \implies u * v \in H = \bigcap_{\alpha \in A} H_\alpha$$

Definition 1.7. (Commutative Semigroup). Let $(S, \cdot)$ be a semigroup. $(S, \cdot)$ is called a Commutative Semigroup, if $\forall x, y \in S : x \cdot y = y \cdot x$.

Definition 1.8. (Cosets). Let S be a semigroup containing the subsemigroup U, and let $c_i \in S$, then by the Left (right) Coset, $c_i U$ $(U c_i)$ of U, we mean the set of elements obtained by multiplying c_i on the right (left) by each element of U. A left coset of U that is also a right coset is simply called a Coset.

2 Intersections of subsemigroups

We now define a fundamental subsemigroup called cyclic subsemigroup.

Definition 2.1. (Cyclic Subsemigroup). Let $\{S,*\}$ be a semigroup with a neutral element $e \in S$. Let $a \in S$. Then $\{\theta^j \mid 0 \leq j \leq \infty,\ e = \theta^0\} =: \langle a \rangle$ is called a cyclic subsemigroup.

Remark 2.2. The subset $\{\theta^j \mid j \in N\} =: \langle\theta\rangle \subset S$ is a commutative subsemigroup (see **Definition 1.7.** for definition of commutative semigroup):

$$\forall\, u, v \in \langle\theta\rangle \Longrightarrow u = \theta^p, \qquad v = a^q$$

$\Longrightarrow u * v = \theta^p * \theta^q = \theta^{p+q} = \theta^{q+p} = \theta^q * \theta^p = v * u \in \langle\theta\rangle$ by $p + q \in N$.

Definition 2.3. (Connected Family of Subsemigroups). Let $H := \{H_\alpha\}_{\alpha\in A}$ be a family of subsemigroups of S. Then H is a connected family of subsemigroups if $(H_\alpha \cap H_\beta)\backslash\{e\} \neq \emptyset$, $\forall \alpha, \beta \in A$, where $e \in S$ is the neutral element of S (Existence of a neutral element may always be assumed by lemma of section 1.4).

Lemma 2.4. Let S be a semigroup, let $H_1, H_2 \subset S$ be subsemigroups such that $H_1, H_2 \notin \{\emptyset, \{e\}\}$. Let H_1, H_2 be related by $H_1 \bowtie H_2$ if $H_1 \cap H_2 \notin \{\emptyset, \{e\}\}$.Then the relation $\bowtie$ is reflexive and symmetric but not necessarily transitive.

Proof. 1. Reflexivity**:** If the subsemigroup $H \subset S$ fulfils $H\backslash\{e\} \neq \emptyset$; then $H \cap H = H \notin \{\emptyset, \{e\}\}$, hence $H \bowtie H$ is true.

2. Symmetry**:** Let $\emptyset \neq H_1, H_2 \subset S$ fulfill $H_1 \bowtie H_2$ then $H_1 \bowtie H_2 \Longrightarrow H_1 \cap H_2 \notin \{\emptyset, \{e\}\} \Longrightarrow H_2 \cap H_1 \notin \{\emptyset, \{e\}\} \Longrightarrow H_2 \bowtie H_1$ that proves the symmetry.

3. Transitivity**:** We give an example that the transitivity does not necessarily hold. Let $S \subset R^2$ be an open convex cone with addition of vectors as its operation. To make it concrete, let the cone be $S := \{(x, y) \mid y > 0\}$. Let

$H_1 := \{(x, y) \mid 0 < y < \frac{\sqrt{3}}{2}(-x),\ -\infty < x < 0\} \subset S$

$H_2 := \{(x, y) \mid y > \frac{\sqrt{2}}{2}\, |x|,\ x \in R\} \subset S$

$H_3 := \{(x, y) \mid 0 < y < \frac{\sqrt{3}}{2}x,\ \infty > x > 0\} \subset S$

The upper borders of H_1 *and* H_3 are segments of straight lines enclosing 120^0 and 60^0 angles respectively with the first (x) axes. The border of H_2 consists of two segments of straight lines enclosing 135^0 and 45^0 angles with the first axes.

$H_1 \cap H_2 = \{(s, t) \mid \frac{\sqrt{2}}{2}|s| < t < \frac{\sqrt{3}}{2}\, |x|,\ -\infty < x < 0\} \notin \{\emptyset, \{e\}\} \Longrightarrow H_1 \bowtie H_2$ holds.

$H_2 \cap H_3 = \{(s, t) \mid \frac{\sqrt{2}}{2}\, |s| < y < \frac{\sqrt{3}}{2}\, |s|,\ \infty > x > 0\} \notin \{\emptyset, \{e\}\} \Longrightarrow H_2 \bowtie H_3$ holds.

$H_1 \cap H_3 = \{(s, t) | 0 < y < \frac{\sqrt{3}}{2}\, |x|,\ -\infty < x < 0\} \cap \{(s, t) \mid 0 < y < \frac{\sqrt{3}}{2}|s|,\ \infty > x > 0\} = \emptyset \Longrightarrow$ the relation $H_1 \bowtie H_3$ is not true. Hence the relation $\bowtie$ is not transitive.

2.1 Intersecting cyclic subsemigroups - their equivalence relation

We now do a quick study of the structure of cyclic subsemigroups as graphs first with infinite cyclic subsemigroups as intersecting chains where the intersection of any two infinite cyclic semigroups is an infinite cyclic subsemigroup of the two intersecting infinite cyclic semigroups. In our graph the nodes will be semigroup elements, and the edges connecting the a^k elements of the cycle to the consecutive a^{k+1} element. In this graph we do not consider multiplications between elements coming from two different chains. The graph structure is available in the original semigroup also and the smallest semigroup containing original cyclic semigroups can be mapped into the free semigroups.

Lemma 2.6. Let $\{S,*\}$ be a semigroup, let $C_1 = \langle a \rangle,\ C_2 = \langle b \rangle \subset S$ be two cyclic semigroups, $a \notin C_2$ and $b \notin C_1$. If $C_1 \cap C_2 \neq \{\emptyset\}$ and $C_1 \cap C_2 \neq \{e\}$; then there exists a free cyclic semigroup $F_{a,b} = \langle s \rangle$ and an isomorphism $i: \langle a \rangle \cup \langle b \rangle \rightarrow F = \langle s \rangle$. In other words, $C_1\ and\ C_2$ can be embedded into one cyclic group.

Proof. Let the first common element $c \in C_1 \cap C_2$ (c is the generator element of the cyclic semigroup: $C_1 \cap C_2 = \langle s \rangle$). Then $\langle a \rangle \ni a^p = c = b^q \in \langle b \rangle$. Then there is a free semigroup $\langle s \rangle$ in which $a = s^q$ and $b = s^p$ thus $a^p = s^{q*p} = s^{p*q} = b^q = c$ holds hence the cyclic group $\langle s \rangle$ contains the image of $C_1,\ C_2$ and $C_1 \cap C_2 \subset S$ by the mapping $i(a^t\ ) \coloneqq s^{pt}$ and $i(b^t\ ) \coloneqq s^{qt},\ \forall t \in N\backslash\{0\}$ which is an isomorphism. ■

Remark 2.7. The isomorphism here does not map from $\langle a, b \rangle \rightarrow \langle c \rangle$. It maps from the $\langle a \rangle \cup \langle b \rangle \rightarrow \langle c \rangle$ where $\langle a \rangle \cup \langle b \rangle \subset \langle a, b \rangle$ using the operations of $\langle a, b \rangle$ along the cyclic groups $\langle a \rangle$ and $\langle b \rangle$. This mapping however, truly represents the intersections between the two subsemigroups.

Remark 2.8. Note that $s \in F$ is determined by $a \in \langle a \rangle$, $b \in \langle b \rangle$ and $c \in \langle a \rangle \cap \langle b \rangle$ which is the generator $\langle c \rangle = \langle a \rangle \cap \langle b \rangle$ of the intersection, the minimal power appearing in the intersection from both $\langle a \rangle$ and $\langle b \rangle$. It can be proved then that the mapping stated in the lemma 2.6 is a unique isomorphism, in other words, the relative positions of the three semigroups are fully determined in the free group.

Lemma 2.9. (Equivalence Relation on the Set of all Cyclic Subsemigroups)
Let S be a semigroup and let $S_C := \{\langle x \rangle\ |\ x\ \in S\}$ be the set of all cyclic subsemigroups. Then the relation $\langle a \rangle \asymp \langle \theta \rangle \Leftrightarrow \langle a \rangle \cap \langle \theta \rangle \neq \{\emptyset\}$ and $\langle a \rangle \cap \langle \theta \rangle \neq \{e\}$ is an equivalence relation $\forall\ \langle \alpha \rangle, \langle \theta \rangle \in S_C$.

Proof. (i) The relation is reflexive:

$\forall \langle a \rangle \in S_C$, $\langle \alpha \rangle \cap \langle \alpha \rangle = \langle \alpha \rangle \neq \langle \alpha \rangle \cap \langle b \rangle \neq \{\emptyset\}$ and $\langle \alpha \rangle \cap \langle b \rangle \neq \{e\} \Rightarrow \langle \alpha \rangle \asymp \langle \alpha \rangle$.

Note that $a \in S$ is an existing element hence $\langle \alpha \rangle \neq \langle a \rangle \cap \langle \alpha \rangle \neq \{\emptyset\}$ and $\langle \alpha \rangle \cap \langle \alpha \rangle \neq \{e\}$.

(ii) The relation is symmetric:

$\forall \langle \alpha \rangle, \langle \alpha \rangle \in S_C$, $\langle \alpha \rangle \asymp \langle b \rangle \Leftrightarrow \langle \alpha \rangle \cap \langle \theta \rangle \neq \emptyset \Leftrightarrow \langle \alpha \rangle \cap \langle \theta \rangle \neq \langle \alpha \rangle \cap \langle \theta \rangle \neq \{\emptyset\}$ and $\langle \alpha \rangle \cap \langle \theta \rangle \neq \{e\} \Leftrightarrow \langle \theta \rangle \asymp \langle \alpha \rangle$ completes the proof of the symmetry.

(iii) Let $\langle \alpha \rangle, \langle \theta \rangle\ and\ \langle c \rangle \in S_C$. Let $\langle \alpha \rangle \asymp \langle \theta \rangle$ and $\langle \theta \rangle \asymp \langle c \rangle$ hold.

By $\langle \alpha \rangle \asymp \langle \theta \rangle \Longrightarrow \langle \alpha \rangle \cap \langle \theta \rangle \neq \{\emptyset\}$ and $\langle \alpha \rangle \cap \langle \theta \rangle \neq \{e\} \Longrightarrow \exists p_1, p_2 > 0$ such that

$$\langle \alpha \rangle \ni \alpha^{p_1} = b^{p_2} \in \langle b \rangle$$

Similarly, $\langle \theta \rangle \asymp \langle \omega \rangle \Longrightarrow \langle \alpha \rangle \cap \langle \theta \rangle \neq \{\emptyset\}$ and $\langle \alpha \rangle \cap \langle \theta \rangle \neq \{e\} \Longrightarrow \exists p_3, p_4 \in N$ such that

$$\langle b \rangle \ni \theta^{p_3} = \omega^{p_4} \in \langle c \rangle$$

Then raising both sides of $\alpha^{p_1} = \theta^{p_2}$ to the power p_3 results in $\alpha^{p_1 p_3} = \theta^{p_2 p_3}$ and raising both sides of $b^{p_3} = c^{p_4}$ to the power p_2 results in $\theta^{p_3 p_2} = \theta^{p_4 p_2}$

Therefore $a^{p_1 p_3} = \theta^{p_2 p_3} = \theta^{p_3 p_2} = \theta^{p_4 p_2} =: h$ fulfills

$$h \in \langle \alpha \rangle \cap \langle \theta \rangle \neq \{\emptyset\} \text{ and } \langle \alpha \rangle \cap \langle \theta \rangle \neq \{e\}$$

Therefore $\langle \alpha \rangle \asymp \langle \omega \rangle$ holds hence the transitivity holds. ■

3 Partitioning of cyclic semigroup

The equivalence relation of Lemma 2.9 partitions a cyclic semigroup into subsemigroups that are independent generators of its oversemigroup. In [6], they obtain a result that relate both concepts (minimal generating set and maximal independent set). There is analogue of this result in group theory. This needs a suitable equivalence relation and we determined one $S/\asymp$.

Corollary 3.2 (Partitioning Semigroup). Let $\{S,*\}$ be a semigroup. Then the system S_C of cyclic subsemigroups of S can be partitioned into a set of pairwise disjoint equivalence classes $S/\asymp \coloneqq \{\hat{C} \mid C \in S_C\}$ where $\hat{C}$ is the equivalence class of cyclic subsemigroups containing the subsemigroup $C \in S_C$.
Proof. S_C be a set and an equivalence class (in this case $S/\asymp$) in S_C divides S_C into pairwise disjoint

Remarks 3.3. The system of all cyclic subsemigroups is $S_C \coloneqq \{\langle a\rangle | a \in S\}$ (see the lemma 2.9). This means that the partitioning splits up the whole original semigroup S into the union of pairwise disjoint components

$$S = \bigcup_{a\in S} \widehat{\langle a\rangle}$$

Each such component contains a maximal set of equivalent sub-cyclic semigroups generated by the points of the component. Let $\hat{S}_C = S/\asymp$ be the set of equivalence classes with respect to the connectedness relation. Then let $\forall a, b \in S, \langle a\rangle \in \hat{a}, \langle b\rangle \in \hat{b}$.

Remarks 3.4. (On Cosets of Cyclic Semigroups). The definition of cosets given in section 1.8. serves as a very useful tool for partitioning sets especially where there is a single operation on the set. In the next section we will use the same concept but specialised for Cyclic Semigroups to present an analogue of the Normal Subgroup for semigroup.

Definition 3.5. (Cosets). Let S be a semigroup and let $S_C = \{\langle a\rangle \mid a \in S\}$ be the set of its cyclic semigroups. Let $c \in S_C,\ C \coloneqq \langle c\rangle$ be a cyclic semigroup and let $b \in S$. Then
$\theta C \coloneqq b * C = \theta\langle c\rangle = \{\theta * c^k \mid k \in \mathbb{N}\}$ is a left coset formed by θ and C. Similarly
$C\theta \coloneqq C * \theta = \langle c\rangle\theta = \{c^k\theta \mid k \in \mathbb{N}\}$ is a right coset formed by θ and C.

Theorem 3.6. (An Analogue of Normal Subgroup [5]). Let S be a semigroup with a neutral element and let $C \subset S$ be subsemigroups with neutral element. Then the following hold:
1. $C * C = \{\beta * \gamma \mid \beta, \gamma \in C\} = C;$
2. Let $C \subset H \subset S$ where H, C are cyclic subsemigroups, $C = \langle\tau\rangle, \tau \in H$. Then the cosets fulfill $\beta * C = C * \beta,\ \forall \beta \in H$. In other words, using terminology of group theory, C is a normal subsemigroup of H.
3. Let $C = \langle\tau\rangle, \tau \in H$. Let $C_k \coloneqq \{(\tau^k)^s = \tau^{k*s} \mid 0 \le s < \infty\} \subset C$. Then

$$(\tau^p * C) \cap (\tau^q * C) = \emptyset\ ;\ \forall\ 0 \le p,\ q < k, p \ne q.$$

Proof. (i). Since $e \in C,\ C = e * C \subset \{a * b \mid a, b \in C\} = C * C \subset C \Longrightarrow C * C = C$.
(ii). Since H is a cyclic subsemigroup of S, it is a commutative semigroup. A subsemigroup

$C \subset H$ is then a commutative subsemigroup of H also. Then the left cosets are equal to the right ones: $a * C = C * \beta, \forall \beta \in H$ by the commutativity of H.

(iii). Let $C = \langle \tau \rangle = \{\tau^p \mid 0 \le p < \infty\}, t \in H$. Let

$$C_k := \{(\tau^k)^s = \tau^{k*s} \mid 0 \le s < \infty\} \subset C.$$

The congruence relation of $mod\ k$ equivalence is an equivalence relation. It partitions the natural numbers into k pairwise disjoint equivalence classes

$$N_{k,q} := \{p \mid q \equiv p \ (mod\ k)\}, 0 \le q < k.$$

Accordingly, the cosets

$\{\tau^p C_k = \{\tau^p (\tau^k)^s \mid \tau^p(\tau^k)^s = \tau^{p+ks}, 0 \le s < \infty\} = \{\tau^\alpha \mid \alpha \in N_{k,p}\}, 0 \le p < k.$

Hence let $\tau^p * C_k = \{\tau^\alpha \mid \alpha \in N_{k,p}\}$, and $\tau^q * C_k = \{\tau^\alpha \mid \alpha \in N_{k,q}\}, \forall 0 \le p, q < k, p \ne q$. Then

$$N_{k,p} \cap N_{k,q} = \emptyset \Longrightarrow (\tau^p * C_k) \cap (\tau^q * C_k) = \emptyset; \forall 0 \le p, q < k, p \ne q. \quad ■$$

3.1 Intersections of subsemigroups with cyclic semigroups

The cyclic subsemigroups were defined in section 2.1 and their commutative property explained in section 2.2. This together with the existence of an explicit formula for its elements extends the study of independence/generating sets to $N - modulus's$.

Lemma 3.8. Let S be a semigroup and let $\emptyset \ne H, C \subset S$ be subsemigroups, let $C = \langle a \rangle$ be a cyclic subsemigroup. Then $T := H \cap C$ is a commutative subsemigroup of S with connected cyclic subsemigroups.

Proof. It was proved in theorem 1.6 that non-empty intersections of subsemigroups are subsemigroups hence T is a subsemigroup of C. Since C is commutative, then the intersection T of H and C is a commutative subsemigroup of $H, a\ subset\ of\ S$. Let $\lambda, \mu \in T \Longrightarrow \lambda = a^p, v = a^q$. Then $\lambda^q = a^{p*q} = a^{q*p} = \mu^p$ proves that $\langle \lambda \rangle \cap \langle \mu \rangle \ne \{\emptyset\}$ and $\langle \lambda \rangle \cap \langle \mu \rangle \ne \{e\}$.

Hence the cyclic subsemigroups of T form a connected system. ■

Remark 3.9. It was mentioned in the Remark 2.7, that the connected families (see definition 2.3) form the equivalence classes as stated in the lemma 2.9, each equivalence class is represented by a free cyclic group. Therefore, in these classes we have a free semigroup operation instead of the original semigroup operation. The case discussed in (the lemma of) 2.13 is different: here we have truly commutative subsemigroups of the original semigroup *S*.

Theorem 3.10. *Let* $\{S, *\}$ be a semigroup. Let $H, C \subset S$, $H \cap C \ne \{\emptyset\}$ and $H \cap C \ne \{e\}$ be subsemigroups and let $C := \langle s \rangle$, $s \in S$ be a cyclic subsemigroup. Then $\emptyset \ne T := H \cap C$ fulfills the following:

1. $\forall \tau \in T, \exists! p \in N$ (there exists ***a unique*** $p\ in\ N$) such that $\tau = \alpha^p$.
2. $\forall (\emptyset \ne A) \subset T, \exists p_{min}(A) := min\{p \mid \alpha^p \in A\}$;

Proof. 1. C is a cyclic subsemigroup, so $\forall a \in C, \exists p \in N$ such that $\tau = \alpha^p$. Hence $\forall \tau \in T := H \cap C \subset C, \exists p \in N$ such that $\tau = \alpha^p$.

2. Since $\emptyset \neq A \subset T,\ \emptyset \neq \tilde{A} := \{p \mid \alpha^p \in A\} \subset N$. Since the set of natural numbers is a well ordered set with its natural ordering, any of its non-empty subset has a minimum. Hence $\tilde{A}$ has a minimum. ■

4 Algorithm for generating systems of semigroup

We have formed enough tools in the previous section for creating algorithms for generating set of semigroups. We conclude with one algorithm for the minimal generating set of a semigroup.

4.1. **Important Corollary** (Algorithm for Minimal Generating Set).
Let S, H, C and T be semigroups as defined in definition 3.7. Let M be initialized as $M \subset N$ and Let $M = \emptyset$; Let $N \subset T$ and let N be initialized as $N = \emptyset$; Let $\Gamma \subset C$ and let $\Gamma = T$. Let $n \in \mathbb{N}$ be the counter, set to $n = 0$ for starting. Then let us consider the following algorithm:
Step 1. If $\Gamma \neq \emptyset$; then goto Step 2 else [exit].
Step 2. Let $n = n + 1$, $k_n = p_{min}(\Gamma)$, $M = M \cup \{k_n\}$ and $N = N \cup \{s^{k_n}\}$ goto step 3.
Step 3. $\Gamma = \Gamma \backslash \langle N \rangle$ goto Step 1.
If this algorithm stops after finite steps then we get a set of powers which are independent

$$\forall p \in M, s^p \notin \langle \{s^t | t \in M, t < p\} \rangle$$

The value of n, $1 \leq n \leq \infty$ gives the total number of minimums recorded. Let a set $A \subset C$ be given. Then the ascending sequence M defined by the algorithm with $\Gamma = A$ is unique.
Proof. 1. Let $P := \{t \mid s^t \in T\} \subset \mathbb{N}, P \neq \emptyset$. Since the set of natural numbers is well ordered with its natural order relation, P has a minimum $m_j \in P$ in the $j - th$ loop of the algorithm. If the minimum is removed from P, $P = P \backslash \langle N \rangle$ and $P \neq \emptyset$, then the new set P has a new minimum $m_{j+1} \in P, m_{j+1} > m_j$. Hence the algorithm will produce a finite or infinite strictly ascending sequence of natural numbers.
2. Let $0 < j \leq n$ be selected. Then let $M_j := \{k_t | \ k_t \in M, 1 \leq t \leq j\} \subset M$ and $N_j := \{s^{k_t} \mid s^{k_t} \in N, 1 \leq t \leq j\} \subset N$. M_j carry the sequence of collected minimum exponent values and N_j carry the set of corresponding sequence of members of T. The sub-semiring $\langle N_j \rangle \supset N_j$ is generated by N_j.

$$\langle N_j \rangle = \left\{ \prod_{t=1}^{j} (s^{p_t})^{\propto_{p_t}} | p_t \in M_j, \propto_{p_t} \in N \right\} = \left\{ \prod_{t=1}^{j} s^{p_t \propto_{p_t}} | p_t \in M_j, \propto_{p_t} \in N \right\}$$

$$= \left\{ s^{\eta} | \ \eta := \sum_{\substack{p \in M_j \\ \alpha_p \in N}} p \times \alpha_p \right\}$$

This shows that the semi-ring $\langle N_j \rangle$ is isomorphic to a positive cone of a sub-module of Z-module over the ring of Z:

Definition 4.1.1. (Positive Cone in the Z −module over Z). Let $M \subset \mathbb{N} \subset Z$ be a countable set of positive coprime numbers (as defined in [8]). Then

$$M_M := \left\{ s^{\eta} | \eta := \sum_{\substack{p \in M_j \\ \alpha_p \in N}} p \times \alpha_p \right\}$$

is a positive cone in the Z −module over Z.

3. Let a set $A \subset C$ be a subsemigroup. The ascending sequence $M := \{k_j\}$ defined by the algorithm in corollary of section 4.1 is unique. We prove this by mathematical induction.

Let $\{a_u\}_{u=1}^{n}$ and $\{b_v\}_{v=1}^{m}$be two sequences such that with $P = A$, $a_1 = b_1 := min\,\{p \mid s^p \in P\}$. Then $\langle\{a_1\}\rangle = \langle\{b_1\}\rangle \Longrightarrow P\backslash\,\langle\{a_1\}\rangle = P\backslash\langle\{b_1\}\rangle = P$ hence $a_2 = min\,\{p \mid s^p \in P\} = b_2$.

Assume that we proved that $a_t = b_t$, $\forall 1 \leq t \leq q$. Then $\langle\{a_t\}_{t=1}^{q}\rangle = \langle\{b_t\}_{t=1}^{q}\rangle$

Hence $P\backslash\{a_t\}_{t=1}^{q} = P\backslash\{b_t\}_{t=1}^{q} \Longrightarrow a_{q+1} = b_{q+1} = min\{p \mid s^p \in P\}$. Hence $a_q = b_q, \forall t$ until $P = \emptyset$.

This happens at the same subscript n since $P\backslash\{a_t\}_{t=1}^{q} = P\backslash\{b_t\}_{t=1}^{q}$, $1 \leq p \leq n$. Hence, we proved the statement.

4. When the algorithm stops, then $A\backslash\langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle = \emptyset$;
   Hence $A = \langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle$ holds.

5. If $\bigcap_{p=1}^{n}(C\backslash\langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle) = \emptyset$ then $T = C \Leftrightarrow C = H$

This follows from the complementation of the intersection with respect to C:

$$\emptyset' = C = \left(\bigcap_{p=1}^{n}(C\backslash\langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle)\right)' = \bigcup_{p=1}^{n}(C\backslash\langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle)'$$

$$\bigcup_{p=1}^{n}\langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle = \langle\{s^{a_t} \mid 1 \leq t \leq n\}\rangle = T$$

■

Remark 4.1.2. As a direct consequence of the statements of the theorem about $M_j \subset M$ and $N_j \subset N$, $1 \leq j \leq n$ and their definition in corollary 4.1, we have the following theorem.

Theorem 4.2. Let S be a semigroup with a neutral element $e \in S$. Let $H, C \subset S$ be subsemigroups, and let C be a cyclic subsemigroup. Let $T := H \cap C =: T \notin \{\emptyset, \{e\}\}$. Let $M \subset \mathbb{N}, N \subset T$ be sequences produced by the algorithm in corollary of section 4.1. Let $n \in \mathbb{N}$ be the number of sequence elements in M *or* N and let $M_j \subset M,\ N_j \subset N,\ 1 \leq j \leq n$ be as defined in the point 2 of proof of the corollary 4.1. Then the following hold.

1. $\bigcup_{j=1}^{n}\langle N_j\rangle = \langle N\rangle = T$

2. $\bigcap_{j=1}^{n}\langle N_j\rangle' = \emptyset \Leftrightarrow C = T \subset H$, hence C is dependent on H

Proof. Set $M_j := \{k_t | k_t \in M, 1 \leq t \leq j\} \subset M$ and $N_j := \{s^{k_t} | s^{k_t} \in N, 1 \leq t \leq j\} \subset N, 1 \leq j \leq n$ as defined in corollary 4.1. The proofs of 1 and 2 follow from the concluding part of the proof of corollary 4.1.

■

5 Conclusion and recommendation

The algorithmic approach for determining minimal generating set discussed here is being improved upon with alternative characterizations to handle the bases (Independent generating set) problem of semigroups. This circumvents the task in utilizing equivalences for partitioning set into generating subsystems as given in section 3 which is also what the use of the popular green's equivalences aims at achieving. In [11], algorithmic approach has been used for the basis of Additive Semigroup of Integers while [3] is about Countable systems of semigroup and [4] about Infinite Semigroups. A newer and interesting characterization in [12] can be applied to Classes of Semigroups. We shall dwell on bases of classes of semigroups in later articles.

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