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Effect of thermal radiation and chemical reaction on unsteady MHD plane – Poiseuille flow of fourth-grade fluid in horizontal parallel plates

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Abstract

A fourth-grade fluid is an important subclass of differential type that is capable of describing shear thinning and shear thickening effects (examples are ketchup, blood, paint, cream, nail polish, etc.). This sort of model is used to explain the flow behaviour of non-Newtonian fluids which are considered vital and applicable in many industrial production processes such as in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films, etc. There is a dearth of knowledge in this area, therefore, this study investigated the thermal radiation and chemical reaction effects on unsteady magnetohydrodynamics plane – poiseuille flow of fourth-grade fluid in horizontal parallel plates channel. The equations that governed the flow were the momentum equation; which is a coupled nonlinear differential equation, the energy equation and concentration equation. The partial differential equations were solved using He – Laplace method which is a combination of Homotopy perturbation method and Laplace transformation method. The findings of the study revealed that: (i) Velocity and temperature fields rise due to the increment of thermal radiation parameter. (ii) For upsurging data of chemical reaction, velocity and concentration fields diminish. (iii) Velocity profile goes up when third and fourth-grade parameters get to raise; Velocity and skin friction fields decline due to the increment of magnetic parameter. (iv) Increasing Prandtl number tend to diminish the velocity and temperature profiles. (v) Strong values of Schmidt number decrease the boundary layer of the Sherwood number field. The results of this work are applicable to industrial processes such polymer extrusion of dye, draining of plastic films etc.

Keywords and phrases: thermal radiation; chemical reaction; Plane-Poiseuille flow; fourth-grade fluid; He-Laplace Scheme

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1 Introduction

The study of thermal radiation and chemical reaction is of most realistic significance to engineers and scientists because of its universal incidence in many branches of science and engineering. This phenomenon plays a significant role in chemical industry, power and cooling industry for dying, evaporation, energy transfer in cooling tower and flow in s desert cooler, etc. [1].

Thermal radiation is the process by which energy in the form of electromagnetic radiation, is emitted by a heated surface in all directions and travels directly to its point of absorption at the speed of light. While, Chemical reaction is a process that involves rearrangement of the molecular or ionic structure of a substance, as distinct from a change in physical form or nuclear reaction. There are two types of such reactions namely homogeneous reaction which occurs uniformly throughout a given phase of a flow and heterogeneous reaction which takes place in a particular region or within the boundary of a phase [2]. And Magneto hydrodynamics (MHD) is the study of the magnetic properties and behaviour of electrically conducting fluids. Example of such magneto fluids include plasma, liquid metals, salt water and electrolytes. The fundamental concept of MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. And, couette flow is the flow of a viscous fluid in the space between two surfaces, one of which is moving tangentially relative to the other. The configuration often takes the form of two parallel plates.

The fourth-grade fluid flow model is an exceptional model which is being used to explain the flow attitude of non-Newtonian fluids, which are considered vital and applicable in many industrial producing processes such in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films etc.

A vast analysis of non-Newtonian fluids problems has been done by many researches, amongst whom are [3 – 14]. Researchers that considered the effects of thermal radiation and chemical reaction include [15], who investigated the unsteady flow of an electrically conducting incompressible non-Newtonian viscoelastic fluid through a porous medium filled in a vertical porous channel in the presence of transverse magnetic field. The fluid and the channel rotate as a solid body with constant angular velocity about an axis perpendicular to the planes of the plates. The effects of thermal radiation and chemical reaction are taken into account embedded with slip boundary condition. The closed-form analytical solutions are obtained for momentum, energy and concentration equations. The influences of the various parameters entering into the problem in the velocity, temperature and concentration field are discussed with the help of graphs. Also, numerical values of physical quantities, such as skin friction coefficient, Nusselt number and Sherwood number are presented in tabular form. [16], who presented an analytical solution to the problem of plane Poiseuille flow in a channel filled with a porous medium. [17], who analysed heat and mass transfer characteristics of naturally corrective hydro-magnetic flows of fourth grade radiative fluid resulting from vertical porous plate. They considered non-linear order chemical reaction and heat generation with thermal diffusion. The complete fundamental equations were transformed into dimensionless equations by implementing finite difference scheme explicitly.

Others and recent investigations were made by [18], who carried out an analysis for unsteady magnetohydrodynamic (MHD) flow of a generalized third grade fluid between two parallel plates. The fluid flow was as a result of the plate oscillating, moving and pressure gradient. Three flow problems were investigated, namely: Couette, Poiseuille and Couette-Poiseuille flows and a number of nonlinear partial differential equations were obtained which were solved using the He-Laplace method. Expressions for the velocity field, temperature and concentration fields were given for each case and finally, effects of physical parameters on the fluid motion, temperature and concentration were plotted and discussed. They found that an increase in the thermal radiation parameter increases the temperature of the fluid and hence reduces the viscosity of the fluid while the concentration of the fluid reduces as the chemical reaction parameter increases. [19] presented an analysis to explore analytical treatment for the computation of Poiseuille flow of a micropolar fluid in a channel placed in between two horizontal parallel plates. Both the plates were placed at constant wall temperatures. Therefore, the flow region was portioned into two different zones named zone I and zone II. Eringen's micropolar fluid flow phenomena took place by assuming no-slip conditions at the interface. Suitable nondimensional variables were imposed for the transformation of governing equations. Analytical treatment was carried out by employing the in-house symbolic command using the MAPLE software. The behavior of several contributing parameters such as material parameters, the couple stresses for both the zones on the velocity, and microrotation profiles were investigated and presented via graphs. The volume flow rate was also calculated and presented in tabular form. The major outcomes of the results were presented as the higher the Reynolds number, the rate increases significantly. The profile was tiled near the central region with a pick starting from the lower plate region to the central region in zone I and retards from the central region to the upper plate in the zone II, and the profiles of angular momentum seem to be symmetric in nature about the central region that was shown in both the zones. And, [20] investigated the unsteady MHD flow of fourth-grade fluid in horizontal parallel plates channel. The upper plate was oscillating and moving while the bottom plate remained stationary. Solutions for momentum, energy and concentration equation were obtained by the He-Laplace scheme. The effect of various flow parameters controlling the physical situation is discussed with the aid of graphs. Significant results from this study, showed that velocity and temperature fields increase with the increase in thermal radiation parameter, while the velocity and concentric fields decrease with increase in chemical reaction parameter. Furthermore, velocity, temperature and concentric fields decrease with the increase in suction parameter.

However, in the above aforementioned investigations, the effect of thermal radiation and chemical reaction on the unsteady MHD plane – poiseuille flow of fourth-grade fluid in a horizontal parallel plates have not been paid attention. We shall therefore, consider this where both the lower plate and the upper plate are fixed.

2 Methodology/Model formulation

We consider an unsteady flow of an electrically incompressible fourth grade fluid between two parallel plates distance h apart and subjected to a uniform transverse magnetic field B_0. Here, both plates are fixed (stationary) and with constant fluid speed u. The thermally radiative and chemically reactive flow is heading x – direction along infinite porous plate with heat generation.

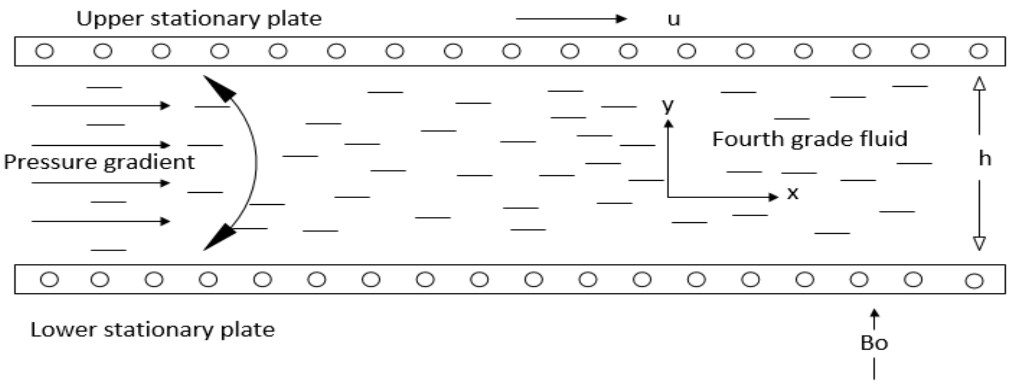

Figure 1: Physical Configuration of the Flow

The state of this fluid is determined by the history of the deformation gradient without a preferred reference configuration. Its constitute equation can be written as

$$T(x,t) = -PI + f_{s=0}^{\infty}\left(F_t^t(s)\right) \tag{2.1}$$

Where PI is the undetermined part of the stress – tensor, F is the deformation gradient and f is the functional.

[21] prescribed different sorts of incompressible fluid category n as viscous fluid agreeing on [22 – 23]. Incompressible fluid of differential type of grade n is the simple fluid obeying the constitutive equation

$$T = -PI + \sum_{j=1}^{n} S_j \tag{2.2}$$

obtained by asymptotic expansion of the functional in equation (2.1) through a retardation parameter α. For $n = 4$ as elucidate by [22], [23] and [17], the first four (4) tensors S_j are given by

$$S_1 = \mu A_1 \tag{2.3}$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{2.4}$$

$$S_3 = \beta_1 A_3 + \beta_2(A_1A_2 + A_2A_1) + \beta_3(trA_1^2)A_1 \tag{2.5}$$

$$S_4 = \gamma_1 A_4 + \gamma_2(A_3A_1 + A_1A_3) + \gamma_3 A_2^2 + \gamma_4(A_2A_1^2 + A_1^2A_2) + \gamma_5(trA_2)A_2 + \gamma_6(trA_2)A_1^2 + [\gamma_7 trA_3 + \gamma_8 tr(A_2A_1)]A_1 \tag{2.6}$$

Where, μ is the coefficient of shear viscosity, $\alpha_i(i = 1,2)$, $\beta_i(i = 1,2,3)$ and $\gamma_i(i = 1(1)8)$ are material constants. The A_n are the Rivlin – Ericksen tensors defined by the recursion relation

$$A_n = \frac{d}{dt}A_{n-1} + A_{n-1}L + L^T A_{n-1}, \; {}_{n>1} \tag{2.7}$$

$$A_1 = L + L^T \tag{2.8}$$

where $L = \nabla V$, $\frac{d}{dt}$ is the material time derivative and $V$is the velocity.

The thermally radiative and chemically reactive flow is heading x – direction along infinite porous plate with heat generation. Here, U_0 is the uniform velocity, T_∞ and C_∞ are the fluid temperature and concentration.

Under the above consideration, the equations that described the physical circumstances are

(I) The Momentum equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} + \frac{\alpha_1 \nu}{\rho}\frac{\partial^3 u}{\partial y^2 \partial t} + \frac{\beta_1 \nu^2}{\rho}\frac{\partial^4 u}{\partial y^2 \partial t^2} + \frac{6(\beta_2+\beta_3)}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \frac{\gamma_1 \nu^3}{\rho}\frac{\partial^5 u}{\partial y^2 \partial t^3} + \frac{2\nu(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_7+\gamma_8)}{\rho C_p}\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2 \partial t}\right] - \frac{\sigma B_0^2}{\rho C_p}u + g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty) - \frac{\nu}{k}u \tag{2.9}$$

(II) The Energy equation:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{k}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial y} \tag{2.10}$$

And,

(III) The Concentration equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} - K_c(C - C_\infty) \tag{2.11}$$

Where u is the fluid velocity, T is the temperature, C is the species concentration equationFrom equation (2.10), q_r is the radiative heat flux define as

$$\frac{\partial q_r}{\partial y} = 4\alpha^2(T_w - T_\infty) \tag{2.11i}$$

We note that, when $\gamma_i = 0$, the fourth grade model reduces to the third grade model. When $\beta_i = 0$, the third grade model reduces to second grade model. When $\alpha_i = 0$, $\beta_i = 0$ and $\gamma_i = 0$ then the model reduces to classical Navier – Stoke equation.

The initial and boundary conditions (Idowu and Sani (2019)) are

$$u(y,t) = h + h\cos(\omega y)\,, \theta(y,t) = h + h\cos(\omega y), C(y,t) = h + h\cos(\omega y)\ \ at\ y > 0, t = 0$$
$$u(y,t) = 1, \theta(y,t) = 1, C(y,t) = 1\ at\ y = 0\ and\ t \geq 0$$
$$u(y,t) \to \infty, \theta(y,t) \to \infty, C(y,t) \to \infty\ as\ y \to \infty\ at\ t > 0 \tag{2.12}$$

Where u is the fluid velocity, T is the temperature and C is the species concentration equation, q_r is the radiative heat flux, ρ is the density of the fluid, C_p is the heat capacity, B_0 is the external magnetic field.

In order to transform equations (2.10) – (2.13), we use the following dimensionless parameters

$$u^* = \frac{u}{U_0},\ p^* = \frac{p}{\mu U_0^2}, t^* = \frac{tU_0^2}{\nu}, G_r = \frac{g\beta_T(T_w - T_\infty)\nu}{U_0^3}, G_c = \frac{g\beta_C(C_w - C_\infty)\nu}{U_0^3}, Ha^2 = \frac{\sigma B_0^2 \nu}{\rho U_0^2}, Da = \frac{KU_0^2}{h^2}, S_c = \frac{D}{\nu}, y^* = \frac{yU_0}{\nu}, x^* = \frac{x}{h}, h = \frac{U_0}{\nu}\ P_r = \frac{kU_0^2}{\nu^2},, \theta = \frac{T - T_0}{T_w - T_\infty}, C^* = \frac{C - C_0}{C_w - C_\infty}, \delta = \frac{4\alpha^2 U_0^2}{\rho c_p \nu}, \alpha = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \beta_a = \frac{\beta_1 U_0^4}{\rho \nu^3}, \beta_b = \frac{(\beta_2 + \beta_3)U_0^4}{\rho \nu^3}, \gamma_a = \frac{\gamma_1 U_0^6}{\rho \nu^3}, \gamma_b = \frac{2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)U_0^6}{\rho \nu^4}, Da = \frac{kU_0^2}{\nu^2}, K_r = \frac{K_c \nu}{U_0^2} \tag{2.13}$$

Substituting equation (2.13) into equations (2.9) – (2.12) and by dropping the asterisks, we have the following:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b \left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - \left(Ha + \frac{1}{Da}\right)u + G_r\theta + G_c C \tag{2.14}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} + \delta\theta \tag{2.15}$$

$$\frac{\partial C}{\partial t} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} - K_r C \tag{2.16}$$

While, the initial and boundary conditions become

$$\left.\begin{array}{c} u(y,t) = 1 + \cos\omega y\,, \theta(y,t) = 1 + \cos\omega y\,, C(y,t) = 1 + \cos\omega y\ \ at\ t = 0\ for\ 0 \leq y \geq 1 \\ u(y,t) = 1, \theta(y,t) = 1, C(y,t) = 1\ at\ y = 0\ for\ t \geq 0 \\ u(y,t) \to \infty, T(y,t) \to \infty, C(y,t) \to \infty\ as\ y \to \infty\ for\ t > 0 \end{array}\right\} \tag{2.17}$$

3 Analysis of the model

In this section we employ the He – Laplace scheme to solve equations (2.14) to (2.16) subjects to the initial and boundary conditions (2.17).
Since equation (2.16) is a coupled non – linear partial differential equation, we have to solve equations (2.15) and (2.16) first.

Now applying Laplace transform on equation (2.16), we have;

$$L\left\{\frac{\partial C}{\partial t}\right\} = \frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\} \tag{3.1}$$

Applying the initial condition and dividing through by s and rearranging, we obtain;

$$L\{C(y,t)\} = \frac{1+\cos(\omega y)}{s} + \frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\} \tag{3.2}$$

Taking the inverse Laplace transform of both sides of equation (3.2), gives;

$$C(y,t) = 1 + \cos(\omega y) + L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\}\right] \tag{3.3}$$

Applying the Homotopy perturbation technique, equation (3.3) yields

$$\sum_{n=0}^{\infty} P^n C_n(y,t) = e^{-y} + P\left[L^{-1}\left\{\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\left\{K_r C\right\}\right\}\right\}\right] \tag{3.4}$$

Comparing the coefficients of the like powers of $'P'$, the following approximations were obtained;

$$P^0: C_0(y,t) = 1 + \cos(\omega y) \tag{3.5}$$

$$P^1: C_1(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_0}{\partial y^2}\right\} - L\{K_r C_0\}\right\}\right] = L^{-1}\left\{\frac{1}{S_c}\left(\frac{\omega^2\cos(\omega y)}{s^2}\right) - K_r\left(\frac{1+\cos(\omega y)}{s^2}\right)\right\} = \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\right)t \tag{3.6}$$

$$P^2: C_2(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_1}{\partial y^2}\right\} - L\{K_r C_1\}\right\}\right] =$$

$$L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\left(\frac{\omega^4}{S_c}\cos(\omega y) + \omega^2 K_r\cos(\omega y)\right)t\right\} + L\left\{K_r\left(\frac{\omega^2}{S_c}\cos(\omega y) + \right.\right.\right.\right.$$

$$K_r^2\left(1+\cos(\omega y)\right)t\Big\}\Big\}\Big] = \left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} \quad (3.7)$$

$$P^3: C_3(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c}L\left\{\frac{\partial^2 C_2}{\partial y^2}\right\} - L\{K_r C_2\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{L\left\{\left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^2}{2!}\right\}\right\}\right] = \left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} \quad (3.8)$$

$$P^4: C_4(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c}L\left\{\frac{\partial^2 C_3}{\partial y^2}\right\} - L\{K_r C_3\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{L\left\{\left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^3}{3!}\right\}\right\}\right] = \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} \quad (3.9)$$

Therefore, in view of equations (3.5), (3.6), (3.7), (3.8) and (3.9) the solution to equation (2.16) is,

$$C(y,t) = C_0(y,t) + C_1(y,t) + C_2(y,t) + C_3(y,t) + C_4(y,t)\cdots \quad (3.10)$$

$$C(y,t) = 1 + \cos(\omega y) + \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\right)t + \left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} + \cdots \quad (3.11)$$

Next, we consider equation (2.15), which is reaaranged to give;

$$\frac{\partial\theta}{\partial t} = \frac{1}{P_r}\frac{\partial^2\theta}{\partial y^2} + \delta\theta \quad (3.11i)$$

Now applying Laplace transform on equation (3.11i).), we get,

$$L\left\{\frac{\partial\theta}{\partial t}\right\} = \frac{1}{P_r}L\left\{\frac{\partial^2\theta}{\partial y^2}\right\} + L\{\delta\theta\} \quad (3.12)$$

Applying the initial condition and dividing through by s and rearranging we obtain;

$$L\{\theta(y,t)\} = \frac{1+\cos(\omega y)}{s} + \frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta}{\partial y^2}\right\} + L\{\delta\theta\}\right\} \quad (3.13)$$

Taking the inverse Laplace transform of both sides of equation (3.13) gives,

$$\theta(y,t) = 1 + \cos(\omega y) + L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta}{\partial y^2}\right\} + +L\{\delta\theta\}\right\}\right] \quad (3.14)$$

Applying the Homotopy perturbation technique on equation (3.14), yields

$$\sum_{n=0}^{\infty}P^n\theta_n(y,t) = e^{-y} + P\left[L^{-1}\left\{\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta}{\partial y^2}\right\} + L\{\delta\theta\}\right\}\right\}\right] \quad (3.15)$$

Comparing the coefficients of the like powers of $'P'$in equation (3.14), the following approximations are obtained;

$$P^0: \theta_0(y,t) = 1 + \cos(\omega y) \quad (3.16)$$

$$P^1: \theta_1(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta_0}{\partial y^2}\right\} + L\{\delta\theta_0\}\right\}\right] = L^{-1}\left\{\frac{1}{s^2 P_r}(-\omega^2\cos(\omega y)) + \frac{\delta}{s^2}(1+\cos(\omega y)\right\} = \left(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1+\cos(\omega y)\right)t \quad (3.17)$$

$$P^2: \theta_2(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta_1}{\partial y^2}\right\} + L\{\delta\theta_1\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\left(\frac{\omega^4}{P_r}\cos(\omega y) - \delta\omega^2\cos(\omega y)\right)t\right\} + \delta L\left\{\left(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1+\cos(\omega y)\right)t\right\}\right\}\right] = \left(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2\right)\frac{t^2}{2!} \quad (3.18)$$

$$P^3: \theta_3(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta_2}{\partial y^2}\right\} + SL\left\{\frac{\partial\theta_2}{\partial y}\right\} + L\{\delta\theta_2\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\left(\frac{-\omega^6}{P_r^2}\cos(\omega y) + \frac{2\delta\omega^4}{P_r}\cos(\omega y) - \delta^2\omega^2\cos(\omega y)\right)\frac{t^2}{2!}\right\} + \delta L\left\{\left(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^4}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2\right)\right\}\frac{t^2}{2!}\right\}\right] = \left(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3\right)\frac{t^3}{3!} \quad (3.19)$$

$$P^4: \theta_4(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\frac{\partial^2\theta_3}{\partial y^2}\right\} + L\{\delta\theta_2\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{P_r}L\left\{\left(\frac{\omega^8}{P_r^3}\cos(\omega y) - \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) + \frac{3\delta^2\omega^4}{P_r^2}\cos(\omega y) - \delta^3\omega^2\cos(\omega y)\right)\frac{t^2}{3!}\right\} + \delta L\left\{\left(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3\right)\right\}\frac{t^2}{3!}\right\}\right] = \left(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4\right)\frac{t^4}{4!} \tag{3.20}$$

Therefore, in view of equations (3.16), (3.17), (3.18), (3.19) and (3.20), the solution to (2.15) is,

$$\theta(y,t) = \theta_0(y,t) + \theta_1(y,t) + \theta_2(y,t) + \theta_3(y,t)\cdots$$

$$\theta(y,t) = 1 + \cos(\omega y) + \left(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1+\cos(\omega y)\right)t + \left(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4\right)\frac{t^4}{4!} + \cdots \tag{3.21}$$

Finally, we now solve equation (2.14), which is rearranged to give

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2\partial t} + \beta_a\frac{\partial^4 u}{\partial y^2\partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_a\frac{\partial^5 u}{\partial y^2\partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\right] - l_2 u + G_r\theta + G_c C \text{ where, } Ha + \frac{1}{Da} = l_2 \tag{3.22}$$

Applying the Laplace transform on both sides of equation (3.22) gives

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2\partial t} + \beta_a\frac{\partial^4 u}{\partial y^2\partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_a\frac{\partial^5 u}{\partial y^2\partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\right] - l_2 u + G_r\theta + G_c C\right\} \tag{3.23}$$

$$\text{But, } L\left\{\frac{\partial u}{\partial t}\right\} = sL\{u(y,t)\} - u(y,0) \tag{3.24}$$

Hence,

$$L\{u(y,t)\} = \frac{u(y,0)}{s} + \frac{1}{s}\, L\Bigg\{-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\,\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a\,\frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a\,\frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + G_r\theta + G_c C\Bigg\} \tag{3.25}$$

Taking the inverse Laplace transform of both sides of equation (3.25), we have;

$$L^{-1}\big\{L\{u(y,t)\}\big\} = L^{-1}\Bigg\{\frac{u(y,0)}{s} - \frac{\partial p}{\partial x} + \frac{1}{s}\, L\Bigg\{\frac{\partial^2 u}{\partial y^2} + \alpha\,\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_{\mathrm{a}}\,\frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_{\mathrm{a}}\,\frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + \frac{G_r}{s}\Bigg(1 + \cos(\omega y) + \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\right) t + \left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!}\Bigg) + \frac{G_c}{s}\Bigg(1 + \cos(\omega y) + \left(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1 + \cos(\omega y)\right) t + \left(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4\right)\frac{t^4}{4!}\Bigg)\Bigg\}\Bigg\} \tag{3.26}$$

or,

$$u(y,t) = 1 + \lambda + \cos(\omega y) + G_c(1+\cos(\omega y)) + G_r(1+\cos(\omega y)) + \Bigg(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1+\cos(\omega y) - \frac{\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\Bigg) t + \Bigg(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2 + \frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\Bigg)\frac{t^2}{2!} + \Bigg(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3 - \frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\Bigg)\frac{t^3}{3!} + \Bigg(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4 + \frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\Bigg)\frac{t^4}{4!} + L^{-1}\Bigg\{\frac{1}{s}\, L\Bigg\{\frac{\partial^2 u}{\partial y^2} + \alpha\,\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a\,\frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_{\mathrm{a}}\,\frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u\Bigg\}\Bigg\} \tag{3.27}$$

Applying the Homotopy perturbation method to equation (3.27), gives

$$\sum_{n=0}^{\infty} P^n u_n(y,t) = 1 + \lambda + \cos(\omega y) + G_c(1 + \cos(\omega y)) + G_r(1 + \cos(\omega y)) + \Big(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1 + \cos(\omega y) - \frac{\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))\Big)t + \Big(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2 + \frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\Big)\frac{t^2}{2!} + \Big(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3 - \frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\Big)\frac{t^3}{3!} + \Big(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4 + \frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\Big)\frac{t^4}{4!} + P\Bigg(L^{-1}\Big\{\frac{1}{s}\,L\Big\{S\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2\partial t} + \beta_a\frac{\partial^4 u}{\partial y^2\partial t^2} + \beta_b H_a(u_n) + \gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3} + \gamma_b\Big[2H_b(u_n) + H_c(u_n)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y} + \Big(\frac{\partial u}{\partial y}\Big)^2\frac{\partial^3 u}{\partial y^2\partial t}\Big] - l_2 u\Big\}\Big\}\Bigg) \tag{3.28}$$

Where, $H_a(u_n), H_b(u_n)$ and $H_c(u_n)$ are the He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2$, $\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}$ and $\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}$ respectively.

The He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2$ are as follows

$$\begin{cases} H_0(u) = (u_0')^2 \\ H_1(u) = 2u_0'u_1' \\ H_2(u) = 2u_0'u_2' + (u_1')^2 \\ H_3(u) = 2u_1'u_2' \\ \vdots \end{cases} \tag{3.29}$$

The He's polynomials for $\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}$ are as follows

$$\begin{cases} H_0(u) = u_0'''u''_{0t} \\ H_1(u) = u_0'''u''_{1t} + u_1'''u''_{1t} \\ H_2(u) = u_0'''u'_{2t} + u_1'''u'_{1t} + u_2'''u'_{0t} \\ H_3(u) = u_1'''u''_{2t} + u_2'''u''_{1t} \\ \vdots \end{cases} \tag{3.30}$$

The He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}$ are as follows

$$\begin{cases} H_0(u) = (u_0')^2(u_0''u_{0t}') \\ H_1(u) = (u_0')^2(u_0''u_{1t}') + (u_0')^2(u_1''u_{0t}') + 2u_0'u_1'(u_0''u_{0t}') \\ H_2(u) = (u_0')^2(u_0''u_{2t}') + (u_0')^2(u_1''u_{1t}') + (u_0')^2(u_2''u_{0t}') + 2u_0'u_1'(u_0''u_{1t}') + 2u_0'u_1'(u_1''u_{0t}') + 2u_0'u_2'(u_0''u_{0t}') \\ \vdots \end{cases} \tag{3.31}$$

Now, comparing the like powers of P in equation (3.28) and equating their coefficients gives

$$P^0; u_0(y,t) = 1 + \lambda + \cos(\omega y) + G_c(1 + \cos(\omega y)) + G_r(1 + \cos(\omega y)) + \left(\frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1 + \cos(\omega y) - \frac{\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))\right)t + \left(\frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2 + \frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3 - \frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4 + \frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} \tag{3.32}$$

$$P^1; u_1(y,t) = L^{-1}\left\{\frac{1}{s}\, L\left\{S\frac{\partial u_0}{\partial y} + \frac{\partial^2 u_0}{\partial y^2} + \alpha\frac{\partial^3 u_0}{\partial y^2 \partial t} + \beta_a\frac{\partial^4 u_0}{\partial y^2 \partial t^2} + \beta_b(u_0')^2(u_0'') + \gamma_a\frac{\partial^5 u_0}{\partial y^2 \partial t^3} + \gamma_b[2u_0'''u''_{0t} + (u_0')^2(u_0''u_{0t}')] - l_2 u_0\right\}\right\} \tag{3.33}$$

Or,

$$u_1(y,t) = (A_{11} + \alpha A_{18} + \beta_b A_{11} + 2\gamma_b A_{15}A_{16} + A_{11}A_{13}A_{17} - l_2A_{19})t + (A_{12} + \beta_b A_{12}A_{13} + \beta_b A_{11}A_{14} + 2\gamma_b A_{16}^2 + A_{13}A_{14}A_{17} - l_2A_{10})\frac{t^2}{2!} + (2\beta_b A_{12}A_{14} + 2A_{11}A_{14}^2 - l_2A_{21})\frac{t^3}{3!} - l_2A_{22}\frac{t^4}{4!} - l_2A_{23}\frac{t^5}{5!} \tag{3.34}$$

Therefore, the solution to equation (2.14) is;

$$u(y,t) = u_0(y,t) + u_1(y,t) + u_2(y,t)\cdots \tag{3.35}$$

$$u(y,t) = A_{19} + (A_{10} + A_{11} + \alpha A_{18} + \beta_b A_{11} + 2\gamma_b A_{15}A_{16} + A_{11}A_{13}A_{17} - l_2A_{19})t + (A_{12} + A_{21} + \beta_b A_{12}A_{13} + \beta_b A_{11}A_{14} + 2\gamma_b A_{16}^2 + A_{13}A_{14}A_{17} - l_2A_{10})\frac{t^2}{2!} + (A_{22} + 2\beta_b A_{12}A_{14} + 2A_{11}A_{14}^2 - l_2A_{21})\frac{t^3}{3!} + \left(A_{23} - l_2A_{22}\frac{t^4}{4!}\right) - l_2A_{23}\frac{t^5}{5!} + \cdots \tag{3.36}$$

where,

$$A_{10} = \frac{-\omega^2}{P_r}\cos(\omega y) + \delta(1 + \cos(\omega y) - \frac{\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))$$

$$A_{11} = -\omega^2\cos(\omega y) - G_c\omega^2\cos(\omega y) - G_r\omega^2\cos(\omega y)$$

$$A_{12} = G_cK_r\omega^2\cos(\omega y) - \delta G_r\omega^2\cos(\omega y) + \frac{G_c\omega^4\cos(\omega y)}{S_c} + \frac{G_r\omega^4\cos(\omega y)}{P_r}$$

$$A_{13} = \omega^2\sin^2(\omega y) + 2G_c\omega^2\sin^2(\omega y) + 2G_r\omega^2\sin^2(\omega y) + 2G_cG_r\omega^2\sin^2(\omega y) + G_c^2\sin^2(\omega y) + G_r^2\sin^2(\omega y)$$

$$A_{14} = \frac{-2G_r\omega^4}{P_r}\sin^2(\omega y) + 2\delta G_r\omega^2\sin^2(\omega y) - \frac{2G_c\omega^4}{S_c}\sin^2(\omega y) - \frac{2G_r\omega^4}{P_r}\sin^2(\omega y) - 2\delta G_r^2\omega^2\sin^2(\omega y) - \frac{4G_cG_r\omega^4}{S_c}\sin^2(\omega y) - 2G_cG_r\omega^2\sin^2(\omega y) - \frac{G_cG_r\omega^2}{P_r}\sin^2(\omega y) - \frac{G_c^2\omega^4}{S_c}\sin^2(\omega y) - K_rG_c^2\omega^2\sin^2(\omega y) + \frac{G_r^2\omega^6}{P_r^2}\sin^2(\omega y) - \frac{2\delta G_r^2\omega^4}{P_r}\sin^2(\omega y) + \frac{2G_cG_r\omega^6}{P_rS_c}\sin^2(\omega y) + \frac{2K_rG_cG_r\omega^4}{P_r}\sin^2(\omega y) + \delta^2G_r^2\omega^2\sin^2(\omega y) + \frac{G_c^2\omega^6}{S_c^2}\sin^2(\omega y) + \frac{2K_rG_c^2\omega^4}{S_c}\sin^2(\omega y) + G_c^2K_r^2\omega^2\sin^2(\omega y)$$

$$A_{15} = \omega^3\sin^2(\omega y) + G_c\omega^3\sin^2(\omega y) + G_r\omega^3\sin^2(\omega y)$$

$$A_{16} = -\frac{G_c\omega^5}{S_c}\sin(\omega y) - \frac{G_r\omega^5}{P_r}\sin(\omega y) - G_cK_r\omega^2\sin(\omega y) - \delta G_r\omega^3\sin(\omega y)$$

$$A_{17} = G_cK_r\omega\sin(\omega y) - \delta G_r\omega\sin(\omega y) + \frac{G_c\omega^3}{S_c}\sin(\omega y) + \frac{G_r\omega^3}{P_r}\sin(\omega y)$$

$$A_{18} = \alpha G_cK_r\omega^2\cos(\omega y) - \alpha\delta G_r\omega^2\cos(\omega y) + \frac{\alpha G_c\omega^2\cos(\omega y)}{S_c} + \frac{\alpha G_r\omega^2\cos(\omega y)}{P_r}$$

$$A_{19} = 1 + \lambda + \cos(\omega y) + G_c(1 + \cos(\omega y)) + G_r(1 + \cos(\omega y))$$

$$A_{21} = \frac{\omega^4}{P_r}\cos(\omega y) - \frac{2\delta\omega^2}{P_r}\cos(\omega y) + \delta^2\cos(\omega y) + \delta^2 + \frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2$$

$$A_{22} = \frac{-\omega^6}{P_r^3}\cos(\omega y) + \frac{3\delta\omega^4}{P_r^2}\cos(\omega y) - \frac{3\delta^2\omega^2}{P_r}\cos(\omega y) + \delta^3\cos(\omega y) + \delta^3 - \frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3$$

$$A_{23} = \frac{\omega^8}{P_r^4}\cos(\omega y) - \frac{4\delta\omega^4}{P_r^3}\cos(\omega y) + \frac{6\delta^2\omega^4}{P_r^2}\cos(\omega y) - \frac{4\delta^2\omega^2}{P_r}\cos(\omega y) - \delta^4\cos(\omega y) + \delta^4 + \frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4 .$$

4 Numerical simulations/sensitivity analysis

Theoretical work on unsteady MHD Poiseuille flow of fourth-grade fluid in horizontal parallel plates channel with thermal radiation and chemical reaction effects has been analyzed and discussed in details. The impact of thermal radiation, chemical reaction, third and fourth-grade parameters along with other pertinent flow parameters are plotted graphically on different flow fields. The default values for the pertinent flow parameters are taken as [20];

$$\lambda = 0.30, \alpha = 0.20, \beta_a = 0.05, \beta_b = 0.05,\ \gamma_a = 0.05, \gamma_b = 0.05,\ S_c = 0.50, G_r = 5, G_c = 5, P_r = 0.71, Ha = 0.30, \delta = 0.05, Da = 1.00, K_r = 0.50.$$

To validate this present work, a numerical comparison is provided in Table 1.

Table 1. Validation of present study against [20] for velocity profile, when the suction parameter $S = 0$

δ	Velocity u (Present work)	Velocity u [20]	Difference
0.05	9.2474	9.2474	0.0001
0.10	9.2675	9.2678	0.0003
0.15	9.2874	9.2885	0.0011
0.20	9.3091	9.3094	0.0003

From Table 1, it can be seen there is no significant difference.

Figs 2 and **3** depict the velocity and temperature fields for increment of thermal radiation parameter δ $(1.00 \leq \delta \geq 4.00)$. Thermal radiation is known as electromagnetic radiation or the conversion of thermal energy which generates the thermal motion of particles in matter. Thermal radiation could be attributed due to thermal excitation. It is observed that both velocity and temperature fields are affected significantly with increase in thermal radiation parameter (δ). Thermal radiation for a medium which contains it inevitably has pressure and density gradients, and the treatment requires the use of hydrodynamics.

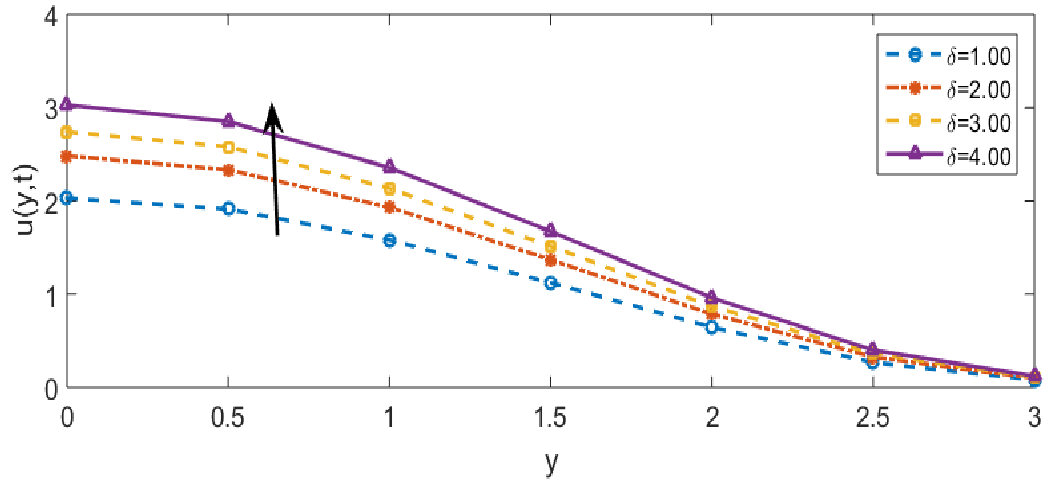

Fig. 2. Effect of thermal radiation parameter δ on Velocity profile u

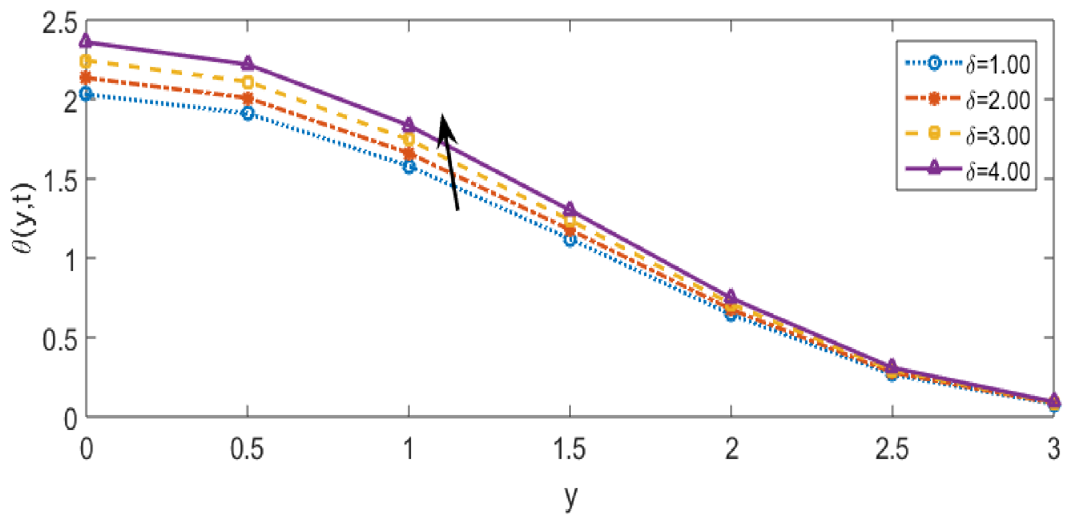

Fig. 3. Effect of thermal radiation parameter δ on temperature distribution θ

The effect of chemical reaction parameter (K_r) on velocity and concentration profiles are depicted in **Figs 4** and **5** respectively. Due to the rise of chemical reaction (K_r) from $1.00 \leq K_r \geq 4.00$, the velocity field decreases, and the concentration field also decreases. Physically, chemical reaction occurs with more disturbance which develops the molecular motion and upsurges the heat transport phenomena and as a result retards the velocity of the flow.

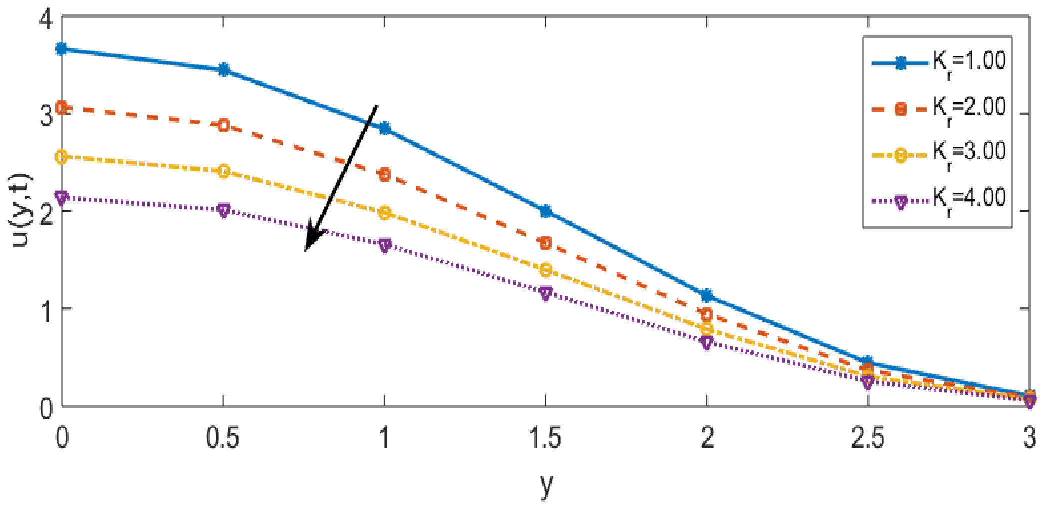

Fig. 4. Effect of chemical reaction parameter K_r on Velocity profile u

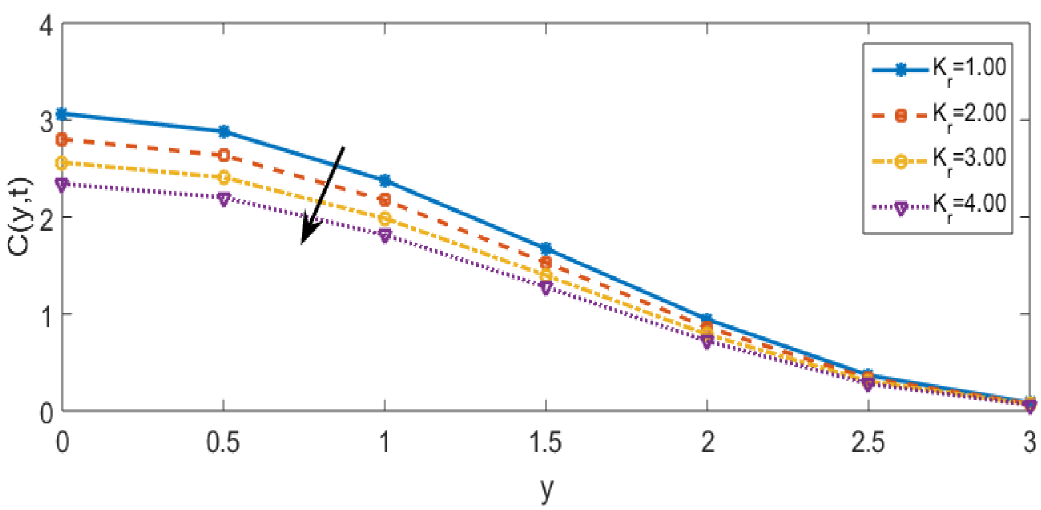

Fig. 5. Effect of chemical reaction parameter K_r on Concentration distribution C

Fig. 6 illustrates the drag force effect on fluid flow. The velocity profile decreases with the increment of Hartmann number $(1.00 \leq Ha \geq 4.00)$. The role of Hartmann number which is the magnetic parameter is to suppress turbulence. Physically, when magnetic field is applied to any fluid, the apparent viscosity of the fluid increases to the point of becoming viscous elastic solid. It is of great interest that yield stress of the fluid can be controlled very accurately through variation of the magnetic field intensity. The result is that the ability of the fluid to transmit force can be controlled with help of electromagnet which give rise to many possible control – based applications, including MHD power generation, electromagnetic casting of metals, MHD propulsion etc.

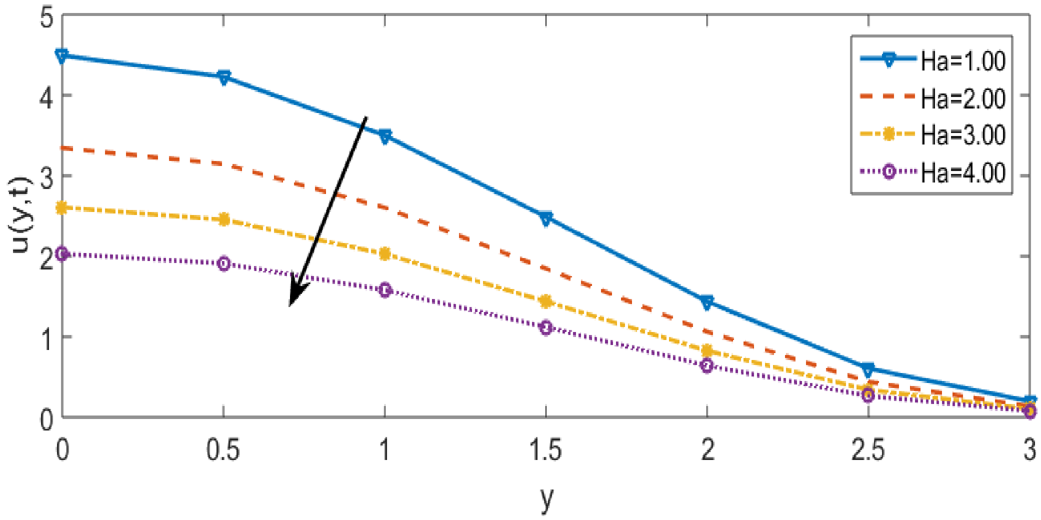

Fig. 6. Effect of Hartmann number Ha on Velocity profile u

The impression of third – grade and fourth – grade parameters on velocity profiles are respectively illustrated in **Figs 7** and **8**. It is observed that the velocity profile develops with the increase in both third – grade $(1.00 \leq (\beta_a = \beta_b) \geq 4.00)$ and fourth – grade $(1.00 \leq (\gamma_a = \gamma_b) \geq 4.00)$ parameters.

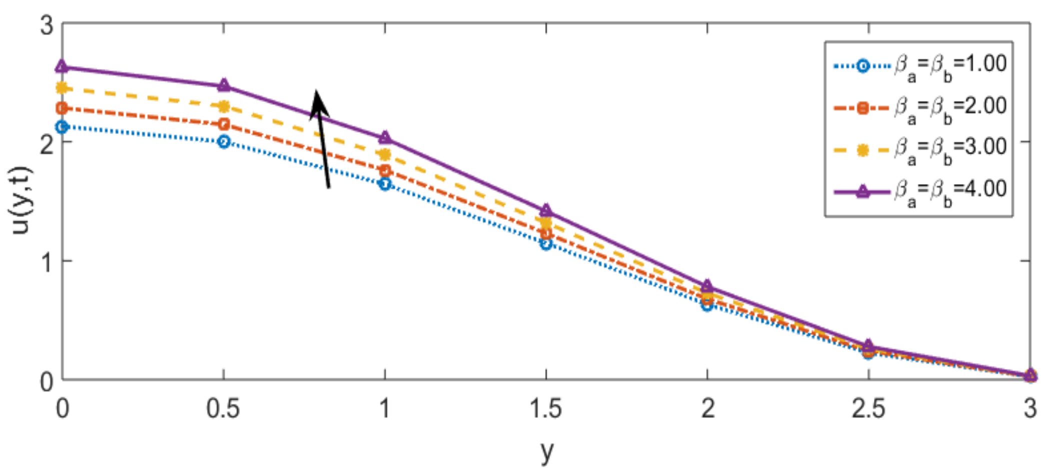

Fig. 7. Effect of β_a and β_b on Velocity profile u

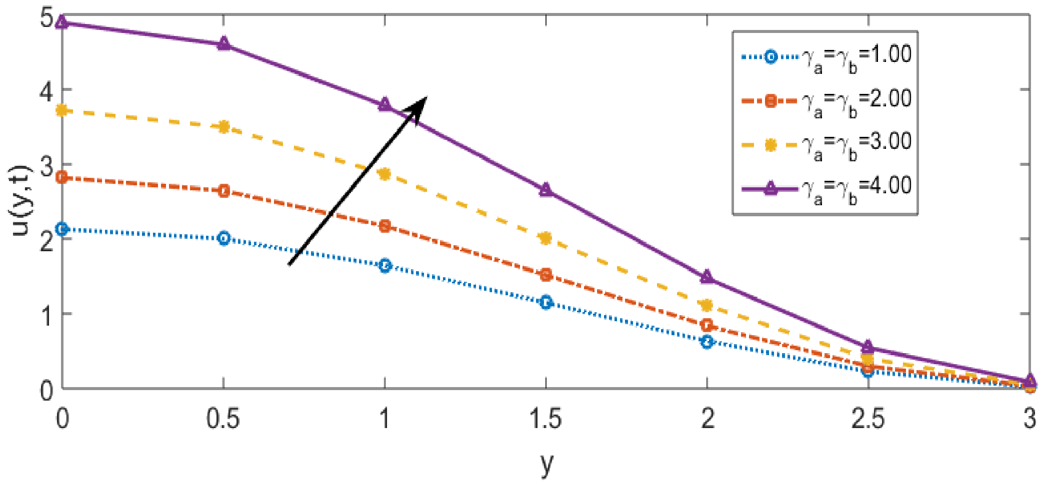

Fig. 8. Effect of γ_a and γ_b on Velocity profile u

Figs 9 and **10** show the impression of Prandtl number (P_r) on velocity and temperatures profiles respectively. The parameter (P_r) is the proportion of kinematic viscosity and thermal diffusivity which changes physically with temperature. For example, water $P_r = 7.0$ (at 20^0C) and Ammonia $P_r = 1.38$ decline more rapidly than air $P_r = 0.71$. However, increase in P_r depict the domination of thermal and momentum diffusivity respectively. Prandtl number is used to determine whether heat transport occurs with either conduction or convection process. Since, Prandtl number is inversely proportional to thermal diffusivity so that increasing P_r led to the decrease in velocity and temperature profiles.

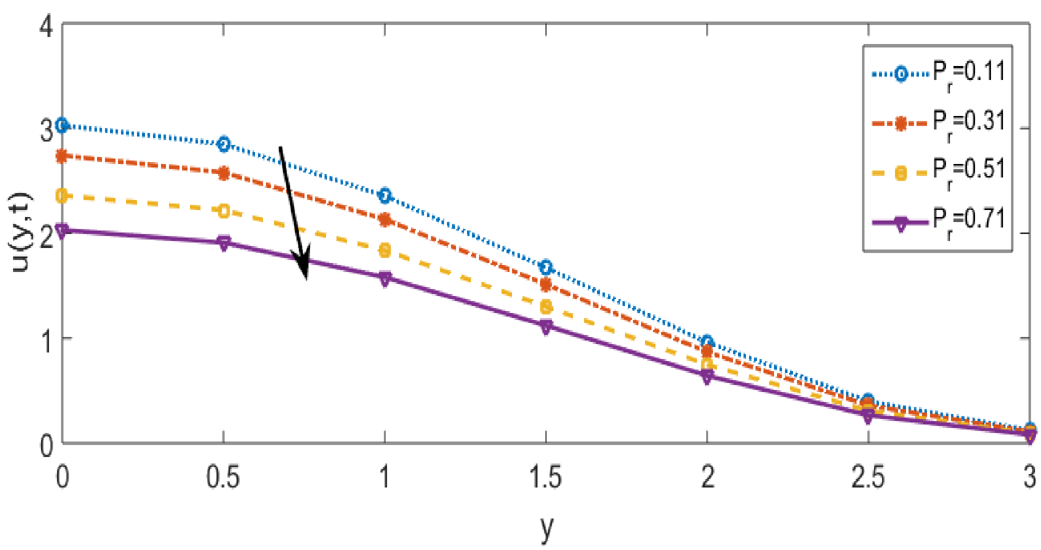

Fig. 9. Effect of Prandtl number P_r on Velocity profile u

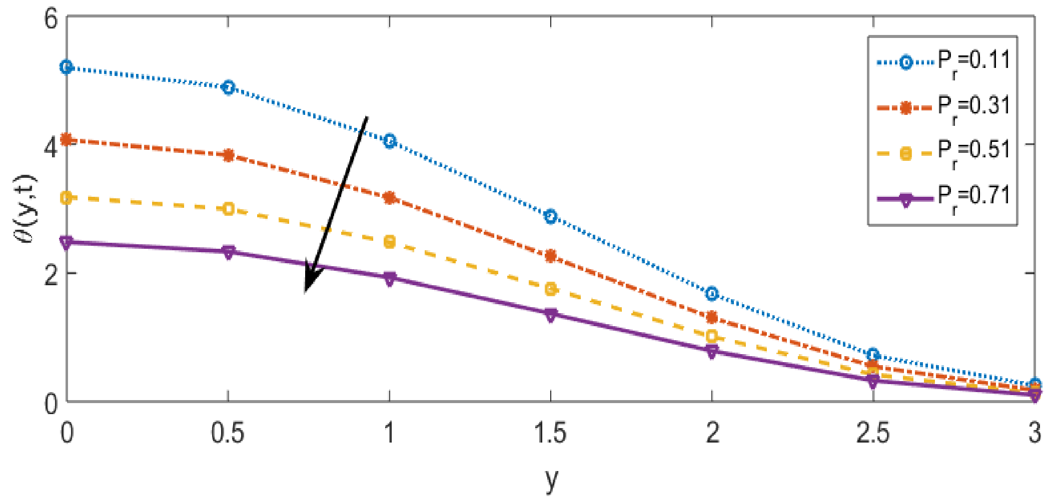

Fig. 10. Effect of Prandtl number P_r on Temperature distribution θ

Figs 11 and **12** display the velocity and concentration profiles respectively for the increment of Schmidt number $(1.00 \leq S_c \geq 4.00)$. Both the velocity and concentration profiles decrease with increase of Schmidt number (S_c). Physically, Schmidt number (S_c) helps to develop fluid concentration and concentration buoyancy force. Furthermore, it can also be used to improve the visualization of fluid fields.

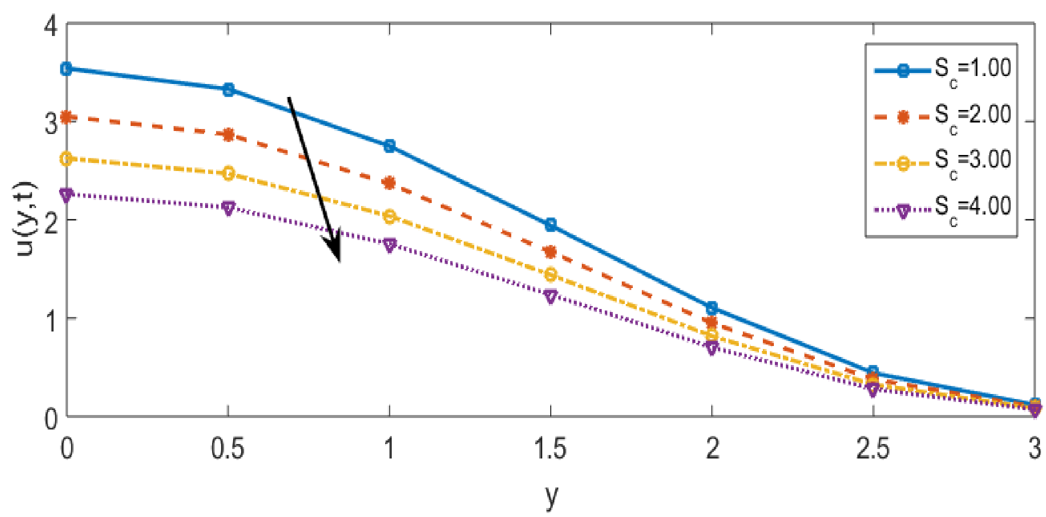

Fig. 11. Effect of Schmidt number S_c on Velocity profile u

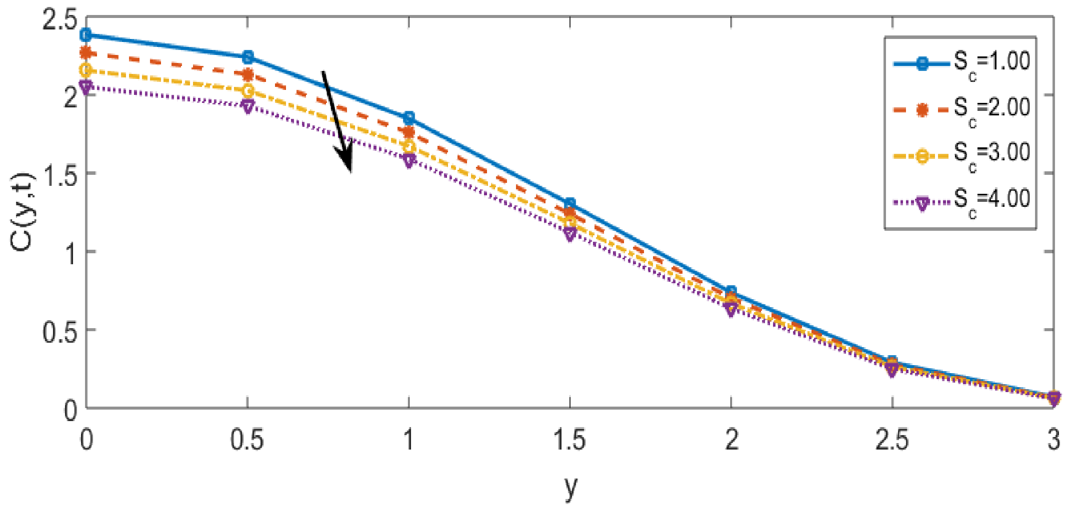

Fig. 12. Effect of Schmidt number S_c on Concentration distribution C

Since, the flow is driven by an externally imposed pressure gradient without motion of either plate, Negative and positive pressure gradients increase and decrease the flow rate, respectively. This is illustrated in **Fig. 13**. It is seen that increase in pressure gradient λ enhances the flow significantly. Pressure Gradient is a physical quantity that describes which direction and at what rate the pressure changes the most rapidly around a particular location.

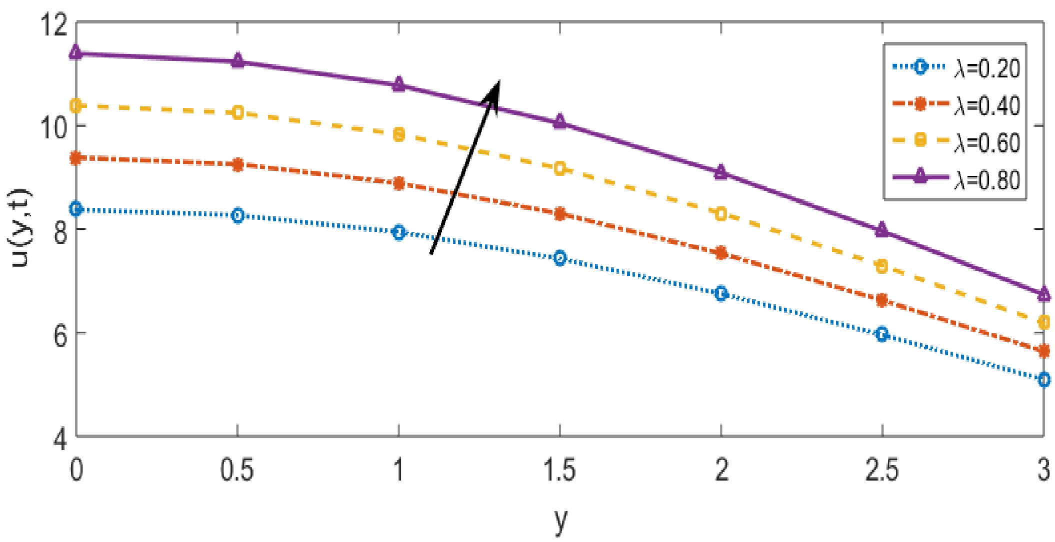

Fig. 13: Effect of Pressure gradient λ on Velocity profile u

5 Conclusion

The effect of thermal radiation and chemical reaction on unsteady MHD flow of fourth-grade fluid in horizontal parallel plates channel has been investigated. The solution for the nonlinear partial differential equations were obtained by the He-Laplace scheme. The effects of flow parameters on velocity, temperature and concentration profiles were depicted in figures and discussed. From the results obtained, the following conclusions are made:

(i) Velocity and temperature fields rise due to the increment of thermal radiation parameter.
(ii) For upsurging data of chemical reaction, velocity and concentration fields diminish.
(iii) Velocity, temperature and concentration fields diminish due to the increment of suction parameter.
(iv) Velocity profile goes up when third and fourth-grade parameters get to raise.
(v) Increasing Prandtl number tend to diminish the velocity and temperature profiles.
(vi) Strong values of Schmidt number decrease the magnitude of velocity and concentration distribution.
(vii) Velocity increases with increase in pressure gradient.

The results of this work are applicable in many industrial producing processes such in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films etc.

References

[1] Umavathi, J.C., Liu, I.C. and Meera, S. (2010). Unsteady flow and heat transfer of porous media sandwiched between viscous fluids. Applied Mathematics and Mechanics 31(12), 1497 – 1516.

[2] Satya, P.V., Venkateswarlu, B., and Devika B. (2015). Chemical reaction and heat source effect on MHD oscillatory flow in an irregular channel. Ain Shams Engineering Journal 7(4), 1 – 10.

[3] Rehan, A.S., Islam, S., and Siddiqui, A.M. (2010). Couette and poiseuille flows for fourth-grade fluids using optimal homotopy asymptotic method. World Applied Science Journal 9(1), 1228 – 1236.

[4] Islam, S., Bano, Z., Siddique, I., and Siddiqui, A.M. (2011). The optimal solutions for the flow of a fourth-grade fluid with partial slip. Computer and Mathematics with Application 6, 1507 – 1516.

[5] Shehzad, N., Zeeshan, A., and Ellahi, R. (2018). Electroosmotic flow of MHD power law Al2O3-PVC nanofluid in a horizontal channel: couette-poiseuille flow model. Commun. Theor. Phys. 6(6), 655 – 666.

[6] Zeeshan, K., Ilyas, K., Murad, U., and Tlili, I. (2018). Effect of thermal radiation and chemical reaction on non-newtonian fluid through a vertically stretching porous plate with suction. Results in Physics 9, 1086 – 1095.

[7] Santhosa, B., Younus S., Kamala G., and Ramana Murthy, MV. (2017). Radiation and chemical effects on MHD free convective heat and mass transfer flow of a viscoelastic fluid past a porous Plate with Heat Generation/Absorption. Int J Chem Sci.15(3), 1 – 16.

[8] Mohan RamiReddy, A., Ramana Reddy, J.V., Sandeep N., and Sugunamma, V., (2017). Effect of nonlinear thermal radiation on MHD chemically reacting Maxwell fluid flow past a linearly stretching sheet. Applications and Applied Mathematics: An International Journal 12(1), 259 – 274.

[9] Opanuga, A.A., Okagbue, H.I., and Agboola, O.O. (2017). Irreversibility analysis of a radiative MHD Poiseuille flow through porous medium with slip condition. Proceedings of the World Congress on Engineering 2017 Vol I WCE 2017, July 5-7, 2017, London, U.K.

[10] Sumathia, K., Arunachalamb, T., and Kavithac, R. (2017). Effect of thermal radiation and chemical reaction on three dimensional MHD fluid flow in porous medium – a numerical study. International Journal of Control Theory and Applications 10(32), 9 – 19.

[11] Hayat, T., Ellahi R., and Mahomed, F.M. (2007). Exact solutions for couette and poiseuille flows for fourth grade Fluids. Acta Mechanica 188, 69 – 78.

[12] Ambreen A.K., Syeda, R.B., Marin, M., Rahmat, E. (2019). Effects of chemical reaction on third-grade mhd fluid flow under the influence of heat and mass transfer with variable reactive index. Journals Heat Transfer Research 50, 1061 – 1080.

[13] Sajid, R., Taza, G., Waris, K., Aamir K., and Zeeshan, K. (2022). Effects of chemical reaction, viscosity, thermal conductivity, heat source, radiation/absorption, on MHD mixed convection nano-fluids flow over an unsteady stretching sheet by HAM and numerical method. Advances in Mechanical Engineering 14(1), 1–20.

[14] Famakinwa, O.A., Koriko, O.K., Adegbie, K.S. and Omowaye, O.J. (2022): Effects of viscous variation, thermal radiation, and Arrhenius reaction: The case of MHD nanofluid flow containing gyrotactic microorganisms over a convectively heated surface. Partial Differential Equations in Applied Mathematics 5, 1 – 7.

[15] Aarti, M., and Gorla, M.G. (2012). The effects of thermal radiation, chemical reaction and rotation on unsteady mhd viscoelastic slip flow. Global Journal of Science Frontier Research Mathematics and Decision Sciences 12(14), 1 – 15.

[16] Joshi, A. (2018). Exact Solution of Plane Poiseuille Fflow in a Channel Filled with a Porous Medium. Fluid Mech Res Int. 2(3), 131–136.

[17] Arifuzzaman, S.M, Shakhaoth, K.Md., Al-Mamum, A., Reza- E-Rabbi S.K, Biswas P., and Karim, I. (2018). Hydrodynamic stability and heat and mass transfer flow analysis of mhd radiative fourth-grade fluid through porous plate with chemical reaction. Journal of King Saud University 31(4), 1388 – 1398.

[18] Idowu, A.S. and Sani, U. (2019). Thermal radiation and chemical reaction effects on unsteady magnetohydrodynamic third-grade fluid flow between stationary and oscillating plates. Int. J. of Applied Mechanics and Engineering 24, 269 – 293.

[19] Priya, M., Mishra, S. R., Mahesh, B., Suthar, D. L., and Purohit, S. D. (2021). Computational behavior of second law poiseuille flow of micropolar fluids in a channel: analytical treatment. Journal of Mathematics, Hindawi 2021, 1 – 13.

[20] Joseph, K.M., Ayankop-Andi, E. and Mohammed, S.U. (2021). Unsteady MHD plane couette-poiseuille flow of fourth-grade fluid with thermal radiation, chemical reaction and suction effects. International Journal of Applied Mechanics and Engineering 26(4), 77 – 98.

[21] Coleman, B.D. and Noll, W. (1960). An approximation theorem for functionals with applications in continuum mechanics. Arch. Ration. Mech. Analy. 6, 355 – 370.

[22] Hayat, T., Wang, Y. and Hutter, K. (2002): Flow of a fourth-grade fluid. Math. Models Methods Appl. Sci. 12, 797 – 811.

[23] Hayat, T., Khan, M., Ayub, M., and Siddiqui, A.M. (2005): The Unsteady Couette Flow of a Second Grade Fluid in a Layer of Porous Medium. Archives of Mechanics 57, 405 – 416.

APPENDIX

Nomenclature

B_0 – external magnetic field
T – temperature of the fluid
C – species concentration
q_r – radiative heat flux
u – fluid velocity
C_p – specific heat capacity
Ha – Hartmann number
P_r – Prandtl number
G_r – Grashof number due to heat transfer
G_c – Grashof number due to mass transfer
K_r – chemical reaction parameter
S_c – Schmidt number
T_w – temperature at the surface
T_∞ – ambient temperature as $y \to \infty$
C_w – concentration at the surface
C_∞ – concentration as $y \to \infty$
x, y – cartesian coordinates

Greek Symbols

μ – coefficient of shear viscosity
α – second grade parameter
β_a, β_b – third grade parameters
γ_a, γ_b – fourth grade parameters
β – thermal expansion coefficient
β_c – concentration expan\sion coefficient
δ – thermal radiation parameter
σ – Stefan – Boltzmann constant
ρ – density of the fluid
ν – kinematic viscosity
λ - pressure gradient